International Journal of Innovative
Computing, Information and Control
Volume **9**, Number **6**, June **2013**

ICIC International ©2013 ISSN 1349-4198
pp. 2393–2404

CRITERIA REDUCTION OF SET-VALUED ORDERED DECISION SYSTEM BASED ON APPROXIMATION QUALITY

Zichun Chen[1], Peng Shi[2,3], Penghui Liu[1] and Zheng Pei[1]

[1]School of Mathematics and Computer Engineering
Xihua University
Chengdu 610039, P. R. China
czclph@163.com

[2]School of Engineering and Science
Victoria University
Melbourne, Vic 8001, Australia
Peng.Shi@vu.edu.au

[3]School of Electrical and Electronic Engineering
The University of Adelaide
Adelaide, SA 5005, Australia

Received March 2012; revised July 2012

Abstract. *The purpose of this paper is to further investigate criteria reduction in the set-valued ordered decision system, in which some objects may have more than one value for an attribute with preference-ordered domains. By considering the uncertainty of criteria values in such information system, we define a similarity dominance relation and give the dominance-based rough approximation of the upward and downward unions. To obtain criteria reduction from the inconsistent set-valued ordered decision system, the upward and downward approximation quality is proposed. Two types of criteria reduction are then formed for preserving the approximation quality. Furthermore, by the significance of the criterion, an algorithm is designed to compute a smallest reduction of a given set-valued ordered decision system. And finally, a numerical example is employed to substantiate the conceptual arguments.*

Keywords: Set-valued ordered decision systems, Similarity dominance relation, Rough set, Approximation quality, Criteria reduction

1. **Introduction.** Rough set theory [13], proposed by Pawlak, is an effective tool for data analysis. It can be used in information systems to describe the dependencies among attributes, evaluate the significance of attributes and derive decision rules. The Pawlak's rough set model is constructed based on equivalence relations which are viewed by many to be one of the main limitations [14, 15]. By relaxing equivalence relations to more general binary relation, the Pawlak's rough set can be extended to a more general model [1, 2, 8, 19, 23, 25, 27]. Attributes reduction is one of the important topics in rough set theory, its aim is to find a minimal attribute subset of the original data sets that is the most informative, and all other attributes can be deleted from the databases with the minimal information loss. Many types of attributes reduction based on the equivalence relation have been proposed. For instance, discernibility matrix [20], consistency of data [12], dependency of attributes [24] and information entropy [11] were employed to find reduction of an information system. In recent years, more attention has been paid to attribute reduction based on the more general binary relation and covering in rough set research [3, 8, 11, 16, 17, 22, 26, 27].

Although the classical rough set approach is a powerful tool for handling many problems, it is not able to deal with inconsistencies originating from the criteria, e.g., attributes with preference-ordered domains like product quality, market share, and debt ratio [7]. To solve this problem, Greco et al. have proposed an extension of Pawlak's rough set approach, which is called the Dominance-based Rough Set Approach (DRSA) to take into account the ordering properties of criteria [5, 6, 7]. This innovation is mainly based on substitution of the equivalence relation by a similarity dominance relation which is reflexive and transitive. In recent years, many studies have been made in DRSA [4, 10, 16, 17, 18, 21, 22, 26, 29].

Set-valued information systems are an important type of data table, and generalized of single-valued information systems. In such information systems, some objects may have more than one values which used to characterize uncertain information and missing information [28]. From the viewpoint of semantics, set-valued information systems can be summarized into two categories: disjunctive and conjunctive systems, and incomplete information systems with some unknown attribute values or partial known attribute values can be viewed as disjunctive set-valued information systems [8, 17]. In recent years, some problems of decision making in the context of set-valued information systems have been studied [8, 9, 17, 27]. For example, in [17], by introducing two dominance relations to the two types of set-valued information systems, Qian et al. investigated the problems of criteria reduction and decision rules extracted from these two types of information systems and decision tables. Nevertheless, the authors only considered the criteria reduction of the consistent set-valued ordered decision table, and the presented dominance relation in the disjunctive set-valued ordered information system was not transitive. Due to the rampant existence of the inconsistent systems in real life, new approach to knowledge reduction in the inconsistent set-valued ordered decision system has become a necessity.

The purpose of this paper is to further study criteria reduction of the inconsistent set-valued ordered decision system interpreted disjunctively from the viewpoint of approximation quality. By introducing a similarity dominance relation in the set-valued ordered information system, we propose the concept of upward and downward approximation quality to characterize the relevance between criteria and decisions of set-valued ordered decision system (SODS), and define two types of criteria reduction by means of approximation quality. Furthermore, by the significance of the criterion, an algorithm is designed to obtain a smallest criteria reduction in inconsistent SODS.

The rest of this paper is organized as follows: we first present some basic concepts of rough set theory and DRSA in Section 2. In Section 3, we introduce a similarity dominance relation in the SODS, by which, a dominance-based rough set approach to SODS is established in Section 4. In Section 5, we define approximation quality to characterize the relevance between criteria and decision of the SODS, and propose the criteria reduction preserving the upward and downward approximation quality. An illustrative example is analyzed in Section 6 to show the feasibility of the conceptual arguments. Results are summarized in Section 7.

2. **Some Basic Concepts of Rough Set Theory and DRSA.** An information system (IS) is a pair $\mathbf{S} = (U, AT)$, where $U = \{x_1, \ldots, x_n\}$ is a nonempty finite set of objects and $AT = \{a_1, \ldots, a_m\}$ a nonempty finite set of attributes. With every subset of attributes $A \subseteq AT$ a binary relation $Ind(A)$, called the A-indiscernibility relation, is defined by $Ind(A) = \{(x, y) \in U \times U : \forall a \in A, a(x) = a(y)\}$, where $a(x)$ is the feature value of object x, then $Ind(A)$ is an equivalence relation. By $[x]_A$ we denote the equivalence class of x. For $X \subseteq U$, the sets $\underline{A}X = \{x \in U : [x]_B \subseteq X\}$ and $\overline{A}X = \{x \in U : [x]_B \cap X \neq \emptyset\}$

are called the A-lower and the A-upper approximations of X, respectively. X is said to be definable if $\underline{A}X = \overline{A}X$; otherwise, X is a rough set.

A decision system (DS) is a pair $\mathbf{S} = (U, C \cup \{d\})$, where d is called the decision attribute, and the elements in C are called condition attributes. We say $a \in A \subseteq C$ is relatively dispensable in A if $Pos_A(d) = Pos_{A-\{a\}}(d)$. Otherwise, a is said to be relatively indispensable in A. Here $Pos_A(d)$ is the union of A-lower approximation of all the equivalence classes of d, i.e., $Pos_A(d) = \bigcup_{X \in U/d} \underline{A}X$. If every attribute in A is relatively indispensable, we say that $A \subseteq C$ is relatively independent in $\mathbf{S}$. A subset $A \subseteq C$ is called a relative reduction in $\mathbf{S}$ if A is relatively independent in $\mathbf{S}$ and $Pos_A(d) = Pos_C(d)$. The collection of all relatively indispensable attributes in C is called the relative core of $\mathbf{S}$. The approximation quality of d with respect to A is defined by

$$\gamma_A(d) = \frac{|Pos_A(d)|}{|U|} \tag{1}$$

The approximation quality is also called dependency, reflecting relevance between condition and decision.

For multi-criteria decision analysis, each object in an IS can be seen as a sample decision, and each condition attribute is a criterion for that decision. A criterion's domain of values is usually ordered according to the decision-maker's preferences. In general, an increasing preference and a decreasing preference are considered by a decision maker. In the following we only consider criteria with increasing preference.

Definition 2.1. [22] *An IS or DS is called a ordered information system (OIS) or ordered decision system (ODS) if all conduction attributes are criteria.*

Within the basic DRSA, the notions of weak preference relation $\succeq_a$ and P-dominance relation D_P are defined as follows. For a criterion a and any $x, y \in U$, $x \succeq_a y$ means that x is at least as good as (is weakly preferred to) y with respect to criterion a. The weak preference relation $\succeq_a$ is supposed to be a complete pre-order, i.e., complete, reflexive, transitive, and antisymmetric relation [4]. Moreover, for a subset of criteria P, the P-dominance relation D_P is defined by $D_P = \{(x, y) \in U \times U : \forall a \in P, x \succeq_a y\}$. When $(x, y) \in D_P$, we say that x P-dominates y, and that y is P-dominated by x. Given the dominance relation D_P, the P-dominating set and P-dominated set of x are defined as $D_P^+(x) = \{y \in U : (y, x) \in D_P\}$ and $D_P^-(x) = \{y \in U : (x, y) \in D_P\}$, respectively. In addition, it is assumed that the decision attribute d makes a partition of U into a finite number of classes $\mathbf{D} = \{D_1, D_2, \ldots, D_r\}$, which are ordered, that is, for all $i, j \le r$, if $i \ge j$, then the objects from D_i are preferred to the objects from D_j. In order to reflect the total order and dominance relations, the upward and downward unions are defined respectively as $D_i^{\geq} = \bigcup_{j \ge i} D_j$ and $D_i^{\leq} = \bigcup_{j \le i} D_j$. Then, we have

$$D_1^{\geq} = D_r^{\leq} = U,\ D_r^{\geq} = D_r,\ D_1^{\leq} = D_1,\ D_i^{\geq} = U - D_{i-1}^{\leq},\ D_i^{\leq} = U - D_{i+1}^{\geq},\ (i \neq 1, r) \tag{2}$$

The statement $x \in D_i^{\geq}$ means "x belongs to at least class D_i", whereas $x \in D_i^{\leq}$ means "x belongs to at most class D_i" [7]. Furthermore, the P-lower and P-upper approximations of $D_i^{\geq}$ and $D_i^{\leq}$ are defined as

$$\underline{P}(D_i^{\geq}) = \{x \in U : D_P^+(x) \subseteq D_i^{\geq}\},\quad \overline{P}(D_i^{\geq}) = \{x \in U : D_P^-(x) \cap D_i^{\geq} \neq \emptyset\} \tag{3}$$

$$\underline{P}(D_i^{\leq}) = \{x \in U : D_P^-(x) \subseteq D_i^{\geq}\},\quad \overline{P}(D_i^{\leq}) = \{x \in U : D_P^+(x) \cap D_i^{\geq} \neq \emptyset\} \tag{4}$$

3. **Similarity Dominance Relation.** In many practical issues, it may happen that some of the attribute values for an object are set-valued, which are always used to characterize uncertain information and missing information in information systems.

Definition 3.1. [17] *An information system* $\boldsymbol{S} = (U, AT)$ *is called a set-valued information system (SIS), if* $\forall a \in AT$, $\forall x \in U$, $a(x) \in 2^{V_a} - \{\emptyset\}$, *where* V_a *is the domain of a attribute* a. *A decision system* $\boldsymbol{S} = (U, C \cup d)$ *is called a set-valued decision system (SDS), if* (U, C) *is a SIS,* (U, d) *is a single-valued information system.*

Table 1 illustrates an SDS, where $U = \{x_1, x_2, \ldots, x_8\}$, $C = \{a_1, a_2, a_3, a_4\}$ is the set of condition attribute, d is a decision attribute.

Table 1. A set-valued ordered decision information

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
a_1	{1}	{1}	{1,2}	{2}	{3}	{1}	{1,2}	{2,3}
a_2	{1}	{1,2}	{2}	{2}	{1}	{1}	{2}	{2}
a_3	{1,2}	{2,3}	{3}	{2,3}	{1}	{1,2}	{2}	{1}
a_4	{2}	{1}	{1,2}	{1}	{1,2}	{1,2}	{1}	{1,2}
d	1	2	2	2	3	1	2	3

From the viewpoint of semantic interpretation, the SIS can be classified into two categories: disjunctive and conjunctive system [8]. In this paper, we focus on the set-valued ordered decision information (SODS) interpreted disjunctively, also denote $\mathbf{S} = (U, C \cup d)$, where all conduction attributes are criteria and $\forall a \in C$, $\forall x \in U$, each value in the subset $a(x)$ is equally possible.

In a set-valued ordered information system, the objects may have more than one evaluations with respect to the condition criteria. The dominance relations between the objects cannot be determined with certainty. Thus, we define a weak preference relation $\succeq_a$ on V_a based on one possibility x dominates y with respect to criterion a as follows:

$x \succeq_a y$ if and only if $\forall v_a^y \in a(y)$, $\exists v_a^x \in a(x)$ such that $v_a^x \geq v_a^y$

If $x \succeq_a y$, then for $\max a(y)$, there $v_a^x \in a(x)$ such that $v_a^x \geq \max a(y)$, so $\max a(x) \geq v_a^x \geq \max a(y)$. Conversely, suppose $\max a(x) \geq \max a(y)$, then $\forall v_a^y \in a(y)$, there $\max a(x) \in a(x)$ such that $\max a(x) \geq \max a(y) \geq v_a^y$, so $x \succeq_a y$. Thus,

$$x \succeq_a y \Leftrightarrow \max a(x) \geq \max a(y) \tag{5}$$

According to Equation (5), It easily obtain that the weak preference relation $\succeq_a$ is a completely pre-ordered on V_a.

Definition 3.2. *Let* $\boldsymbol{S} = (U, C \cup d)$ *be a SODS,* $A \subseteq C$, *we define a dominance relation* $R_A^{\succeq}$ *as follows:*

$$R_A^{\succeq} = \{(x, y) \in U^2 : \forall a \in A, x \succeq_a y\}. \tag{6}$$

Lemma 3.1. *Let* $\boldsymbol{S} = (U, C \cup d)$ *be a SODS,* $A \subseteq C$, *we have that*

(1) $R_A^{\succeq}$ *is a similarity dominance relation, i.e.,* $R_A^{\succeq}$ *is reflexive and transitive;*

(2) $R_A^{\succeq} = \bigcap_{a \in A} R_a^{\succeq}$, *where* $R_a^{\geq} = R_{\{a\}}^{\geq}$;

(3) *if* $B \subseteq A \subseteq C$, *then* $R_C^{\succeq} \subseteq R_A^{\succeq} \subseteq R_B^{\succeq}$.

Remark 3.1. *In* [17], *Qian et al. defined a binary dominance relation in the disjunctive set-valued ordered information system as follows:* $R_A^{\vee\geq} = \{(x, y) \in U^2 : \forall a \in A, \max a(x) \geq \min a(y)\}$. *However,* $R_A^{\vee\geq}$ *is only reflexive, not transitive, i.e., it is not pre-ordered.*

By dominance relation $R_A^{\succeq}$, we can obtain the A-dominating set and the A-dominated of x as follows:

$$R_A^{\succeq +}(x) = \{y \in U : (y, x) \in R_A^{\succeq}\} = \{y \in U : \forall a \in A, \max a(y) \geq \max a(x)\} \tag{7}$$

$$R_A^{\succeq -}(x) = \{y \in U : (x, y) \in R_A^{\succeq}\} = \{y \in U : \forall a \in A, \max a(x) \geq \max a(y)\} \tag{8}$$

Lemma 3.2. *Let $\boldsymbol{S} = (U, C \cup d)$ be a SODS, $A, B \subseteq C$, $x, y \in U$, then we have that*
(1) $B \subseteq A \Rightarrow R_A^{\succeq +}(x) \subseteq R_B^{\succeq +}(x)$, $R_A^{\succeq -}(x) \subseteq R_B^{\succeq -}(x)$;
(2) $y \in R_A^{\succeq +}(x) \Leftrightarrow R_A^{\succeq +}(y) \subseteq R_A^{\succeq +}(x)$, $y \in R_A^{\succeq -}(x) \Leftrightarrow R_A^{\succeq -}(y) \subseteq R_A^{\succeq -}(x)$;
(3) $R_A^{\succeq +}(x) = \bigcup\{R_C^{\succeq +}(y) : y \in R_A^{\succeq +}(x)\}$, $R_A^{\succeq -}(x) = \bigcup\{R_C^{\succeq -}(y) : y \in R_A^{\succeq -}(x)\}$.

Example 3.1. *Let us compute the C-dominating set and the C-dominated based on dominance relation $R_C^{\succeq}$. In Table 1, where $C = \{a_1, a_2, a_3, a_4\}$.*

By $R_C^{\succeq +}(x) = \{y \in U : \forall a \in C, \max a(y) \geq \max a(x)\}$ and $R_C^{\succeq -}(x) = \{y \in U : \forall a \in C, \max a(y) \leq \max a(x)\}$, one can get that
$R_C^{\succeq +}(x_1) = \{x_1, x_3, x_6\}$, $R_C^{\succeq +}(x_2) = \{x_2, x_3, x_4\}$, $R_C^{\succeq +}(x_3) = \{x_3\}$, $R_C^{\succeq +} = \{x_3, x_4\}$,
$R_C^{\succeq +}(x_5) = \{x_5, x_8\}$, $R_C^{\succeq +}(x_6) = \{x_1, x_3, x_6\}$, $R_C^{\succeq +}(x_7) = \{x_3, x_4, x_7\}$, $R_C^{\succeq +}(x_8) = \{x_8\}$.
$R_C^{\succeq -}(x_1) = \{x_1, x_6\}$, $R_C^{\succeq -}(x_2) = \{x_2\}$, $R_C^{\succeq -}(x_3) = \{x_1, x_2, x_3, x_4, x_6, x_7\}$,
$R_C^{\succeq -}(x_4) = \{x_2, x_4, x_7\}$, $R_C^{\succeq -}(x_5) = \{x_5\}$, $R_C^{\succeq -}(x_6) = \{x_1, x_6\}$, $R_C^{\succeq -}(x_7) = \{x_7\}$,
$R_C^{\succeq -}(x_8) = \{x_5, x_8\}$.

4. **Dominance-Based Rough Approximations.** In this section, we will investigate dominance-based rough approximations of the upward and downward unions based on the similarity dominance relation and gives its some important properties.

Let $\mathbf{S} = (U, C \cup \{d\})$ be a given SODS, $\mathbf{D} = \{D_1, D_2, \cdots, D_r\}$ be ordered decision classes induced by d, $\mathbf{D}^{\geq} = \{D_1^{\geq}, D_2, \cdots, D_r^{\geq}\}$ and $\mathbf{D}^{\leq} = \{D_1^{\leq}, D_2, \cdots, D_r^{\leq}\}$ be the upward unions and downward unions classes, respectively.

Definition 4.1. *Let $\boldsymbol{S} = (U, C \cup \{d\})$ be a SODS, $A \subseteq C$. Then, $\forall D_i^{\geq} \in \boldsymbol{D}^{\geq}$, $\forall D_i^{\leq} \in \boldsymbol{D}^{\leq}$, the A-lower approximations and A-upper approximations of $D_i^{\geq}$ and $D_i^{\leq}$ are defined as follows:*

$$\underline{A}(D_i^{\geq}) = \bigcup\{R_A^{\succeq +}(x) : R_A^{\succeq +}(x) \subseteq D_i^{\geq}\}, \quad \overline{A}(D_i^{\geq}) = \bigcup\{R_A^{\succeq +}(x) : x \in D_i^{\geq}\} \tag{9}$$

$$\underline{A}(D_i^{\leq}) = \bigcup\{R_A^{\succeq -}(x) : R_A^{\succeq -}(x) \subseteq D_i^{\leq}\}, \quad \overline{A}(D_i^{\leq}) = \bigcup\{R_A^{\succeq -}(x) : x \in D_i^{\leq}\} \tag{10}$$

The lower and upper approximations of the upward and downward unions have the following important properties.

Theorem 4.1. *Let $\boldsymbol{S} = (U, C \cup \{d\})$ be a SODS, $A, B \subseteq C$, $\forall D_i^{\geq} \in \boldsymbol{D}^{\geq}$, $\forall D_i^{\leq} \in \boldsymbol{D}^{\leq}$, we have that*

$$\underline{A}(D_i^{\geq}) = U - \overline{A}(D_{i-1}^{\leq}), \quad (i \geq 2), \quad \underline{A}(D_i^{\leq}) = U - \overline{A}(D_{i+1}^{\geq}), \quad (i \leq r-1) \tag{11}$$

$$\underline{A}(D_1^{\geq}) = \overline{A}(D_1^{\geq}) = U, \quad \underline{A}(D_r^{\leq}) = \overline{A}(D_r^{\leq}) = U \tag{12}$$

$$\underline{A}(D_i^{\geq}) \subseteq D_i^{\geq} \subseteq \overline{A}(D_i^{\geq}), \quad \underline{A}(D_i^{\leq}) \subseteq D_i^{\leq} \subseteq \overline{A}(D_i^{\leq}) \tag{13}$$

$$\underline{A}(D_i^{\geq}) = \{x \in U : R_A^{\geq +}(x) \subseteq D_i^{\geq}\} \tag{14}$$

$$\overline{A}(D_i^{\geq}) = \{x \in U : R_A^{\geq -}(x) \cap D_i^{\geq} \neq \emptyset\} \tag{15}$$

$$\underline{A}(D_i^{\leq}) = \{x \in U : R_A^{\geq -}(x) \subseteq D_i^{\leq}\} \tag{16}$$

$$\overline{A}(D_i^{\leq}) = \{x \in U : R_A^{\geq +}(x) \cap D_i^{\leq} \neq \emptyset\} \tag{17}$$

$$B \subseteq A \Rightarrow \underline{B}(D_i^{\geq}) \subseteq \underline{A}(D_i^{\geq}), \quad \overline{A}(D_i^{\geq}) \subseteq \overline{B}(D_i^{\geq}) \tag{18}$$

$$B \subseteq A \Rightarrow \underline{B}(D_i^{\leq}) \subseteq \underline{A}(D_i^{\leq}), \quad \overline{A}(D_i^{\leq}) \subseteq \overline{B}(D_i^{\leq}) \tag{19}$$

Proof: These proof can be concluded similarly in [10, 26].

Definition 4.2. *Let $\boldsymbol{S} = (U, C \cup \{d\})$ be a SODS, and denote $R_d^{\geq} = \{(x, y) \in U^2 : d(x) \geq d(y)\}$. If $R_C^{\succeq} \subseteq R_d^{\geq}$, then $\boldsymbol{S}$ is called consistent; otherwise it is called inconsistent.*

According to Definition 4.2, in a consistent SODS, if x is better than y in terms of criterion a, then x's decision should not be worse than y's.

Theorem 4.2. *Let $\boldsymbol{S} = (U, C \cup \{d\})$ be a SODS, then the following statements are equivalent:*

(1) $S = (U, C \cup \{d\}, V, f)$ *is consistent.*
(2) $\forall x \in U$, $R_C^{\succeq +}(x) \subseteq D_j^{\geq}$, *where* $j = d(x)$.
(3) $\forall i \geq r$, $\underline{C}(D_i^{\geq}) = D_i^{\geq} = \overline{C}(D_i^{\geq})$.
(4) $\forall x \in U$, $R_C^{\succeq -}(x) \subseteq D_j^{\leq}$, *where* $j = d(x)$.
(5) $\forall i \geq r$, $\underline{C}(D_i^{\leq}) = D_i^{\leq} = \overline{C}(D_i^{\leq})$.

Proof: (1) $\Rightarrow$ (2): $\forall x \in U$, if $\forall y \in R_C^{\succeq +}(x)$, then $(y, x) \in R_C^{\succeq}$. Since $\mathbf{S}$ is consistent, so we have $(y, x) \in R_d^{\geq}$, it follows that $d(y) \geq d(x) = j$. Therefore, $R_C^{\succeq +}(x) \subseteq D_j^{\geq}$ holds.
(2) $\Rightarrow$ (1): $\forall (y, x) \in R_C^{\succeq}$, i.e., $y \in R_C^{\succeq +}(x)$, by (2) we obtain $y \in R_C^{\succeq +}(x) \subseteq D_j^{\geq}$, where $j = d(x)$. Hence, $d(y) \geq j = d(x)$, that is, $(y, x) \in R_d^{\geq}$. It follows that $\mathbf{S}$ is consistent.
(2) $\Rightarrow$ (3): $\forall i \geq r$, if $y \in \overline{C}(D_i^{\geq})$, then by Definition 4.1, there exist $x \in D_i^{\geq}$ such that $y \in R_C^{\succeq +}(x)$. By (2) we obtain $y \in D_j^{\geq}$, where $j = d(x)$, this indicates $d(y) \geq d(x)$. However, $x \in D_i^{\geq}$, $d(x) \geq i$, so $d(y) \geq i$, it follows that $y \in D_i^{\geq}$; hence, $\overline{C}(D_i^{\geq}) \subseteq D_i^{\geq}$. On the other hand, $D_i^{\geq} \subseteq \overline{C}(D_i^{\geq})$ by Equation (13), so we have $D_i^{\geq} = \overline{C}(D_i^{\geq})$. Similarly, we can prove $\underline{C}(D_i^{\geq}) = D_i^{\geq}$.
(3) $\Rightarrow$ (2): $\forall x \in U$. Since $x \in D_j^{\geq}$, where $j = d(x)$, so we have $x \in \underline{C}(D_j^{\geq})$ by (3). Thus, by Definition 4.1, we can obtain $R_C^{\succeq +}(x) \subseteq D_j^{\geq}$, where $j = d(x)$.

The proofs of the rest (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) are similar to the proof of (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

5. **Criteria Reduction in SODS.** Attribute reduction is one of major topics in the rough set approach. By the method, superfluous condition attributes are removed so that we may find condition attributes related to the decision attribute. Through using discernibility matrices, Qian et al. have proposed criteria reduction to the consistent set-valued ordered decision system [17]. Due to the rampant existence of the inconsistent systems in real life, new approach to criteria reduction in the inconsistent set-valued ordered decision system has become a necessity. In this section, we will introduce approximation quality to characterize the relevance between condition criteria and decision of the SODS, and propose the criteria reduction preserving the upward and downward approximation quality.

Definition 5.1. *Let $\boldsymbol{S} = (U, C \cup \{d\})$ be a SODS, $A \subseteq C$, and $\boldsymbol{D} = \{D_1, D_2, \cdots, D_r\}$ be ordered decision classes induced by d, denote by*

$$\gamma_A^{\geq}(d) = \frac{1}{r-1} \sum_{i=2}^{r} \frac{|\underline{A}(D_i^{\geq})|}{|D_i^{\geq}|}; \tag{20}$$

$$\gamma_A^{\leq}(d) = \frac{1}{r-1} \sum_{i=1}^{r-1} \frac{|\underline{A}(D_i^{\leq})|}{|D_i^{\leq}|} \tag{21}$$

then $\gamma_A^{\geq}(d)$ and $\gamma_A^{\leq}(d)$ are called upward and downward approximation quality of the subset of condition criteria A relative to the decision d, respectively.

Obviously, $0 \leq \gamma_A^{\geq}(d) \leq 1$ and $0 \leq \gamma_A^{\leq}(d) \leq 1$.

Remark 5.1. *It is noticed that $\underline{A}(D_1^{\geq}) = \underline{A}(D_r^{\leq}) = U$ for each $A \subseteq C$, we cannot obtain any information about approximation quality from the lower of $D_1^{\geq}$ and $D_r^{\leq}$. Thus, the* *upward approximation quality is defined based on* $D_i^{\geq}$ $(2 \leq i \leq r)$ *and the downward approximation quality based on* $D_i^{\leq}$ $(1 \leq i \leq r-1)$.

Theorem 5.1. *Let* $\boldsymbol{S} = (U, C \cup \{d\})$ *be a SODS. If* $B \subseteq A \subseteq C$, *then we have* $\gamma_B^{\geq}(d) \leq \gamma_A^{\geq}(d)$ *and* $\gamma_B^{\leq}(d) \leq \gamma_A^{\leq}(d)$.

Proof: Since $B \subseteq A$, by Equations (18) and (19), we have $\underline{B}(D_i^{\geq}) \subseteq \underline{A}(D_i^{\geq})$ and $\underline{B}(D_i^{\leq}) \subseteq \underline{A}(D_i^{\leq})$ for $1 \leq i \leq r$. Hence, $\gamma_B^{\geq}(d) \leq \gamma_A^{\geq}(d)$, $\gamma_B^{\leq}(d) \leq \gamma_A^{\leq}(d)$.

By Theorem 5.1, $\forall a \in A \subseteq C$, we have $\gamma_{A-\{a\}}^{\geq}(d) \leq \gamma_A^{\geq}(d)$ and $\gamma_{A-\{a\}}^{\leq}(d) \leq \gamma_A^{\leq}(d)$.

Theorem 5.2. *Let* $\boldsymbol{S} = (U, C \cup \{d\})$ *be a SODS, then we have that*

(1) $\boldsymbol{S}$ *is a consistent SODS* $\Leftrightarrow$ $\gamma_C^{\geq}(d) = 1$;

(2) $\boldsymbol{S}$ *is a consistent SODS* $\Leftrightarrow$ $\gamma_C^{\leq}(d) = 1$.

Proof: (1) If $\mathbf{S}$ is a consistent SODS, then for each $D_i^{\geq}$, we have $\underline{C}(D_i^{\geq}) = D_i^{\geq}$ by Theorem 4.2. Hence, $\gamma_C^{\geq}(d) = 1$. Conversely, suppose that $\gamma_C^{\geq}(d) = 1$ and $\mathbf{S}$ is a inconsistent SODS, then there is $D_j^{\geq}$ $(2 \leq j \leq r)$ such that $\underline{C}(D_j^{\geq}) \subset D_j^{\geq}$ by Theorem 4.2, i.e., $|\underline{C}(D_j^{\geq})| < |D_j^{\geq}|$. So we have $\gamma_C^{\geq}(d) < 1$, which is a contradiction. Hence, $\mathbf{S}$ is a consistent.

(2) The proof of (2) is similar to the proof of (1).

Definition 5.2. *Let* $\boldsymbol{S} = (U, C \cup \{d\})$ *be a SODS. For* $a \in C$, *if* $\gamma_{C-\{a\}}^{\geq}(d) = \gamma_C^{\geq}(d)$, *then we say a is upward dispensable, otherwise a is upward indispensable and* $\gamma_C^{\geq}(d) > \gamma_{C-\{a\}}^{\geq}(d)$. *The collection of all the upward indispensable elements is called the upward core, denoted by* $Core^{\geq}(\boldsymbol{S})$. *Similarly, if* $\gamma_{C-\{a\}}^{\leq}(d) = \gamma_C^{\leq}(d)$, *then we say a is downward dispensable, otherwise a is downward indispensable and* $\gamma_C^{\leq}(d) > \gamma_{C-\{a\}}^{\leq}(d)$. *The collection of all the downward indispensable elements is called the downward core, denoted by* $Core^{\leq}(\boldsymbol{S})$.

Definition 5.3. *Let* $\boldsymbol{S} = (U, C \cup \{d\})$ *be a SODS,* $A \subseteq C$. *We say A is an upward reduction if*

(1) $\gamma_A^{\geq}(d) = \gamma_C^{\geq}(d)$;

(2) $\forall B \subsetneq A$, $\gamma_A^{\geq}(d) > \gamma_B^{\geq}(d)$.

Correspondingly, we can define the downward reduction as follows.

Definition 5.4. *Let* $\boldsymbol{S} = (U, C \cup \{d\})$ *be a SODS,* $A \subseteq C$. *We say A is an downward reduction if*

(1) $\gamma_A^{\leq}(d) = \gamma_C^{\leq}(d)$;

(2) $\forall B \subsetneq A$, $\gamma_A^{\leq}(d) > \gamma_B^{\leq}(d)$.

Theorem 5.3. *Let* $\boldsymbol{S} = (U, C \cup \{d\})$ *be a SODS,* $A \subseteq C$, *then we have that*

(1) *if A is an upward reduction, then* $Core^{\geq}(\boldsymbol{S}) \subseteq A$;

(2) *if A is an downward reduction, then* $Core^{\leq}(\boldsymbol{S}) \subseteq A$.

Proof: (1) $\forall a \in Core^{\geq}(\mathbf{S})$, i.e., $\gamma_C^{\geq}(d) > \gamma_{C-\{a\}}^{\geq}(d)$. If $a \notin A$ then $A \subseteq C - \{a\}$. Since A is an upward reduction, we have $\gamma_C^{\geq}(d) \geq \gamma_{C-\{a\}}^{\geq}(d) \geq \gamma_A^{\geq}(d) = \gamma_C^{\geq}(d)$ by Definition 5.3, that is $\gamma_C^{\geq}(d) = \gamma_{C-\{a\}}^{\geq}(d)$, which is a contradiction. Hence, $Core^{\geq}(\mathbf{S}) \subseteq A$.

(2) The proof of (2) is similar to the proof of (1).

Theorem 5.4. *Let* $\boldsymbol{S} = (U, C \cup \{d\})$ *be a SODS,* $A \subseteq C$, *then we have that*

(1) *A is an upward reduction if and only if* $\forall i$ $(2 \leq i \leq r)$, $\underline{A}(D_i^{\geq}) = \underline{C}(D_i^{\geq})$, *and* $\forall B \subsetneq A$, $\exists j$ $(2 \leq j \leq r)$, $\underline{B}(D_j^{\geq}) \subsetneq \underline{A}(D_j^{\geq})$;

(2) *A is an downward reduction if and only if* $\forall i$ $(2 \leq i \leq r)$, $\underline{A}(D_i^{\leq}) = \underline{C}(D_i^{\leq})$, *and* $\forall B \subsetneq A$, $\exists j$ $(2 \leq j \leq r)$, $\underline{B}(D_j^{\leq}) \subsetneq \underline{A}(D_j^{\leq})$.

Proof: (1) Let A is an upward reduction. Since $\underline{A}(D_i^{\geq}) \subseteq \underline{C}(D_i^{\geq})$ for all $i \leq r$ by Equation (18), if there is i_0 such that $\underline{A}(D_{i_0}^{\geq}) \subsetneq \underline{C}(D_{i_0}^{\geq})$, that is $|\underline{A}(D_{i_0}^{\geq})| < |\underline{C}(D_{i_0}^{\geq})|$, then we have $\gamma_A^{\geq}(\mathbf{D}) < \gamma_C^{\geq}(\mathbf{D})$ by Theorem 5.1, which is a contradiction. Hence, $\forall i$ $(2 \leq i \leq r)$, $\underline{A}(D_i^{\geq}) = \underline{C}(D_i^{\geq})$. Furthermore, suppose that there is a $B \subsetneq A$ such that $\forall i$ $(2 \leq i \leq r)$, $\underline{B}(D_i^{\geq}) = \underline{A}(D_i^{\geq})$, then we have $\gamma_A^{\leq}(d) = \gamma_B^{\leq}(d)$, which is a also contradiction to $\gamma_A^{\leq}(d) > \gamma_B^{\leq}(d)$. Conversely, It is straightforward by Definition 4.1.

(2) The proof of (2) is similar to the proof of (1).

Theorem 5.4 shows that an upward/downward reduction is a minimal condition criterion set preserving the lower approximations of all the upper/downward unions.

Lemma 5.1. *Let $\boldsymbol{S} = (U, C \cup \{d\})$ be a SODS, $A \subseteq C$, then we have that*

(1) $\forall i \leq r, \underline{A}(D_i^{\geq}) = \underline{C}(D_i^{\geq}) \Leftrightarrow \overline{A}(D_i^{\leq}) = \overline{C}(D_i^{\leq})$;

(2) $\forall i \leq r, \underline{A}(D_i^{\leq}) = \underline{C}(D_i^{\leq}) \Leftrightarrow \overline{A}(D_i^{\geq}) = \overline{C}(D_i)^{\geq})$.

Proof: We can easily derive these conclusions according to Equations (11) and (12).

Theorem 5.5. *Let $\boldsymbol{S} = (U, C \cup \{d\})$ be a SODS, $A \subseteq C$, then we have that*

(1) *A is an upward reduction if and only if* $\forall i$ $(2 \leq i \leq r)$, $\overline{A}(D_i^{\leq}) = \overline{C}(D_i^{\leq})$, *and* $\forall B \subsetneq A$, $\exists j$ $(2 \leq j \leq r)$, $\overline{B}(D_j^{\leq}) \subsetneq \overline{A}(D_j^{\leq})$;

(2) *A is an downward reduction if and only if* $\forall i$ $(2 \leq i \leq r)$, $\overline{A}(D_i^{\geq}) = \overline{C}(D_i^{\geq})$, *and* $\forall B \subsetneq A$, $\exists j$ $(2 \leq j \leq r)$, $\overline{B}(D_j^{\geq}) \subsetneq \overline{A}(D_j^{\geq})$.

Proof: These conclusions are straightforward according to Theorem 5.4 and Lemma 5.1.

Theorem 5.5 shows that an upward/downward reduction is also a minimal condition criterion set preserving the upper approximations of all the downward/upper unions.

Definition 5.5. *Let $\boldsymbol{S} = (U, C \cup \{d\})$ be a SODS, $a \in C$, denote by*

$$Sig^{\geq}(a, C, d) = \gamma_C^{\geq}(d) - \gamma_{C-\{a\}}^{\geq}(d) \tag{22}$$

$$Sig^{\leq}(a, C, d) = \gamma_C^{\leq}(d) - \gamma_{C-\{a\}}^{\leq}(d) \tag{23}$$

then we say $Sig^{\geq}(a, C, d)$ and $Sig^{\leq}(a, C, d)$ are the upward and downward significance of a to d, respectively.

The upward/downward significance of a to d characterize the capacity of the upward/downward approximation quality when A is deleted from C.

By Definitions 5.2 and 5.5, we have the following result.

Theorem 5.6. *Let $\boldsymbol{S} = (U, C \cup \{d\})$ be a SODS, then we have that*

(1) $Core^{\geq}(\boldsymbol{S}) = \{a \in C : Sig^{\geq}(a, C, d) > 0\}$;

(2) $Core^{\leq}(\boldsymbol{S}) = \{a \in C : Sig^{\leq}(a, C, d) > 0\}$.

Definition 5.6. *Let $\boldsymbol{S} = (U, C \cup \{d\})$ be a SODS, $A \in C$, $a \in C$, $a \notin A$, denote by*

$$Sig_A^{\geq}(a, d) = \gamma_{A \cup \{a\}}^{\geq}(d) - \gamma_A^{\geq}(d) \tag{24}$$

$$Sig_A^{\leq}(a, d) = \gamma_{A \cup \{a\}}^{\leq}(d) - \gamma_A^{\leq}(d) \tag{25}$$

then we say $Sig_A^{\geq}(a, d)$ and $Sig_A^{\leq}(a, d)$ are the relative upward and downward significance of criterion a to A, respectively.

The relative upward/downward significance of criterion a to A can be used to measure the capacity of the upward/downward approximation quality when a is added to A. The larger the value of $Sig_A^{\geq}(a,d)/Sig_A^{\leq}(a,d)$ is, the more important a to A is.

Now we can use the upward/downward significance and relative upward/downward significance of a criterion as heuristic knowledge to design an algorithm to compute all the relative reduction. Firstly, by Theorem 5.6, we can calculate $Core^{\geq}(\mathbf{S})$ and $Core^{\leq}(\mathbf{S})$. Secondly, since the upward/downward core of criteria is a subset of all upward/downward reduction by Theorem 5.3, we can take the $Core^{\geq}(\mathbf{S})/Core^{\leq}(\mathbf{S})$ for original upward/downward reduction set and extend it one by one. According to the relative upward/downward significance of criterion, we can select the criterion with the biggest relative upward/downward significance to the upward/downward core and add it to the upward/downward core until its upward/downward approximation quality relative to d is equal to the upward/downward approximation quality of C to d.

Algorithm 1. Search of the upward core and the upward reduction of a given SODS.
Input: a given SODS $\mathbf{S} = (U, C \cup \{d\})$
Output: a upward reduction of the ordered decision table
Step 1: $\emptyset \to Core^{\geq}(\mathbf{S})$
Step 2: For each $a_i \in C$, compute $Sig^{\geq}(a_i, C, d)$. If $Sig^{\geq}(c_i, C) > 0$ then
          $Core^{\geq}(\mathbf{S}) \cup \{a_i\}$
          end
     end
Step 3: if $\gamma^{\geq}_{Core^{\geq}(\mathbf{S})}(d) = \gamma^{\geq}_{C}(d)$ then
Step 4:   return $Core^{\geq}(\mathbf{S})$ // Now, $Core^{\geq}(\mathbf{S})$ is the smallest upward reduction
Step 5: else $Core^{\geq}(\mathbf{S}) \to Red$
Step 6: for each $a_j \in \{C - Core^{\geq}(\mathbf{S})\}$, compute $Sig^{\geq}_{Core^{\geq}(\mathbf{S})}(a_j, d)$
      end
Step 7: if $Sig^{\geq}_{Core^{\geq}(\mathbf{S})}(a_k, d) = \max_{a_j \in \{C - Core^{\geq}(\mathbf{S})\}} Sig^{\geq}_{Core^{\geq}(\mathbf{S})}(a_j, d)$
      $Red \bigcup \{a_k\} \to Red$, go to Step 6
      else return Red
Step 8: end

Let $|C| = n$ and $|U| = m$, the time complexity of Algorithm 1 is $O(n^3m^2)$. Similarly, we can also develop an algorithm of an downward reduction. In the following, we use an example to illustrate feasibility of the algorithm.

Example 5.1. (Continued from Example 3.1). *We calculate the smallest upward reduction and core by Algorithm 1 in Table 1.*

From Table 1, we have that $D_1 = \{x_1, x_6\}$, $D_2 = \{x_2, x_3, x_4, x_7\}$, $D_3 = \{x_5, x_8\}$. $D_2^{\geq} = \{x_2, x_3, x_4, x_5, x_7, x_8\}$, $D_3^{\geq} = \{x_5, x_8\}$.

First, we calculate the upward significance of each criterion. Put $C = \{a_1, a_2, a_3, a_4\}$, $A_1 = \{a_2, a_3, a_4\}$, $A_2 = \{a_1, a_3, a_4\}$, $A_3 = \{a_1, a_2, a_4\}$, $A_4 = \{a_1, a_2, a_3\}$.

By Definition 5.1, we can obtain that $\gamma_C^{\geq}(d) = 1$, $\gamma_{A_1}^{\geq}(d) = \frac{5}{12}$, $\gamma_{A_2}^{\geq}(d) = 1$,$\gamma_{A_3}^{\geq}(d) = 1$, $\gamma_{A_4}^{\geq}(d) = 1$. *Then, we can calculate the upward significance of each criterion by Definition 5.5 as follows:*

$Sig^{\geq}(a_1, C, d) = \frac{7}{12}$, $Sig^{\geq}(a_2, C, d) = Sig^{\geq}(a_3, C, d) = Sig^{\geq}(a_4, C, d) = 0$.

Hence, $Core^{\geq}(\boldsymbol{S}) = \{a_1\}$. *Since* $\gamma^{\geq}_{Core^{\geq}(\boldsymbol{S})}(d) = \frac{11}{12}$, $\gamma_C^{\geq}(d) = 1$ *and* $\gamma^{\geq}_{Core^{\geq}(\boldsymbol{S})}(d) \neq \gamma_C^{\geq}(d)$, *we calculate the relative upward significance of other criteria to the upward core.*

Put $B_1 = \gamma_{Core^{\geq}(\boldsymbol{S})} = \{a_1\}$, *for* $C - B_1 = \{a_2, a_3, a_4\}$, *we have* $Sig^{\geq}_{B_1}(a_2, d) = \frac{1}{12}$, $Sig^{\geq}_{B_1}(a_3, d) = \frac{1}{12}$, $Sig^{\geq}_{B_1}(a_4, d) = 0$ *and* $\gamma^{\geq}_{\{a_1,a_2\}}(d) = \gamma_C^{\geq}(d)$, $\gamma^{\geq}_{\{a_1,a_3\}}(d) = \gamma_C^{\geq}(d)$. *So*

$\{a_1, a_2\}$ and $\{a_1, a_3\}$ are the set of the upward reduction in Table 1. Similarly, we can obtain $\{a_1, a_2\}$ and $\{a_1, a_3\}$ are also the set of the downward reduction in Table 1.

6. Numerical Example.

In many practical decision-making issues, set-valued information systems and set-valued decision tables have very wide applications, which can be used in intelligent decision-making and knowledge discovery from information systems with uncertain information and set-valued information [17]. A decision-maker may need to adopt a better one from some possible projects or find some directions from existing successful projects before making a decision. The purpose of this section is to illustrate how to obtain criteria reduction from the inconsistent SODS by using the approaches proposed in this paper.

Let us consider a practical decision problem of a summary of reviewer's reports for papers submitted to a journal. Suppose $U = \{x_1, \ldots, x_{10}\}$ be a set of ten evaluated papers, $C = \{a_1, a_2, a_3, a_4\}$ be a set of condition criteria, d is the decision attribute, where a_1 means "originality", a_2 means "presentation", a_3 means "technical soundness", a_4 means "motivation", d means "overall evaluation". The values of these criteria are

$$V_{a_1} = V_{a_2} = V_{a_3} = V_{a_4} = \{0(\textit{poor}),\ 1(\textit{fair}),\ 2(\textit{good})\},$$
$$V_d = \{1(\textit{reject}),\ 2(\textit{revision}),\ 3(\textit{accept})\}.$$

Assume that we have three specialists to evaluate the criterion values for these papers, and the evaluations given by these specialists are of the same importance, then it is possible that their evaluation results in the same criterion are not the same as one another. If we want to combine these evaluations together without losing information, we should union the evaluations given by every specialist for every attribute value. Furthermore, if we suppose here that "*poor* $<$ $fair$ $<$ *good*", "$reject$ $<$ $revision$ $<$ $ccept$" are two preference, then this problem can be stated by a set-valued ordered decision system as shown in Table 2.

TABLE 2. A summary of reviewer's reports for ten papers

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}
a_1	{1}	{0,1}	{0}	{0}	{2}	{0,2}	{1}	{0}	{1}	{1}
a_2	{1,2}	{0}	{0,1}	{1}	{0}	{0}	{1}	{0}	{1}	{0,1}
a_3	{1}	{2}	{1,2}	{1}	{1}	{1}	{0,2}	{2}	{0,1}	{1}
a_4	{2}	{0}	{0,2}	{0,2}	{1}	{1}	{2}	{0,1}	{2}	{2}
d	3	1	1	2	2	2	3	1	3	2

From Table 2, Equations (7) and (8), we have that
$R_C^{\succeq+}(x_1) = \{x_1\}$, $R_C^{\succeq+}(x_2) = \{x_2, x_7\}$, $R_C^{\succeq+}(x_3) = \{x_3, x_7\}$,
$R_C^{\succeq+}(x_4) = \{x_1, x_3, x_4, x_7, x_9, x_{10}\}$, $R_C^{\succeq+}(x_5) = R_C^{\succeq+}(x_6) = \{x_5, x_6\}$,
$R_C^{\succeq+}(x_7) = \{x_7\}$, $R_C^{\succeq+}(x_8) = \{x_3, x_7, x_8\}$, $R_C^{\succeq+}(x_9) = R_C^{\succeq+}(x_{10}) = \{x_1, x_7, x_9, x_{10}\}$;
$R_C^{\succeq-}(x_1) = \{x_1, x_4, x_9, x_{10}\}$, $R_C^{\succeq-}(x_2) = \{x_2\}$, $R_C^{\succeq-}(x_3) = \{x_3, x_4, x_8\}$, $R_C^{\succeq-}(x_4) = \{x_4\}$,
$R_C^{\succeq-}(x_5) = R_C^{\succeq-}(x_6) = \{x_5, x_6\}$, $R_C^{\succeq-}(x_7) = \{x_2, x_3, x_4, x_7, x_8, x_9, x_{10}\}$,
$R_C^{\succeq-}(x_8) = \{x_8\}$, $R_C^{\succeq-}(x_9) = R_C^{\succeq-}(x_{10}) = \{x_4, x_9, x_{10}\}$.

The decision attribute d makes a partition of U into a finite number of classes such that $\mathbf{D} = \{D_1, D_2, D_3\}$, where $D_1 = \{\textit{reject}\} = \{x_2, x_3, x_8\}$, $D_2 = \{\textit{revision}\} = \{x_4, x_5, x_6, x_{10}\}$, $D_3 = \{\textit{accept}\} = \{x_1, x_7, x_9\}$. Thus,
$D_2^{\geq} = D_2 \bigcup D_3 = \{x_1, x_4, x_5, x_6, x_7, x_9, x_{10}\}$, $D_3^{\geq} = D_3 = \{x_1, x_7, x_9\}$,
$D_1^{\leq} = D_1 = \{x_2, x_3, x_8\}$, $D_2^{\leq} = D_1 \bigcup D_2 = \{x_2, x_3, x_4, x_5, x_6, x_8, x_{10}\}$.

It must be noticed that $(x_3, x_4) \in R_C^{\geq}$ and $(x_3, x_4) \notin R_d^{\geq}$, then Table 2 is inconsistent.

By Definitions 4.1 and 5.1, we have
$\underline{C}(D_2^{\geq}) = \{x_1, x_5, x_6, x_7, x_9, x_{10}\}$, $\underline{C}(D_3^{\geq}) = \{x_1, x_7\}$;
$\underline{C}(D_1^{\leq}) = \{x_2, x_8\}$, $\underline{C}(D_2^{\leq}) = \{x_2, x_3, x_4, x_5, x_6, x_8\}$;
$\gamma_C^{\geq}(d) = \frac{16}{21}$, $\gamma_C^{\leq}(d) = \frac{16}{21}$.
Based on Definition 5.5, by computation, we have
$Sig^{\geq}(a_1, C, d) = \frac{11}{21}$, $Sig^{\geq}(a_2, C, d) = Sig^{\geq}(a_3, C, d) = \frac{1}{6}$, $Sig^{\geq}(a_4, C, d) = 0$.

Hence, $Core^{\geq}(\mathbf{S}) = \{a_1, a_2, a_3\}$. Since $\gamma^{\geq}_{Core^{\geq}(\mathbf{S})}(d) = \frac{16}{21} = \gamma_C^{\geq}(d)$, By Algorithm 1, $\{a_1, a_2, a_3\}$, i.e., $\{originality, presentation, technical \;\; soundness\}$ is the upward reduction of Table 2. Similarly, it is not difficult to obtain $\{a_1, a_2\}$ and $\{a_1, a_3\}$ are the downward reduction of Table 2.

7. **Conclusions.** Set-valued information systems are an important type of data table, and have very wide applications in many practical decision-making issues. In this paper, we have investigated criteria reduction in the inconsistent set-valued ordered decision system. By considering the uncertainty of criteria values in the set-valued ordered information system, we defined a similarity dominance relation. To obtain criteria reduction from the inconsistent set-valued ordered decision system, the upward and downward approximation quality are proposed and two types of criteria reduction are defined. And finally, by the significance of the criterion, an algorithm is designed to compute a smallest reduction of a given set-valued ordered decision system. In further research, we will consider to extract and simplify the dominance decision rules in the inconsistent set-valued ordered decision system.

Acknowledgment. This work is partially supported by the research fund of Sichuan key laboratory of intelligent network information processing (SGXZD1002-10), the National Natural Science Foundation (61175055, 61105059), Sichuan Key Technology Research and Development Program (2012GZ0019, 2011FZ0051), the research fund of education department of Sichuan province (10ZC058) and Supported by the open research fund of key laboratory of intelligent network information processing, Xihua University (SZJJ2012-026).

REFERENCES

[1] L. An, Z. Chen and L. Tong, Rough set analysis of interval-valued information systems, *ICIC Express Letters*, vol.5, no.8(B), pp.2951-2956, 2011.
[2] R. Slowinski and D. Vanderpooten, A generalized definition of rough approximations based on similarity, *IEEE Trans. Data Knowl. Eng.*, vol.2, pp.331-336, 2000.
[3] D. G. Chen, C. Z. Wang and Q. H. Hu, A new approach to attribute reduction of consistent and inconsistent covering decision systems with covering rough sets, *Information Sciences*, vol.177, pp.3500-3518, 2007.
[4] K. Dembcznski, S. Greco and R. Slowinski, Rough set approach to multiple criteria classification with imprecise evaluations and assignments, *European Journal of Operational Research*, vol.198, pp.626-636, 2009.
[5] S. Greco, B. Matarazzo and R. Slowinski, Theory and methodology rough approximation of a preference relation by dominance relations, *European Journal of Operational Research*, vol.117, pp.63-83, 1999.
[6] S. Greco, B. Matarazzo and R. Slowinski, Rough sets theory for multi-criteria decision analysis, *European Journal of Operational Research*, vol.129, pp.1-47, 2001.
[7] S. Greco, B. Matarazzo and R. Slowinski, Rough approximation by dominance relation, *International Journal of Intelligent Systems*, vol.17, pp.153-171, 2002.
[8] Y. Y. Guan and H. K. Wang, Set-valued information systems, *Information Sciences*, vol.176, pp.2507-2525, 2006
[9] Y. Leung, W. Z. Wu and W. X. Zhang, Knowledge acquisition in incomplete information systems: A rough set approach, *European Journal of Operational Research*, vol.168, pp.164-180, 2006.

[10] M. Inuiguchi, Y. Yoshioka and Y. Kusunoki, Variable-precision dominance-based rough set approach and attribute reduction, *International Journal of Approximate Reasoning*, 2009.

[11] F. Li and Y. Q. Yin, Approaches to knowledge reduction of covering decision systems based on information theory, *Information Sciences*, vol.179, pp.1694-1704, 2009.

[12] J. S. Mi, W. Z. Wu and W. X. Zhang, Approaches to knowledge reduction based on variable precision rough set model, *Information Sciences*, vol.159, pp.255-272, 2004.

[13] Z. Pawlak, *Rough Sets: Theoretical Aspects of Reasoning about Data*, Kluwer Academic Publishers, Boston, 1991.

[14] Z. Pawlak, Rough sets: Some extensions, *Information Sciences*, vol.177, pp.28-40, 2007.

[15] Z. Pawlak and A. Skowron, Rough sets and boolean reasoning, *Information Sciences*, vol.177, pp.41-73, 2007.

[16] Y. Qian, J. Liang and C. Dang, Interval ordered information systems, *Computers and Mathematics with Applications*, vol.56, pp.1994-2009, 2008.

[17] Y. Qian, C. Dang, J. Liang and D. Tang, Set-valued ordered information systems, *Information Sciences*, vol.179, pp.2809-2832, 2009.

[18] J. J. H. Liou and G. H. Tzeng, A dominance-based rough set approach to customer behavior in the airline market, *Information Sciences*, vol.180, pp.2230-2238, 2010.

[19] A. Skowron and J. Stepaniuk, Tolerance approximation spaces, *Fundamenta Information*, vol.27, pp.245-253, 1996.

[20] A. Skowron and C. Rauszer, The discernibility matrices and functions in information systems, in *Intelligent Decision Support, Handbook of Applications and Advances of the Rough Sets Theory*, R. Slowinsk (ed.), Kluwer Academic Publishers, 1992.

[21] R. Susmaga, R. Slowinski, S. Greco and B. Matarazzo, Generation of reducts and rules in multi-attribute and multi-criteria classification, *Control and Cybernetics Series*, vol.29, no.4, pp.969-988, 2000.

[22] M. W. Shao and W. X. Zhang, Dominance relation and rules in an incomplete ordered information system, *International Journal of Intelligent Systems*, vol.20, pp.13-27, 2005.

[23] K. Watjanapong and S. Yan, The characters recognition based on tolerant rough sets with fuzzy c-mean, *ICIC Express Letters*, vol.3, no.4, pp.1399-1404, 2009.

[24] J. Wang and D. Q. Miao, Analysis on attribute reduction strategies of rough set, *J. Comput. Sci. Technol*, vol.13, pp.189-193, 1998.

[25] Y. Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, *Information Sciences*, vol.101, pp.239-259, 1998.

[26] X. Yang, J. Yang, C. Wu and D. Yu, Dominance-based rough set approach and knowledge reductions in incomplete ordered information system, *Information Sciences*, vol.178, pp.1219-1234, 2008.

[27] W. X. Zhang, Incomplete information system and its optimal selections, *Computers and Mathematics with Applications*, vol.48, no.5, pp.691-698, 2004.

[28] W. X. Zhang, Y. Leung and W. Z. Wu, *Information Systems and Knowledge Discovery*, Science Press, Beijing, China, 2003.

[29] L. Y. Zhai, L. P. Khoo and Z. W. Zhong, A dominance-based rough set approach to Kansei Engineering in product development, *Expert Systems with Applications*, vol.36, pp.393-402, 2009.