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EDUCATIONAL EVALUATION APPLYING APPROXIMATE REASONING WITH TRIANGULAR AND QUADRATIC FUNCTIONS

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Abstract. *In our previous studies regarding application of approximate reasoning, fuzzy numbers have been applied to obtain educational evaluation results. However, the reasoning results were not always appropriate since the conventional extension principle would sometimes prevent the symmetry of the results regardless of its symmetric input. The study aims to solve the above predicament; the author established an improved extension principle to implement approximate reasoning with fuzzy numbers more naturally. The focus of this paper was the comparison between the conventional extension principle and the improved extension principle by applying them to two education evaluation cases. The findings showed that the improved extension principle performed better than the conventional one and it would ensure the symmetry of the output provided that the input was symmetric.*
Keywords: Approximate reasoning, Characteristic function, Educational evaluation, Extension principle, Fuzzy number

1. Introduction.

In recent years, approximate reasoning has been developed quickly and has begun to be applied as a tool for educational evaluation. For example, it is reasonable for teachers to evaluate study results of Student S_1's (Examination = 95, Report = 65) and Student S_2's (Examination = 100, Report = 30) by taking their weighted means with the parameter $\alpha = 0.7$; however, this method is not suitable for some cases like Student S_3's (Examination = 40, Report = 100):

	Grade
Student S_1:	
$h(95, 65; \alpha = 0.7) = 0.7 \cdot 95 + (1 - 0.7) \cdot 65 = 86$	(A / Excellent)
Student S_2:	
$h(100, 30; \alpha = 0.7) = 0.7 \cdot 100 + (1 - 0.7) \cdot 30 = 79$	(B / Good)
Student S_3:	
$h(40, 100; \alpha = 0.7) = 0.7 \cdot 40 + (1 - 0.7) \cdot 100 = 58$	(D / Fail)

For some teachers who hardly ignore Student S_3's high scores at Report and are likely to give this student a C[1] instead of a D[1], approximate reasoning can always help these teachers reach their ideal evaluation results by adjusting reasoning rules and membership functions until the reasoning results reach their ideals. In the articles [1, 2], the authors elaborate on the way how they apply approximate reasoning to evaluating calligraphy and art works. Furthermore, if we want to apply approximate reasoning to educational evaluation more effectively, then educational evaluation should not be limited to regular (crisp) numbers. Therefore, the execution of approximate reasoning with fuzzy numbers

[1] A / Excellent, B / Good, C / Fair and D / Fail.

is proposed. It is known that the extension principle proposed by Prof. Lotfi Zadeh is valid for all fuzzy numbers operations and especially good at triangular fuzzy numbers operation [3]. This paper will provide a new choice to execute approximate reasoning with fuzzy numbers. As an educational evaluation tool, the new method given in this article is especially developed to avoid some unnatural evaluation results and simplify the computation of the execution.

2. Preliminaries.

Before giving the explanation about the new way to execute approximate reasoning with fuzzy numbers, we have to know how the inverse functions of fuzzy numbers are defined here.

Definition 2.1. *A fuzzy number $\tilde{A} =< a-t, a, a+t >_{tri}$ is called an isosceles triangular fuzzy number, if it has the characteristic function with positive t as follows.*

$$\chi_{\tilde{A}}(x) = \max\left\{0, 1 - \frac{|x-a|}{t}\right\} \tag{1}$$

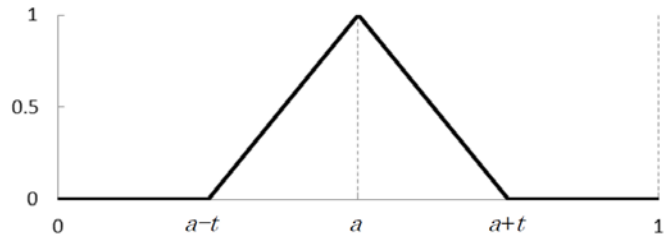

FIGURE 1. Characteristic function of an isosceles triangular fuzzy number

Definition 2.2. *A fuzzy number $\tilde{A} =< a-t, a, a+t >_{qua}$ is called a quadratic-curved fuzzy number, if it has the characteristic function with positive t as follows.*

$$\chi_{\tilde{A}}(x) = \begin{cases} \frac{2}{t^2}(x-(a-t))^2, & a-t \le x \le a - \frac{t}{2} \\ 1 - \frac{2}{t^2}(x-a)^2, & a - \frac{t}{2} < x \le a + \frac{t}{2} \\ \frac{2}{t^2}(x-(a+t))^2, & a + \frac{t}{2} < x \le a+t \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

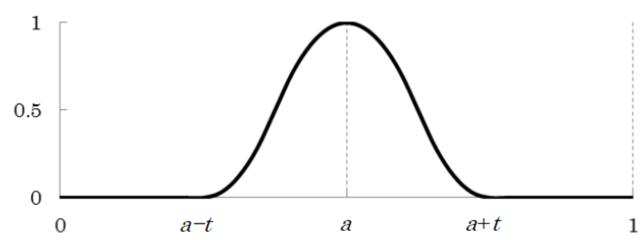

FIGURE 2. Characteristic function of a quadratic-curved fuzzy number

By Figure 1 and Figure 2, we know that both of the isosceles triangular fuzzy number and the quadratic-curved fuzzy number are fuzzy singletons.

Definition 2.3. *For any fuzzy number $\tilde{A}$ the inverse function of its characteristic function is defined as follows.*

$$\chi^{-1}_{\tilde{A}_{left}}(x) = \inf\{y \mid \chi_{\tilde{A}}(y) = x\}; \text{ and} \tag{3}$$

$$\chi^{-1}_{\tilde{A}_{right}}(x) = \sup\{y \mid \chi_{\tilde{A}}(y) = x\} \tag{4}$$

Example 2.1. *Let $\tilde{A}$ be the isosceles triangular fuzzy number defined by $< a-t, a, a+t >_{tri}$ (see Definition 2.1). Then*

$$\chi_{\tilde{A}_{left}}(x) = a + t(x-1); \text{ and} \tag{5}$$

$$\chi_{\tilde{A}_{right}}(x) = a - t(x-1). \tag{6}$$

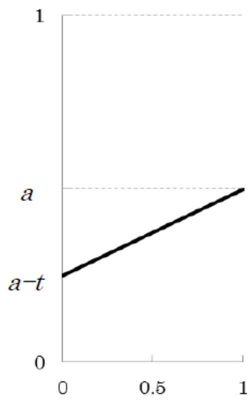

FIGURE 3. Inverse function $\chi^{-1}_{\tilde{A}_{left}}(x)$

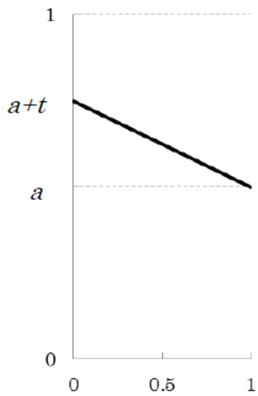

FIGURE 4. Inverse function $\chi^{-1}_{\tilde{A}_{right}}(x)$

3. A Tool for Educational Evaluation Applying Approximate Reasoning.

To apply approximate reasoning, we usually follow these five steps:

Step 1: Organize the evaluation structure;
Step 2: Establish the grading scales and the reasoning rules (matrix);
Step 3: Establish the membership functions based on grading scales;
Step 4: Analyze the facts and execute the reasoning; and
Step 5: Investigate the results of the reasoning.

Step 1: Based on the goal of the lesson concerned, we establish two evaluation structures for Japanese and Taiwanese calligraphy works, shown as Figures 5 and 6.

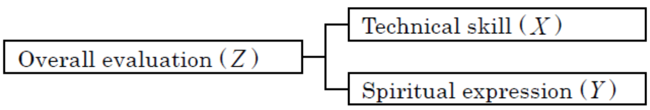

FIGURE 5. Evaluation structure for a calligraphy work in Japan

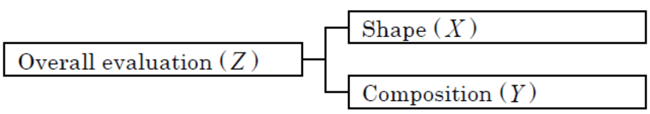

FIGURE 6. Evaluation structure for a calligraphy work in Taiwan

Step 2: Since both of the evaluation structures set above consist of two evaluation items, we may apply the following grading scales and the reasoning rules (matrix) to them:

$$\begin{aligned}
\text{Overall evaluation } (Z)\text{:} &\quad A_Z,\ B_Z,\ C_Z,\ D_Z,\ E_Z \\
\text{Technical skill / Shape } (X)\text{:} &\quad A_X,\ B_X,\ C_X \\
\text{Spiritual expression / Composition } (Y)\text{:} &\quad A_Y,\ B_Y,\ C_Y
\end{aligned}$$

$$\begin{aligned}
\text{Rule 1:} &\quad A_X, A_Y \Rightarrow A_Z \\
\text{Rule 2:} &\quad B_X, A_Y \Rightarrow B_Z \\
&\quad \vdots \\
\text{Rule 9:} &\quad C_X, C_Y \Rightarrow E_Z
\end{aligned}$$

Table 1. Reasoning rules matrix

Evaluation item X	Evaluation item Y		
	A_Y	B_Y	C_Y
A_X	A_Z	B_Z	C_Z
B_X	B_Z	C_Z	D_Z
C_X	C_Z	D_Z	E_Z

Step 3: Establish membership functions based on grading scales given at Step 2. According to the preference of the teacher, these membership functions can be established in different shapes like triangles or bells.

$$\mu_{A_Z}(z) = \max\{0, 1 - 4|z-1|\}; \tag{7}$$

$$\mu_{B_Z}(z) = \max\left\{0, 1 - 4\left|z - \frac{3}{4}\right|\right\}; \tag{8}$$

$$\mu_{C_Z}(z) = \max\left\{0, 1 - 4\left|z - \frac{1}{2}\right|\right\}; \tag{9}$$

$$\mu_{D_Z}(z) = \max\left\{0, 1 - 4\left|z - \frac{1}{4}\right|\right\}; \text{ and} \tag{10}$$

$$\mu_{E_Z}(x) = \max\{0, 1 - 4|z|\}. \tag{11}$$

$$\mu_{A_X}(x) = \max\{0, 1 - 2|x-1|\}; \tag{12}$$

$$\mu_{B_X}(x) = \max\{0, 1 - 2|x-0.5|\}; \text{ and} \tag{13}$$

$$\mu_{C_X}(x) = \max\{0, 1 - 2|x|\}. \tag{14}$$

$$\mu_{A_Y}(y) = \begin{cases} 8(y-0.5)^2, & 0.5 \le y \le 0.75 \\ 1 - 8(y-1)^2, & 0.75 < x \le 1.25 \\ 8(y-1.5)^2, & 1.25 < x \le 1.5 \\ 0, & \text{otherwise} \end{cases}; \tag{15}$$

$$\mu_{B_Y}(y) = \begin{cases} 8y^2, & 0 \le y \le 0.25 \\ 1 - 8(y-0.5)^2, & 0.25 < x \le 0.75 \\ 8(y-1)^2, & 0.75 < x \le 1 \\ 0, & \text{otherwise} \end{cases}; \text{ and} \tag{16}$$

$$\mu_{C_Y}(y) = \begin{cases} 8(y+0.5)^2, & -0.5 \le y \le -0.25 \\ 1 - 8y^2, & -0.25 < x \le 0.25 \\ 8(y-0.5)^2, & 0.25 < x \le 0.5 \\ 0, & \text{otherwise} \end{cases}. \tag{17}$$

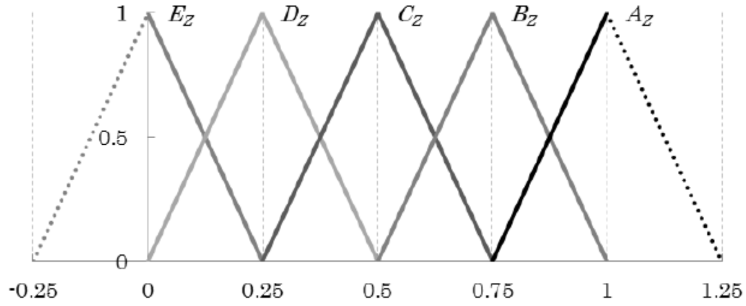

FIGURE 7. Membership functions on Z

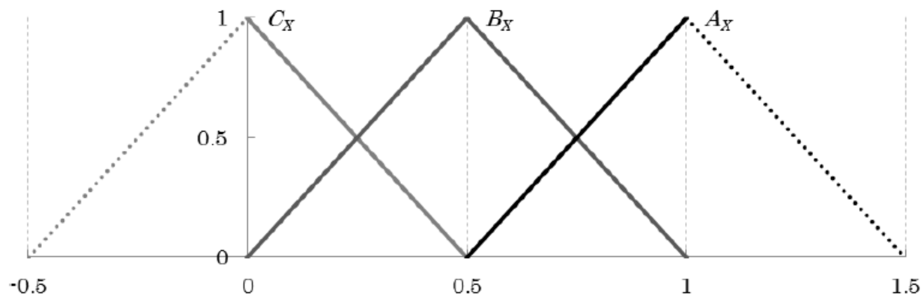

FIGURE 8. Membership functions on X

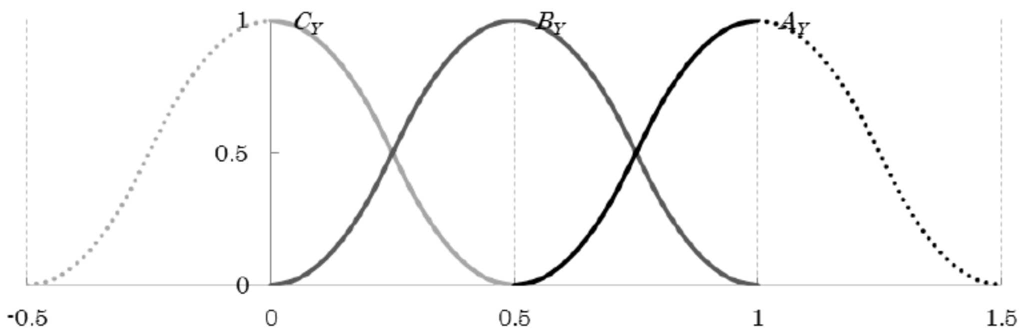

FIGURE 9. Membership functions on Y

FIGURE 10. Japanese student's calligraphy work

Step 4: Analyze the fact about two calligraphy works: one is Japanese student's, and the other is Taiwanese student's.

FIGURE 11. Taiwanese student's calligraphy work

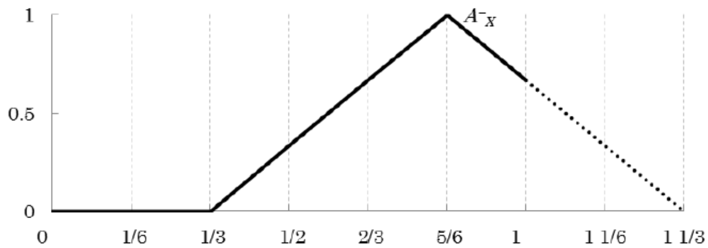

FIGURE 12. Fuzzy number $\tilde{A}^-_X$

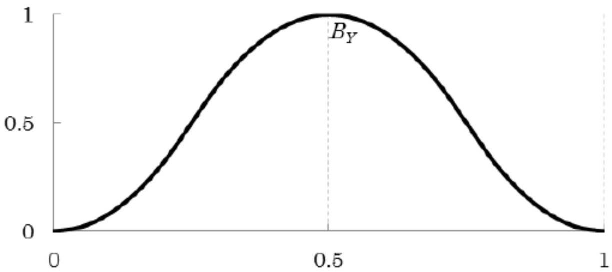

FIGURE 13. Fuzzy number $\tilde{B}_Y$

To analyze these works along the evaluation items given above, we collected some comments about them. In Japanese teachers' perspectives, Figure 10 is a fine brushwork and written well. Teachers also said that the shape, balance and the size of the characters are controlled well. Then we have analysis results as follows:

Technical skill: A^-_X approximately, we represent it by fuzzy number $\tilde{A}^-_X =< \frac{1}{3}, \frac{5}{6}, \frac{4}{3} >_{tri}$, shown as Figure 12.

Spiritual expression: B_Y approximately, we represent it by fuzzy number $\tilde{B}_Y =< 0, \frac{1}{2}, 1 >_{qua}$, shown as Figure 13.

In Taiwanese teachers' perspectives on Figure 11, the placement of the letters reaches an average level, but the shapes of the letters are not exquisite enough. We analyze the fact in accordance to these comments and use some fuzzy numbers to represent them as follows:

Shape: B_X approximately, we represent it by fuzzy number $\tilde{B}_X =< 0, \frac{1}{2}, 1 >_{tri}$, shown as Figure 14.

Composition: B^-_Y approximately, we represent it by fuzzy number $\tilde{B}^-_Y =< \frac{1}{6}, \frac{1}{3}, \frac{5}{6} >_{qua}$, shown as Figure 15.

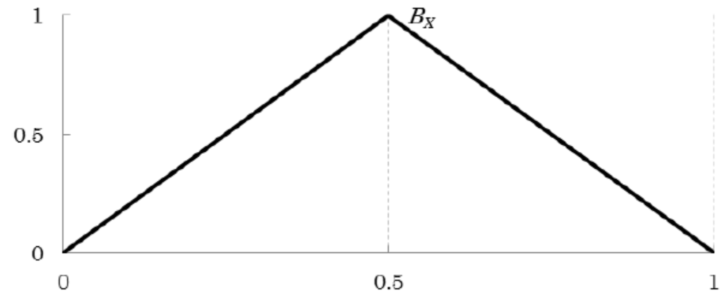

FIGURE 14. Fuzzy number $\tilde{B}_X$

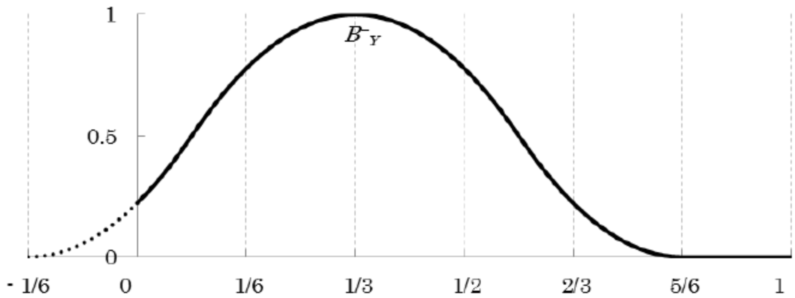

FIGURE 15. Fuzzy number $\tilde{B}^-_Y$

If we just want to execute approximate reasoning with input $(A^-_X, B_Y) = (\frac{5}{6}, \frac{1}{2})$, then we may obtain the output by following sub-steps i) – iii).

i) *Calculate the input's membership grade along the grading scales:*

$$\mu_{A_X}(A^-_X) = \mu_{A_X}\left(\frac{5}{6}\right) = \frac{2}{3}; \tag{18}$$

$$\mu_{B_X}(A^-_X) = \mu_{B_X}\left(\frac{5}{6}\right) = \frac{1}{3}; \tag{19}$$

$$\mu_{C_X}(A^-_X) = \mu_{C_X}\left(\frac{5}{6}\right) = 0; \tag{20}$$

$$\mu_{A_Y}(B_Y) = \mu_{A_Y}\left(\frac{1}{2}\right) = 0; \tag{21}$$

$$\mu_{B_Y}(B_Y) = \mu_{B_Y}\left(\frac{1}{2}\right) = 1; \text{ and} \tag{22}$$

$$\mu_{C_Y}(B_Y) = \mu_{C_Y}\left(\frac{1}{2}\right) = 0. \tag{23}$$

ii) *Based on the reasoning rules, compute the membership grade for each grade on Z:*

Since $\mu_{C_X}(A^-_X) = \mu_{A_Y}(B_Y) = \mu_{C_Y}(B_Y) = 0$, we need to consider only two reasoning rules here.

$$A_X, B_Y \Rightarrow B_Z: \quad \mu_{A_X}(A^-_X) \cdot \mu_{B_Y}(B_Y) = \frac{2}{3} \cdot 1 = \frac{2}{3}; \text{ and} \tag{24}$$

$$B_X, B_Y \Rightarrow C_Z: \quad \mu_{B_X}(A^-_X) \cdot \mu_{B_X}(B_Y) = \frac{1}{3} \cdot 1 = \frac{1}{3}. \tag{25}$$

iii) *According to the membership grade obtained above, compute the reasoning result by the product-sum-gravity method:*

For the membership grades for B_Z and C_Z obtained above, we may say the reasoning output $f(A_X^-, B_Y)$ is $\frac{2}{3} \cdot B_Z + \frac{1}{3} \cdot C_Z$. That is,

$$f(A_X^-, B_Y) = \frac{2}{3} \cdot B_Z + \frac{1}{3} \cdot C_Z = B_Z^-. \tag{26}$$

If we want to execute approximate reasoning with more complicated inputs like $(\tilde{A}_X^-, \tilde{B}_Y)$ and $(\tilde{B}_X, \tilde{B}_Y^-)$, where $\tilde{A}_X^-$, $\tilde{B}_Y$, $\tilde{B}_X$ and $\tilde{B}_Y^-$ are fuzzy numbers, then we may obtain the characteristic functions of the outputs $f(\tilde{A}_X^-, \tilde{B}_Y)$ and $f(\tilde{B}_X, \tilde{B}_B^-)$ by Zadeh's extension principle as Equation (27) and Equation (28) shown:

$$\chi_{f(\tilde{A}_X^-, \tilde{B}_Y)} = \bigvee_{f(x,y)} (\chi_{\tilde{A}_X^-}(x) \wedge \chi_{\tilde{B}_Y}(y)); \text{ and} \tag{27}$$

$$\chi_{f(\tilde{B}_X, \tilde{B}_Y^-)} = \bigvee_{f(x,y)} (\chi_{\tilde{B}_X}(x) \wedge \chi_{\tilde{B}_Y^-}(y)). \tag{28}$$

In the case that the supports of the fuzzy number $\tilde{A}_X^- =< \frac{1}{3}, \frac{5}{6}, \frac{4}{3} >_{tri}$ and fuzzy number $\tilde{B}_Y^- =< -\frac{1}{6}, \frac{1}{3}, \frac{5}{6} >_{qua}$ go beyond the interval $[0, 1]$, the paper [4] suggest that grading scales and some reasoning rules should be extended before the execution of approximate reasoning. For this reason, we extend grading scales and reasoning rules as the method suggested in [4] and then compute the reasoning results by Zadeh's extension principle. Here, we may illustrate the reasoning results as following graphs shown:

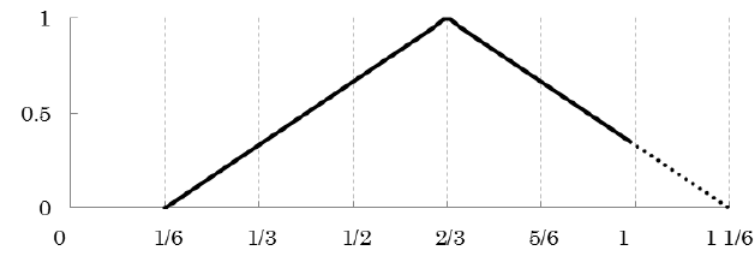

FIGURE 16. Reasoning result $f(\tilde{A}_X^-, \tilde{B}_Y)$ obtained by extension principle

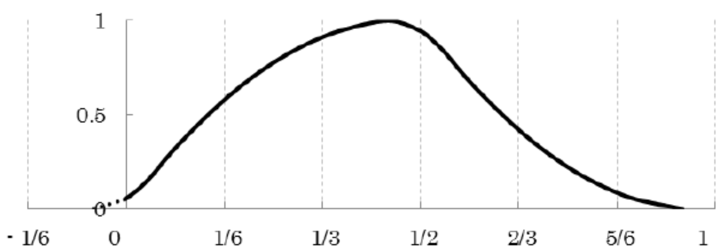

FIGURE 17. Reasoning result $f(\tilde{B}_X, \tilde{B}_Y^-)$ obtained by extension principle

Obviously, $f(\tilde{A}_X^-, \tilde{B}_Y)$ is an isosceles triangular fuzzy number, and $f(\tilde{B}_X, \tilde{B}_Y^-)$ is an asymmetric fuzzy numbers. However, teachers are likely to expect to obtain a symmetric reasoning result while executing reasoning with symmetric input. Therefore, we give another extension principle to ensure the symmetry and to simplify the computation. In order to introduce how to apply this improved extension principle to approximate reasoning, we implement approximate reasoning on the same cases and execute the 4th step of the reasoning again by following new sub-steps I) – V):

Step 4 (improved): For the inputs $(\tilde{A}_X^-, \tilde{B}_Y)$ and $(\tilde{B}_X, \tilde{B}_Y^-)$, we calculate the membership functions of their reasoning results by new procedures.

I) *Analyze the input (Consider their representative values* $rep(\cdot)$ *and their inverse function):*

We can say that the representative value of fuzzy number is located at its peak if it is a fuzzy singleton; therefore, we have the representative values for all inputs as follows.

$$rep(\tilde{A}_X^-) = rep\left(< \frac{1}{3}, \frac{5}{6}, \frac{4}{3} >_{tri}\right) = \frac{5}{6}; \tag{29}$$

$$rep(\tilde{B}_Y) = rep\left(< 0, \frac{1}{2}, 1 >_{qua}\right) = \frac{1}{2}; \tag{30}$$

$$rep(\tilde{B}_X) = rep\left(< 0, \frac{1}{2}, 1 >_{tri}\right) = \frac{1}{2}; \text{ and} \tag{31}$$

$$rep(\tilde{B}_Y^-) = rep\left(< -\frac{1}{6}, \frac{1}{3}, \frac{5}{6} >_{tri}\right) = \frac{1}{3}. \tag{32}$$

II) *Calculate each representative value's membership grade along the corresponding grading scales:*

$$\mu_{A_X}(rep(\tilde{A}_X^-)) = \mu_{A_X}\left(\frac{5}{6}\right) = \frac{2}{3}; \tag{33}$$

$$\mu_{B_X}(rep(\tilde{A}_X^-)) = \mu_{B_X}\left(\frac{5}{6}\right) = \frac{1}{3}; \tag{34}$$

$$\mu_{C_X}(rep(\tilde{A}_X^-)) = \mu_{C_X}\left(\frac{5}{6}\right) = 0; \tag{35}$$

$$\mu_{A_Y}(rep(\tilde{B}_Y)) = \mu_{A_Y}\left(\frac{1}{2}\right) = 0; \tag{36}$$

$$\mu_{B_Y}(rep(\tilde{B}_Y)) = \mu_{B_Y}\left(\frac{1}{2}\right) = 1; \tag{37}$$

$$\mu_{C_Y}(rep(\tilde{B}_Y)) = \mu_{C_Y}\left(\frac{1}{2}\right) = 0; \tag{38}$$

$$\mu_{A_X}(rep(\tilde{B}_X)) = \mu_{A_X}\left(\frac{1}{2}\right) = 0; \tag{39}$$

$$\mu_{B_X}(rep(\tilde{B}_X)) = \mu_{B_X}\left(\frac{1}{2}\right) = 1; \tag{40}$$

$$\mu_{C_X}(rep(\tilde{B}_X)) = \mu_{C_X}\left(\frac{1}{2}\right) = 0; \tag{41}$$

$$\mu_{A_Y}(rep(\tilde{B}_Y^-)) = \mu_{A_Y}\left(\frac{1}{3}\right) = 0; \tag{42}$$

$$\mu_{B_Y}(rep(\tilde{B}_Y^-)) = \mu_{B_Y}\left(\frac{1}{3}\right) = \frac{2}{3}; \text{ and} \tag{43}$$

$$\mu_{C_Y}(rep(\tilde{B}_Y^-)) = \mu_{C_Y}\left(\frac{1}{3}\right) = \frac{1}{3}. \tag{44}$$

III) *Based on reasoning rules, calculate the parameter α by following definition:*

$$\alpha = \sum_X \sum_Y \alpha_{X,Y} \mu_X(rep(\cdot))\mu_Y(rep(\cdot)), \tag{45}$$

where $\alpha_{X,Y}$ are parameters satisfying $\alpha_{X,Y}X + (1 - \alpha_{X,Y})Y = Z$.

Here, we get α_{A_X,B_Y} from (reasoning rule: $A_X, B_Y \Rightarrow B_Z$)

$$\alpha_{A_X,B_Y} A_X + (1 - \alpha_{A_X,B_Y})B_Y = B_Z. \tag{46}$$

Referring to Figures 7, 8 and 9, we rewrite Equation (46) into Equation (47).

$$\alpha_{A_X,B_Y} \times 1 + (1 - \alpha_{A_X,B_Y}) \times \frac{1}{2} = \frac{3}{4}. \tag{47}$$

Thus, $\alpha_{A_X,B_Y} = \frac{1}{2}$. Furthermore, we obtain $\alpha_{X,Y} = \frac{1}{2}$ for all $X \neq Y$ by the similar way, and define $\alpha_{X,Y}$ by the average of its neighbors while $X = Y$. That is,

$$\alpha_{A_X,A_Y} = \frac{\alpha_{A_X,B_Y} + \alpha_{B_X,A_Y}}{2} = \frac{1}{2}; \tag{48}$$

$$\alpha_{B_X,B_Y} = \frac{\alpha_{A_X,B_Y} + \alpha_{B_X,A_Y} + \alpha_{B_X,C_Y} + \alpha_{C_X,B_Y}}{4} = \frac{1}{2}; \text{ and} \tag{49}$$

$$\alpha_{A_X,A_Y} = \frac{\alpha_{B_X,C_Y} + \alpha_{C_X,B_Y}}{2} = \frac{1}{2}. \tag{50}$$

Therefore,

$$\alpha = \sum_X \sum_Y \alpha_{X,Y} \mu_X(rep(\tilde{A}_X^-))\mu_Y(rep(\tilde{B}_Y)) = \frac{1}{2}; \text{ and} \tag{51}$$

$$\alpha = \sum_X \sum_Y \alpha_{X,Y} \mu_X(rep(\tilde{B}_X))\mu_Y(rep(\tilde{B}_Y^-)) = \frac{1}{2}. \tag{52}$$

IV) *Calculate the inverse functions of the characteristic functions of the output fuzzy number:*

$$\chi^{-1}_{Z_{left}}(x) = \alpha\chi^{-1}_{X_{left}}(x) + (1 - \alpha)\chi^{-1}_{Y_{left}}(x); \text{ and} \tag{53}$$

$$\chi^{-1}_{Z_{right}}(x) = \alpha\chi^{-1}_{X_{right}}(x) + (1 - \alpha)\chi^{-1}_{Y_{right}}(x). \tag{54}$$

Hence, we have the inverse functions of the characteristic functions of $f(\tilde{A}_X^-, \tilde{B}_Y)$ and $f(\tilde{B}_X, \tilde{B}_Y^-)$ by

$$\chi^{-1}_{f(\tilde{A}_X^-,\tilde{B}_Y)_{left}}(x) = \alpha\chi^{-1}_{\tilde{A}^-_{X_{left}}}(x) + (1 - \alpha)\chi_{\tilde{B}_{Y_{left}}}(x); \tag{55}$$

$$\chi^{-1}_{f(\tilde{A}_X^-,\tilde{B}_Y)_{right}}(x) = \alpha\chi^{-1}_{\tilde{A}^-_{X_{right}}}(x) + (1 - \alpha)\chi_{\tilde{B}_{Y_{right}}}(x); \tag{56}$$

$$\chi^{-1}_{f(\tilde{B}_X,\tilde{B}_Y^-)_{left}}(x) = \alpha\chi^{-1}_{\tilde{B}_{X_{left}}}(x) + (1 - \alpha)\chi_{\tilde{B}^-_{Y_{left}}}(x); \text{ and} \tag{57}$$

$$\chi^{-1}_{f(\tilde{B}_X,\tilde{B}_Y^-)_{right}}(x) = \alpha\chi^{-1}_{\tilde{B}_{X_{right}}}(x) + (1 - \alpha)\chi_{\tilde{B}^-_{Y_{right}}}(x). \tag{58}$$

V) *Pull-back the inverse functions obtained above and combine them:*

The reasoning results of $f(\tilde{A}_X^-, \tilde{B}_Y)$ and $f(\tilde{B}_X, \tilde{B}_Y^-)$ are illustrated as following Figures 18 and 19.

Here, the inverse functions of $f(\tilde{A}_X^-, \tilde{B}_Y)$ and $f(\tilde{B}_X, \tilde{B}_Y^-)$ are symmetrical with respect to $y = \frac{2}{3}$ and $y = \frac{5}{12}$ respectively; hence, the membership functions of $f(\tilde{A}_X^-, \tilde{B}_Y)$ and $f(\tilde{B}_X, \tilde{B}_Y^-)$ are symmetric with respect to $x = \frac{2}{3}$ and $x = \frac{5}{12}$ respectively as well.

Step 5: Investigate the reasoning results obtained above as follows:

FIGURE 18. Reasoning result $f(\tilde{A}_X^-, \tilde{B}_Y)$ obtained by improved extension principle

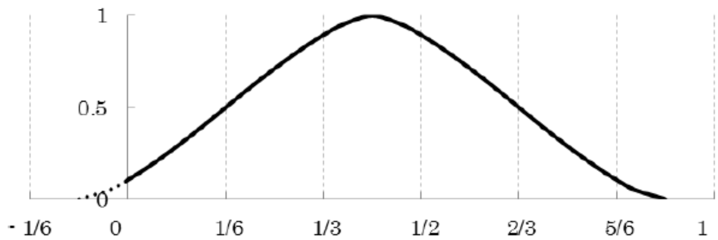

FIGURE 19. Reasoning result $f(\tilde{B}_X, \tilde{B}_Y^-)$ obtained by improved extension principle

The characteristic functions of $f(\tilde{A}_X^-, \tilde{B}_Y)$ and $f(\tilde{B}_X, \tilde{B}_Y^-)$ have their peaks at $\frac{2}{3}$ and $\frac{5}{12}$ respectively (see Figure 18 and Figure 19). That is, we can say that the reasoning results are B_Z^- and C_Z^- approximately in accordance to Figure 7.

4. **Conclusions.** In this paper, the author proposed another method for the operation of fuzzy numbers in two shapes. The new method for the operation of fuzzy numbers, improved extension principle, was compared with the conventional extension principle while considering how to implement approximate reasoning with fuzzy numbers. For the educational evaluation cases considered in this study, the author showed that the improved extension principle can help obtain more reasonable reasoning output than before (Figure 20), because the improved one always ensures the symmetry of the output (see Figure 21) while conducting reasoning with symmetric input.

Moreover, it also especially simplified the computation of the execution of the reasoning on quadratic-curved fuzzy numbers and blended the shapes of the input more naturally.

In this study, the author's discussion about the extension principle was confined to a binary operation $f(X, Y)$, so we want to take the study further to deal with more functions like a function involving three or more variables in the future.

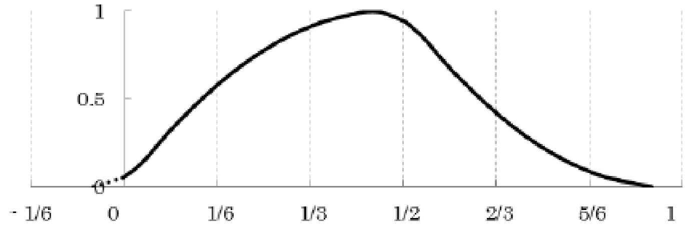

FIGURE 20. Reasoning result obtained by conventional extension principle

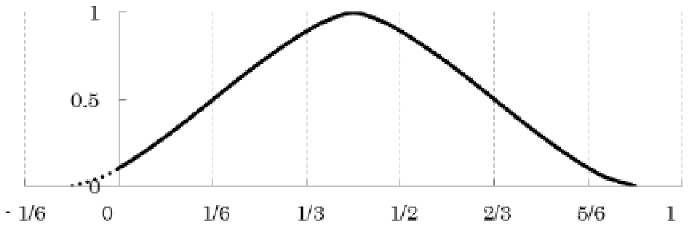

Figure 21. Reasoning result obtained by improved extension principle

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