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SOLVING ALGEBRAIC RICCATI EQUATION FOR SINGULAR SYSTEM BASED ON MATRIX SIGN FUNCTION

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Abstract. *The objective of this paper is to propose a constructive methodology for determining the appropriate weighting matrices $\{Q, R\}$, which guarantees the solvability of the generalized algebraic Riccati equation and for solving the generalized Riccati equation via the matrix sign function for the stabilizable singular system. A decomposition technique is developed to decompose the singular system into a controllable reduced-order regular subsystem and a non-dynamic subsystem. As a result, the well-developed analysis and synthesis methodologies developed for a regular system can be applied to the reduced-order regular subsystem. Finally, we transform the results obtained for the reduced-order regular subsystem back to those for the original singular system. Illustrative examples are presented to show the effectiveness and accuracy of the proposed methodology.*
Keywords: Riccati equation, Singular system, Matrix sign function

1. **Introduction.** Singular systems are often encountered in many fields of science and engineering systems, including circuits, economic systems, boundary control systems and chemical processes [1]. Over the past decades, much effort has been invested in the analysis, synthesis and applications of singular systems due to the fact that singular systems appear more nature to represent the real systems than the regular systems (state-space systems) [1-5]. The real singular systems usually consist of the non-dynamic subsystems and the dynamic subsystems, which are mathematically governed by the mixed representation of algebraic and differential equations. The complex nature of the singular systems often encounters many difficulties in finding the analytical and numerical solutions to such systems, particularly when there is a need for their control.

Over the past decades, the theory and design of linear quadratic regulator (LQR) for optimal control of the regular systems have been well-developed and successfully applied to many practical design problems [6-10]. Instead of tuning the controllers to satisfy the desirable classical control specifications for regular systems, the optimal controller can be easily designed by tuning the weighting matrices $\{Q, R\}$ in the algebraic Riccati equation, for which many analytical and numerical solutions are available. The methodologies to find specific weighting matrices $\{Q, R\}$ for optimal control of regular systems have been well-developed in the literature but not for singular systems, which is an open problem to be solved.

The motivation of this paper is to propose a constructive methodology for determining the appropriate weighting matrices $\{Q, R\}$, which guarantees the solvability of the generalized algebraic Riccati equation and for solving the Riccati equation via the matrix sign function method for the singular systems. A decomposition technique is developed to decompose the singular system into a reduced-order regular subsystem and a non-dynamic subsystem. As a result, the well-known analysis and synthesis methodologies developed for a regular system can be applied to the reduced-order regular subsystem. Finally, we transform the results obtained for the reduced-order regular subsystem back to those for the original singular system. The computationally fast and numerically stable matrix sign function method is used to obtain the solution of the generalized algebraic Riccati equation for optimal control of the linear continuous-time singular system.

Consider the stabilizable [1] n-th order generalized linear, time-invariant system characterized by

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) \in \Re^n$ is the states, $u \in \Re^m$ is the control, $E \in \Re^{n\times n}$, $A \in \Re^{n\times n}$ and $B \in \Re^{n\times m}$ are real constant matrices, and E is possibly singular. In recent studies, the algebraic Riccati equation (ARE) for the regular system [11-19] has been generalized to the ARE [18,19] with the nonsingular matrix E in (1). The generalized Riccati equation [19] is given by

$$A^TPE + E^TPA - E^TPBR^{-1}B^TPE + Q = O_{n\times n}, \tag{2}$$

where $Q \in \Re^{n\times n}$, $R \in \Re^{m\times m}$ and $P \in \Re^{n\times n}$ are real constant matrices. It should remark that the generalized Riccati Equation (2) might have no solution, even if the selected Q and R are positive-definite matrices, and E is a singular matrix.

For instance, let

$$E = \left[\begin{array}{c:c} I_\kappa & O \\ \hdashline O & E_f \end{array}\right]_{n\times n}, \quad A = \left[\begin{array}{c:c} A_s & O \\ \hdashline O & I_{n-\kappa} \end{array}\right]_{n\times n}, \quad B = \left[\begin{array}{c} B_s \\ \hdashline B_f \end{array}\right]_{n\times m},$$

$$Q = \left[\begin{array}{c:c} Q_s & 0 \\ \hdashline 0 & Q_f \end{array}\right]_{n\times n}, \quad R_{m\times m} > O,$$

and $P = \left[\begin{array}{c:c} P_s & 0 \\ \hdashline 0 & P_f \end{array}\right]_{n\times n}$, where I_κ denotes the $\kappa \times \kappa$ identity matrix and E_f is in the Jordan canonical form. From (2), we have

$$\left[\begin{array}{c:c} A_s^TP_s & O \\ \hdashline O & P_fE_f \end{array}\right] + \left[\begin{array}{c:c} P_sA_s & O \\ \hdashline O & E_f^TP_f \end{array}\right] - \left[\begin{array}{c:c} P_sB_sR^{-1}B_s^TP_s & P_sB_sR^{-1}B_f^TP_fE_f \\ \hdashline E_f^TP_fB_fR^{-1}B_s^TP_s & E_f^TP_fB_fR^{-1}B_s^TP_fE_f \end{array}\right]$$
$$+ \left[\begin{array}{c:c} Q_s & O \\ \hdashline O & Q_f \end{array}\right] = \left[\begin{array}{c:c} O_k & O \\ \hdashline O & O_{n-k} \end{array}\right],$$

which implies

$$A_s^TP_s + P_sA_s + P_sB_sR^{-1}B_s^TP_s + Q_s = O_\kappa, \tag{3}$$

$$P_sB_sR^{-1}B_f^TP_fE_f = O_{\kappa\times(n-\kappa)}, \tag{4}$$

$$E_f^TP_fB_fR^{-1}B_s^TP_s = O_{(n-\kappa)\times\kappa}, \tag{5}$$

$$P_f E_f + E_f^T P_f + E_f^T P_f B_f R^{-1} B_s^T P_f E_f + Q_f = O_{(n-\kappa)}. \quad (6)$$

For $P_s > 0$ and any non-null matrices B_f and B_s, (4) yields $P_f \times E_f = O_{(n-\kappa)}$, which induces, for example,

$$\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = O_3, \quad (7)$$

where "$*$" denotes free variables. Similarly, (5) gives $E_f^T \times P_f = O_{(n-\kappa)}$, which induces, for example,

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix} = O_3. \quad (8)$$

As a result, the pairs of (7) and (8) indicate that P_f is a null matrix, where the last-right-bottom element denotes as a free variable. This also implies that P is not a positive-definite matrix.

In general, the respective E_f and P_f can be given by

$$E_f = \text{block diagonal } \{E_{f_1}, E_{f_2}, \cdots, E_{f_l}\} \quad (9a)$$

and

$$P_f = \text{block diagonal } \{P_{f_1}, P_{f_2}, \cdots, P_{f_l}\}. \quad (9b)$$

For example, let

$$E_{f_i} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_{f_j} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad E_{f_k} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $1 \le i < j < k \le l$, which gives

$$P_{f_i} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \end{bmatrix}, \quad P_{f_j} = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}, \quad \text{and} \quad P_{f_k} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * \end{bmatrix},$$

where "$*$" denotes free variables. The triple (4)-(6) also gives $Q_f = O$. From the above illustrative examples, we can conclude that P and Q are not positive-definite matrices. Therefore, even if the selected Q and R are positive-definite matrices, and E is a singular matrix, the generalized Riccati Equation (2) might have no solution.

By utilizing the neural network approaches [20-23] but without explicitly providing a constructive way for determining the weighting matrices $\{Q, R\}$, various solution methods for the generalized Riccati equation in (2) can be found in [20-23]. This paper proposes a constructive method to determine the weighting matrices $\{Q, R\}$ for the solution of the generalized Riccati equation in (2) for singular systems via the computationally fast and numerically stable matrix sign function method.

2. **Problem Formulation and Main Result.**

Consider the controllable linear continuous-time singular system

$$E_r \dot{x}(t) = A_r x(t) + B_r u(t), \quad (10)$$

where $x(t) \in \Re^n$ is the states, $u(t) \in \Re^m$ is the control, $E_r \in \Re^{n\times n}$ is a singular matrix, and $A_r \in \Re^{n\times n}$ and $B_r \in \Re^{n\times m}$ are real constant matrices. The singular system is assumed to be controllable at finite and impulsive modes. The singular system can be transformed into the slow and fast subsystem models [24], such as (Appendix A)

$$\hat{E}\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \tag{11}$$

where

$$\hat{x} = \left[\begin{array}{c} \hat{x}_s \\ \hline \hat{x}_f \end{array}\right]_{n\times 1}, \quad \hat{E} = \left[\begin{array}{c|c} I_\kappa & O \\ \hline O & \hat{E}_f \end{array}\right]_{n\times n}, \quad \hat{A} = \left[\begin{array}{c|c} \hat{A}_s & O \\ \hline O & I_{n-\kappa} \end{array}\right]_{n\times n}, \quad \hat{B} = \left[\begin{array}{c} \hat{B}_s \\ \hline \hat{B}_f \end{array}\right]_{n\times m},$$

the Os denote null matrices with appropriate sizes, $\hat{E}_f$ is in the Jordan canonical form with d blocks of sizes $u_1, u_2, \cdots, u_d$, and $\sum_{i=1}^{d} u_i$ = column (row) number of $\hat{E}_f$.

Lemma 2.1. *Given the linear controllable continuous-time singular system (10), the generalized algebraic Riccati equation for the steady-state linear quadratic regulator is*

$$A_r^T P_r E_r + E_r^T P_r A_r - E_r^T P_r B_r R_r^{-1} P_r E_r + Q_r = O_n. \tag{12}$$

Proof: For the finite-time linear quadratic regulator (LQR) problem, let the quadratic cost function for the singular system (10) be chosen as

$$\min_{u(t)} J_c = \frac{1}{2}\int_0^{T_f} \left[x^T(t)Q_r x(t) + u^T(t)R_r u(t)\right]dt, \tag{13}$$

where $Q_r \geq O$, $R_r > O$, and T_f is the final time. Here, the Pontryagin's maximum principle [9] is used to solve this optimization problem. Define a Hamiltonian as

$$H(t) = \frac{1}{2}\left(x^T(t)Q_r x(t) + u^T(t)R_r u(t)\right) + \lambda^T \left(A_r x(t) + B_r u(t)\right),$$

where $\lambda(t) \in \Re^{n\times 1}$ is an un-determined multiplier function. The state and costate equations are respectively given as

$$\frac{\partial H(t)}{\partial \lambda(t)} = A_r x(t) + B_r u(t) = E_r \dot{x}(t),$$

$$\frac{\partial H(t)}{\partial x(t)} = Q_r x(t) + A_r^T \lambda(t) = -E_r^T \dot{\lambda}(t),$$

and the stationary condition is

$$\frac{\partial H(t)}{\partial u(t)} = R_r u(t) + B_r^T \lambda(t) = O.$$

Solving the last equation yields the optimal control law in terms of the costate equation as

$$u(t) = -R_r^{-1} B_r^T \lambda(t). \tag{14}$$

Substituting (14) into (10) yields

$$E_r \dot{x} = A_r x(t) - B_r R_r^{-1} B_r^T \lambda(t),$$

which can be combined with the costate equation to give the homogeneous Hamiltonian system as

$$\begin{bmatrix} E_r \dot{x}(t) \\ E_r \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A_r & -B_r R_r^{-1} B_r^T \\ -Q_r & -A_r^T \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}. \tag{15}$$

The coefficient matrix in (15) is called the Hamiltonian matrix. Let

$$\lambda(t) = P_r(t) E_r x(t),$$

which implies $E_r^T \lambda(t) = E_r^T P_r(t) E_r x(t)$ and

$$u(t) = -R_r^{-1} B_r^T P_r E_r x(t), \tag{16}$$

with an unknown $n \times n$ auxiliary matrix function $P_r(t)$. To find the auxiliary function $P_r(t)$, we differentiate the costate equation in (16) and use the state equation in (10) with the control law in (16) to get

$$\begin{aligned} E_r^T \dot{\lambda}(t) &= E_r^T \dot{P}_r(t) E_r x(t) + E_r^T P_r(t) E_r \dot{x}(t) \\ &= E_r^T \dot{P}_r(t) E_r x(t) + E_r^T P_r(t)[A_r x(t) - B_r R_r^{-1} B_r^T P_r(t) E_r x(t)]. \end{aligned} \tag{17}$$

Now, from the costate equation, for all t, we have

$$-E_r^T \dot{P}_r(t) E_r x(t) = \left[Q_r + A_r^T P_r(t) E_r + E_r^T P_r(t) A_r - E_r^T P_r(t) B_r R_r^{-1} B_r^T P_r(t) E_r\right] x(t),$$

$$-E_r^T \dot{P}_r(t) E_r = Q_r + A_r^T P_r(t) E_r + E_r^T P_r(t) A_r - E_r^T P_r(t) B_r R_r^{-1} B_r^T P_r(t) E_r. \tag{18}$$

The $\dot{P}_r(t)$ in (18) is a null matrix in steady state. Hence, we have

$$A_r^T P_r E_r + E_r^T P_r A_r - E_r^T P_r B_r R_r^{-1} P_r E_r + Q_r = O_n. \tag{19}$$

This is the generalized algebraic Riccati equation used to determine the steady-state linear quadratic regulator for the linear continuous-time singular system (10). This completes the proof.

Lemma 2.2. *Let $\hat{P}_f$ and $\hat{E}_f$ be two matrices, where $\hat{E}_f$ is a singular matrix of the single Jordan canonical form. The following semi-positive definite matrix*

$$\hat{P}_f = \left[\begin{array}{c:c} O_{(n-1)\times(n-1)} & O_{(n-1)\times 1} \\ \hdashline O_{1\times(n-1)} & *_{1\times 1} \end{array}\right]_{n\times n} \tag{20}$$

satisfies the constraints $\hat{P}_f \times \hat{E}_f = O$ and $\hat{E}_f^T \times \hat{P}_f = O$, where the "$*$" denotes a free variable.

Proof: Let

$$\hat{E}_f = \left[\begin{array}{ccccc:c} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ddots & & 1 \\ \hdashline 0 & 0 & 0 & \cdots & & 0 \end{array}\right]_{n\times n}.$$

From the constraint $\hat{P}_f \times \hat{E}_f = O$, we have

$$\hat{P}_f = \left[\begin{array}{c:c} O_{n\times(n-1)} & *_{n\times 1} \end{array}\right]_{n\times n}.$$

Similarly, let

$$\hat{E}_f^T = \left[\begin{array}{cccc:c} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots \\ \hdashline 0 & 0 & \cdots & 1 & 0 \end{array}\right]_{n\times n}$$

and by the constraint $\hat{E}_f^T \times \hat{P}_f = O$, we have

$$\hat{P}_f = \left[\begin{array}{c} O_{(n-1)\times n} \\ \hdashline *_{1\times n} \end{array}\right]_{n\times n}.$$

Hence, from above results we have

$$\hat{P}_f = \left[\begin{array}{c:c} O_{(n-1)\times(n-1)} & O_{(n-1)\times 1} \\ \hdashline O_{1\times(n-1)} & *_{1\times 1} \end{array}\right]_{n\times n}.$$

This completes the proof.

Remark 2.1. *Let $\hat{E}_f$ be a null matrix. The matrix $\hat{P}_f = [*]_{n\times n}$ would satisfy the constraints $\hat{P}_f \times \hat{E}_f = O$ and $\hat{E}_f^T \times \hat{P}_f = O$, where "$*$"s denote free variables.*

Theorem 2.1. *Given the singular system in (10), which is assumed to be controllable at finite and impulsive modes and can be decomposed into a reduced-order regular subsystem and a non-dynamic subsystem by the approach shown in Appendix A. Then, consider the generalized algebraic Riccati equation for the steady-state linear quadratic regulator, which is optimal in the sense of the quadratic cost function (13) for the controllable linear continuous-time singular system in (10), as*

$$A_r^T P_r E_r + E_r^T P_r A_r - E_r^T P_r B_r R_r^{-1} P_r E_r + Q_r = O_{n\times n}. \tag{21}$$

The solution P_r of (21) is given by

$$P_r = \left(((\alpha E_r + \beta A_r)MWV)^{-1}\right)^T \hat{P}((\alpha E_r + \beta A_r)MWV)^{-1}, \tag{22}$$

where

$$\hat{P} = \left[\begin{array}{c:c} \hat{P}_s & O \\ \hdashline O & O_{n-\kappa} \end{array}\right]_{n\times n}, \tag{23}$$

$\hat{P}_s$ in (23) is a solution of the following Riccati equation:

$$\hat{A}_s\hat{P}_s + \hat{P}_s\hat{A}_s - \hat{P}_s\hat{B}_s\hat{R}^{-1}\hat{P}_s + \hat{Q}_s = O_\kappa, \tag{24}$$

$\hat{Q}_s$ and $\hat{R}$ in (24) are both selected positive-definite matrices, and $\{M, W, V\}$ in (22) are constant matrices and $\{\alpha, \beta\}$ in (22) are real constants (see Appendix A). The resulting weighting matrices in the original cost function in (13) become

$$Q_r = \left((MV)^{-1}\right)^T \hat{Q}\,(MV)^{-1}, \tag{25}$$

where

$$\hat{Q} = \left[\begin{array}{c:c} \hat{Q}_s & O \\ \hdashline O & O_{n-k} \end{array}\right]_{n\times n}, \tag{26}$$

and

$$R_r = \hat{R}. \tag{27}$$

The solution of the Riccati equation $\hat{P}_s$ in (24) guarantees the stability of the reduced-order regular subsystem in (A.17) as well as the stability of the singular system without having the impulsive mode in (10).

Proof: Let $\hat{Q} = \left[\begin{array}{c:c} \hat{Q}_s & O \\ \hdashline O & \hat{Q}_f \end{array}\right]_{n\times n}$, where $\hat{Q}_s \in R^{\kappa\times\kappa}$ and $\hat{R} \in R^{m\times m}$ are positive-definite matrices.

From (12), we have

$$\left[\begin{array}{c:c} \hat{A}_s^T\hat{P}_s & O \\ \hdashline O & \hat{P}_f\hat{E}_f \end{array}\right] + \left[\begin{array}{c:c} \hat{P}_s\hat{A}_s & O \\ \hdashline O & \hat{E}_f^T\hat{P}_f \end{array}\right] + \left[\begin{array}{c:c} \hat{P}_s\hat{B}_s\hat{R}^{-1}\hat{B}_s^T\hat{P}_s & \hat{P}_s\hat{B}_s\hat{R}^{-1}\hat{B}_f^T\hat{P}_f\hat{E}_f \\ \hdashline \hat{E}_f^T\hat{P}_f\hat{B}_f\hat{R}^{-1}\hat{B}_s^T\hat{P}_s & \hat{E}_f^T\hat{P}_f\hat{B}_f\hat{R}^{-1}\hat{B}_s^T\hat{P}_f\hat{E}_f \end{array}\right]$$

$$+ \left[\begin{array}{c:c} \hat{Q}_s & O \\ \hdashline O & \hat{Q}_f \end{array}\right] = \left[\begin{array}{c:c} O_k & O \\ \hdashline O & O_{n-k} \end{array}\right],$$

which gives

$$\hat{A}_s^T\hat{P}_s + \hat{P}_s\hat{A}_s + \hat{P}_s\hat{B}_s\hat{R}^{-1}\hat{B}_s^T\hat{P}_s + \hat{Q}_s = O_\kappa, \tag{28}$$

$$\hat{P}_s\hat{B}_s\hat{R}^{-1}\hat{B}_f^T\hat{P}_f\hat{E}_f = O_{\kappa\times(n-\kappa)}, \tag{29}$$

$$\hat{E}_f^T\hat{P}_f\hat{B}_f\hat{R}^{-1}\hat{B}_s^T\hat{P}_s = O_{(n-\kappa)\times\kappa}, \tag{30}$$

$$\hat{P}_f\hat{E}_f + \hat{E}_f^T\hat{P}_f + \hat{E}_f^T\hat{P}_f\hat{B}_f\hat{R}^{-1}\hat{B}_s^T\hat{P}_f\hat{E}_f + \hat{Q}_f = O_{(n-\kappa)}. \tag{31}$$

From (29), (30), and Lemma 2.2, we have the $\hat{E}_f$ and $\hat{P}_f$ shown in (9a) and (9b). For simplicity in analysis, we let the free variable be zero, which yields

$$\hat{P} = \left[\begin{array}{c:c} \hat{P}_s & O \\ \hdashline O & O_{n-\kappa} \end{array}\right]_{n\times n}. \tag{32}$$

By (29), (30), and (21), we have

$$\hat{Q}_f = O_{n-\kappa}, \tag{33}$$

which gives

$$\hat{Q} = \left[\begin{array}{c:c} \hat{Q}_s & O \\ \hdashline O & O_{n-\kappa} \end{array}\right]_{n\times n}.$$

In addition, from (16) and Appendices A and B, we have the linear quadratic regulator

$$\begin{aligned} u(t) &= -\hat{R}^{-1}\hat{B}^T\hat{P}\hat{E}\hat{x}(t) \\ &= -\hat{R}^{-1}(V^{-1}\bar{B})^T\hat{P}(V^{-1}\bar{E}V)V^{-1}\bar{x}(t) \\ &= -\hat{R}^{-1}((WV)^{-1}\tilde{B})^T\hat{P}((WV)^{-1}\tilde{E}V)V^{-1}M^{-1}x(t) \\ &= -\hat{R}^{-1}((MWV)^{-1}B_n)^T\hat{P}((MWV)^{-1}E_nMV)V^{-1}M^{-1}x(t) \\ &= -\hat{R}^{-1}(((\alpha E_r + \beta A_r)MWV)^{-1}B_r)^T\hat{P}(((\alpha E_r + \beta A_r)MWV)^{-1}E_rMV) \\ &\quad V^{-1}M^{-1}x(t) \\ &= -\hat{R}^{-1}B_r^T(((\alpha E_r + \beta A_r)MWV)^{-1})^T\hat{P}((\alpha E_r + \beta A_r)MWV)^{-1}E_rx(t) \\ &\triangleq -\hat{R}^{-1}B_r^TP_rE_rx(t), \end{aligned}$$

which yields

$$P_r = \left(((\alpha E_r + \beta A_r)MWV)^{-1}\right)^T\hat{P}((\alpha E_r + \beta A_r)MWV)^{-1},$$

where $\{M, W, V\}$ are constant matrices and $\{\alpha, \beta\}$ are real constants (see Appendix A). Furthermore, from (13), we have

$$\min_{u(t)} J_c = \frac{1}{2}\int_0^{T_f}\left[x^T(t)Q_rx(t) + u^T(t)R_ru(t)\right]dt,$$

where

$$\begin{aligned} \hat{x}^T(t)\hat{Q}\hat{x}(t) &= \bar{x}^T(t)(V^{-1})^T\hat{Q}V^{-1}\bar{x}(t) \\ &= x^T(t)(M^{-1})^T(V^{-1})^T\hat{Q}V^{-1}M^{-1}x(t) \\ &= x^T(t)(M^{-1})^T(V^{-1})^T\hat{Q}V^{-1}M^{-1}x(t) \\ &= x^T(t)\left((MV)^{-1}\right)^T\hat{Q}\,(MV)^{-1}x(t) \\ &\triangleq x^T(t)Q_rx(t), \end{aligned}$$

which gives

$$Q_r = \left((MV)^{-1}\right)^T\hat{Q}(MV)^{-1},$$

where V and M are matrices (see Appendix A). Notice that $rank(Q_r) = rank(\hat{Q}) = rank(\hat{Q}_s)$ and $R_r = \hat{R}$.

The proof of the claim that the solution of the generalized Riccati equation guarantees the stability of the reduced-order regular subsystem can be found in literature [9,10]. Since the singular system can be transformed into a reduced-order regular subsystem and a non-dynamic subsystem, the stability of the reduced-order regular subsystem assures the stability of the singular system without having impulsive mode. Besides, the Q_r developed in (25) is not an arbitrary matrix. This completes the proof.

3. **Illustrative Examples.** To show the effectiveness and accuracy of the proposed methodology, a pure mathematical model is utilized in Example 3.1 and a practical system is adapted in Example 3.2, where the sub-matrix $\bar{E}_2$ in (A.11) in Example 3.1 has the Jordan-type eigenstructure and the sub-matrix $\bar{E}_2$ in Example 3.2 is a null matrix.

Example 3.1. *Consider the controllable linear continuous-time singular system [24] described in (10) with*

$$E_r = \begin{bmatrix} 1 & 2 & 1 & 1 & -3 & -2 \\ 0 & 2 & 2 & 1 & -3 & -3 \\ 1 & 2 & 1 & 1 & -3 & -2 \\ 1 & 2 & 1 & 3 & -5 & -4 \\ 0 & 2 & 1 & 1 & -2 & -2 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad A_r = I_6, \quad B_r^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}.$$

Taking $\alpha = 0$ and $\beta = 1$, then $E_n = E_r, A_n = A_r$ and $B_n = B_r$, which induces $0E_n + A_n = I_6$. By definition of the standard form, $\{E_r, A_r\}$ is in the standard form. Because E_n is singular, i.e., E_n includes some zero eigenvalues, we can utilize the bilinear transform to find the similarity transformation matrix M of E_n is necessary. Taking $\rho = 0.5$ and using the algorithm described in Appendix A, we have

$$\tilde{E} = (E_n - \rho I_6)(E_n + \rho I_6)^{-1} = \begin{bmatrix} 0.3333 & 1.6 & -2.4 & 0.16 & 0.9067 & 2.24 \\ 0 & 0.6 & 1.6 & 0.16 & -1.76 & -1.76 \\ 1.3333 & 1.6 & -3.4 & 0.16 & 0.9067 & 2.24 \\ 1.3333 & 1.6 & -2.4 & 0.76 & -0.6933 & 0.64 \\ 0 & 1.6 & -2.4 & 0.16 & 1.24 & 2.24 \\ 1.3333 & 0 & 0 & 0 & -1.3333 & -1 \end{bmatrix},$$

$$sign\left(\tilde{E}\right) = \begin{bmatrix} 1 & 2 & 2 & 0 & -4 & -2 \\ 0 & 1 & 2 & 0 & -2 & -2 \\ 2 & 2 & 1 & 0 & -4 & -2 \\ 2 & 2 & 2 & 1 & -6 & -4 \\ 0 & 2 & 2 & 0 & -3 & -2 \\ 2 & 0 & 0 & 0 & -2 & -1 \end{bmatrix},$$

$$sign^+\left(\tilde{E}\right) = \begin{bmatrix} 1 & 1 & 1 & 0 & -2 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 & -2 & -1 \\ 1 & 1 & 1 & 1 & -3 & -2 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad sign^-\left(\tilde{E}\right) = \begin{bmatrix} 0 & -1 & -1 & 0 & 2 & 1 \\ 0 & 0 & -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 & 2 & 1 \\ -1 & -1 & -1 & 0 & 3 & 2 \\ 0 & -1 & -1 & 0 & 2 & 1 \\ -1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

$$M = \left[ind\left(sign^{+}\left(\tilde{E}\right)\right) ind\left(sign^{-}\left(\tilde{E}\right)\right)\right] = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & -1 & -1 & 0 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 \end{bmatrix}. \tag{34}$$

From (A.10), we obtain

$$M^{-1}E_nM = \left[\begin{array}{c:c} \bar{E}_1 & O \\ \hdashline O & \bar{E}_2 \end{array}\right] = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$M^{-1}A_nM = \left[\begin{array}{c:c} I_3 & O \\ \hdashline O & I_3 \end{array}\right], \quad M^{-1}B_n = \left[\begin{array}{c:c} \bar{B}_1^T & \bar{B}_2^T \end{array}\right]^T = \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & -1 \end{array}\right].$$

From (A.12), we have

$$W = \left[\begin{array}{c:c} \overline{E}_1 & O \\ \hdashline O & \frac{1}{\beta}(I_{n-\kappa} - \alpha\overline{E}_2) \end{array}\right] = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \tag{35}$$

which gives

$$W^{-1} = \left[\begin{array}{c:c} \overline{E}_1^{-1} & O \\ \hdashline O & \beta(I_{n-\kappa} - \alpha\overline{E}_2)^{-1} \end{array}\right] = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & -0.25 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \tag{36}$$

$$W^{-1}\bar{E}_n = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad W^{-1}\bar{A}_n = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & -0.25 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

$$W^{-1}\bar{B}_n = \left[\begin{array}{ccc:ccc} 1 & 0.25 & 0.5 & 2 & 0 & 1 \\ 0 & -0.75 & 0.5 & 0 & 0 & -1 \end{array}\right]^T.$$

Based on (A.14) and the fact that $\bar{E}_f$ is in the Jordan form, we have

$$V = I_6, \tag{37}$$

$$\hat{E}_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{A}_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & -0.25 \\ 0 & 0 & 0.5 \end{bmatrix},$$

$$\hat{B}_s = \begin{bmatrix} 1 & 0 \\ 0.25 & -0.75 \\ 0.5 & 0.5 \end{bmatrix}, \quad \hat{B}_f = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

Solving the algebraic Riccati equation $\hat{A}_s\hat{P}_s + \hat{P}_s\hat{A}_s - \hat{P}_s\hat{B}_s\hat{R}^{-1}\hat{P}_s + \hat{Q}_s = O_3$*, where* $\hat{Q}_s = 10^5 \times I_3$ *and* $\hat{R} = I_2$*, yields*

$$\hat{P}_s = \begin{bmatrix} 366083.3850 & -365309.5782 & -547705.9160 \\ -365309.5782 & 365165.5649 & 546938.6356 \\ -547705.9160 & 546938.6356 & 820335.5830 \end{bmatrix},$$

where some fractional bits are truncated at here. By (22)-(27) and (36)-(37), we have

$$\hat{P} = \left[\begin{array}{c:c} \hat{P}_S & O_3 \\ \hdashline O_3 & O_3 \end{array}\right]$$

$$\begin{aligned} P_r &= \left(((\alpha E_r + \beta A_r)MWV)^{-1}\right)^T \hat{P}\left((\alpha E_r + \beta A_r)MWV\right)^{-1} \\ &= \begin{bmatrix} 3.6608 & -1.8265 & -1.8265 & -1.8253 & -0.0090 & 3.6518 \\ -1.8265 & 0.9129 & 0.9129 & 0.9109 & 0.0027 & -1.8238 \\ -1.8265 & 0.9129 & 0.9129 & 0.9109 & 0.0027 & -1.8238 \\ -1.8253 & 0.9109 & 0.9109 & 0.9117 & 0.0026 & -1.8226 \\ -0.0090 & 0.0027 & 0.0027 & 0.0026 & 0.0036 & -0.0054 \\ 3.6518 & -1.8238 & -1.8238 & -1.8226 & -0.0054 & 3.6464 \end{bmatrix} \times 10^5, \end{aligned}$$

$$\hat{Q} = \left[\begin{array}{c:c} \hat{Q}_s & O_3 \\ \hdashline O_3 & O_3 \end{array}\right] = \left[\begin{array}{c:c} I_3 & O_3 \\ \hdashline O_3 & O_3 \end{array}\right] \times 10^5,$$

$$Q_r = ((MV)^{-1})^T\hat{Q}(MV)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 3 & 2 \\ 0 & -1 & -1 & -1 & 2 & 2 \end{bmatrix} \times 10^5,$$

$$R_r = \hat{R} = I_2.$$

Substituting the computed P_r *and* Q_r *into (21) yields*

$$\begin{aligned} &A_r^T P_r E_r + E_r^T P_r A_r - E_r^T P_r B_r R_r^{-1} P_r E_r + Q_r \\ &= \begin{bmatrix} 1.6440 & -1.5023 & -1.5069 & -2.3537 & 2.3334 & 3.9701 \\ -1.5024 & 1.3916 & 1.4039 & 2.2332 & -2.2344 & -3.7129 \\ -1.5071 & 1.4039 & 1.4162 & 2.2354 & -2.2441 & -3.7272 \\ -2.3535 & 2.2332 & 2.2354 & 3.4922 & -3.4717 & -5.8226 \\ 2.3334 & -2.2345 & -2.2441 & -3.4717 & 3.4624 & 5.7769 \\ 3.9701 & -3.7127 & -3.7272 & -5.8227 & 5.7770 & 9.7207 \end{bmatrix} \times 10^{-6} \\ &\cong O_6, \end{aligned}$$

which shows that the solution is quite satisfactory.

Example 3.2. *Consider the controllable linear continuous-time singular circuit system (Figure 1) [1], where* $R = 1,000\Omega$*, inductance* $L = 1H$*, capacitances* $C_1 = 0.002F$*,* $C_2 = 0.2F$*, and voltage* $u(t)$ *is the control input.*

The state vector is $x(t) = \begin{bmatrix} v_{c_1}(t) & v_{c_2}(t) & i_2(t) & i_1(t) \end{bmatrix}^T$*, where the* $v_{c_1}(t)$*,* $v_{c_2}(t)$*,* $i_2(t)$*, and* $i_1(t)$ *are voltages of capacitors and amperages of the currents flowing over them,*

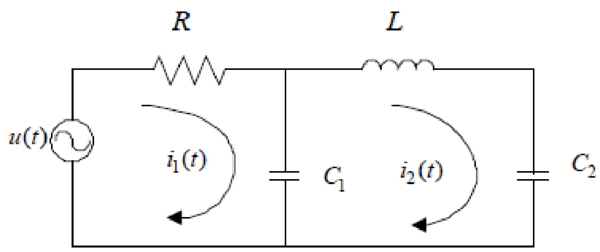

FIGURE 1. The singular circuit system

respectively. According to Kirchoff's second law, we may establish the following state equation

$$E_r\dot{x}(t) = A_r x(t) + B_r u(t),$$

where

$$E_r = \begin{bmatrix} 0.002 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_r = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1000 \end{bmatrix}, \quad B_r = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

Taking $\alpha = 0$ and $\beta = 1$ to have

$$E_n = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0.001 & 0 & 0 & 0 \end{bmatrix}, \quad A_n = I_4, \quad B_n^T = \begin{bmatrix} -1 & -1 & 0 & 0 \end{bmatrix},$$

which induces $0E_n + A_n = I_4$. By definition of the standard form, $\{E_r, A_r\}$ is in the standard form. Because E_n is singular, i.e., E_n includes some zero eigenvalues, we can utilize the bilinear transform to find the similarity transformation matrix M of E_n. Taking $\rho = 0.05$ and using the algorithm described in Appendix A, we have

$$\tilde{E} = (E_n - \rho I_6)(E_n + \rho I_6)^{-1} = \begin{bmatrix} 1.105 & 0 & 0 & 0 \\ 0.256 & 0.975 & -0.494 & 0 \\ -0.104 & 0.099 & 0.975 & 0 \\ -0.002 & 0 & 0 & -1 \end{bmatrix},$$

$$sign\left(\tilde{E}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.002 & 0 & 0 & -1 \end{bmatrix},$$

$$sign^+\left(\tilde{E}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.001 & 0 & 0 & 0 \end{bmatrix},$$

$$sign^-\left(\tilde{E}\right) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.001 & 0 & 0 & 1 \end{bmatrix},$$

$$M = \left[ind\left(sign^{+}\left(\tilde{E}\right)\right) ind\left(sign^{-}\left(\tilde{E}\right)\right)\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.001 & 0 & 0 & 0.001 \end{bmatrix}. \tag{38}$$

From (A.10) and (A.12), we have

$$M^{-1}E_nM = \left[\begin{array}{c:c} \bar{E}_1 & O \\ \hdashline O & \bar{E}_2 \end{array}\right] = \left[\begin{array}{ccc:c} -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0.2 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right],$$

$$M^{-1}A_nM = \left[\begin{array}{c:c} I_3 & O \\ \hdashline O & I_1 \end{array}\right], \quad M^{-1}B_n = \left[\begin{array}{c:c} \bar{B}_1^T & \bar{B}_2^T \end{array}\right]^T = \left[\begin{array}{ccc:c} -1 & -1 & 0 & -1 \end{array}\right],$$

$$W = \left[\begin{array}{c:c} \overline{E}_1 & O \\ \hdashline O & \frac{1}{\beta}(I_{n-\kappa} - \alpha\overline{E}_2) \end{array}\right] = \left[\begin{array}{ccc:c} -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0.2 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right], \tag{39}$$

which implies

$$W^{-1} = \left[\begin{array}{c:c} \overline{E}_1^{-1} & O \\ \hdashline O & \beta(I_{n-\kappa} - \alpha\overline{E}_2)^{-1} \end{array}\right] = \left[\begin{array}{ccc:c} -1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 1 & -1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right], \tag{40}$$

$$W^{-1}\bar{E}_n = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 \end{array}\right], \quad W^{-1}\bar{A}_n = \left[\begin{array}{ccc:c} -1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 1 & -1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right],$$

$$W^{-1}\bar{B}_n = \left[\begin{array}{ccc:c} 1 & 0 & 0 & -1 \end{array}\right]^T.$$

Based on (A.14) and the fact that $\bar{E}_f$ is null, we have

$$V = I_4, \tag{41}$$

$$\hat{E}_f = [0], \quad \hat{A}_s = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 5 \\ 1 & -1 & 0 \end{bmatrix}, \quad \hat{B}_s = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{B}_f = [-1].$$

Solving the algebraic Riccati equation $\hat{A}_s\hat{P}_s + \hat{P}_s\hat{A}_s - \hat{P}_s\hat{B}_s\hat{R}^{-1}\hat{P}_s + \hat{Q}_s = O_3$, where $\hat{Q}_s = 10^5 \times I_3$ and $\hat{R} = I_1$, yields

$$\hat{P}_s = \begin{bmatrix} 317.4953 & 128.7194 & 719.1360 \\ 128.7194 & 64192.9211 & 41715.6595 \\ 719.1360 & 41715.6595 & 228397.8584 \end{bmatrix}$$

where some fractional bits are truncated at here. By (22)-(27) and (40)-(41), we have

$$\hat{P} = \left[\begin{array}{c:c} \hat{P}_s & O_{3\times 1} \\ \hdashline O_{1\times 3} & O_1 \end{array}\right]$$

$$P_r = \left(((\alpha E_r + \beta A_r)MWV)^{-1}\right)^T \hat{P}\left((\alpha E_r + \beta A_r)MWV\right)^{-1}$$

$$= \begin{bmatrix} 227.2271 & -227.6787 & 207.9347 & 0 \\ -227.6787 & 228.3979 & -208.5783 & 0 \\ 207.9347 & -208.5783 & 1604.8230 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times 10^3,$$

$$\hat{Q} = \left[\begin{array}{c:c} \hat{Q}_s & O_{3\times1} \\ \hdashline O_{1\times3} & O_1 \end{array}\right] = \left[\begin{array}{c:c} I_3 & O_{3\times1} \\ \hdashline O_{1\times3} & O_1 \end{array}\right] \times 10^5,$$

$$Q_r = \left((MV)^{-1}\right)^T \hat{Q}\,(MV)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times 10^5,$$

$$R_r = \hat{R} = I_1.$$

Substituting the computed P_r and Q_r into (21) yields

$$\begin{aligned} & A_r^T P_r E_r + E_r^T P_r A_r - E_r^T P_r B_r R_r^{-1} P_r E_r + Q_r \\ &= \begin{bmatrix} 0.03449 & 0.01055 & 0.10652 & -0.00421 \\ 0.01069 & 0.00262 & 0.03492 & -0.02948 \\ 0.10594 & 0.03492 & 0.30355 & -1.44101 \\ -0.00421 & -0.02935 & -1.44101 & 0 \end{bmatrix} \times 10^{-7} \\ &\cong O_4, \end{aligned}$$

which shows that the solution is very satisfactory.

Here, we would like to point out that no toolbox in MATLAB can be used to solve this problem.

4. **Conclusion.** This paper shows that even if the selected Q and R are positive-definite matrices, and E is a singular matrix, the generalized Riccati Equation (2) might have no solution. Therefore, this paper aims to propose a constructive methodology for determining the appropriate weighting matrices $\{Q, R\}$, which guarantees the solvability of the generalized algebraic Riccati equation for the controllable linear continuous-time singular system based on the matrix sign function method. A decomposition technique is developed to decompose the singular system into a reduced-order regular subsystem and a non-dynamic subsystem, so that the singular problem can be converted into a standard regular problem. As a result, the computationally fast and numerically stable matrix sign function method [25] can be utilized to solve the generalized algebraic Riccati equation for the singular system. Finally, we transform the obtained results obtained for the regular system back to those for the original singular system. The developed design methodology for finding the LQR can be extended to find the optimal tracker for singular systems. Illustrative examples are presented to show the effectiveness and accuracy of the proposed methodology.

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Appendix A: Singular System Descriptions [24].

A.1 Preliminaries for decomposition of singular systems. Consider the linear continuous-time singular system as follows

$$E_r\dot{x}(t) = A_r x(t) + B_r u(t), \tag{A.1}$$

where $x(t) \in R^n$ is the state vector and $u(t) \in R^m$ is the input. These constant matrices E_r, A_r and B_r all have appropriate dimensions, and E_r is a singular matrix. The matrix sign function of a square matrix $A \in C^{n\times n}$ with $\mathrm{Re}(\sigma(A)) \neq 0$ is defined [26] as follows

$$sign(A) = 2sign^+(A) - I_n, \tag{A.2}$$

where I_n is an $n \times n$ identity matrix and

$$sign^+(A) = \frac{1}{2\pi i}\int_{c_+} (\lambda I_n - A)^{-1} d\lambda, \tag{A.3}$$

where c_+ is a simple closed contour in right-half plane of λ and encloses all right-half plane eigenvalues of A. For another thing, the matrix sign function [27,28] is also defined as

$$sign(A) = A(\sqrt{A^2})^{-1} = A^{-1}(\sqrt{A^2}), \tag{A.4}$$

where the matrix $\sqrt{A^2}$ denotes the principal square root of A^2.

Preserving the eigenvectors of the original matrix is a main feature of the matrix sign function. This property is useful for engineering problem to study the eigenstructures of matrices and develop applications. The singular matrix E_r can be modified by using bilinear transformation.

$$\tilde{E}_r = (E_r - \rho I_n)(E_r + \rho I_n)^{-1}, \tag{A.5}$$

where ρ is the radius of a circle with center at the origin so that the circle only contains zero eigenvalues and no eigenvalues of E_r located on the circle. Therefore, the eigenvalues within the circle are mapped into the left-half plane of the complex s-plane, and the others are mapped into the right-half plane of the complex s-plane.

A.2 The regular pencil and the standard pencil.

Definition A.2.1 [29]. *Let E_r and A_r be two square constant matrices if* $\det(sE_r - A_r) \neq 0$*, for all s, then $(sE_r - A_r)$ is called a regular pencil.*

Definition A.2.2 [30]. *Let $(sE_n - A_n)$ be a regular pencil. If there exists scalars α and β such that $\alpha E_n + \beta A_n = I_n$, then $(sE_n - A_n)$ is called a standard pencil. Note that for any regular pencil, $(sE_r - A_r)$ can be easily transformed into a standard one by multiplying $(\alpha E_r + \beta A_r)^{-1}$ to E_r and A_r respectively, where α and β are scalars such* *that* $\det(\alpha E_r + \beta A_r) \neq 0$. Hence, the matrix coefficients of a standard pencil $(sE_n - A_n)$ becomes

$$E_n = (\alpha E_r + \beta A_r)^{-1} E_r, \tag{A.6}$$

$$A_n = (\alpha E_r + \beta A_r)^{-1} A_r. \tag{A.7}$$

The modified system retains its state vector $x(t)$ and the matrices (E_n, A_n) have the following properties.

Lemma A.2.1 [31].

1) $E_n A_n = A_n E_n$, *which means that* E_n *and* A_n *commute each other.*

2) E_n *and* A_n *have the same eigenspaces.*

The above properties enable us to decompose a singular system into a reduced-order regular subsystem (slow subsystem) and a nondynamic subsystem (fast subsystem).

A.3 Decomposition of singular systems. Consider the continuous-time singular system (A.1). It is well known that the singular system can be decomposed into slow and fast subsystem. From (A.6) and (A.7), the regular pencil $(sE_r - A_r)$ can be transformed into a standard one $(sE_n - A_n)$. Note that since E_r is a singular matrix, which has at least one zero eigenvalue, β cannot be equal zero. Hence, multiply (A.1) by $(\alpha E_r + \beta A_r)^{-1}$ can get the following equation

$$E_n \dot{x}(t) = A_n x(t) + B_n u(t), \tag{A.8}$$

where $E_n = (\alpha E_r + \beta A_r)^{-1} E_r$, $A_n = (\alpha E_r + \beta A_r)^{-1} A_r$ and $B_n = (\alpha E_r + \beta A_r)^{-1} B_r$. Because $\alpha E_n + \beta A_n = I_n$, the pencil $(sE_n - A_n)$ is a standard one which has the properties mentioned in Lemma A.2.1. In order to decompose system (A.8), we convert state space $x(t)$ into $\bar{x}(t)$ by

$$x(t) = M\bar{x}(t) \tag{A.9}$$

where the constant matrix M is a block modal matrix of E_n and determined by means of the extended matrix sign function. The M matrix of state space transformation is given as follows:

Step 1: Find $sign(\tilde{E}_n)$ using the extended matrix sign function with an adequate ρ, where

$$\tilde{E}_n = (E_n - \rho I_n)(E_n + \rho I_n)^{-1}.$$

Step 2: Find $sign^+(\tilde{E}_n) = \frac{1}{2}[I_n + sign(\tilde{E}_n)]$ and $sign^-(\tilde{E}_n) = \frac{1}{2}[I_n - \mathrm{sign}(\tilde{E}_n)]$.

Step 3: Construct the matrix

$$M = [ind(sign^+(\tilde{E}_n))ind(sign^-(\tilde{E}_n))], \tag{A.10}$$

where $ind(\cdot)$ represents the collection of the linearly independent column vectors of $(\cdot)$.

Substituting (A.9) into (A.10) and multiplying M^{-1} on the left, the equation can be rewritten as

$$\begin{aligned} M^{-1} E_n M \dot{\bar{x}}(t) &= M^{-1} A_n M \bar{x}(t) + M^{-1} B_n u(t) \\ &= \frac{1}{\beta}(I_n - \alpha E_n)\bar{x}(t) + M^{-1} B_n u(t) \end{aligned}$$

i.e.,

$$\begin{bmatrix} \bar{E}_1 & O \\ O & \bar{E}_2 \end{bmatrix} \dot{\bar{x}}(t) = \begin{bmatrix} \frac{1}{\beta}\left(I_k - \alpha \bar{E}_1\right) & O \\ O & \frac{1}{\beta}\left(I_{n-\kappa} - \alpha \bar{E}_2\right) \end{bmatrix} \bar{x}(t) + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} u(t), \tag{A.11}$$

where $\bar{x}(t) = [\bar{x}_s^T(t), \bar{x}_f^T(t)]^T$, and $M^{-1} E_n M =$ block diagonal $\left\{\bar{E}_1, \bar{E}_2\right\}$. $\bar{E}_1$ is invertible with $\mathrm{rank}(\bar{E}_1) = \deg\{\det(sE_r - A_r)\}$, $\left[\bar{B}_1^T, \bar{B}_2^T\right]^T = M^{-1} B_n$ and $\bar{E}_2$ is a nilpotent matrix with dimension $(n-\kappa)\times(n-\kappa)$. Notice that since $\det(I_{n-\kappa}-\alpha\overline{E}_2)=1$, it is invertible. Let

$$W=\left[\begin{array}{c:c}\overline{E}_1 & O\\ \hdashline O & \frac{1}{\beta}(I_{n-\kappa}-\alpha\overline{E}_2)\end{array}\right], \tag{A.12}$$

which implies

$$W^{-1}=\left[\begin{array}{c:c}\overline{E}_1^{-1} & O\\ \hdashline O & \beta\left(I_{n-\kappa}-\alpha\overline{E}_2\right)^{-1}\end{array}\right]. \tag{A.13}$$

Simplifying (A.11) by multiplying W^{-1} on both sides, one has

$$\left[\begin{array}{c:c}I_\kappa & O\\ \hdashline O & \beta\left(I_{n-\kappa}-\alpha\bar{E}_2\right)^{-1}\bar{E}_2\end{array}\right]\dot{\bar{x}}(t)=\left[\begin{array}{c:c}\frac{1}{\beta}(\bar{E}_1^{-1}-\alpha I_\kappa) & O\\ \hdashline O & I_{n-\kappa}\end{array}\right]\bar{x}(t)$$
$$+\left[\begin{array}{c}\bar{E}_1^{-1}\bar{B}_1\\ \hdashline \beta\left(I_{n-\kappa}-\alpha\bar{E}_2\right)^{-1}\bar{B}_2\end{array}\right]u(t),$$

$$\left[\begin{array}{c:c}I_\kappa & O\\ \hdashline O & \bar{E}_f\end{array}\right]\dot{\bar{x}}(t)=\left[\begin{array}{c:c}\overline{A}_s & O\\ \hdashline O & I_{n-\kappa}\end{array}\right]\bar{x}(t)+\left[\begin{array}{c}\bar{B}_s\\ \hdashline \bar{B}_f\end{array}\right]u(t), \tag{A.14}$$

where $\bar{E}_f=\beta\left(I_{n-\kappa}-\alpha\bar{E}_2\right)^{-1}\bar{E}_2$, $\overline{A}_s=\frac{1}{\beta}\left(\bar{E}_1^{-1}-\alpha I_\kappa\right)$, $\bar{B}_s=\bar{E}_1^{-1}\bar{B}_1$, $\bar{B}_f=\beta(I_{n-\kappa}-\alpha\bar{E}_2)^{-1}\bar{B}_2$.

Let

$$\bar{x}(t)=V\hat{x}(t), \tag{A.15}$$

where $\hat{x}(t)=\left[\hat{x}_s^T(t),\hat{x}_f^T(t)\right]^T=\left[\bar{x}_s^T(t),(U^{-1}\bar{x}_f^T(t))\right]^T$ and

$$V=\left[\begin{array}{c:c}I_\kappa & O\\ \hdashline O & U\end{array}\right]_{n\times n}. \tag{A.16}$$

U is a modal matrix of $\bar{E}_f$ with dimension $(n-\kappa)\times(n-\kappa)$ such that $U^{-1}\bar{E}_fU$ is in the Jordan canonical form. The function JORDAN in MATLAB can be utilized to compute the generalized eigenvectors and the Jordan canonical form of a Jordan matrix. Substituting (A.15) into (A.14) and multiplying it by V^{-1}, we obtain

$$\left[\begin{array}{c:c}I_k & O\\ \hdashline O & \hat{E}_f\end{array}\right]\dot{\hat{x}}(t)=\left[\begin{array}{c:c}\hat{A}_s & O\\ \hdashline O & I_{(n-\kappa)}\end{array}\right]\hat{x}(t)+\left[\begin{array}{c}\hat{B}_s\\ \hdashline \hat{B}_f\end{array}\right]u(t), \tag{A.17}$$

where $\hat{E}_f=U^{-1}\bar{E}_fU$, $\hat{A}_s=\bar{A}_s$, $\hat{B}_s=\bar{B}_s$ and $\hat{B}_f=U^{-1}\bar{B}_f$. Notice that $\hat{E}_f$ is in the Jordan block form with d blocks of sizes $u_1,u_2,\cdots,u_d$, where $\sum_{i=1}^d u_i=$ column (row) number of $\hat{E}_f$.

Appendix B: Solving Riccati Equation via Matrix Sign Function [32]. The Riccati equation for the controllable continuous-time system $(\hat{A}_s,\hat{B}_s)$ with weighting matrices $\hat{Q}_s(>O)$ and $\hat{R}(>O)$ is given by

$$\hat{A}_s\hat{P}_s+\hat{P}_s\hat{A}_s-\hat{P}_s\hat{B}_s\hat{R}^{-1}\hat{P}_s+\hat{Q}_s=O. \tag{B.1a}$$

The steady state solution of this Riccati equation, $\hat{P}_s(>O)$ with $(\hat{Q}_s,\hat{A}_s)$ detectable, can be easily computed using the properties of the matrix sign function [25,33] and the eigenvalue-eigenvector approach [34]. Consider the Hamiltonian associated with the given system

$$H = \begin{bmatrix} \hat{A}_s & -\hat{B}_s \hat{R} B_s^T \\ -\hat{Q}_s & -\hat{A}_s^T \end{bmatrix}. \tag{B.1b}$$

The following algorithm can be utilized to obtain the solution $\hat{P}_s$,

$$\left. \begin{aligned} & H_{k+1} = \tfrac{1}{2}\left[H_k + H_k^{-1}\right], H_0 = H \\ \text{and} \quad & \\ & \lim_{k\to\infty} H_k = sign(H) \end{aligned} \right\}. \tag{B.2a}$$

Let

$$sign^+(H) = \frac{1}{2}\left[I_{2n} + sign(H)\right]. \tag{B.2b}$$

Construct a block modal matrix X as

$$X = \left[ind\left(sign^+(H)\right), ind\left(I_{2n} - sign^+(H)\right)\right] \triangleq \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \tag{B.3a}$$

where $ind(\cdot)$ represents the collection of the linearly independent column vectors of $(\cdot)$. Then, we have

$$\hat{P}_s = X_{22}(X_{12})^{-1}. \tag{B.3b}$$

To alleviate the problems of computing H_k^{-1}, the Hamiltonnain can transformed into a symmetric form as follows [25]

$$\hat{H} = \hat{J}H = \begin{bmatrix} O_n & -I_n \\ I_n & O_n \end{bmatrix} H = \begin{bmatrix} \hat{Q}_s & \hat{A}_s^T \\ \hat{A}_s & -\hat{B}_s \hat{R} B_s^T \end{bmatrix}. \tag{B.4a}$$

Then, the algorithm in (B.2) becomes

$$\left. \begin{aligned} & \hat{H}_{k+1} = \tfrac{1}{2}\left[\hat{H}_k + \hat{J}\hat{H}_k^{-1}\right], \hat{H}_0 = \hat{J}H \\ \text{and} \quad & \\ & \lim_{k\to\infty} \left(-\hat{J}\hat{H}_k\right) = sign(H) \end{aligned} \right\}. \tag{B.4b}$$

The computation of the inverse of the symmetric matrix $\hat{H}_k$ is much simpler than computing the inverse of H_k. The Riccati solution $\hat{P}_s$ is again given by (B.3).