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NEW PEAK-TO-PEAK STATE-SPACE REALIZATION OF DIRECT FORM INTERFERED DIGITAL FILTERS FREE OF OVERFLOW LIMIT CYCLES

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Abstract. *This paper introduces an induced l_∞ approach (also known as a peak-to-peak approach) to derive a new criterion for the realization of state-space direct form digital filters free of overflow limit cycles. The criterion guarantees asymptotic stability and reduces the effect of external interference to an induced l_∞ norm constraint. The criterion can overcome some disadvantages of existing criteria. Finally, a numerical example is presented to demonstrate the efficacy of the proposed stability criterion.*
Keywords: Induced l_∞ approach (or peak-to-peak approach), Asymptotic stability, Direct form digital filters, Finite wordlength effects, Linear matrix inequality (LMI)

1. **Introduction.** When concerned with the realization of a digital filter using saturation arithmetic, it is necessary to have a criterion to choose the values of the filter coefficients so that the nonexistence of zero-input limit cycles is ensured. The criteria for the elimination of zero-input limit cycles in digital filters employing saturation overflow arithmetic have attracted much attention [1, 2, 3, 4, 5, 6, 7, 8]. In this paper, direct form digital filters involving single overflow nonlinearity [1, 2, 6, 7, 8] are considered. The quantization and overflow nonlinearities may interact with each other. However, if the total number of quantization steps is large, then the effects of these nonlinearities can be decoupled and investigated separately. Under this decoupling approximation, the quantization effects in digital filters may be neglected when the effects of overflow are examined [9].

When a high-order digital filter is implemented in hardware, it is usually split into several biquad filters before implementation. Interferences may then exist between some biquad filters. These interferences cause malfunction as well as destruction in the last [10, 11]. However, most existing criteria for the stability analysis of digital filters are only available under specific conditions. In unfavorable environments with modeling uncertainties or external interferences, these existing stability criteria for digital filters are useless. Therefore, it is important to study an alternative stability criterion that can overcome the weaknesses of existing stability criteria for digital filters. Recently, Ahn proposed some robust stability criteria for digital filters with external interferences [12, 13, 14, 15, 16].

Some disturbances or noises, which cause instability or poor performance, always exist in real physical systems. Thus, the effect of disturbances or noises should also be reduced in a stability analysis of physical systems. The induced l_∞ approach (also known as the peak-to-peak approach), which was introduced in [17, 18], is an attractive method for dealing with several dynamic systems with disturbances or noises, because some general stability conclusions can be obtained using only inputs and outputs measurements. Recently, an induced l_∞ stability criterion for fixed-point digital filters was proposed in [19]. A natural question arises: can an induced l_∞ stability criterion for direct form digital filters with saturation arithmetic and external interference be obtained? To the best of the authors' knowledge, no solution has been presented in the literature thus far for the induced l_∞ stability of direct form digital filters with saturation arithmetic and external interference, which means the issue remains unresolved and challenging.

In this paper, we use the induced l_∞ approach to obtain a new criterion for the realization of state-space direct form digital filters free of overflow limit cycles with saturation arithmetic and external interference. The criterion is a new contribution to the topic of stability analysis for digital filters. The proposed stability criterion ensures that the direct form digital filter is asymptotically stable, and the induced l_∞ norm from the external interference to the state vector is reduced within an interference attenuation level. For a fixed positive scalar variable, the criterion can be easily checked using existing convex optimization algorithms [20, 21].

This paper is organized as follows. In Section 2, a new criterion for the induced l_∞ stability of direct form digital filters with saturation arithmetic and external interference is proposed. In Section 3, a numerical example is given, and finally, conclusions are presented in Section 4.

2. **New Peak-to-Peak Realization Criterion.** The direct form digital filter under consideration has a linear part described by the transfer function $G(z)$:

$$G(z) = h_1 z^{-n} + h_2 z^{-(n-1)} + h_3 z^{-(n-2)} + \cdots + h_n z^{-1}. \tag{1}$$

The output of $G(z)$ is $y(r)$ and the input to $G(z)$ is $f(y(r))$. The following condition is imposed:

$$z^n - h_n z^{n-1} - h_{n-1} z^{n-2} \cdots - h_2 z - h_1 \neq 0 \tag{2}$$

for all $|z| \geq 1$, which implies that the infinite-precision counterpart of the filter is stable. It is assumed that the saturation overflow arithmetic is given by

$$f(y(r)) = \begin{cases} 1, & \text{if } y(r) > 1, \\ y(r), & \text{if } -1 \leq y(r) \leq 1, \\ -1, & \text{if } y(r) < -1. \end{cases} \tag{3}$$

Note that the saturation overflow arithmetic is confined to the sector $[0, 1]$, i.e.,

$$f(0) = 0, \quad 0 \leq \frac{f(y(r))}{y(r)} \leq 1. \tag{4}$$

The direct form digital filter (1) can be represented by the following equation:

$$\begin{aligned} x_1(r+1) &= x_2(r), \\ x_2(r+1) &= x_3(r), \\ &\vdots \\ x_{n-1}(r+1) &= x_n(r), \\ x_n(r+1) &= f\left(h_1 x_1(r) + h_2 x_2(r) + \cdots + h_n x_n(r)\right), \end{aligned} \tag{5}$$

This can also be represented by the following matrix form:

$$x(r+1) = Ax(r) + Bf(H^T x(r)), \tag{6}$$

where

$$A = \begin{bmatrix} 0 & & & \\ 0 & & I_{n-1} & \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \ H = \begin{bmatrix} h_1 \\ \vdots \\ h_{n-1} \\ h_n \end{bmatrix}, \ x(r) = \begin{bmatrix} x_1(r) \\ \vdots \\ x_{n-1}(r) \\ x_n(r) \end{bmatrix}, \tag{7}$$

and I_{n-1} denotes the $(n-1) \times (n-1)$ identity matrix. This paper considers the following direct form digital filter with external interference:

$$x(r+1) = Ax(r) + Bf(H^T x(r)) + w(r), \tag{8}$$

where $w(r) \in R^n$ is the external interference. Let $z(r) \in R^p$ be a linear combination of the states, which is given by

$$\begin{aligned} z(r) &= [z_1(r) \ z_2(r) \ \cdots \ z_p(r)]^T \\ &= Kx(r), \end{aligned} \tag{9}$$

where $K \in R^{p \times n}$ is a known constant matrix.

Given a level $\gamma > 0$, in this paper, a new stability criterion is found such that the direct form digital filter (8)-(9) with $w(r) = 0$ is asymptotically stable and

$$\sup_{r \geq 0} \{z^T(r) z(r)\} < \gamma^2 \sup_{r \geq 0} \{w^T(r) w(r)\} \tag{10}$$

under zero-initial conditions for all non-zero $w(r)$. In (10), the parameter γ is called the interference attenuation level. In this case, the direct form digital filter (8)-(9) is said to be asymptotically stable with induced l_∞ performance γ.

A new induced l_∞ stability criterion for direct form digital filters with saturation arithmetic and external interference is proposed in the following theorem:

Theorem 2.1. *For a given level $\gamma > 0$, if there exist a symmetric positive definite matrix P and positive scalars δ, m, λ, μ such that*

$$\begin{bmatrix} \delta HH^T + A^T PA + (\lambda - 1)P & A^T PB + mH & A^T P \\ B^T PA + mH^T & B^T PB - 2m - \delta & B^T P \\ PA & PB & P - \mu I \end{bmatrix} < 0, \tag{11}$$

$$\begin{bmatrix} \lambda P & 0 & K^T \\ 0 & (\gamma - \mu) I & 0 \\ K & 0 & \gamma I \end{bmatrix} > 0, \tag{12}$$

then the direct form digital filter (8)-(9) is asymptotically stable with induced l_∞ performance γ.

Proof: A Lyapunov function $V(x(r)) = x^T(r) P x(r)$ is considered. Along the trajectory of the direct form digital filter (8), we have

$$\begin{aligned} &\Delta V(x(r)) \\ =& V(x(r+1)) - V(x(r)) \\ =& [Ax(r) + Bf(H^T x(r)) + w(r)]^T P [Ax(r) + Bf(H^T x(r)) + w(r)] - x^T(r) P x(r) \\ =& x^T(r)[A^T PA - P]x(r) + x^T(r) A^T PBf(H^T x(r)) + x^T(r) A^T Pw(r) \\ &+ f(H^T x(r)) B^T PAx(r) + \{f(H^T x(r))\}^2 B^T PB + f(H^T x(r)) B^T Pw(r) \\ &+ w^T(r) PAx(r) + w^T(r) PBf(H^T x(r)) + w^T(r) Pw(r) + f(H^T x(r))[2mH^T x(r) \\ &- 2mf(H^T x(r))] - 2mf(y(r))[y(r) - f(y(r))]. \end{aligned}$$

From (4), we have $\{f(H^T x(r))\}^2 \le \{H^T x(r))\}^2 = x^T(r)HH^T x(r)$. Then, for a positive scalar δ, we obtain

$$\delta[x^T(r)HH^T x(r) - \{f(H^T x(r))\}^2] \ge 0. \tag{13}$$

Using (13), an upper bound for $\Delta V(x(r))$ is obtained, which is

$$\begin{aligned}
&\Delta V(x(r)) \\
&\le x^T(r)[A^T PA - P]x(r) + x^T(r)[A^T PB + mH]f(H^T x(r)) + x^T(r)A^T Pw(r) \\
&\quad + f(H^T x(r))[B^T PA + mH^T]x(r) + \{f(H^T x(r))\}^2[B^T PB - 2m] \\
&\quad + f(H^T x(r))B^T Pw(r) + w^T(r)PAx(r) + w^T(r)PBf(H^T x(r)) + w^T(r)Pw(r) \\
&\quad - 2mf(y(r))[y(r) - f(y(r))] + \delta[x^T(r)HH^T x(r) - \{f(H^T x(r))\}^2] \\
&= \begin{bmatrix} x(r) \\ f(H^T x(r)) \\ w(r) \end{bmatrix}^T \begin{bmatrix} \delta HH^T + A^T PA + (\lambda - 1)P & A^T PB + mH & A^T P \\ B^T PA + mH^T & B^T PB - 2m - \delta & B^T P \\ PA & PB & P - \mu I \end{bmatrix} \\
&\quad \times \begin{bmatrix} x(r) \\ f(H^T x(r)) \\ w(r) \end{bmatrix} - \lambda x^T(r)Px(r) + \mu w^T(r)w(r) + \Phi(r),
\end{aligned} \tag{14}$$

where $\Phi(r) = -2mf(y(r))[y(r) - f(y(r))] \le 0$ in view of (3). If (11) is satisfied, then

$$\begin{aligned}
\Delta V(x(r)) &< -\lambda x^T(r)Px(r) + \mu w^T(r)w(r) \\
&= -\lambda V(x(r)) + \mu w^T(r)w(r).
\end{aligned} \tag{15}$$

Hence, $\Delta V(x(r)) < 0$ holds whenever $V(x(r)) \ge \frac{\mu}{\lambda} w^T(r)w(r)$ [19]. Since $V(x(0)) = 0$ under the zero-initial condition, this shows that $V(x(r))$ cannot exceed the value $\frac{\mu}{\lambda} w^T(r)w(r)$

$$x^T(r)Px(r) = V(x(r)) < \frac{\mu}{\lambda} w^T(r)w(r) \tag{16}$$

for $r \ge 0$. From (16), the following is obtained [19]:

$$\begin{aligned}
&\frac{1}{\gamma} x^T(r)K^T Kx(r) - \gamma w^T(r)w(r) \\
&= \frac{1}{\gamma} x^T(r)K^T Kx(r) - (\gamma - \mu)w^T(r)w(r) - \mu w^T(r)w(r) \\
&< \frac{1}{\gamma} x^T(r)K^T Kx(r) - (\gamma - \mu)w^T(r)w(r) - \lambda x^T(r)Px(r).
\end{aligned} \tag{17}$$

The matrix inequality (12) gives

$$\frac{1}{\gamma} \begin{bmatrix} K^T \\ 0 \end{bmatrix} \begin{bmatrix} K & 0 \end{bmatrix} < \begin{bmatrix} \lambda P & 0 \\ 0 & (\gamma - \mu)I \end{bmatrix}. \tag{18}$$

If (18) is pre- and post-multiplied by $[x^T(r)\ w^T(r)]$ and $[x^T(r)\ w^T(r)]^T$, respectively, then

$$\frac{1}{\gamma} x^T(r)K^T Kx(r) - (\gamma - \mu)w^T(r)w(r) - \lambda x^T(r)Px(r) < 0, \tag{19}$$

which ensures

$$\frac{1}{\gamma} x^T(r)K^T Kx(r) - \gamma w^T(r)w(r) < 0 \tag{20}$$

from (17) [19]. Thus, we have

$$\begin{aligned} z^T(r)z(r) &= x^T(r)K^TKx(r) \\ &< \gamma^2 w^T(r)w(r). \end{aligned} \tag{21}$$

Taking the supremum over $r \geq 0$ leads to (10). When $w(r) = 0$, then

$$\Delta V(x(r)) < -\lambda V(x(r)) < 0 \tag{22}$$

from (15). This guarantees

$$\lim_{r\to\infty} x(r) = 0 \tag{23}$$

from the Lyapunov stability theory, and thereby completes the proof.

Remark 2.1. *For a fixed positive scalar λ, (11) and (12) are linear matrix inequalities (LMIs)* [20]*. Various efficient convex optimization algorithms can be used to check whether these LMIs are feasible. In this paper, in order to solve these LMIs, the MATLAB LMI Control Toolbox* [21] *is utilized, which implements state-of-the-art interior-point algorithms.*

Remark 2.2. *Most existing works on stability for direct form digital filters in the literature have never considered the worst-case peak value of the state vector for all bounded peak values of the interference signals. Unfortunately, with the existing works, it is impossible to check the induced l_∞ stability for direct form digital filters. For the first time, this paper proposes an induced l_∞ stability criterion for direct form digital filters. The result proposed in this paper opens a new path for application of the induced l_∞ approach to robust stability analysis for direct form digital filters. In contrast to the existing works on direct form digital filters, the advantage of our criterion is that we can consider the worst-case peak value of the state vector for all bounded peak values of the interference signals.*

Remark 2.3. *The criterion proposed in this paper can be used in several signal processing applications. For example, we design a state-space direct form digital filter and its robust stability can be checked to achieve the induced l_∞ performance by the proposed criterion. Then, we can implement this robustly stable digital filter using hardware. Therefore, from the point of view of signal processing and circuit implementation, the proposed criterion for direct form digital filters is of significance for many applications.*

Remark 2.4. *The l_∞ induced norm* [17, 18, 19] *is defined as*

$$\|T_{zw}\|_{l_\infty} = \frac{\sqrt{\sup_{r\geq 0}\{z^T(r)z(r)\}}}{\sqrt{\sup_{r\geq 0}\{w^T(r)w(r)\}}}$$

where T_{zw} is a transfer function matrix from $w(r)$ to $z(r)$. For a given level $\gamma > 0$, $\|T_{zw}\|_{l_\infty} < \gamma$ can be restated in the equivalent form (10). If $L(r)$ is defined

$$L(r) = \frac{\sup_{0\leq k\leq r}\{z^T(k)z(k)\}}{\sup_{0\leq k\leq r}\{w^T(k)w(k)\}}, \tag{24}$$

the performance (10) can be represented by $L(\infty) < \gamma^2$, which is verified in Section 3.

3. **Numerical Example.** Consider a second-order direct form digital filter (8) with

$$h_1 = -0.5,\ h_2 = 1,\ w(r) = \begin{bmatrix} \cos(5r) \\ 3\sin(10r) \end{bmatrix},\ K = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}. \quad (25)$$

Let the induced l_∞ performance be specified by $\gamma = 0.35$. In addition, $\lambda = 0.01$ is fixed. Solving (11)-(12) by the convex optimization technique of MATLAB software gives

$$P = \begin{bmatrix} 0.0502 & 0.0359 \\ 0.0359 & 0.0971 \end{bmatrix},\ m = 0.0930,\ \delta = 0.0052,\ \mu = 0.3380.$$

Figure 1 shows the plot of $L(r)$, which is defined in (24). This figure verifies $L(\infty) < \gamma^2 = 0.1225$, which means the induced l_∞ norm from $w(r)$ to $z(r)$ is reduced within the interference attenuation level γ. It can be seen that each of the criteria in previous works [1, 2, 6, 7, 8, 23] fails in the direct form digital filter given by (8)-(9) with the parameters (25). However, the proposed criterion (11)-(12) ensures asymptotic stability with the induced l_∞ performance.

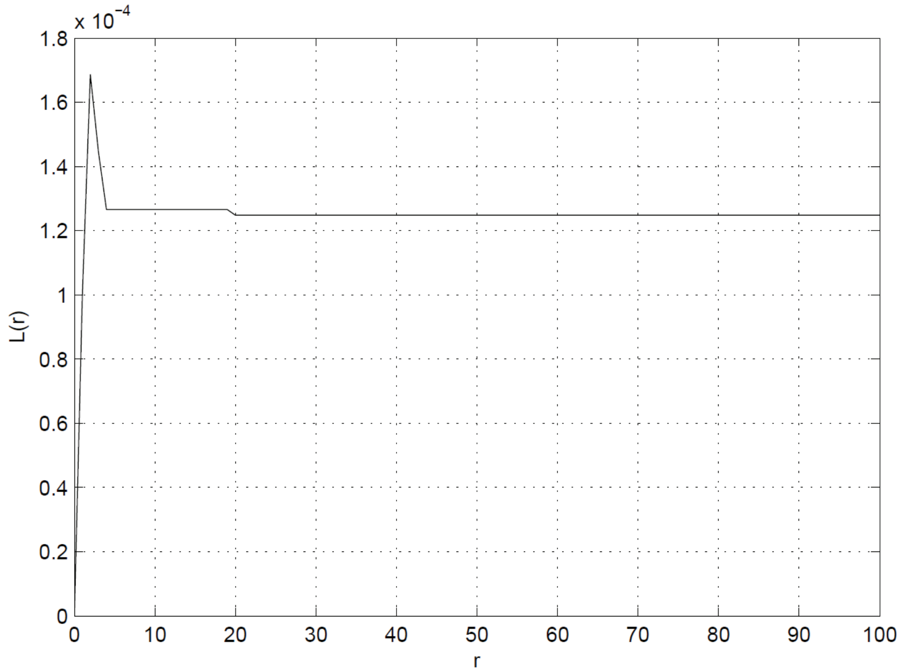

FIGURE 1. The plot of $L(r)$

4. **Conclusions.** This paper proposes a new criterion for the induced l_∞ stability of direct form digital filters free of overflow limit cycles with saturation arithmetic and external interference. The proposed stability criterion for direct form digital filters can ensure a reduction in the effect of external interference within an interference attenuation level. Thus, it can overcome the weaknesses of the existing stability criteria for digital filters. Finally, a numerical example is presented to demonstrate the validity of the proposed criterion.

REFERENCES

[1] P. M. Ebert, J. E. Mazo and M. G. Taylor, Overflow oscillations in recursive digital filters, *Bell System Tech. J.*, vol.48, pp.2999-3020, 1969.
[2] I. W. Sandberg, The zero-input response of digital filters using saturation arithmetic, *IEEE Trans. Circuits Syst.*, vol.26, pp.911-915, 1979.

[3] V. Singh, Elimination of overflow oscillations in fixed-point state-space digital filters using saturation arithmetic, *IEEE Trans. Circuits Syst.*, vol.37, pp.814-818, 1990.
[4] H. Kar and V. Singh, A new criterion for the overflow stability of second-order state-space digital filters using saturation arithmetic, *IEEE Trans. Circuits Syst. I*, vol.45, pp.311-313, 1998.
[5] H. Kar and V. Singh, Stability analysis of discrete-time systems in a state-space realisation with partial state saturation nonlinearities, *IEE Proc. of Control Theory Appl.*, vol.150, pp.205-208, 2003.
[6] H. Kar and V. Singh, Elimination of overflow oscillations in digital filters employing saturation arithmetic, *Digital Signal Processing*, vol.15, pp.536-544, 2005.
[7] V. Singh, LMI approach to stability of direct form digital filters utilizing single saturation overflow nonlinearity, *IEEE Trans. Circuits Syst. II*, vol.54, pp.878-881, 2007.
[8] V. Singh, Stability analysis of a class of digital filters utilizing single saturation nonlinearity, *Automatica*, vol.44, pp.282-285, 2008.
[9] T. A. C. M. Classen, W. F. G. Mecklenbrauker and J. B. H. Peek, Effects of quantization and overflow in recursive digital filters, *IEEE Trans. Acoust. Speech Signal Process.*, vol.24, pp.517-529, 1976.
[10] Y. Tsividis, *Mixed Analog-Digital VLSI Devices and Technology*, World Scientific Publishing, 2002.
[11] J. Monteiro and R. V. Leuken, *Integrated Circuit and System Design: Power and Timing Modeling, Optimization and Simulation*, Springer, 2010.
[12] C. K. Ahn, Criterion for the elimination of overflow oscillations in fixed-point digital filters with saturation arithmetic and external disturbance, *International Journal of Electronics and Communications*, vol.65, pp.750-752, 2011.
[13] C. K. Ahn and S. T. Jung, A passivity criterion for fixed-point state-space digital filters, *International Journal of the Physical Sciences*, vol.6, no.5, pp.982-986, 2011.
[14] C. K. Ahn, A new LMI criterion for the realization of limit cycle-free direct form digital filters with saturation arithmetic, *International Journal of Innovative Computing, Information and Control*, vol.8, no.2, pp.1453-1461, 2012.
[15] C. K. Ahn, A new condition for the elimination of overflow oscillations in direct form digital filters, *International Journal of Electronics*, vol.99, no.11, pp.1581-1588, 2012.
[16] C. K. Ahn, l_2-l_∞ stability criterion for fixed-point state-space digital filters with saturation arithmetic, *International Journal of Electronics*, DOI: 10.1080/00207217.2012.743083, 2013 (in press).
[17] J. Abedor, K. Nagpal and K. Poolla, A linear matrix inequality approach to peak-to-peak gain minimization, *Int. J. Robust Nonlinear Contr.*, vol.6, pp.899-927, 1996.
[18] C. Scherer, P. Gahinet and M. Chilali, Multiobjective output-feedback control via LMI optimization, *IEEE Trans. Automat. Contr.*, vol.42, pp.896-911, 1997.
[19] C. K. Ahn and Y. S. Lee, Induced l_∞ stability of fixed-point digital filters without overflow oscillations and instability due to finite word length effects, *Advances in Difference Equations*, vol.2012, no.51, 2012.
[20] S. Boyd, L. E. Ghaoui, E. Feron and V. Balakrishinan, *Linear Matrix Inequalities in Systems and Control Theory*, SIAM, Philadelphia, PA, 1994.
[21] P. Gahinet, A. Nemirovski, A. J. Laub and M. Chilali, *LMI Control Toolbox*, The Mathworks Inc., 1995.
[22] C. K. Ahn, IOSS criterion for the absence of limit cycles in digital filters employing saturation overflow arithmetic, *Circuits, Systems, and Signal Processing*, DOI: 10.1007/s00034-012-9520-0, 2013 (in press).
[23] V. Singh, Elimination of overflow oscillations in direct form digital filters using saturation arithmetic, *Automatica*, pp.2989-2991, 2008.