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THE RELATIONSHIP BETWEEN GAP METRIC AND TIME-VARYING GAP METRIC FOR LINEAR TIME-VARYING SYSTEMS

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Abstract. *This paper is concerned with the relationship between the gap metric and the time-varying gap metric for linear time-varying systems in the feedback configuration. It is shown that these two kinds of gaps are identical to each other when we measure the distance between corresponding graphs of a plant and a controller in a feedback system. This result supplies a simple way for computing the supremum appearing in the definition of the time-varying gap. Furthermore, the developed criteria are also applied to compute the optimal minimal angles of stabilizable linear time-varying systems.*
Keywords: Linear time-varying system, Stabilization, Gap metric, Time-varying gap metric, Optimal minimal angle

1. Introduction.

The notion of gap metric was originated in the field of functional analysis in [1] to measure the distance between two closed subspaces of a Hilbert space. This concept was first introduced into control theory by Zames and El-Sakkary [2] for measuring the difference between the graphs of two (possibly) unstable linear time-invariant systems. And Georgiou [3] related this metric to a particular two-block H^∞ problem for linear time-invariant systems. In [4], El-Sakkary pointed out that the gap metric is better suited for approximating unstable systems and studying the robustness of the stability of their feedback interconnections than the metric based on norms. Furthermore, the graph topology induced by gap metric is the weakest topology such that the feedback stability is a robust property [3, 4]. Based on these results, the gap metric is widely used as an appropriate geometric tool in the study of stability, robustness of feedback systems in the H^∞ control theory [5-8] and also in practical applications [9].

In [10], Feintuch extended the notion of gap metric to the time-varying systems in the framework of nest algebra. And he also gave a generalization of the connection between gap and the two-block optimization problem obtained in [3] to the time-varying case. It is pointed out in [10, 11] that the appropriate metric in the connection for the time-varying systems is not the gap between two linear systems themselves anymore, but the supremum of the sequence of gaps between them measured at every instant of time. Such a supremum is defined to be the time-varying gap for time-varying systems [10-12]. Moreover, it is proved in [11] that the time-varying gap is no larger than the gap. So it is natural to ask whether the time-varying gap is strictly smaller than the gap. If the answer is positive, then one can derive that when a system achieves the robustness boundary given by the time-varying gap, it must satisfy the boundary given by the gap; that is, the time-varying gap metric offers a wider robust stability margin than the gap metric does. This motivates our interest in investigating the deep relationship between the gap metric and the time-varying gap metric. However, to the best of our knowledge, no studies concerning this topic have been performed until now.

In this paper, the relationship between the gap metric and the time-varying gap metric for linear time-varying systems is studied. By using the operator-theoretic approaches, we show that when measuring the distance between the orthogonal complement of the inverse graph of a plant and the graph of a controller in a feedback configuration, the gap metric and the time-varying gap metric are in fact identical. This result reflects that the time-varying gap metric is not better than the gap metric in the feedback stabilization and robust problems. Furthermore, there are two important applications of the developed criteria. One is that the equivalence of these two metrics supplies a simple way for computing the supremum appearing in the definition of the time-varying gap. The other is that the derived result offers a new way of computing the optimal minimal angles of stabilizable linear time-varying systems. And we conclude that the value of the cosine of the optimal minimal angle is in fact the norm of a time-varying Hankel operator.

This paper is organized as follows. In Section 2, some basic definitions and auxiliary properties of linear time-varying systems are recalled. In Section 3, the gap metric and the time-varying gap metric are introduced. In Section 4, the main result of this paper about the relationship between these two metrics in the feedback configuration is proved and the result is used to compute the optimal minimal angles of stabilizable linear time-varying systems. Section 5 contains the conclusion.

2. **Stabilization and Strong Representation.** In this section we introduce some basic concepts and results for linear time-varying systems. More details can be found in [11-13].

Let $\mathbb{C}$ denote the set of complex numbers. Let $\mathcal{H}$ be the complex infinite-dimensional sequence space

$$\ell^2 = \left\{ (x_1, x_2, x_3, \cdots) : x_i \in \mathbb{C}, \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\},$$

where $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{C}$. Obviously, $\mathcal{H}$ is a Hilbert space with the standard inner product $(x, y) = (\{x_i\}_{i=1}^{\infty}, \{y_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} x_i \bar{y}_i$.

Let $\mathcal{B}(\mathcal{H})$ denote the space of bounded linear operators on $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, $RanT$ denotes the set $\{Tx, x \in \mathcal{H}\}$ and $KerT$ denotes the set $\{x \in \mathcal{H}, Tx = 0\}$. T^* stands for the adjoint of operator T. The induced norm of T is defined by

$$\|T\| = \sup_{x \in \mathcal{H}, x \neq 0} \frac{\|Tx\|}{\|x\|},$$

and the minimum modulus of T, denoted as $\tau(T)$, is defined by

$$\tau(T) = \inf_{x \in \mathcal{H}, x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

Let $\mathcal{H}_e = \{(x_1, x_2, x_3, \cdots) : x_i \in \mathbb{C}\}$ be the extended space of $\mathcal{H}$.

For each $n \geq 0$, we denote by P_n the standard truncation projection on $\mathcal{H}$ and $\mathcal{H}_e$ as

$$P_n(x_1, x_2, \cdots, x_n, x_{n+1}, \cdots) = (x_1, x_2, \cdots, x_n, 0, 0, \cdots)$$

with $P_0 = 0$, $P_\infty = I$. P_n sets all outputs after time n to zero, so the projection sequence $\{P_n\}_{n=0}^{\infty}$ is crucial to the physical notion of causality for linear systems.

In the operator-theoretic framework, linear time-varying systems are defined as follows.

Definition 2.1. [12] *A linear transformation T on $\mathcal{H}_e$ is causal if $P_n T = P_n T P_n$ for each n. A linear time-varying system on $\mathcal{H}_e$ is a causal linear transformation on $\mathcal{H}_e$, which is continuous with respect to the resolution topology.*

We denote the set of all linear time-varying systems on $\mathcal{H}_e$ by $\mathcal{L}$. It is clear that $\mathcal{L}$ is an algebra with standard addition and multiplication. And any element of $\mathcal{L}$ is an infinite-dimensional lower triangular matrix (with respect to the standard basis of $\mathcal{H}$, see Chapter 5 of [12]).

For $T \in \mathcal{L}$, the linear manifold $\mathcal{D}(T) = \{x \in \mathcal{H} : \ Tx \in \mathcal{H}\}$ denotes the domain of operator T. The graph of T is the set

$$\mathcal{G}(T) = \left\{ \begin{bmatrix} x \\ Tx \end{bmatrix} : \ x \in \mathcal{D}(T) \right\} \subset \mathcal{H} \oplus \mathcal{H}.$$

And the inverse graph of T is

$$\mathcal{G}^{-1}(T) = \left\{ \begin{bmatrix} Tx \\ x \end{bmatrix} : \ x \in \mathcal{D}(T) \right\} \subset \mathcal{H} \oplus \mathcal{H}.$$

Here we use the symbol $\mathcal{H} \oplus \mathcal{H}$ to denote the direct sum of spaces $\mathcal{H}$ and $\mathcal{H}$. It is shown in Theorem 5.3.4 of [12] that $\mathcal{G}(T)$ is a closed subspace of the Hilbert space $\mathcal{H} \oplus \mathcal{H}$.

Definition 2.2. [12] *A linear time-varying system T is stable if its restriction on $\mathcal{H}$ is a bounded operator. That is, there exists a positive constant c such that for every x in $\mathcal{H}$, Tx is in $\mathcal{H}$ and $\|Tx\| \leq c\|x\|$.*

The set of stable linear time-varying systems, denoted by $\mathcal{S}$, is a weak operator closed subalgebra of $\mathcal{B}(\mathcal{H})$ containing the identity [12].

For $L, C \in \mathcal{L}$, we consider the standard feedback configuration in [12] with plant L and controller C. The closed-loop system equations are

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} I & C \\ L & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

where $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is the externally applied input and $e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ is the internal input.

The feedback system $\{L, C\}$ is well-posed if the linear transformation

$$\begin{bmatrix} I & C \\ L & I \end{bmatrix} : \mathcal{D}(L) \oplus \mathcal{D}(C) \to \mathcal{H} \oplus \mathcal{H}$$

is invertible. And its inverse is given by the transfer matrix

$$H(L, C) = \begin{bmatrix} (I - CL)^{-1} & -C(I - LC)^{-1} \\ -L(I - CL)^{-1} & (I - LC)^{-1} \end{bmatrix}.$$

Definition 2.3. [12] *The closed-loop system $\{L, C\}$ is stable if $\begin{bmatrix} I & C \\ L & I \end{bmatrix}$ has a bounded causal inverse defined on $\mathcal{H} \oplus \mathcal{H}$. This is equivalent to requiring that all the entries of $H(L, C)$ are in $\mathcal{S}$. A plant L is stabilizable if there exists a controller $C \in \mathcal{L}$ such that $\{L, C\}$ is stable. If it is the case, C is called a stabilizing controller for L.*

Remark 2.1. *In our context, the transfer matrix is a 2×2-block operator on $\mathcal{H}_e \oplus \mathcal{H}_e$ or $\mathcal{H} \oplus \mathcal{H}$. It is a generalized matrix and each entry of it is an operator in $\mathcal{L}$ or $\mathcal{S}$. (It is different from the characterization of a transfer matrix by the state space approach.)*

In order to characterize the stabilizable plants, we need the following notions of representations for a linear system. Notice that they are also crucial to the definitions of gap and time-varying gap metrics for time-varying systems in Section 3.

Definition 2.4. [12, 13] *A plant $L \in \mathcal{L}$ has a strong right representation $\begin{bmatrix} M \\ N \end{bmatrix}$ with $M, N \in \mathcal{S}$ if*

- $\mathcal{G}(L) = Ran \begin{bmatrix} M \\ N \end{bmatrix}$;
- $\begin{bmatrix} M \\ N \end{bmatrix}$ *has a causal bounded left inverse; that is, there exist* $X, Y \in \mathcal{S}$ *such that*

$$\begin{bmatrix} Y & -X \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = I.$$

L has a strong left representation $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$ *with* $\hat{M}, \hat{N} \in \mathcal{S}$ *if*

- $\mathcal{G}(L) = Ker \begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$;
- $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$ *has a causal bounded right inverse; that is, there exist* $\hat{X}, \hat{Y} \in \mathcal{S}$ *such that*

$$\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix} \begin{bmatrix} \hat{X} \\ \hat{Y} \end{bmatrix} = I.$$

As shown in [12], strong right and strong left representations are alternate, but equivalent to the more familiar notions of right and left coprime factorizations. And it is proved in Theorem 6.3.5 of [12] that strong representations are not unique.

Proposition 2.1. [12] *If* $\begin{bmatrix} M \\ N \end{bmatrix}$ *is a strong right representation of* $L \in \mathcal{L}$*, then any strong right representation of L is of the form* $\begin{bmatrix} M \\ N \end{bmatrix} S$ *with S invertible in* $\mathcal{S}$.

A dual statement holds for the strong left representations.

The stabilizability of linear systems is closely related to the existence of strong right and strong left representations. The most famous theorem characterizing this relationship is the Youla parametrization .

Theorem 2.1. ***(Youla parametrization theorem)*** [11, 12] *A plant* $L \in \mathcal{L}$ *is stabilizable if and only if there exist* M, N, X, Y, $\hat{M}$, $\hat{N}$, $\hat{X}$, $\hat{Y} \in \mathcal{S}$ *such that* $\begin{bmatrix} M \\ N \end{bmatrix}$ *and* $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$ *are, respectively, strong right and strong left representations for L and the following double Bezout identity*

$$\begin{bmatrix} Y & -X \\ -\hat{N} & \hat{M} \end{bmatrix} \begin{bmatrix} M & \hat{X} \\ N & \hat{Y} \end{bmatrix} = \begin{bmatrix} M & \hat{X} \\ N & \hat{Y} \end{bmatrix} \begin{bmatrix} Y & -X \\ -\hat{N} & \hat{M} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

holds. Moreover, $C \in \mathcal{L}$ *stabilizes L if and only if it has a strong right representation* $\begin{bmatrix} \hat{Y} + NQ \\ \hat{X} + MQ \end{bmatrix}$ *and a strong left representation* $\begin{bmatrix} -(X + Q\hat{M}) & Y + Q\hat{N} \end{bmatrix}$ *for some* $Q \in \mathcal{S}$.

Among all the strong representations, we are interested in the particular ones called the normalized strong representations.

Definition 2.5. [11, 12] *The strong right representation* $\begin{bmatrix} M \\ N \end{bmatrix}$ *of L is normalized if it is an isometry, i.e.,* $M^*M + N^*N = I$*. The strong left representation* $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$ *is normalized if it is a co-isometry, i.e.,* $\hat{N}\hat{N}^* + \hat{M}\hat{M}^* = I$.

And the following result shows that normalized strong representations always exist.

Proposition 2.2. [11, 12] *If* $L \in \mathcal{L}$ *has a strong right (or left) representation, it has a normalized one.*

In the sequel, we focus on the normalized strong representations of time-varying systems. And they do supply us with perfect properties.

Proposition 2.3. [12] *Suppose that* $\begin{bmatrix} M \\ N \end{bmatrix}$ *and* $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$ *are normalized strong right and left representations for* $L \in \mathcal{L}$. *Then* $Z = \begin{bmatrix} M^* & N^* \\ -\hat{N} & \hat{M} \end{bmatrix}$ *is a unitary operator, i.e.,* $Z^*Z = ZZ^* = I$.

Proposition 2.4. [11] *If* $\begin{bmatrix} M \\ N \end{bmatrix}$ *and* $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$ *are normalized strong right and left representations for* L, *respectively, then the orthogonal projection from* $\mathcal{H} \oplus \mathcal{H}$ *onto* $\mathcal{G}(L) = \begin{bmatrix} M \\ N \end{bmatrix}\mathcal{H}$ *is given by* $P_{\begin{bmatrix} M \\ N \end{bmatrix}\mathcal{H}} = \begin{bmatrix} M \\ N \end{bmatrix}\begin{bmatrix} M^* & N^* \end{bmatrix}$ *and the orthogonal projection from* $\mathcal{H} \oplus \mathcal{H}$ *onto* $\mathcal{G}(L)^{\perp} = \left(\begin{bmatrix} M \\ N \end{bmatrix}\mathcal{H}\right)^{\perp} = \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix}\mathcal{H}$ *is given by* $P_{\left(\begin{bmatrix} M \\ N \end{bmatrix}\mathcal{H}\right)^{\perp}} = \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix}\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$.

3. **Gap and Time-Varying Gap.** In this section, the definitions of the gaps and the time-varying gaps from [10-12] for linear time-varying systems are introduced. Some useful results are also listed.

As shown in [1], the gap metric is originally defined on the closed subspaces of a Hilbert space.

Definition 3.1. *Suppose that* $\mathcal{M}_1$, $\mathcal{M}_2$ *are two closed subspaces of Hilbert space* $\mathcal{K}$ *and* P_1, P_2 *are the corresponding orthogonal projections. The directed gap from* $\mathcal{M}_1$ *to* $\mathcal{M}_2$ *is defined by*

$$\vec{\delta}(\mathcal{M}_1, \mathcal{M}_2) = \|(I - P_2)P_1\|.$$

And $\vec{\delta}(\mathcal{M}_2, \mathcal{M}_1) = \|(I - P_1)P_2\|$. *Then the gap between* $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *is defined to be*

$$\delta(\mathcal{M}_1, \mathcal{M}_2) = \max\left\{\vec{\delta}((\mathcal{M}_1, \mathcal{M}_2), \vec{\delta}(\mathcal{M}_2, \mathcal{M}_1)\right\} = \|P_1 - P_2\|.$$

It has been proved that $0 \leq \delta \leq 1$ is a metric and is called the gap metric. And the following property shows when the two directed gaps are equal.

Proposition 3.1. [12] *If* $\delta(\mathcal{M}_1, \mathcal{M}_2) < 1$, *then*

$$\delta(\mathcal{M}_1, \mathcal{M}_2) = \vec{\delta}(\mathcal{M}_1, \mathcal{M}_2) = \vec{\delta}(\mathcal{M}_2, \mathcal{M}_1).$$

If we identify a linear time-varying system $T \in \mathcal{L}$ with its graph $\mathcal{G}(T)$, then we can consider the gap metric as a measure of distance between two linear systems.

For $L_1, L_2 \in \mathcal{L}$, $\begin{bmatrix} M_1 \\ N_1 \end{bmatrix}$ and $\begin{bmatrix} M_2 \\ N_2 \end{bmatrix}$ are their normalized strong right representations, respectively. Denote by $P_{\begin{bmatrix} M_i \\ N_i \end{bmatrix}\mathcal{H}}$ the orthogonal projection from $\mathcal{H} \oplus \mathcal{H}$ on the graph $\mathcal{G}(L_i)$ of L_i, $i = 1, 2$. Notice that the orthogonal projection can be computed by Proposition 2.4. Then the directed gap from linear time-varying system L_1 to L_2 is defined by

$$\begin{aligned}\vec{\delta}(L_1, L_2) &= \vec{\delta}\left(\mathcal{G}(L_1), \mathcal{G}(L_2)\right) \\ &= \left\| \left(I - P_{\left[\begin{smallmatrix} M_2 \\ N_2 \end{smallmatrix}\right]\mathcal{H}} \right) P_{\left[\begin{smallmatrix} M_1 \\ N_1 \end{smallmatrix}\right]\mathcal{H}} \right\| = \left\| P_{\left(\left[\begin{smallmatrix} M_2 \\ N_2 \end{smallmatrix}\right]\mathcal{H}\right)^{\perp}} P_{\left[\begin{smallmatrix} M_1 \\ N_1 \end{smallmatrix}\right]\mathcal{H}} \right\|.\end{aligned}$$

The other directed gap from L_2 to L_1 is defined similarly. And the gap between L_1 and L_2 is

$$\delta(L_1, L_2) = \max\left\{\vec{\delta}(L_1, L_2), \vec{\delta}(L_2, L_1)\right\} = \left\| P_{\left[\begin{smallmatrix} M_1 \\ N_1 \end{smallmatrix}\right]\mathcal{H}} - P_{\left[\begin{smallmatrix} M_2 \\ N_2 \end{smallmatrix}\right]\mathcal{H}} \right\|.$$

Normalized strong representation $\begin{bmatrix} M_i \\ N_i \end{bmatrix}$ implies that $\begin{bmatrix} M_{in} \\ N_{in} \end{bmatrix} = \begin{bmatrix} M_i \\ N_i \end{bmatrix}(I - P_n)$ is an isometry on the subspace $\mathcal{H}_n = (I - P_n)\mathcal{H}$ of $\mathcal{H}$ with range in $\mathcal{H}_n \oplus \mathcal{H}_n$ for each n, $i = 1, 2$ [12]. Let $P_{\left[\begin{smallmatrix} M_{in} \\ N_{in} \end{smallmatrix}\right]\mathcal{H}_n}$ be the orthogonal projection on the range of $\begin{bmatrix} M_{in} \\ N_{in} \end{bmatrix}$ for $i = 1, 2$. Then we can define a special direct gap

$$\vec{\delta}_n(L_1, L_2) = \left\| \left(I_n - P_{\left[\begin{smallmatrix} M_{2n} \\ N_{2n} \end{smallmatrix}\right]\mathcal{H}_n} \right) P_{\left[\begin{smallmatrix} M_{1n} \\ N_{1n} \end{smallmatrix}\right]\mathcal{H}_n} \right\| = \left\| P_{\left(\left[\begin{smallmatrix} M_{2n} \\ N_{2n} \end{smallmatrix}\right]\mathcal{H}_n\right)^{\perp}} P_{\left[\begin{smallmatrix} M_{1n} \\ N_{1n} \end{smallmatrix}\right]\mathcal{H}_n} \right\|,$$

where $I_n = \begin{bmatrix} I - P_n & 0 \\ 0 & I - P_n \end{bmatrix}$ is the identity operator on $\mathcal{H}_n \oplus \mathcal{H}_n$. Following [10, 11, 12], the directed time-varying gap from L_1 to L_2 is defined as

$$\vec{\alpha}(L_1, L_2) = \vec{\alpha}\left(\mathcal{G}(L_1), \mathcal{G}(L_2)\right) = \sup_{n \geq 0} \vec{\delta}_n(L_1, L_2).$$

And the time-varying gap between L_1 and L_2 is

$$\alpha(L_1, L_2) = \max\left\{\vec{\alpha}(L_1, L_2), \vec{\alpha}(L_2, L_1)\right\} = \sup_{n \geq 0} \left\| P_{\left[\begin{smallmatrix} M_{1n} \\ N_{1n} \end{smallmatrix}\right]\mathcal{H}_n} - P_{\left[\begin{smallmatrix} M_{2n} \\ N_{2n} \end{smallmatrix}\right]\mathcal{H}_n} \right\|.$$

α is a metric which is called the time-varying gap metric [10]. And it is proved in [11] that

$$\vec{\alpha}(L_1, L_2) \leq \vec{\delta}(L_1, L_2)$$

and

$$\alpha(L_1, L_2) \leq \delta(L_1, L_2).$$

In [12], Feintuch related the definition of the directed time-varying gap to a two-block optimization problem, which generalized the result of [3]. The generalization is stated as follows.

Proposition 3.2. [12] *With notations as above,*

$$\vec{\alpha}(L_1, L_2) = \sup_{n \geq 0} \vec{\delta}_n(L_1, L_2) = \inf_{Q \in \mathcal{S}} \left\| \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} Q \right\|.$$

It is clear that the gaps and the time-varying gaps can also be defined on the inverse graphs, orthogonal complement of graphs or orthogonal complement of inverse graphs of linear systems as long as proper orthogonal projections are chosen. And the following proposition shows that the gap is an indicator of the degree of the stability of a closed loop feedback system.

Proposition 3.3. [12] *Consider a plant $L \in \mathcal{L}$ with normalized strong right representation $\begin{bmatrix} M \\ N \end{bmatrix}$ and normalized strong left representation $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$, a controller $C \in \mathcal{L}$ with normalized strong right representation $\begin{bmatrix} V \\ U \end{bmatrix}$ and normalized strong left representation $\begin{bmatrix} -\hat{U} & \hat{V} \end{bmatrix}$. The following are equivalent to the stability of $\{L, C\}$:*

(i) $\delta(\mathcal{G}^{-1}(L), (\mathcal{G}(C))^{\perp}) = \delta\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}, \left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}\right)^{\perp}\right)$

$$= \left\| P_{\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}\right)^{\perp}} P_{\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}} \right\| < 1.$$

(ii) $\hat{V}M - \hat{U}N$ *is invertible in* $\mathcal{S}$.

(iii) $\hat{M}V - \hat{N}U$ *is invertible in* $\mathcal{S}$.

We end this section with a result about the connection between slightly different two-block type problems and the time-varying gaps from [11].

Proposition 3.4. [11] *The notations are as those in Proposition 3.3. Then we have*

1.
$$\inf_{Q \in \mathcal{S}} \left\| \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\| = \inf_{Q \in \mathcal{S}} \left\| \begin{bmatrix} -\hat{U}^* \\ \hat{V}^* \end{bmatrix} - \begin{bmatrix} N \\ M \end{bmatrix} Q \right\|$$

$$= \sup_{n \geq 0} \left\| P_{\left(\begin{bmatrix} V_n \\ U_n \end{bmatrix}\mathcal{H}_n\right)^{\perp}} P_{\left(\begin{bmatrix} N_n \\ M_n \end{bmatrix}\mathcal{H}_n\right)^{\perp}} \right\| = \vec{\alpha}((\mathcal{G}^{-1}(L))^{\perp}, \mathcal{G}(C)).$$

2.
$$\inf_{Q \in \mathcal{S}} \left\| \begin{bmatrix} V \\ U \end{bmatrix} - \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} Q \right\| = \inf_{Q \in \mathcal{S}} \left\| \begin{bmatrix} N \\ M \end{bmatrix} - \begin{bmatrix} -\hat{U}^* \\ \hat{V}^* \end{bmatrix} Q \right\|$$

$$= \sup_{n \geq 0} \left\| P_{\begin{bmatrix} V_n \\ U_n \end{bmatrix}\mathcal{H}_n} P_{\begin{bmatrix} N_n \\ M_n \end{bmatrix}\mathcal{H}_n} \right\| = \vec{\alpha}(\mathcal{G}(C), (\mathcal{G}^{-1}(L))^{\perp})).$$

3.
$$\inf_{Q \in \mathcal{S}} \left\| \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\| = \sup_{n \geq 0} \left\| P_{\begin{bmatrix} V_n \\ U_n \end{bmatrix}\mathcal{H}_n} - P_{\left(\begin{bmatrix} N_n \\ M_n \end{bmatrix}\mathcal{H}_n\right)^{\perp}} \right\|$$

$$= \alpha(\mathcal{G}(C), (\mathcal{G}^{-1}(L))^{\perp}).$$

In next section, we will show that the time-varying gaps appeared in Proposition 3.4 are in fact all gaps.

4. **The Relationship Between Two Kinds of Gaps.**

The purpose of this section is to discuss the relationship between the gaps and the time-varying gaps when we measure the distance between the orthogonal complement of the inverse graph of a plant and the graph of a controller in a feedback system.

We assume that the strong right representation $\begin{bmatrix} M \\ N \end{bmatrix}$ and the strong left representation $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$ for plant $L \in \mathcal{L}$ are both normalized in this section. So are the strong representations $\begin{bmatrix} V \\ U \end{bmatrix}$ and $\begin{bmatrix} -\hat{U} & \hat{V} \end{bmatrix}$ for controller $C \in \mathcal{L}$.

Before we state the main results, the following lemmas are needed.

Lemma 4.1. [12] *If* $\left\| P_{\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}\right)^{\perp}} P_{\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}} \right\| < 1$, *then* $\left\| P_{\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}} P_{\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}} \right\| < 1$.

Definition 4.1. *The minimal angle $\theta_{\min} \in [0, \frac{\pi}{2}]$ between two closed subspaces $\mathcal{M}_1$, $\mathcal{M}_2$ of Hilbert space $\mathcal{K}$ is given by*

$$\cos\theta_{\min}(\mathcal{M}_1, \mathcal{M}_2) = \sup_{0 \neq u \in \mathcal{M}_1, 0 \neq v \in \mathcal{M}_2} \frac{|(u,v)|}{\|u\| \cdot \|v\|} = \|P_{\mathcal{M}_1} P_{\mathcal{M}_2}\|,$$

where $P_{\mathcal{M}_i}$ is the orthogonal projection onto $\mathcal{M}_i$ for $i = 1, 2$.

Lemma 4.2. [12] *Suppose the feedback system $\{L, C\}$ is stable, with the symbols mentioned above, we have*

$$\begin{aligned} \cos\theta_{\min}\left(\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}\right)^{\perp}, \left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}\right) &= \left\| P_{\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}\right)^{\perp}} P_{\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}} \right\| \\ &= \delta\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}, (\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H})^{\perp}\right) \\ &= \|N^*V + M^*U\| \\ &= \|\hat{M}\hat{U}^* + \hat{N}\hat{V}^*\|. \end{aligned}$$

Lemma 4.3. [13] *If $A, X \in \mathcal{S}$ such that AX is invertible in $\mathcal{S}$, then A and X are both invertible in $\mathcal{S}$.*

Lemma 4.4. [12] ***(Small gain theorem)*** *If $T \in \mathcal{B}(\mathcal{H})$ and $\|I - T\| < 1$, then T is invertible and*

$$\|T^{-1}\| \leq \frac{1}{1 - \|I - T\|}.$$

The following lemmas exhibit some basic properties of the minimum modulus of an operator.

Lemma 4.5. [14] *Let $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$ be Hilbert spaces. If the bounded linear operator* $\begin{bmatrix} X \\ Y \end{bmatrix} : \mathcal{K}_1 \to \mathcal{K}_2 \times \mathcal{K}_3$ *is an isometry, then*

$$\|X\|^2 = 1 - \inf_{0 \neq u \in \mathcal{K}_1} \frac{\|Yu\|^2}{\|u\|^2} = 1 - \tau(Y)^2.$$

Similarly, if the bounded linear operator $\begin{bmatrix} \tilde{X} & \tilde{Y} \end{bmatrix} : \mathcal{K}_1 \times \mathcal{K}_2 \to \mathcal{K}_3$ *is a co-isometry, then*

$$\|\tilde{X}\|^2 = 1 - \inf_{0 \neq u \in \mathcal{K}_1} \frac{\|\tilde{Y}^* u\|^2}{\|u\|^2} = 1 - \tau(\tilde{Y}^*)^2.$$

Lemma 4.6. [12] *If $X \in \mathcal{B}(\mathcal{H})$ is invertible, then $\tau(X) = \frac{1}{\|X^{-1}\|}$.*

Now we are in the position to show our results. We first give the following proposition.

Proposition 4.1. *Let* $\begin{bmatrix} M \\ N \end{bmatrix}$ *and* $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$ *be the normalized strong right and left representations for plant L, respectively, and* $\begin{bmatrix} V \\ U \end{bmatrix}$, $\begin{bmatrix} -\hat{U} & \hat{V} \end{bmatrix}$ *be the same for the controller C. If $\{L, C\}$ is stable and $\|N^*V + M^*U\| = \delta$, then we have*

$$\|(N^*V + M^*U)(\hat{M}V - \hat{N}U)^{-1}\| = \delta(1 - \delta^2)^{-\frac{1}{2}}.$$

Proof: Since $\begin{bmatrix} M \\ N \end{bmatrix}$ and $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$ are normalized, by Proposition 2.3, $\begin{bmatrix} M^* & N^* \\ -\hat{N} & \hat{M} \end{bmatrix}$ is a unitary operator. The normalization of $\begin{bmatrix} V \\ U \end{bmatrix}$ ensures that it is an isometry. From these two facts, we have that $\begin{bmatrix} M^* & N^* \\ -\hat{N} & \hat{M} \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} M^*U + N^*V \\ \hat{M}V - \hat{N}U \end{bmatrix}$ is isometric; that is,

$$(M^*U + N^*V)^*(M^*U + N^*V) + (\hat{M}V - \hat{N}U)^*(\hat{M}V - \hat{N}U) = I. \tag{1}$$

According to Lemma 4.5, we can get the following equation

$$\|M^*U + N^*V\|^2 + \tau(\hat{M}V - \hat{N}U)^2 = 1. \tag{2}$$

From the hypothesis that $\{L, C\}$ is stable and Proposition 3.3, $\hat{M}V - \hat{N}U$ is invertible in $\mathcal{S}$. Then by Lemma 4.6,

$$\tau(\hat{M}V - \hat{N}U)^2 = \frac{1}{\|(\hat{M}V - \hat{N}U)^{-1}\|^2}. \tag{3}$$

By multiplying $((\hat{M}V - \hat{N}U)^{-1})^*$ on the left-hand side of (1) and $(\hat{M}V - \hat{N}U)^{-1}$ on its right-hand side, we obtain

$$\begin{bmatrix} (M^*U + N^*V)(\hat{M}V - \hat{N}U)^{-1} \\ I \end{bmatrix}^* \begin{bmatrix} (M^*U + N^*V)(\hat{M}V - \hat{N}U)^{-1} \\ I \end{bmatrix}$$
$$= ((\hat{M}V - \hat{N}U)^{-1})^*(\hat{M}V - \hat{N}U)^{-1}.$$

It then follows that

$$\left\| \begin{bmatrix} (M^*U + N^*V)(\hat{M}V - \hat{N}U)^{-1} \\ I \end{bmatrix} \right\|^2 = \|(M^*U + N^*V)(\hat{M}V - \hat{N}U)^{-1}\|^2 + 1 \tag{4}$$

$$= \|(\hat{M}V - \hat{N}U)^{-1}\|^2, \tag{5}$$

where (4) comes from direct computation of the induced norm of the operator.

If $\|N^*V + M^*U\| = \delta$, then combining (2), (3) and (5), we derive that

$$\|(N^*V + M^*U)(\hat{M}V - \hat{N}U)^{-1}\| = \delta(1-\delta^2)^{-\frac{1}{2}}.$$

The following main result reveals the equivalence between the gaps and time-varying gaps in the feedback configuration.

Theorem 4.1. *Suppose $L \in \mathcal{L}$ is a plant and $C \in \mathcal{L}$ is a controller. The symbols M, N, $\hat{M}$, $\hat{N}$, V, U, $\hat{V}$, $\hat{U}$ in $\mathcal{S}$ are as those in Proposition 4.1. Then*

(a).

$$\begin{aligned}&\left\| P_{\left(\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}\right)^{\perp}} P_{\left(\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]\mathcal{H}\right)^{\perp}} \right\| \\ &= \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\| = \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} -\hat{U}^* \\ \hat{V}^* \end{bmatrix} - \begin{bmatrix} N \\ M \end{bmatrix} Q \right\| \\ &= \vec{\delta}((\mathcal{G}^{-1}(L))^{\perp}, \mathcal{G}(C)) = \|\hat{M}\hat{U}^* + \hat{N}\hat{V}^*\|.\end{aligned}$$

(b).

$$\begin{aligned}&\left\| P_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}} P_{\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]\mathcal{H}} \right\| \\ &= \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} V \\ U \end{bmatrix} - \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} Q \right\| = \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} N \\ M \end{bmatrix} - \begin{bmatrix} -\hat{U}^* \\ \hat{V}^* \end{bmatrix} Q \right\| \\ &= \vec{\delta}(\mathcal{G}(C), (\mathcal{G}^{-1}(L))^{\perp}) = \|V^*N + U^*M\|.\end{aligned}$$

(c).

$$\begin{aligned}&\left\| P_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}} - P_{\left(\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]\mathcal{H}\right)^{\perp}} \right\| \\ &= \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\| = \delta(\mathcal{G}(C), (\mathcal{G}^{-1}(L))^{\perp}).\end{aligned}$$

The proof of (a)**.**

Since $\left\| P_{\left(\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}\right)^{\perp}} P_{\left(\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]\mathcal{H}\right)^{\perp}} \right\| \leq 1$, we divide the proof of the first equation into two steps.

First, we assume that $\left\| P_{\left(\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}\right)^{\perp}} P_{\left(\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]\mathcal{H}\right)^{\perp}} \right\| = 1$. From Proposition 3.3, we can get that $\hat{M}V - \hat{N}U$ is not invertible in $\mathcal{S}$. So, for any $Q \in \mathcal{S}$, $(\hat{M}V - \hat{N}U)Q$ is not invertible in $\mathcal{S}$; otherwise, there will be a contradiction to Lemma 4.3. By Lemma 4.4,

$$\|I - (\hat{M}V - \hat{N}U)Q\| \geq 1$$

holds for any $Q \in \mathcal{S}$, from which we get that

$$\inf_{Q\in\mathcal{S}} \|I - (\hat{M}V - \hat{N}U)Q\| \geq 1.$$

Because $\begin{bmatrix} \hat{M} & -\hat{N} \end{bmatrix}$ is a co-isometry, it holds that

$$\left\| \begin{bmatrix} \hat{M} & -\hat{N} \end{bmatrix} \right\| = \left\| \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} \right\| = 1.$$

Hence,

$$\begin{aligned} \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\| &\geq \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} \hat{M} & -\hat{N} \end{bmatrix} \left(\begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right) \right\| \\ &= \inf_{Q\in\mathcal{S}} \|I - (\hat{M}V - \hat{N}U)Q\| \geq 1. \end{aligned}$$

On the other hand, if we take $Q_0 = 0 \in \mathcal{S}$, then

$$\inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\| \leq \left\| \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q_0 \right\| = \left\| \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} \right\| = 1.$$

So, in this situation, we have

$$\left\| P_{\left(\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H} \right)^{\perp}} P_{\left(\begin{bmatrix} N \\ M \end{bmatrix} \mathcal{H} \right)^{\perp}} \right\| = 1 = \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\|.$$

Second, we assume that $\left\| P_{\left(\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H} \right)^{\perp}} P_{\left(\begin{bmatrix} N \\ M \end{bmatrix} \mathcal{H} \right)^{\perp}} \right\| = \delta < 1$. By Proposition 3.3 and Lemma 4.2, this assumption ensures that $\{L, C\}$ is stable, $\hat{M}V - \hat{N}U$ is invertible in $\mathcal{S}$ and

$$\left\| P_{\left(\begin{bmatrix} V \\ U \end{bmatrix} \mathcal{H} \right)^{\perp}} P_{\left(\begin{bmatrix} N \\ M \end{bmatrix} \mathcal{H} \right)^{\perp}} \right\| = \|N^*V + M^*U\| = \delta.$$

Taking $Q_0 = (1-\delta^2)(\hat{M}V - \hat{N}U)^{-1} \in \mathcal{S}$, then we can get

$$\begin{aligned} &\inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\| \\ &\leq \left\| \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - (1-\delta^2) \begin{bmatrix} V \\ U \end{bmatrix} (\hat{M}V - \hat{N}U)^{-1} \right\| \\ &= \left\| \begin{bmatrix} \hat{M} & -\hat{N} \\ N^* & M^* \end{bmatrix} \left(\begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - (1-\delta^2) \begin{bmatrix} V \\ U \end{bmatrix} (\hat{M}V - \hat{N}U)^{-1} \right) \right\| \\ &= \left\| \begin{bmatrix} \delta^2 \\ -(1-\delta^2)(M^*U + N^*V)(\hat{M}V - \hat{N}U)^{-1} \end{bmatrix} \right\| \\ &= \left(\delta^4 + (1-\delta^2)^2 \left\| (M^*U + N^*V)(\hat{M}V - \hat{N}U)^{-1} \right\|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where the first equation follows from the fact that

$$\begin{bmatrix} \hat{M} & -\hat{N} \\ N^* & M^* \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} M^* & N^* \\ -\hat{N} & \hat{M} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

is a unitary operator which preserves the norm and the last equation follows from direct computation of the induced norm. Since $\|N^*V + M^*U\| = \delta$, from Proposition 4.1, we have

$$\|(N^*V + M^*U)(\hat{M}V - \hat{N}U)^{-1}\| = \delta(1-\delta^2)^{-\frac{1}{2}}.$$

Then,

$$\begin{aligned} \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\| &\le \left(\delta^4 + (1-\delta^2)^2 \|(M^*U + N^*V)(\hat{M}V - \hat{N}U)^{-1}\|^2\right)^{\frac{1}{2}} \\ &= \delta = \left\| P_{\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}\right)^{\perp}} P_{\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}} \right\|. \end{aligned}$$

For the opposite direction, by using the unitarity of the operator $\begin{bmatrix} V^* & U^* \\ -\hat{U} & \hat{V} \end{bmatrix}$, we obtain that

$$\begin{aligned} \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\| &= \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} V^* & U^* \\ -\hat{U} & \hat{V} \end{bmatrix} \left(\begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right) \right\| \\ &= \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} (V^*\hat{M}^* - U^*\hat{N}^*) - Q \\ -\hat{U}\hat{M}^* - \hat{V}\hat{N}^* \end{bmatrix} \right\| \\ &\ge \|\hat{U}\hat{M}^* + \hat{V}\hat{N}^*\|. \end{aligned}$$

From Lemma 4.2, $\|\hat{U}\hat{M}^* + \hat{V}\hat{N}^*\| = \|M^*U + N^*V\|$. So,

$$\inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\| \ge \|M^*U + N^*V\| = \left\| P_{\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}\right)^{\perp}} P_{\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}} \right\|.$$

The above arguments show that

$$\inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix} Q \right\| = \left\| P_{\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}\right)^{\perp}} P_{\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}} \right\|.$$

This completes the proof of the first equality.

The second equality in (a) can be proved in an analogous way.

Notice that, by Proposition 2.4,

$$\begin{aligned} \vec{\delta}((\mathcal{G}^{-1}(L))^{\perp}, \mathcal{G}(C)) &= \left\| P_{\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}\right)^{\perp}} P_{\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}} \right\| \\ &= \left\| \begin{bmatrix} -\hat{U}^* \\ \hat{V}^* \end{bmatrix} \begin{bmatrix} -\hat{U} & \hat{V} \end{bmatrix} \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} \begin{bmatrix} \hat{M} & -\hat{N} \end{bmatrix} \right\| \\ &= \|\hat{M}\hat{U}^* + \hat{N}\hat{V}^*\| \end{aligned}$$

holds whether $\{L, C\}$ is stable or not.

The proof of (b)**.**

Following Proposition 2.4, we can see

$$\begin{aligned}
\left\| P_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}} P_{\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]\mathcal{H}} \right\| &= \left\| \begin{bmatrix} V \\ U \end{bmatrix} \begin{bmatrix} V^* & U^* \end{bmatrix} \begin{bmatrix} N \\ M \end{bmatrix} \begin{bmatrix} N^* & M^* \end{bmatrix} \right\| \\
&= \|V^*N + U^*M\| \\
&\le \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} (\hat{M}V - \hat{N}U) - Q \\ M^*U + N^*V \end{bmatrix} \right\| \\
&= \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} \hat{M} & -\hat{N} \\ N^* & M^* \end{bmatrix} \left(\begin{bmatrix} V \\ U \end{bmatrix} - \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} Q \right) \right\| \\
&= \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} V \\ U \end{bmatrix} - \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} Q \right\|.
\end{aligned}$$

Take $Q_0 = \hat{M}V - \hat{N}U \in \mathcal{S}$. It follows that

$$\begin{aligned}
\inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} V \\ U \end{bmatrix} - \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} Q \right\| &= \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} (\hat{M}V - \hat{N}U) - Q \\ M^*U + N^*V \end{bmatrix} \right\| \\
&\le \left\| \begin{bmatrix} (\hat{M}V - \hat{N}U) - Q_0 \\ M^*U + N^*V \end{bmatrix} \right\| \\
&= \|M^*U + N^*V\| \\
&= \left\| P_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}} P_{\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]\mathcal{H}} \right\|.
\end{aligned}$$

So,

$$\left\| P_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}} P_{\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]\mathcal{H}} \right\| = \|M^*U + N^*V\| = \inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} V \\ U \end{bmatrix} - \begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} Q \right\|.$$

The equality between $\left\| P_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}} P_{\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]\mathcal{H}} \right\|$ and $\inf_{Q\in\mathcal{S}} \left\| \begin{bmatrix} N \\ M \end{bmatrix} - \begin{bmatrix} -\hat{U}^* \\ \hat{V}^* \end{bmatrix} Q \right\|$ can be proved similarly.

The proof of (c)**.**

If

$$\left\| P_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]\mathcal{H}} - P_{\left(\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]\mathcal{H}\right)^\perp} \right\| = \delta\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}, \left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H} \right)^\perp \right) < 1,$$

then by Proposition 3.1,

$$\delta\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}, \left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}\right) = \vec{\delta}\left(\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}, \begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}\right) = \inf_{Q\in\mathcal{S}}\left\|\begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix}Q\right\|,$$

where the last equation follows from the result (a).

If

$$\begin{aligned}
&\delta\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}, \left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}\right)\\
&= \max\left\{\vec{\delta}\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}, \left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}\right), \vec{\delta}\left(\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}, \begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}\right)\right\}\\
&= \vec{\delta}\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}, \left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}\right) = \left\|P_{\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}} P_{\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}}\right\|\\
&= 1,
\end{aligned}$$

due to Lemma 4.1, we can get that

$$\vec{\delta}\left(\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}, \begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}\right) = \left\|P_{\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}\right)^{\perp}} P_{\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}}\right\| = 1.$$

So,

$$\delta\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}, \left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}\right) = \inf_{Q\in\mathcal{S}}\left\|\begin{bmatrix} \hat{M}^* \\ -\hat{N}^* \end{bmatrix} - \begin{bmatrix} V \\ U \end{bmatrix}Q\right\|.$$

If

$$\begin{aligned}
&\delta\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}, \left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}\right)\\
&= \max\left\{\vec{\delta}\left(\begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}, \left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}\right), \vec{\delta}\left(\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}, \begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}\right)\right\}\\
&= \vec{\delta}\left(\left(\begin{bmatrix} N \\ M \end{bmatrix}\mathcal{H}\right)^{\perp}, \begin{bmatrix} V \\ U \end{bmatrix}\mathcal{H}\right)\\
&= 1,
\end{aligned}$$

the consequence follows directly. This completes the whole proof.

Remark 4.1. *The main idea of the proof of Theorem 4.1 is inspired by the work of* [6]*. However, the technique used here for linear time-varying systems is operator theoretic, which is quite different from the function theoretic technique used for linear time-invariant systems in* [6]*.*

Remark 4.2. *Comparing the results of Theorem 4.1 with that of Proposition 3.4, we can find that the time-varying gaps between the graph of a controller and the orthogonal complement of the inverse graph of a plant are in fact the gaps between them. That is,* *when we consider the feedback stabilization problem, the gap metric and the time-varying gap metric are in fact the same. This result also reflects that the time-varying gap metric is not better than the gap metric in the feedback stabilization and robust problems.*

Considering from the standpoint of computation, we say that Theorem 4.1 offers a solution to the supermum in Proposition 3.4. And in this situation, the supermum is just achieved at $n = 0$.

Remark 4.3. *From the result (c) of Theorem 4.1, we conclude that in order to compute the gap between the graph of a controller and the orthogonal complement of the inverse graph of a plant, it is enough to solve a single two-block problem rather than the previous two two-block problems.*

In the end, we conclude our paper by showing how our results can be used to compute the optimal minimal angles of time-varying systems.

For $L \in \mathcal{L}$ stabilizable, we denote by $\mathcal{S}(L)$ the set of all the stabilizing controllers $C \in \mathcal{L}$ for L. Then by the Youla parametrization Theorem 2.1, the set $\mathcal{S}(L)$ can be characterized as follows:

$$\mathcal{S}(L) = \left\{ \begin{array}{l} \text{controllers with strong right representations} \begin{bmatrix} \hat{Y} + NQ \\ \hat{X} + MQ \end{bmatrix} \\ \text{and strong left representations } \begin{bmatrix} -(X + Q\hat{M}) & Y + Q\hat{N} \end{bmatrix},\ Q \in \mathcal{S} \end{array} \right\}.$$

And the optimal minimal angles of time-varying systems are defined by the time-varying gaps as follows.

Definition 4.2. [11, 12] *Suppose plant L is stabilizable. Then the optimal minimal angle for L, $\Theta_{\min}^{opt}$ is*

$$\cos \Theta_{\min}^{opt} = \inf \left\{ \alpha \left(\mathcal{G}^{-1}(L), (\mathcal{G}(C))^{\perp} \right) : C \in \mathcal{S}(L) \right\}.$$

If C_0 attains this infimum, we say that C_0 is a maximally stabilizing controller for L.

The following theorem shows that the cosine of the optimal minimal angle is equal to the norm of a time-varying Hankel operator and the maximally stabilizing controller always exists whenever the plant L is stabilizable.

Theorem 4.2. *Suppose $\begin{bmatrix} M \\ N \end{bmatrix}$ and $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$ are normalized strong right and left representations for plant L, respectively. $\hat{X}, \hat{Y} \in \mathcal{S}$ satisfy the double Bezout identity in Theorem 2.1. Then,*

$$\cos \Theta_{\min}^{opt} = \inf_{C \in \mathcal{S}(L)} \delta \left(\mathcal{G}(C), (\mathcal{G}^{-1}(L))^{\perp} \right) = \| H_{M^*\hat{X} + N^*\hat{Y}} \|,$$

where $H_{M^*\hat{X}+N^*\hat{Y}}$ is a time-varying Hankel operator with the symbol $M^*\hat{X} + N^*\hat{Y} \in \mathcal{B}(\mathcal{H})$. Moreover, this infimum can always be achieved by a stabilizing controller C_0.

Proof: From the definition of the time-varying gap, it follows that

$$\alpha \left(\mathcal{G}^{-1}(L), (\mathcal{G}(C))^{\perp} \right) = \alpha \left(\mathcal{G}(C), (\mathcal{G}^{-1}(L))^{\perp} \right).$$

For any $C \in \mathcal{S}(L)$, the controller has a strong right representation $\begin{bmatrix} \hat{Y} + NQ \\ \hat{X} + MQ \end{bmatrix}$ and a strong left representation $\begin{bmatrix} -(X + Q\hat{M}) & Y + Q\hat{N} \end{bmatrix}$, $Q \in \mathcal{S}$. Then, by Theorem 4.1, we can get that

$$\begin{aligned}\alpha\left(\mathcal{G}^{-1}(L),(\mathcal{G}(C))^{\perp}\right) &= \delta\left(\mathcal{G}(C),(\mathcal{G}^{-1}(L))^{\perp}\right)\\ &= \inf_{\tilde{Q}\in\mathcal{S}}\left\|\begin{bmatrix}\hat{M}^*\\ -\hat{N}^*\end{bmatrix}-\begin{bmatrix}\hat{Y}+NQ\\ \hat{X}+MQ\end{bmatrix}\tilde{Q}\right\|\\ &= \|M^*(\hat{X}+MQ)+N^*(\hat{Y}+NQ)\|\\ &= \|M^*\hat{X}+N^*\hat{Y}+Q\|.\end{aligned}$$

Taking the infimum over all the controllers in the set $\mathcal{S}(L)$, we have

$$\inf_{C\in\mathcal{S}(L)}\delta\left(\mathcal{G}(C),(\mathcal{G}^{-1}(L))^{\perp}\right)=\inf_{Q\in\mathcal{S}}\|M^*\hat{X}+N^*\hat{Y}+Q\|=dist\left(M^*\hat{X}+N^*\hat{Y},\mathcal{S}\right).$$

Here, $dist\left(M^*\hat{X}+N^*\hat{Y},\mathcal{S}\right)$ denotes the distance from the operator $M^*\hat{X}+N^*\hat{Y}\in\mathcal{B}(\mathcal{H})$ to the space $\mathcal{S}$. By the commutant lifting technique for the optimization problem of linear time-varying systems (see [15]), there exists a $Q_0\in\mathcal{S}$ such that

$$\begin{aligned}\cos\Theta_{\min}^{opt} &= \inf_{C\in\mathcal{S}(L)}\alpha\left(\mathcal{G}^{-1}(L),(\mathcal{G}(C))^{\perp}\right)\\ &= \inf_{C\in\mathcal{S}(L)}\delta\left(\mathcal{G}(C),(\mathcal{G}^{-1}(L))^{\perp}\right)\\ &= \inf_{Q\in\mathcal{S}}\|M^*\hat{X}+N^*\hat{Y}+Q\|\\ &= \|H_{M^*\hat{X}+N^*\hat{Y}}\|\\ &= \|M^*\hat{X}+N^*\hat{Y}+Q_0\|,\end{aligned}$$

where $\|H_{M^*\hat{X}+N^*\hat{Y}}\|$ is the norm of the time-varying Hankel operator $H_{M^*\hat{X}+N^*\hat{Y}}$ with the symbol $M^*\hat{X}+N^*\hat{Y}\in\mathcal{B}(\mathcal{H})$. The above arguments also imply that the maximally stabilizing controller C_0 for L exists. And the strong right representation for C_0 is given by $\begin{bmatrix}\hat{Y}+NQ_0\\ \hat{X}+MQ_0\end{bmatrix}$, where the Youla parameter Q_0 achieves the infimum $\inf_{Q\in\mathcal{S}}\|M^*\hat{X}+N^*\hat{Y}+Q\|$.

Remark 4.4. *For more details about commutant lifting technique and time-varying Hankel operator, please refer to* [15].

5. **Conclusions.** In this paper, the relationship between the gap metric and the time-varying gap metric for linear time-varying systems has been considered. By using the operator theoretic technique, we showed that in the feedback configuration, the gaps and the time-varying gaps between the graph of a controller and the orthogonal complement of the inverse graph of a plant are in fact identical. From this fact, we got that the time-varying gap metric is actually not better than the gap metric in the feedback stabilization and robust problems. Moreover, two important applications of the developed criteria were obtained. One was that the equivalence of these two metrics supplied a simple way for computing the supremum appearing in the definition of the time-varying gap. The other one was that the criterion was applied to compute the optimal minimal angles of stabilizable linear time-varying systems. And it was obtained that the value of the cosine of the optimal minimal angle is in fact the norm of a time-varying Hankel operator.

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