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AN OPTIMAL CONTROL APPROACH TO ROBUST CONTROL OF NONLINEAR SPACECRAFT RENDEZVOUS SYSTEM WITH θ-D TECHNIQUE

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Abstract. *In this paper, a new optimal robust control design approach is proposed for the task of spacecraft rendezvous such that both the stabilization and optimal performance are achieved. An approximate closed-form solution to the nonlinear optimal control problem is obtained by employing the θ-D technique, which is derived based on an approximate solution to the Hamilton-Jacobi-Bellman equation obtained via a perturbation technique. This approach provides a simple design procedure, and the feedback control law obtained is easy for on-board implementation. The effectiveness of the proposed approach is demonstrated through simulation studies.*
Keywords: Spacecraft rendezvous, Optimal control, Robust control, Uncertainties

1. **Introduction.** The efficient execution of a rendezvous manoeuvre has been well recognized as an essential component of various types of space missions, such as intercepting, repair, rescue, docking, large-scale structure assembling and satellite networking [1]. During the last few decades, the spacecraft rendezvous control problems have attracted considerable attention and many design approaches have been developed. For example, Lyapunov differential equation approach is used in [2] to solve spacecraft rendezvous on elliptical orbit, where the control is subject to constraints; the annealing algorithm is utilized in [3] to design an orbital controller for the spacecraft rendezvous system; the nondominated sorting genetic algorithm is employed in [4] to study the optimal impulsive control of a rendezvous manoeuvre; the linear matrix inequality approach is proposed in [5, 6] to solve the robust H_∞ control problem of spacecraft rendezvous. However, in most of the papers mentioned above, linearized models are used to describe the relative motion of the spacecraft rendezvous. These linearized equations are valid only if the inter-spacecraft distance is small. If the inter-spacecraft distance or the duration for the execution of rendezvous manoeuvre is large, the linearized models are no longer valid. In these situations, the use of nonlinear models is inevitable. For a nonlinear spacecraft rendezvous model, it depends critically on the angular velocity of the target spacecraft.

However, its accuracy can be affected by many uncertainty factors such as external perturbations and detection errors. On the other hand, mass variation and/or detection errors of the thrusters being applied will cause inaccuracy in the calculation of the control input. Due to these uncertainties, the stability or even the safety of the rendezvous mission could be compromised. Some studies on problems of uncertain systems have been reported in the literature (see, for example, [4, 5, 6, 7, 8]). However, the parameter uncertainties have not been taken into consideration in the existing literature on the nonlinear control problem of spacecraft rendezvous. In [9], a robust control problem in state space representation is formulated under the matching condition, where the uncertainty is state dependent. This robust control problem is transformed into an optimal control problem, where the uncertainty is incorporated in the cost function. It is shown in [9] that if the solution to the optimal control problem exists, then it is also a solution to the robust control problem. This approach has two main advantages: (i) The solution obtained is guaranteed to be robust; and (ii) it allows for a trade-off between fast response time and small control input through appropriate adjustment of the relative weights of the state and control in the cost function. The approach has been successfully applied to robust control of robot manipulators in [10] and robust hovering control of a planar vertical take-off and landing (PVTOL) aircraft in [11]. Unfortunately, as it is pointed out in [12], the resulting nonlinear optimal control problem obtained by using this approach is still very difficult to solve. Linearization of the nonlinear PVTOL aircraft dynamics about a small roll angle is carried out in [11]. In this way, a closed form solution can be obtained by using linear quadratic regulator (LQR) method. For nonlinear uncertain systems, the inverse optimal control approach is used in [13], where a modified cost function, instead of the given cost function, is to be minimized. Thus, the task of solving a complicated optimal problem is being avoided. However, the modified cost function does not have any physical meaning, and there is no systematic way to construct the Lyapunov function that is needed in this inverse optimal control design process.

In this paper, we consider a nonlinear robust control problem of spacecraft rendezvous with parametric and control input uncertainties. The robust control approach proposed in [9] is used to find a robust control for the nonlinear rendezvous spacecraft system. This robust control problem is then transformed into an optimal control problem, where the uncertainty is incorporated into the cost function. As it is mentioned above, this optimal control problem is still very difficult to solve. In this paper, a recently proposed approach, called the θ-D approach, reported in [14] is employed to design the optimal robust control for the spacecraft rendezvous. The θ-D technique is derived based on an approximate solution to the Hamilton-Jacobi-Bellman (HJB) equation obtained via a perturbation process. This approach has three advantages: (i) It enhances the system performance, where robustness and optimality are being integrated in a unified optimal control framework; (ii) we can obtain a nonlinear feedback controller by taking only a finite number of terms in the series expansion of the solution to the HJB equation; and (iii) this approach does not require excessive computational load, such as in the case of the state dependent Riccati equation (SDRE) approach proposed in [15], and hence it is easy for on-board implementation.

The rest of this paper is organized as follows. In Section 2, the dynamic model for spacecraft rendezvous is presented. Then, the robust control design problem is formulated. In Section 3, an optimal control approach to robust control problem is proposed. Based on the results obtained in Section 3, the θ-D technique is employed for the design of the robust control for spacecraft rendezvous in Section 4. To illustrate the applicability of the approach proposed, an example is given and solved in Section 5. Finally, some concluding remarks are made in Section 6.

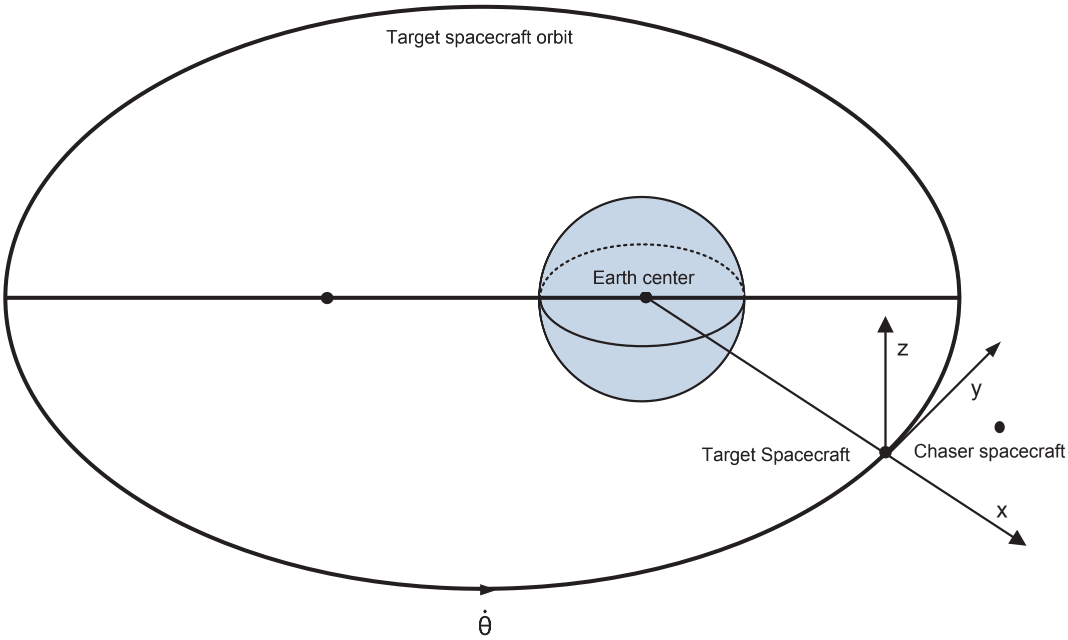

FIGURE 1. Spacecraft rendezvous system

2. **Dynamic Model and Problem Formulation.** The spacecraft rendezvous system is depicted in Figure 1. We assume that the target spacecraft is moving in a circular orbit, and that the orbital coordinate frame is a right-handed Cartesian coordinate, where the origin is attached to the center of the mass of the target spacecraft. The x-axis is along the vector from the earth center to the center of the mass of the target spacecraft, the y-axis is along the target orbit circumference, and the z-axis is obtained by taking the cross product $y \times x$. The relative dynamic model is described by the nonlinear equations (see [16]) given below:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 2\dot{\theta}\dot{y} + \dot{\theta}^2 x + a(x,y,z)(R+x) + \frac{\mu}{R^2} \\ \dot{\theta}^2 y - 2\dot{\theta}\dot{x} + a(x,y,z)y \\ a(x,y,z)z \end{bmatrix} + \boldsymbol{a}_f \tag{1}$$

where x, y and z are the components of the relative position; $\dot{x}$, $\dot{y}$ and $\dot{z}$ denote, respectively, the first order derivatives of x, y and z; $\ddot{x}$, $\ddot{y}$ and $\ddot{z}$ denote their respective second order time derivatives; R is the orbital radius of the target spacecraft; μ is the gravitational constant of the earth; $\dot{\theta} = \sqrt{\frac{\mu}{R^3}}$ is the angular velocity of the target spacecraft moving in its orbit around the earth; $\boldsymbol{a}_f$ is referred to as the control applied to the chaser spacecraft; and $a(x,y,z) = -\frac{\mu}{[(r+x)^2+y^2+z^2]^{\frac{3}{2}}}$. For brevity, $a(x,y,z)$ will be abbreviated as a.

In practice, there exist continuous external perturbations and errors in detection. Thus, $\dot{\theta}$ is difficult to be determined accurately, and hence the parametric uncertainties $\Delta\dot{\theta}$ should not be ignored. On the other hand, mass variation and uncertainty of the thrusters being applied will cause inaccuracy in the calculation of the control input. Thus, the uncertainty $\Delta\boldsymbol{a}_f$ of the control input $\boldsymbol{a}_f$ should also be considered. When these uncertainties are incorporated in system (1), we obtain

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} 2n\dot{y} + n^2 x + a(R+x) + \frac{\mu}{R^2} \\ n^2 y - 2n\dot{x} + ay \\ az \end{bmatrix} + \boldsymbol{u}_f \tag{2}$$

where $n = \dot{\theta} + \Delta\dot{\theta}$, $\boldsymbol{u}_f = \boldsymbol{a}_f + \Delta\boldsymbol{a}_f$, while $\Delta\dot{\theta}$ and $\Delta\boldsymbol{a}_f$ denote the respective bounded disturbances.

To ensure that the origin is an equilibrium point of system (2), we introduce a new variable s, satisfying

$$\dot{s} = -\lambda s,$$

where λ is a positive number.

Let

$$\boldsymbol{x} = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} & s \end{bmatrix}^T, \quad \boldsymbol{u} = \begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^T.$$

Then,

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + B\boldsymbol{u} + B\Delta_b(\boldsymbol{x})\boldsymbol{u} + \Delta_f(\boldsymbol{x}), \tag{3}$$

where

$$f(\boldsymbol{x}) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \dot{\theta}^2 + a & 0 & 0 & 0 & 2\dot{\theta} & 0 & b \\ 0 & \dot{\theta}^2 + a & 0 & -2\dot{\theta} & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda \end{bmatrix} \boldsymbol{x},$$

$$\Delta_f(\boldsymbol{x}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \Delta\delta & 0 & 0 & 0 & 2\Delta\dot{\theta} & 0 & 0 \\ 0 & \Delta\delta & 0 & -2\Delta\dot{\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{x},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T, \quad \Delta_b(\boldsymbol{x}) = \begin{bmatrix} \Delta u_x & 0 & 0 \\ 0 & \Delta u_y & 0 \\ 0 & 0 & \Delta u_z \end{bmatrix},$$

$b = \frac{R^3 a + \mu}{sR^2}$ and $\Delta\delta = 2\dot{\theta}\Delta\dot{\theta} + (\Delta\dot{\theta})^2$.

Based on system (3), our objective is to determine a control input $\boldsymbol{u}$, such that the resulting system (3) is robustly stable.

3. **An Optimal Control Approach to Robust Control Problem.**

In [9], it is shown that finding a robust stabilizing controller under uncertainties can be achieved via solving an optimal control problem with an appropriately modified cost function. However, the results obtained in [9] do not apply to the case of the robust spacecraft control problem under consideration. The reason is that the control input uncertainty is not taken into consideration in [9]. In this paper, we shall derive new results, which extend those obtained in [9], for a more general case, where both parameter uncertainties and control input uncertainties are incorporated in the system of the robust spacecraft control problem.

Optimal Control Problem (OCP): Given the nonlinear system

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + B(\boldsymbol{x})\boldsymbol{u} + (I - B(\boldsymbol{x})B^+(\boldsymbol{x}))\boldsymbol{v} \tag{4}$$

find a feedback control law $\boldsymbol{u}$ and $\boldsymbol{v}$ such that the cost function:

$$J = \int_0^{+\infty} (2f_{\max}(\boldsymbol{x}) + 2b_{\max}(\boldsymbol{u}) + \boldsymbol{x}^T\Theta\boldsymbol{x} + \boldsymbol{u}^T\Omega\boldsymbol{u} + \rho^2\|\boldsymbol{v}\|^2)\mathrm{d}t \tag{5}$$

is minimized, where Θ and Ω are positive definite matrices, while $f_{\max}(\boldsymbol{x})$ and $b_{\max}(\boldsymbol{x})$ are given upper bounds for uncertainties $\Delta_f(\boldsymbol{x})$ and $\Delta_b(\boldsymbol{x})$ appearing in the following conditions.

$$\Delta_f^T(\boldsymbol{x})[\rho^2 I + (B^+(\boldsymbol{x}))^T\Omega B^+(\boldsymbol{x})]\Delta_f(\boldsymbol{x}) \le f_{\max}(\boldsymbol{x}) \tag{6a}$$

$$\boldsymbol{u}^T\Delta_b^T(\boldsymbol{x})C^T(\boldsymbol{x})[\rho^2 I + (B^+(\boldsymbol{x}))^T\Omega B^+(\boldsymbol{x})]C(\boldsymbol{x})\Delta_b(\boldsymbol{x})\boldsymbol{u} \le b_{\max}(\boldsymbol{x}) \tag{6b}$$

$$2\rho^2\|\boldsymbol{v}\|^2 < \boldsymbol{x}^T\Theta\boldsymbol{x} \tag{6c}$$

Robust Control Problem (RCP): Consider the nonlinear system

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + B(\boldsymbol{x})\boldsymbol{u} + C(\boldsymbol{x})\Delta_b(\boldsymbol{x})\boldsymbol{u} + \Delta_f(\boldsymbol{x}) \tag{7}$$

where $\Delta_b(\boldsymbol{x})$, $\Delta_f(\boldsymbol{x})$ are bounded uncertainties satisfying (6a) and (6b), and $f(0) = 0$, $\Delta_f(0) = 0$. Find a feedback control law $\boldsymbol{u}$ such that the closed-loop system (7) is globally asymptotically stable for all admissible uncertainties (i.e., for all bounded uncertainties $\Delta_b(\boldsymbol{x})$ and $\Delta_f(\boldsymbol{x})$ satisfying (6a) and (6b).

The following theorem establishes the relationship between (OCP) and (RCP).

Theorem 3.1. *If $\boldsymbol{u}$, $\boldsymbol{v}$ are an optimal feedback control law to (OCP), then $\boldsymbol{u}$ is a robust globally asymptotically stabilizing feedback control law to (RCP).*

Proof: For notational simplicity, the explicit reference to $\boldsymbol{x}$ for a function of $\boldsymbol{x}$ will be dropped when no confusion can arise. Consider (OCP). Define the value function $V(\boldsymbol{x})$ given by

$$V(\boldsymbol{x}) = \min_{\boldsymbol{u},\boldsymbol{v}} \int_0^\infty (2f_{\max}(\boldsymbol{x}) + 2b_{\max}(\boldsymbol{x}) + \boldsymbol{x}^T\Theta\boldsymbol{x} + \boldsymbol{u}^T\Omega\boldsymbol{u} + \rho^2\|\boldsymbol{v}\|^2)\mathrm{d}t.$$

From the HJB equation and the optimality condition (see [17]), it follows that

$$2f_{\max} + 2b_{\max} + \boldsymbol{x}^T\Theta\boldsymbol{x} + \boldsymbol{u}^T\Omega\boldsymbol{u} + \rho^2\|\boldsymbol{v}\|^2 + V_{\boldsymbol{x}}^T[f + B\boldsymbol{u} + (I - BB^+)\boldsymbol{v}] = 0 \tag{8a}$$

$$2\boldsymbol{u}^T\Omega + V_{\boldsymbol{x}}^T B = 0 \tag{8b}$$

$$2\rho^2\boldsymbol{v}^T + V_{\boldsymbol{x}}^T(I - BB^+) = 0 \tag{8c}$$

It remains to show that $\boldsymbol{u}$ is a solution to (RCP), i.e., the equilibrium $\boldsymbol{x} = 0$ of system (7) under the control $\boldsymbol{u}$ is globally asymptotically stable for all admissible uncertainties. To do this, we shall show that $V(\boldsymbol{x})$ is a Lypunov function. Clearly, $V(\boldsymbol{x}) = 0$ for $\boldsymbol{x} = 0$ and $V(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \ne 0$. We need to show that $\dot{V}(\boldsymbol{x}) < 0$ for $\boldsymbol{x} \ne 0$. From (8), it follows that

$$\begin{aligned}
\dot{V}(\boldsymbol{x}) &= V_{\boldsymbol{x}}^T\dot{\boldsymbol{x}} \\
&= V_{\boldsymbol{x}}^T(f + B\boldsymbol{u} + C\Delta_b\boldsymbol{u} + \Delta_f) \\
&= V_{\boldsymbol{x}}^T[f + B\boldsymbol{u} + (I - BB^+)\boldsymbol{v}] - V_{\boldsymbol{x}}^T(I - BB^+)\boldsymbol{v} + V_{\boldsymbol{x}}^T C\Delta_b\boldsymbol{u} + V_{\boldsymbol{x}}^T\Delta_f \\
&= V_{\boldsymbol{x}}^T[f + B\boldsymbol{u} + (I - BB^+)\boldsymbol{v}] + 2\rho^2\|\boldsymbol{v}\|^2 + V_{\boldsymbol{x}}^T C\Delta_b\boldsymbol{u} + V_{\boldsymbol{x}}^T\Delta_f \\
&= V_{\boldsymbol{x}}^T[f + B\boldsymbol{u} + (I - BB^+)\boldsymbol{v}] + 2\rho^2\|\boldsymbol{v}\|^2 + V_{\boldsymbol{x}}^T(I - BB^+)C\Delta_b\boldsymbol{u} \\
&\quad + V_{\boldsymbol{x}}^T BB^+C\Delta_b\boldsymbol{u} + V_{\boldsymbol{x}}^T(I - BB^+)\Delta_f + V_{\boldsymbol{x}}^T BB^+\Delta_f \\
&= V_{\boldsymbol{x}}^T[f + B\boldsymbol{u} + (I - BB^+)\boldsymbol{v}] + 2\rho^2\|\boldsymbol{v}\|^2 - 2\rho^2\boldsymbol{v}^T C\Delta_b\boldsymbol{u} \\
&\quad - 2\boldsymbol{u}^T\Omega B^+C\Delta_b\boldsymbol{u} - 2\rho^2\boldsymbol{v}^T\Delta_f - 2\boldsymbol{u}^T\Omega B^+\Delta_f \\
&= -2f_{\max} - 2b_{\max} - \boldsymbol{x}^T\Theta\boldsymbol{x} - \boldsymbol{u}^T\Omega\boldsymbol{u} + \rho^2\|\boldsymbol{v}\|^2 - 2\rho^2\boldsymbol{v}^T C\Delta_b\boldsymbol{u} \\
&\quad - 2\boldsymbol{u}^T\Omega B^+C\Delta_b\boldsymbol{u} - 2\rho^2\boldsymbol{v}^T\Delta_f - 2\boldsymbol{u}^T\Omega B^+\Delta_f.
\end{aligned} \tag{9}$$

Since $\Omega > 0$, there exists a $\Phi > 0$ such that $\Omega = \Phi^2$. It is clear that

$$0 \le \left(\frac{1}{\sqrt{2}}\Phi\boldsymbol{u} + \sqrt{2}\Phi B^+C\Delta_b\boldsymbol{u}\right)^T\left(\frac{1}{\sqrt{2}}\Phi\boldsymbol{u} + \sqrt{2}\Phi B^+C\Delta_b\boldsymbol{u}\right),$$

i.e.,

$$-2(\Phi\boldsymbol{u})^T\Phi B^+C\Delta_b\boldsymbol{u} \leq \frac{1}{2}(\Phi\boldsymbol{u})^T(\Phi\boldsymbol{u}) + 2(\Phi B^+C\Delta_b\boldsymbol{u})^T(\Phi B^+C\Delta_b\boldsymbol{u}). \tag{10}$$

Thus, (10) can be written as:

$$-2\boldsymbol{u}^T\Omega B^+C\Delta_b\boldsymbol{u} \leq \frac{1}{2}\boldsymbol{u}^T\Omega\boldsymbol{u} + 2\boldsymbol{u}^T\Delta_b^TC^T(B^+)^T\Omega B^+C\Delta_b\boldsymbol{u}. \tag{11}$$

Similarly,

$$-2\boldsymbol{u}^T\Omega B^+\Delta_f \leq \frac{1}{2}\boldsymbol{u}^T\Omega\boldsymbol{u} + 2\Delta_f^T(B^+)^T\Omega B^+\Delta_f. \tag{12}$$

From (11) and (12), we obtain

$$-2\boldsymbol{u}^T\Omega B^+C\Delta_b - 2\boldsymbol{u}^T\Omega B^+\Delta_f \leq \boldsymbol{u}^T\Omega\boldsymbol{u} + 2\boldsymbol{u}^T\Delta_b^TC^T(B^+)^T\Omega B^+C\Delta_b\boldsymbol{u} + 2\Delta_f^T(B^+)^T\Omega B^+\Delta_f. \tag{13}$$

In view of the proof given to establish the validity (13), it is easy to deduce that

$$-2\boldsymbol{v}^TC\Delta_b\boldsymbol{u} \leq \frac{1}{2}\|\boldsymbol{v}\|^2 + 2\boldsymbol{u}^T\Delta_b^TC^TC\Delta_b\boldsymbol{u}, \quad -2\boldsymbol{v}^T\Delta_f \leq \frac{1}{2}\|\boldsymbol{v}\|^2 + 2\Delta_f^T\Delta_f. \tag{14}$$

From (14), it follows that

$$-2\boldsymbol{v}^T\Delta_f - 2\boldsymbol{v}^TC\Delta_b\boldsymbol{u} \leq \|\boldsymbol{v}\|^2 + 2\boldsymbol{u}^T\Delta_b^TC^TC\Delta_b\boldsymbol{u} + 2\Delta_f^T\Delta_f \tag{15}$$

According to (6), (9), (13), (15), and $\Theta > 0$, we have

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &= -2f_{\max}(\boldsymbol{x}) - 2b_{\max}(\boldsymbol{x}) - \boldsymbol{x}^T\Theta\boldsymbol{x} - \boldsymbol{u}^T\Omega\boldsymbol{u} + \rho^2\|\boldsymbol{v}\|^2 - 2\rho^2\boldsymbol{v}^TC\Delta_b\boldsymbol{u} \\ &\quad -2\boldsymbol{u}^T\Omega B^+C\Delta_b\boldsymbol{u} - 2\rho^2\boldsymbol{v}^T\Delta_f - 2\boldsymbol{u}^T\Omega B^+\Delta_f \\ &\leq -2f_{\max}(\boldsymbol{x}) - 2b_{\max}(\boldsymbol{x}) - \boldsymbol{x}^T\Theta\boldsymbol{x} + 2\boldsymbol{u}^T\Delta_b^TC^T(B^+)^T\Omega B^+C\Delta_b\boldsymbol{u} \\ &\quad +2\Delta_f^T(B^+)^T\Omega B^+\Delta_f + 2\rho^2\boldsymbol{u}^T\Delta_b^TC^TC\Delta_b\boldsymbol{u} + 2\rho^2\Delta_f^T\Delta_f + 2\rho^2\|\boldsymbol{v}\|^2 \\ &< 0.\end{aligned}$$

This implies that $V(\boldsymbol{x})$ is a Lyapunov function. Thus, the closed-loop system (7) is globally asymptotically stable for all admissible uncertainties.

Corollary 3.1. *Consider the nonlinear system*

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + B(\boldsymbol{x})\boldsymbol{u} + B(\boldsymbol{x})\Delta_b(\boldsymbol{x})\boldsymbol{u} + \Delta_f(\boldsymbol{x}) \tag{16}$$

where $f(0) = 0$ and $\Delta_f(0) = 0$. The feedback control law $\boldsymbol{u}$, under which the closed-loop system (16) is globally asymptotically stable with uncertainties $\Delta_b(\boldsymbol{x})$ and $\Delta_f(\boldsymbol{x})$, can be obtained from solving the optimal control problem given below:

Find a feedback control law $\boldsymbol{u}$ such that the cost function:

$$J = \int_0^{+\infty}(2f_{\max}(\boldsymbol{x}) + 2b_{\max}(\boldsymbol{u}) + \boldsymbol{x}^T\Theta\boldsymbol{x} + \boldsymbol{u}^T\Omega\boldsymbol{u})\mathrm{d}t \tag{17}$$

is minimized subject to

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + B\boldsymbol{u}, \tag{18}$$

where Θ and Ω are positive definite matrices, while $f_{\max}(\boldsymbol{x})$ and $b_{\max}(\boldsymbol{x})$ are given upper bounds for the uncertainties $\Delta_f(\boldsymbol{x})$ and $\Delta_b(\boldsymbol{x})$ satisfying

$$\Delta_f^T(\boldsymbol{x})\Omega\Delta_f(\boldsymbol{x}) \leq f_{\max}(\boldsymbol{x}) \quad \text{and} \quad \boldsymbol{u}^T\Delta_b^T(\boldsymbol{x})\Omega\Delta_b(\boldsymbol{x})\boldsymbol{u} \leq b_{\max}(\boldsymbol{x}). \tag{19}$$

Proof: The proof is similar to that given for Theorem 3.1.

Remark 3.1. *It is worth pointing out that Theorem 3.1 is presented in its most general form, which covers the specific spacecraft rendezvous problem considered in this paper as a special case. It is applicable to many other applications, such as spacecraft orbit transfer and satellite attitude control.*

4. **Robust Control Design via $\boldsymbol{\theta}$-$\boldsymbol{D}$ Technique.** In Section 3, we see that finding a robust control law for a nonlinear (RCP) can be achieved via solving a nonlinear (OCP). However, the optimal control problem involving a nonlinear dynamic system is very difficult to solve by using the optimality condition. The reason is that the form of the optimal feedback control depends on the solution of the Hamilton-Jacobi-Bellman (HJB) equation (see [17]). The HJB equation is known to be extremely difficult to solve when the system dynamic is nonlienar, even for problems with moderate dimension. In this paper, we employ the θ-D technique to find an approximate analytical solution via a perturbation process. This approach will provide a closed-form suboptimal feedback control law. It overcomes the undesirable situation of large-control for large-initial-state which is encountered when Taylor series expansion approach is used. Furthermore, the computational load for this approach is much less excessive when compared with the popular state dependent Riccati equation (SDRE) technique reported in [15]. Consequently, it is much more suitable for on-board implementation.

From (19) and the forms of $\Delta_f(\boldsymbol{x})$ and $\Delta_b(\boldsymbol{x})$ given in (3), we assume that

$$f_{\max}(\boldsymbol{x}) = \boldsymbol{x}^T\Lambda_f\boldsymbol{x}, \quad b_{\max}(\boldsymbol{x}) = \boldsymbol{u}^T\Lambda_b\boldsymbol{u}, \tag{20}$$

where Λ_f and Λ_b are positive definite matrices. Let $Q = 4\Lambda_f + 4\Lambda_b + 2\Theta$, and let $R = 2\Omega$. Then, (OCP) given by (16) and (17) can be rewritten as:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + B\boldsymbol{u} \tag{21}$$

with the cost function

$$J = \frac{1}{2}\int_0^\infty \boldsymbol{x}^TQ\boldsymbol{x} + \boldsymbol{u}^TR\boldsymbol{u}\mathrm{d}t, \tag{22}$$

where $\boldsymbol{x} \in \Omega \subset R^7$ and Ω is a compact subset, $f : \Omega \to R^7$ is continuously differentiable and $f(0) = 0$, $B \in R^{n\times m}$ is a constant matrix, $u : \Omega \to R^3$ is a control input, and $Q \in R^{7\times 7}$ and $R \in R^{3\times 3}$ are positive definite matrices.

The optimal solution to this infinite time nonlinear regulator problem can be obtained via solving the following HJB equation (see [17]):

$$\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}f(\boldsymbol{x}) - \frac{1}{2}\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}}^T BR^{-1}B^T\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} + \frac{1}{2}\boldsymbol{x}^TQ\boldsymbol{x} = 0 \tag{23}$$

where $V(\boldsymbol{x})$ is the value function, i.e.,

$$V(\boldsymbol{x}) = \min_{\boldsymbol{u}} \frac{1}{2}\int_0^\infty \boldsymbol{x}^TQ\boldsymbol{x} + \boldsymbol{u}^TR\boldsymbol{u}\mathrm{d}t.$$

From the optimality condition (8b), it follows that the optimal control is given by

$$\boldsymbol{u} = -R^{-1}B^T\left(\frac{\partial V}{\partial \boldsymbol{x}}\right)^T \tag{24}$$

However, the HJB Equation (23) for a nonlinear optimal control problem is known to be very difficult to solve even for the case when the dynamic system is of moderate dimension. The θ-D technique will give an approximate closed-form solution by introducing a perturbation to the cost function as follows:

$$J_p = \frac{1}{2}\int_0^\infty \boldsymbol{x}^T\left(Q + \sum_{i=1}^\infty D_i\theta^i\right)\boldsymbol{x} + \boldsymbol{u}^TR\boldsymbol{u}\mathrm{d}t, \tag{25}$$

where $\sum\limits_{i=1}^\infty D_i\theta^i$ is a perturbation series in terms of an instrumental variable θ. The construction of this series will be discussed later. Rewrite the state Equation (21) in a linear factorization structure,

$$\begin{aligned}\dot{\boldsymbol{x}} &= f(\boldsymbol{x}) + B\boldsymbol{u} = F(\boldsymbol{x})\boldsymbol{x} + B\boldsymbol{u} \\ &= \left(A_0 + \theta\frac{A(\boldsymbol{x})}{\theta}\right)\boldsymbol{x} + B\boldsymbol{u}\end{aligned} \tag{26}$$

where $A_0 = F(\boldsymbol{x}_0) - \gamma I$ is a constant matrix, $\gamma > 0$ is an adjustable parameter, and $A(\boldsymbol{x}) = F(\boldsymbol{x}) - A_0$. Then, the new optimal control problem given by (25) and (26) can be solved through solving the perturbed HJB equation:

$$\frac{\partial V^T}{\partial \boldsymbol{x}}\left[A_0 + \theta\frac{A(\boldsymbol{x})}{\theta}\right]\boldsymbol{x} - \frac{1}{2}\frac{\partial V^T}{\partial \boldsymbol{x}}BR^{-1}B^T\frac{\partial V}{\partial \boldsymbol{x}} + \frac{1}{2}\boldsymbol{x}^T\left(Q + \sum_{i=1}^{\infty} D_i\theta^i\right)\boldsymbol{x} = 0 \tag{27}$$

By applying a power series expansion given below:

$$\frac{\partial V}{\partial \boldsymbol{x}} = \sum_{i=0}^{\infty} T_i\theta^i\boldsymbol{x},$$

the optimal control can be written as:

$$\boldsymbol{u} = -R^{-1}B^T\sum_{i=0}^{\infty} T_i\theta^i\boldsymbol{x} \tag{28}$$

where for each $i = 0, 1, \ldots, n, \ldots, T_i$ is a symmetric matrix. They are solved recursively as described below. First, substitute

$$\frac{\partial V}{\partial \boldsymbol{x}} = \sum_{i=0}^{\infty} T_i\theta^i\boldsymbol{x}$$

in the perturbed HJB Equation (27) and equating the coefficients of θ^i to zero. This gives

$$T_0A_0 + A_0^TT_0 - T_0BR^{-1}B^TT_0 + Q = 0 \tag{29a}$$

$$T_1\Psi + \Psi^TT_1 = \Xi_0(\theta, \boldsymbol{x}) - D_1 \tag{29b}$$

$$T_2\Psi + \Psi^TT_2 = \Xi_1(\theta, \boldsymbol{x}) - D_2 \tag{29c}$$

$$\vdots$$

$$T_n\Psi + \Psi^TT_n = \Xi_{n-1}(\theta, \boldsymbol{x}) - D_n \tag{29d}$$

where

$$\Psi = A_0 - BR^{-1}B^TT_0, \quad \Xi_i(\theta, \boldsymbol{x}) = -\frac{T_iA(\boldsymbol{x})}{\theta} - \frac{A^T(\boldsymbol{x})T_i}{\theta} + \sum_{j=1}^{i} T_jBR^{-1}B^TT_{i+1-j}.$$

From (29), we see that T_i is a function of $\boldsymbol{x}$ and θ. In order to ensure that T_i is linear in $\frac{1}{\theta^i}$, i.e., $\frac{\partial V}{\partial \boldsymbol{x}}$ is independent of θ, we construct the following expression for the perturbation matrices D_i,

$$D_1 = k_1e^{l_1t}\Xi_0(\theta, \boldsymbol{x}), \quad D_2 = k_2e^{l_2t}\Xi_1(\theta, \boldsymbol{x}), \quad \ldots \quad, D_n = k_ne^{l_nt}\Xi_{n-1}(\theta, \boldsymbol{x}), \tag{30}$$

where $k_i > 0$ and $l_i > 0$, $i = 1, 2, \ldots, n$, are adjustable design parameters.

Remark 4.1. *From (29), it is easy to see that (29a) is an algebraic Riccati equation. The rest of the equations are Lyapunov equations that are linear in terms of* T_i*,* $i = 1, \ldots, n$*. For (29a), once* A_0*,* B*,* Q *and* R *are determined, a positive definite matrix solution* T_0 *can be obtained. For (29b), it can be rewritten as:*

$$T_1\Psi + \Psi^TT_1 = (I - \Lambda_1)\Xi_0(\theta, \boldsymbol{x}). \tag{31}$$

TABLE 1. The orbital parameters of the target spacecraft

Parameters	Symbol	Values
Chaser mass	m	1000 kg
Orbit radius	r	42241 km
Gravity constant	μ	3.986×10^{14} m^3/s^2
Angular velocity	ω	7.2722×10^{-5} rads/s

Thus, we can solve (31) from matrix algebra theory. Similar to how (31) is solved, $T_2, \ldots, T_n$ can also be obtain from (29b)-(29d), recursively. In practice, retaining the first three terms T_0, T_1 and T_2 in the expression of the control (29) is sufficient to achieve satisfactory performance (see [18]*).*

5. **Numerical Example.** In this section, we provide an example to illustrate the usefulness of the controller design approach proposed in previous sections. Here, the target spacecraft is moving along a geosynchronous orbit with an orbital period of 24 h. Parameters are listed in Table 1. Assume that the initial relative position $(x_0, y_0, z_0) = (40$ km, -10 km, -30 km$)$ at time $t = 0$. Furthermore, for simplicity, we assume that the initial state $\boldsymbol{x}_0 = (40$ km, -10 km, -30 km, 0, 0, 0, 0$)$, which means that the spacecraft are relatively static before time $t = 0$. Choose $\gamma = 800$ and $\lambda = 0.001$, we obtain $A_0 = F(x_0) - \gamma I$ and B from (26). Let

$$\Theta = I, \quad \Omega = 1000I.$$

The uncertain parameter bounds in (20) are

$$\Lambda_f = 25I, \quad \Lambda_b = 12I,$$

where I is an identity matrix with appropriate dimension. Then, the matrices Q and R in the cost function (22) are

$$Q = 4\Lambda_f + 4\Lambda_b + 2\Theta, \quad R = 2\Omega,$$

respectively. We select

$$D_1 = e^{-0.0001t}\Xi_0(\theta, \boldsymbol{x}), \quad D_2 = e^{-0.01t}\Xi_1(\theta, \boldsymbol{x}), \quad D_3 = e^{-0.003t}\Xi_2(\theta, \boldsymbol{x}).$$

By using the proposed approach in Remark 4.1 to solve (29), we obtain the controller (24), which is an approximate solution to the HJB Equation (23). With the obtained controller, the required control input acceleration during the rendezvous process is depicted in Figure 2. The maximum input accelerations in x-axis, y-axis and z-axis are $|u_x|_{\max} = 0.0046$ m/s^2, $|u_y|_{\max} = 0.0049$ m/s^2 and $|u_z|_{\max} = 0.0044$ m/s^2, respectively. For a chaser spacecraft weighting 1000 kg, the maximum input thrusts in x-axis, y-axis and z-axis are, respectively, only 4.6 N, 4.9 N and 4.4 N. The relative position of the two spacecrafts is depicted in Figure 3. From Figure 3, we can see that the two spacecrafts are accomplishing rendezvous mission asymptotically.

6. **Conclusions.** In this work, robust nonlinear control problem of spacecraft rendezvous was formulated. To solve this problem, some results obtained in previous work was extended to a more general case such that they are applicable to the nonlinear control problem of spacecraft rendezvous under consideration. The robust nonlinear control problem was then transformed into an optimal control problem involving a properly modified cost function. An approximate closed-form solution to the nonlinear optimal control problem

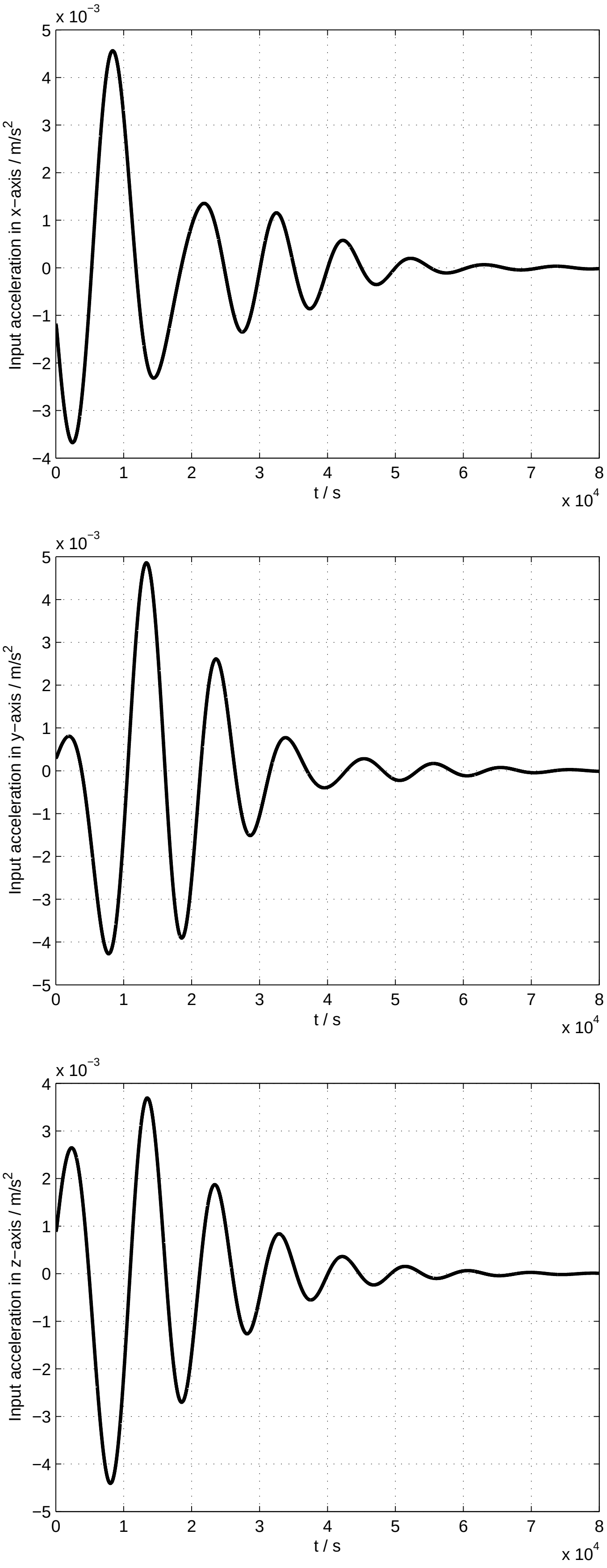

Figure 2. Control input acceleration

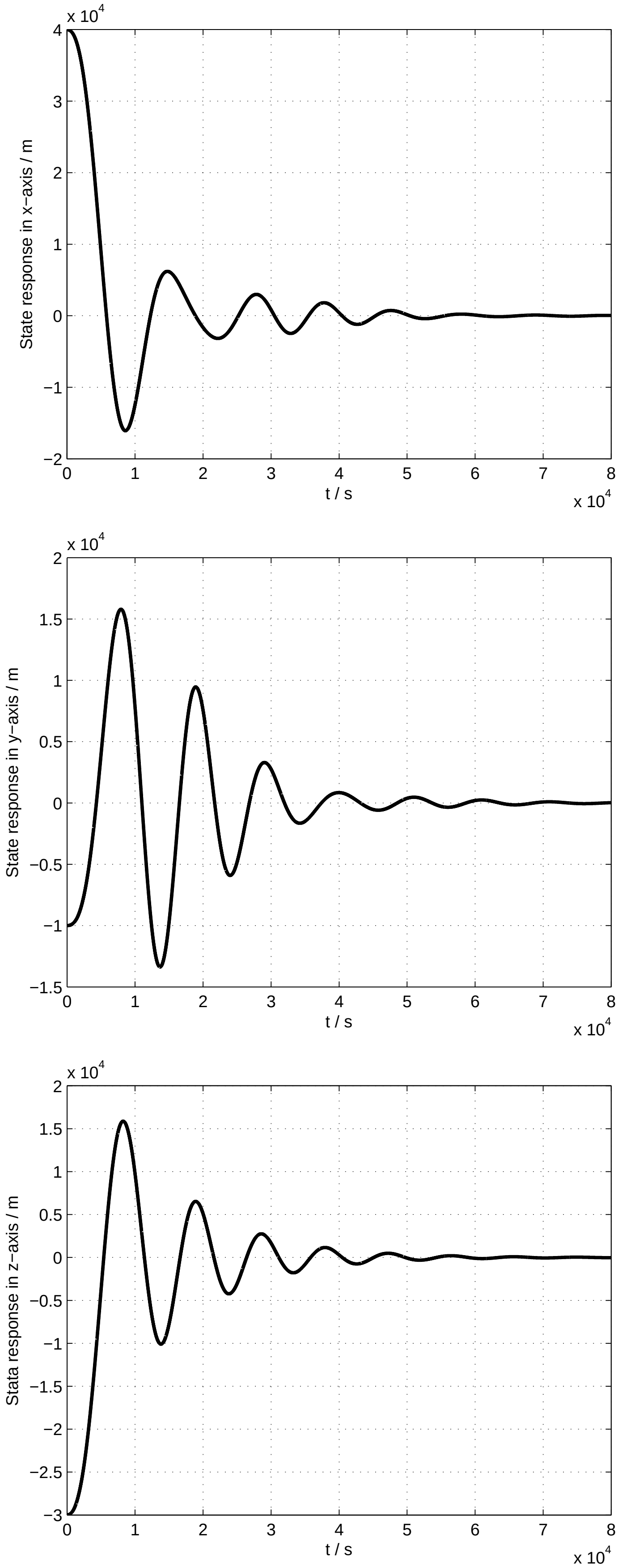

FIGURE 3. Relative position of the two spacecrafts

is obtained by applying the θ-D technique. This approach provides a simple design procedure which is easy for on-board implementation. From the simulation results, we observe the effectiveness of the proposed approach to the spacecraft rendezvous problem.

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