VARIABLE FRACTIONAL DELAY FILTER
WITH SUB-EXPRESSION COEFFICIENTS

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ABSTRACT. *Variable fractional delay (VFD) filters are useful for various signal processing and communication applications with frequency characteristics such as fractional delays to be varied online. In this paper, we investigate the design of VFD filters with discrete coefficients as a means of achieving low complexity and efficient hardware implementation. The optimization problem with minimax criterion is formulated as a mixed integer programming problem with a non-linear cost function and continuous constraints. An efficient optimization procedure is proposed to tackle the design problem that includes a combination of the branch and bound method and an adaptive scheme for discretization. Design examples are given to demonstrate the effectiveness of the proposed algorithm.*
Keywords: Variable fractional delay filters, Discrete coefficients, Subexpression, Optimization, Branch and bound method

1. **Introduction.** Stochastic and deterministic systems have been active fields of research in many scientific and technical disciplines [1-11]. In stochastic systems [1-4], attention is focused on the design of filters subject to system errors. Full-order and reduced-ordered filter design approaches are developed in terms of linear matrix inequalities [3]. A great number of results concerning stochastic systems have been reported which include stochastic stability, stochastic control and estimation. It is well-known that Kalman filtering is effective in dealing with the state estimation problems. However, the drawback with Kalman filters is that their practical applications are limited to those for which system dynamics and noise statistics are precisely known.

In this paper, we investigate the design of fractional delay Farrow digital filters. These filters have applications in many different areas of signal processing and communications, including timing offset recovery in digital receivers, comb filter design, sampling rate conversion [8], speech coding, time delay estimation, one-dimensional digital signal interpolation and image interpolation [7-14]. Fractional delay digital filters provide controllable fractional delay for digital signals which can be efficiently adjusted online. Several papers have developed for the design of the variable fractional delay (VFD) Farrow-based structure with both finite and infinite precision coefficients [5-7, 11-14]. Here, we concentrate on investigating the design of the VFD filters for the efficient modified Farrow structure [15] with discrete coefficients and minimax criterion to reduce the complexity associated with the filters. Since the coefficients of the modified VFD structure have either even or odd symmetry, the number of multiplications in the filter can be reduced approximately by half.

During the past decade, the design of low complexity FIR filters has attracted considerable attention. One of the most common strategies is to optimize the coefficients

in signed power-of-two (SPT) space where each coefficient is represented as a sum of a limited number of SPT terms. Thus, coefficient multiplication can be replaced by shifters and adders [16]. The total number of adders used to implement the filter, which is a good measure of the complexity of the filter, can be greatly reduced by extracting common subexpressions that can be shared among different filter coefficients. In most cases, the FIR filters are designed in two stages. First, the filters are designed in a discrete space or a power-of-two (SPT) space, such that a given specification is met. In the second stage, various optimization techniques are employed to identify and extract the common subexpressions from the filter coefficients and so reduce the number of adders. Many optimization techniques are based on a heuristic search or a combined heuristics exhaustive search. As the two stages are optimized independently, the results obtained are often local solutions.

Here, we develop a technique that directly optimizes the coefficients of the modified Farrow structure given a set of subexpressions. The motivation of employing the subexpression set instead of the traditional SPT space is that the filter coefficients can use the terms from a large set base without incurring extra computational complexity. By combining the branch and bound method with an adaptive scheme for quantization, the optimal solution for the problem is readily obtained. The results obtained are compared with those using other techniques.

The rest of the paper is organized as follows. In Section 2, the VFD filters in a modified Farrow structure with an adjustable fractional delay are outlined and the optimization problem with infinite precision solution is formulated. In Section 3, the design of the VFD filters with coefficients belonging to a subexpression space is developed. A design example is presented in Section 4 and conclusion remarks are made in Section 5.

2. **A Modified Farrow Structure with an Adjustable Fractional Delay.** Consider the design of VFD filter with minmax criteria and infinite precision coefficients. The VFD filter in a modified Farrow structure (see [15]) is depicted in Figure 1. The structure consists of $M + 1$ parallel FIR filters with transfer functions

$$G_m(z) = \sum_{n=0}^{N} h(n,m) z^{-n}, \quad 0 \leq m \leq M. \tag{1}$$

The transfer function for the VFD filter is given as

$$H(\mathbf{h}, z, p) = \sum_{m=0}^{M} (1 - 2p)^m G_m(z) \tag{2}$$

where $\mathbf{h}$ is the coefficient vector of the VFD filter,

$$\mathbf{h} = \{h(0,0), \ldots, h(0,M), \ldots, h(N,0), \ldots, h(N,M)\}$$

and p is the control or tuning parameter, varied in the range $[0, 1]$. In addition, the coefficients $h(n,m)$ satisfy the following symmetry conditions:

$$h(n,m) = \begin{cases} h(N-n,m), & \text{if } m \text{ is even} \\ -h(N-n,m), & \text{if } m \text{ is odd.} \end{cases} \tag{3}$$

As such, the frequency response for the VFD filter in (2) can be reduced to

$$\begin{aligned} H(\mathbf{h}, \omega, p) &= \sum_{n=0}^{N} \sum_{m=0}^{M} (1 - 2p)^m h(n,m) e^{-jn\omega} \\ &= e^{-j\omega N/2} \sum_{n=0}^{(N-1)/2} \sum_{m=0}^{M} (1 - 2p)^m \beta(n,m) h(n,m) \end{aligned} \tag{4}$$
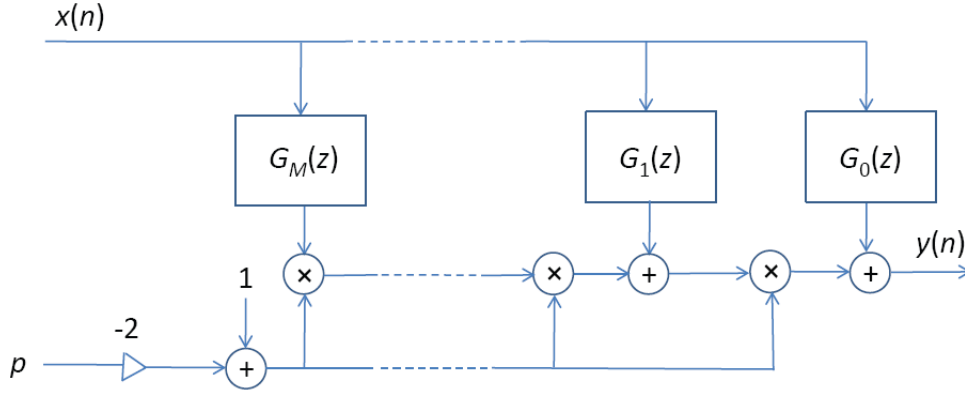
FIGURE 1. Modified Farrow structure with an adjustable fractional delay p

where $\beta(n, m)$ is a cosine or sine function depending on whether n is even or odd. The desired frequency response $H_d(\omega, p)$ is specified by

$$H_d(\omega, p) = e^{-j\omega\tau_d(p)}, \quad \omega \in \Omega_p = [0, \omega_p], \quad \omega_p \leq \pi.$$

The desired group delay $\tau_d(p)$ is given by

$$\tau_d(p) = \frac{N-1}{2} + p. \tag{5}$$

The design of a VFD filter with minimax criterion can be formulated as

$$\begin{cases} \min_{\mathbf{h} \in \mathbb{R}^{(N+1)(M+1)}} \epsilon \\ W(\omega, p)|H(\mathbf{h}, \omega, p) - H_d(\omega, p)| \leq \epsilon, \quad \forall \omega \in \Omega_p, \quad p \in [0, 1] \end{cases} \tag{6}$$

where $W(\omega, p)$ denotes the weighting function depending on ω and p. By using the Real Rotation Theorem [20], the non-linear optimization problem (6) can be reformulated as a semi-infinite linear programming problem

$$\begin{cases} \min_{\mathbf{h} \in \mathbb{R}^{(N+1)(M+1)}} \epsilon \\ W(\omega, p)R\left\{(H(\mathbf{h}, \omega, p) - H_d(\omega, p))\, e^{j2\pi\lambda}\right\} \leq \epsilon, \quad \forall \omega \in \Omega_p, \quad p \in [0, 1], \quad \lambda \in [0, 1] \end{cases} \tag{7}$$

where $R\{\cdot\}$ denotes the real part of a complex number.

The problem (7) can be solved by using semi-infinite programming techniques or discretization. For discretization, if λ is restricted to a finite set between 0 and 1, e.g., $\lambda \in \Lambda = \left\{0, \frac{1}{8}, \frac{2}{8}, \ldots, \frac{7}{8}\right\}$, then the complex error magnitude is at worst 0.68 dB more than the optimum error. As the optimization problem is discretized over three variables λ, ω and p, the number of discretized constraints can be large. Thus, a near-active constraint scheme [12] can be employed to solve this problem by reducing the number of constraints to those around the active points.

3. **VFD with Subexpression Space Coefficients.** We now formulate the design of a VFD filter in modified Farrow structure with discrete coefficients as a means of achieving low complexity associated with the filter. The filter coefficients are taken from a subexpression space B generated by a set S,

$$B = \left\{\sum_{i=0}^{L-1} y(i)2^{-q(i)}, \ y(i) \in S, \ 1 \leq q(i) \leq b\right\} \tag{8}$$

where L is the maximum allowable number of terms and b is the maximum allowable number of bits [17, 18]. Here, the base set S is obtained by extending the traditional SPT

set $S = \{0, \pm 1\}$ to a larger set, for example $S = \{0, \pm 1, \pm (2^{-1} + 2^{-2})\}$. The advantage of employing a larger set base is that the terms in S can be used for many filter coefficients without incurring extra computational complexity. As such, the VFD filter coefficients can be designed with a smaller number of multiplications when compared with those with a smaller base given the same number of allowable bits.

For the design of the VFD filter with discrete coefficient $\bar{\mathbf{h}}$, a floating gain $\beta > 0$ is included. The design of the filter with discrete coefficients and minimax criterion can be formulated as

$$
\begin{cases}
\min\limits_{\bar{\mathbf{h}} \in B^{(N+1)(M+1)}, \ \beta \in \mathbb{R}^+} \epsilon \\
W(\omega, p) \left| \frac{H(\bar{\mathbf{h}}, \omega, p)}{\beta} - H_d(\omega, p) \right| \le \epsilon, \ \forall \omega \in \Omega_p, \ p \in [0, 1]
\end{cases}
\tag{9}
$$

or

$$
\begin{cases}
\min\limits_{\bar{\mathbf{h}} \in B^{(N+1)(M+1)}, \ \beta \in \mathbb{R}^+} \epsilon \\
W(\omega, p) \left| H(\bar{\mathbf{h}}, \omega, p) - \beta H_d(\omega, p) \right| \le \epsilon \beta, \ \forall \omega \in \Omega_p, \ p \in [0, 1].
\end{cases}
\tag{10}
$$

The right hand side of the constraints in (10) is non-linear in terms of the variables ϵ and β. To overcome the non-linearity, a new variable is introduced by letting $\delta = \epsilon\beta$. The optimization problem (10) can be rewritten as

$$
\begin{cases}
\min\limits_{\bar{\mathbf{h}} \in B^{(N+1)(M+1)}, \ \beta \in \mathbb{R}^+} \delta/\beta \\
W(\omega, p) \left| H(\bar{\mathbf{h}}, \omega, p) - \beta H_d(\omega, p) \right| \le \delta, \ \forall \omega \in \Omega_p, \ p \in [0, 1].
\end{cases}
\tag{11}
$$

The problem (11) is a mixed integer optimization problem with non-linear objective function and continuous infinite linear constraints. The problem can be solved by introducing a new variable α and reducing the objective function in (11) to a linear function for a fixed value of α,

$$
\begin{cases}
\min\limits_{\bar{\mathbf{h}} \in B^{(N+1)(M+1)}, \ \beta, \delta \in \mathbb{R}^+} f(\alpha, \delta, \beta) = \delta - \alpha\beta \\
W(\omega, p) | H(\bar{\mathbf{h}}, \omega, p) - \beta H_d(\omega, p) | \le \delta, \ \forall \omega \in \Omega_p, \ p \in [0, 1].
\end{cases}
\tag{12}
$$

An iterative procedure can be employed to update the value of α [16]. Initially, α is set at 0. This approach, however, results in high computational complexity as the mixed-integer discrete optimization problem (12) is required to be solved multiple times. Here, we observe that acceptable results are obtained by solving the first case with $\alpha = 0$. The optimization problem (12) becomes a mixed integer semi-infinite linear optimization problem. This problem can be solved by combining the global branch and bound technique for discrete optimization problem with the near-active constraint scheme for a semi-infinite programming problem. Note that while the idea of the branch and bound algorithm is generally known, the process of formulating the relaxation problem and the procedure of cutting/dividing the branches are problem dependent.

We now employ branch and bound with a depth-first search to optimize the discrete filter coefficients in (12). The problem is divided into a number of sub-problems, referred to as branches or nodes, by restricting a subset of coefficients to fixed finite precisions. For each node, a lower bound is then obtained. This lower bound can then be used to decide whether to immediately cut the node or to allow further branching. The procedure of the branch and bound algorithm is outlined as follows:

Procedure 3.1. *Branch and bound algorithm in conjunction with a near-active constraint procedure for solving optimizing problem (12).*

- Step 0: Initialize a set D containing the discrete branching coefficients, $D = \emptyset$, and a set C defined the branching constraints, $C = \emptyset$. Initialize the optimum

discrete solution $(\mathbf{h}^*, \beta^*)$ as the quantized solution $(\mathbf{h}_q, \beta_q)$ of the infinite precision solution $\mathbf{h}_{\text{inf}}$. Set $\mathbf{h}_{\text{inf}}^C = \mathbf{h}_{\text{inf}}$. Initialize the maximum number of nodes visited for the algorithm.

- Step 1: Select a coefficient $h(n, m)$, $h(n, m) \notin D$, with the largest magnitude in $\mathbf{h}_{\text{inf}}^C$ for branching.
- Step 2: Set $D = D \cup h(n, m)$ and start the branching process from $h(n, m)$. Denote by $h^\ell(n, m)$ and $h^u(n, m)$ the largest and smallest discrete values in B so that

$$h^\ell(n, m) \leq h_{inf}^C(n, m) \leq h^u(n, m).$$

Two branches are added at the node $h(n, m)$ with extra constraints $h(n, m) \leq h^\ell(n, m)$ or $h(n, m) \geq h^u(n, m)$. Select the first branch and add the constraint $h(n, m) \leq h^\ell(n, m)$ to the constraint set C.

- Step 3: Make a decision on the current branch and the current node. This is done by obtaining a lower bound for the branch by solving for the optimum solution of the following relaxation problem for the branch

$$
\begin{cases}
\min\limits_{\mathbf{h} \in \mathbb{R}^{(N+1)(M+1)}, \ \beta \in \mathbb{R}^+} \delta \\
W(\omega, p)|H(\mathbf{h}, \omega, p) - \beta H_d(\omega, p)| \leq \delta, \ \forall \omega \in \Omega_p, \ p \in [0, 1] \\
\text{subject to additional constraints in } C.
\end{cases}
\tag{13}
$$

This problem is solved by using discretization and the near-active constraints [12]. Denote by $\{\mathbf{h}^C, \beta^C\}$ the optimum solution for (13) corresponding to the constraint set C. Quantized $\mathbf{h}^C$ to the closest finite precision solution $\mathbf{h}_q^C$ with coefficients in the subexpression space B. We now consider solving the following optimization problem for β_q^C

$$
\begin{cases}
\min\limits_{\beta \in \mathbb{R}^+} \delta \\
W(\omega, p)|H(\mathbf{h}_q^C, \omega, p) - \beta H_d(\omega, p)| \leq \delta, \ \forall \omega \in \Omega_p, \ p \in [0, 1].
\end{cases}
\tag{14}
$$

If the quantized solution is better than the current discrete optimal solution,

$$\frac{\delta_q}{\beta_q} < \frac{\delta_{\text{opt}}}{\beta_{\text{opt}}}$$

then update the optimal discrete solution by setting $(\mathbf{h}^*, \beta^*) = (\mathbf{h}_q^C, \beta_q^C)$. If the relaxation cost is within a small tolerance τ of the current optimal discrete solution,

$$\left| \frac{\delta^C}{\beta^C} - \frac{\delta_{\text{opt}}}{\beta_{\text{opt}}} \right| \leq \tau$$

or the current relaxation cost is greater than the optimal discrete solution,

$$\frac{\delta^C}{\beta^C} \geq \frac{\delta_{\text{opt}}}{\beta_{\text{opt}}}$$

then the current node is removed. Then, backtrack up the tree to the next branching node. Update the node set D and the constraint set C accordingly by removing constraints related to the omitted nodes. If the node set is $\emptyset$ or the number of node visited greater than the allowable limit, then stop the procedure: the optimum discrete solution is set as $(\mathbf{h}^*, \beta^*)$. Otherwise, continue with branching by returning to Step 1.

4. **Design Example.** Consider the design of a VFD filter with the frequency range $\Omega_p = [0, 0.75\pi]$ and the number of bits for discrete coefficients is $b = 10$. The filter coefficients are restricted as the sum of two powers-of-two, i.e., $L = 2$. The number of discretization points for p is 21, while the number of points for ω is $20N$ [15]. We consider the maximum frequency response deviation error

$$\Delta_{FR} = \max_{0 \le p \le 1} \left[\max_{\omega \in \Omega_p} |H(\mathbf{h}, \omega, p) - H_d(\omega, p)| \right].$$

Table 1 shows the maximum frequency response error deviation for the filters with coefficients obtained as a sum of two powers of two with the traditional base $S = \{0, \pm 1\}$ [11] and an extended base

$$\hat{S} = \{0, \pm 1, \pm \left(2^{-1} + 2^{-2}\right)\}.$$

The value of N increases from 11 to 19 while M increases from 3 to 4. The table also shows the maximum frequency response error for the infinite precision solution. It can be seen from the table that the minimum error is significantly reduced by using an extended base. In many cases, the solutions obtained from an extended base are much closer to the infinite precision solution than those obtained from the standard base.

Table 2 gives the minimax error for $N = 11$ and $M = 3$. The table also shows the (i) infinite precision solution; (ii) finite precision solution with the standard base; (iii) finite precision solution with an extended base; (iv) infinite precision solution with a few coefficients rounded to 0; and finite precision solution obtained in [15]. The filters obtained in

TABLE 1. Maximum frequency response deviation errors for M and N

N	M	Desired Delay τ_d	Traditional base $S = \{0, \pm 1\}$	Extended base $S = \{0, \pm 1, \pm (2^{-1} + 2^{-2})\}$	Infinite precision solution
11	3	5	0.0160	0.0097	0.0094
11	4	5	0.0220	0.0051	0.0039
13	3	6	0.0133	0.0096	0.0094
13	4	6	0.0142	0.0024	0.0016
15	3	7	0.0159	0.0110	0.0094
15	4	7	0.0259	0.0130	0.0011
17	3	8	0.0189	0.0145	0.0094
17	4	8	0.0262	0.0096	0.0011
19	3	9	0.0215	0.0130	0.0094
19	4	9	0.0236	0.0066	0.0011

TABLE 2. Maximum frequency response deviation errors with $N = 11$ and $M = 3$

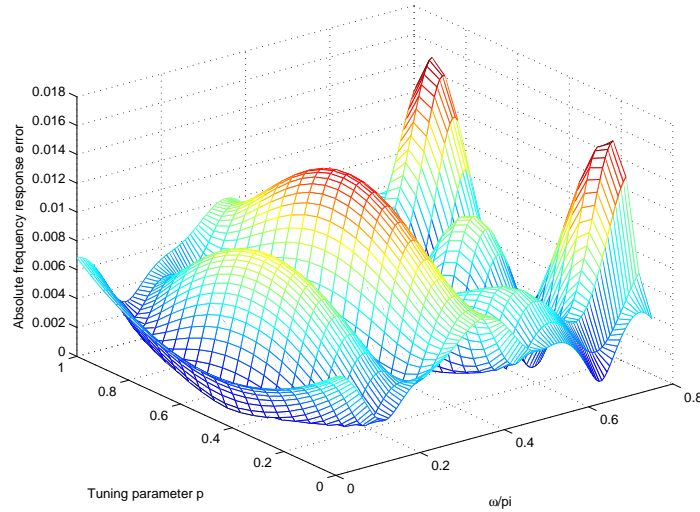|	Δ_{FR}
Infinite precision minmax solution	0.0094
Sum of two power of two with $S = \{0, \pm 1\}$	0.0160
Sum of two terms with $\hat{S} = \{0, \pm 1, \pm (2^{-1} + 2^{-2})\}$	0.0097
Infinite precision solution in [15]	0.0127
Infinite precision solution in [15] with a few coefficients rounded to 0	0.0139
Finite precision in [15]	0.0208

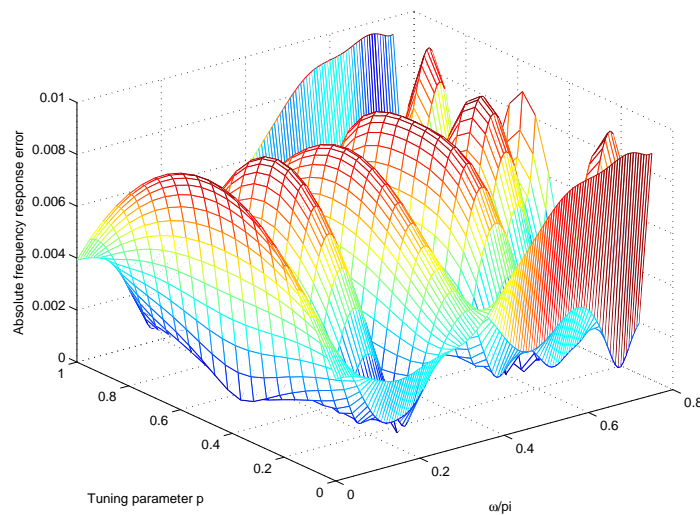FIGURE 2. Magnitude frequency response error for VFD filters with traditional base

FIGURE 3. Magnitude frequency response error for VFD filters with modified base

[15] have larger frequency response errors than filters designed by using minimax criteria. From the table, the minimax error is significantly reduced by employing the subexpression with a larger base.

Figures 2 and 3 plot the magnitude frequency response deviation errors for the VFD filters with the traditional and modified bases and discrete coefficients. The value of N is 11 while $M = 3$. Figure 4 shows the magnitude frequency response error for the VFD filter with infinite precision coefficients. It can be seen from the figures that the frequency response error for the VFD filter with discrete coefficients and the modified base is much lower than that with the traditional base. In fact, the frequency response error for the VFD filter with the modified base is very closed to the infinite precision solution.
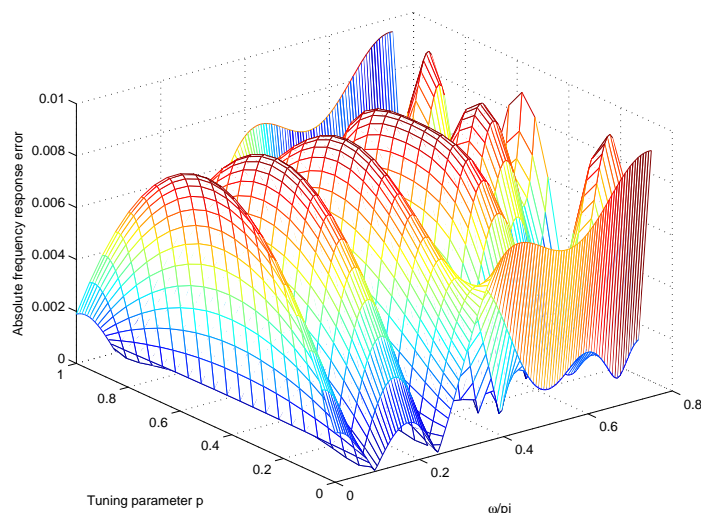
FIGURE 4. Magnitude frequency response error for the infinite precision VFD filter

5. **Conclusions.** This paper investigates the design of the modified Farrow structure with coefficients restricted in the discrete space constructed by a given set of subexpressions. The design problem is formulated as a mixed integer programming problem with non-linear cost function and continuous constraints. An efficient two-stage optimization procedure is proposed to tackle the design problem that includes an iterative procedure in combination with the branch and bound method and an adaptive scheme for grid quantization. The design example shows that by adding an additional term in the subexpression, the minimax error for the discrete filters is significantly reduced when compared with the filters with traditional sum of two-powers-of-two coefficients while requiring approximately the same number of additions and multiplications.

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