NON-MONOTONIC LYAPUNOV FUNCTIONS FOR STABILITY ANALYSIS AND STABILIZATION OF DISCRETE TIME TAKAGI-SUGENO FUZZY SYSTEMS

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ABSTRACT. *This paper presents a new approach for the stability analysis and controller synthesis of discrete-time Takagi-Sugeno fuzzy dynamic systems. In this paper, non-monotonic Lyapunov function is utilized to relax the monotonic requirement of Lyapunov theorem which renders larger class of functions to provide stability. To this end, three new sufficient conditions are proposed to establish global asymptotic stability. In this regard, the Lyapunov function decreases every few steps; however, it can be increased locally. Moreover, a new method is proposed to design the state feedback controller. It is shown that the Lyapunov function and the state feedback control law can be obtained by solving a set of Linear Matrix Inequalities (LMI) or Iterative Linear Matrix Inequalities (ILMI) which are numerically feasible with commercially available softwares. Finally, the exhausted numerical examples manifest the effectiveness of our proposed approach and that it is less conservative compared with the available schemes.*
Keywords: Discrete-time Takagi-Sugeno fuzzy dynamic systems, Non-monotonic Lyapunov function, Stability analysis, Stabilization, Linear matrix inequalities (LMI), Iterative linear matrix inequalities (ILMI)

1. **Introduction.** Introduction of fuzzy logic by Zadeh opens a new era on the design of controllers for industrial plants. However, analysis of fuzzy systems using the conventional analysis methods is difficult which is essentially because of the structure of fuzzy systems and definitions of membership functions. The Takagi-Sugeno (T-S) fuzzy system is a breakthrough in the stability analysis and controller design of fuzzy control systems. The stability analysis [1,2], robust stability [3,4], optimality [5], disturbance attenuation [6], and adaptive control [7] can be effectively addressed under the framework of this T-S fuzzy system.

In the past three decades, the analysis and synthesis of T-S fuzzy systems has drawn a great deal of attention [8-18]. There exist two major reasons for this attention: the first one is the fact that T-S fuzzy systems can be used to approximate a large class of nonlinear systems and so they are called universal approximators [19]; the second main reason is that one can analyze and design nonlinear systems by taking full advantage of modern linear control theory.

There has been a vast amount of literature on the successful and effective use of T-S fuzzy systems in modeling, synthesis, and control of complex nonlinear systems which are ubiquitous in chemical processes [20-22], robotics systems [23-25], automotive systems

[26], and many manufacturing processes [27]. The aforementioned fact necessitates the use of T-S fuzzy systems for industrial purposes.

Since stability analysis is one of the principal mathematical tools in the controller design of fuzzy control systems, much attention is focused on obtaining less conservative conditions in order to guarantee their stability. It is clear that Lyapunov's direct method is the most well known stability analysis of T-S fuzzy systems. A single quadratic Lyapunov function was first employed to prove the stability of fuzzy systems [1,2]. To guarantee the global stability of the system, a common positive definite symmetric matrix $\boldsymbol{P}$ must be found which can satisfy Lyapunov inequalities for all local linear subsystems and can ensure the system global stability. It should be noted that because of the inherent conservatism in the current method, finding a common positive definite matrix $\boldsymbol{P}$ is problematic when the number of fuzzy rules of a T-S fuzzy system is growing larger.

Since common quadratic Lyapunov functions tend to be conservative, it is thus desirable to develop less conservative stability results for T-S fuzzy systems. In this regard piecewise quadratic Lyapunov functions [8-10] and fuzzy Lyapunov functions [12-16] are introduced to replace the single quadratic Lyapunov function. For the derived stability conditions from the fuzzy Lyapunov function, it is necessary to find r positive definite matrices $\boldsymbol{P_1}$, $\boldsymbol{P_2}$, ..., $\boldsymbol{P_r}$, which can satisfy r Lyapunov inequalities in each subsystem. This results in r^2 inequalities to be satisfied. The stability analysis and stabilization results were improved in the sense of less conservatism by representing the interactions among the fuzzy submodels in a single matrix [28,29], and also by introducing extra slack variables in LMIs as in [30]. However, conservatism still remains as an important and challenging problem. In addition, in a wide variety of practical applications such as translational oscillator with an eccentric rotational proof mass actuator (TORA) [31] and inverted pendulum on a cart [32], the less conservative methods show a great improvement related to stability and performance criteria in fuzzy control with previous techniques. Thus, to achieve less conservative stability conditions should be of both theoretical and practical importance. This fact motivates the present study.

It should be emphasized that the primary tool to establish the aforementioned stability of T-S fuzzy systems is the well-known Lyapunov's direct method first published in 1892, which is based on the existence of a scalar function of the states and can monotonically decrease along trajectories. Weakening the requirements of a suitable Lyapunov function is of great interest, since finding a suitable Lyapunov function is the main challenge in the stability analysis based on Lyapunov theory.

According to authors' knowledge, there are few works addressing this approach on stability analysis. In [33], Butz replaced $\dot{V} < 0$ with a condition on $\dot{V}$, $\ddot{V}$, and $\dddot{V}$ to establish global asymptotic stability; however, his condition could not be verified to be convex. Although Ahmadi and Parrilo in [34] replaced $V_{k+1} - V_k < 0$ with $\tau(V_{k+2} - V_k) + (V_{k+1} - V_k) < 0$ and established global asymptotic stability in discrete time, their results could not be either used to derive less conservative stability conditions or to design the state feedback controller for discrete time T-S fuzzy systems.

In this paper, we propose a new approach for stability analysis of discrete time T-S fuzzy systems based on non-monotonic Lyapunov criteria. Our goal is to further reduce the conservatism in the existing results by relaxing the monotonicity requirement of Lyapunov's theorem. As a result the class of functions which can be utilized to prove stability is enlarged. Then the analysis results are utilized for the synthesis purpose of T-S fuzzy controllers and can be expressed as a set of Linear Matrix Inequalities (LMI) or Iterative Linear Matrix Inequalities (ILMI), which are numerically feasible with commercially available software.

The rest of this paper contains five more sections. The preliminary definitions and existing stability conditions of discrete time T-S fuzzy system are reviewed in Section 2. The main theorems on stability based on non-monotonic Lyapunov function are presented in Section 3. A new method for designing the state feedback controller is proposed in Section 4. Numerical examples are given in Section 5 to present the superiority and the effectiveness of the proposed criteria. Finally, some conclusions are drawn in Section 6.

2. Problem Statement and Preliminaries.

2.1. Discrete time T-S fuzzy system.
A discrete time T-S fuzzy system is described by fuzzy *If-Then* rules to represent local linear input-output relations of a system. Then l-th rule of this fuzzy system is of the following form:

$$
\begin{aligned}
R^l : \ & \text{If } z_1 \text{ is } F_1^l, z_2 \text{ is } F_2^l, \ldots, z_v \text{ is } F_v^l \\
\text{Then} \quad & \boldsymbol{x}(k+1) = \boldsymbol{A}_l \boldsymbol{x}(k) + \boldsymbol{B}_l \boldsymbol{u}(k) \\
& \boldsymbol{y}(k) = \boldsymbol{C}_l \boldsymbol{x}(k) \\
& l \in L : \{1, 2, \ldots, r\}
\end{aligned}
\tag{1}
$$

where $\boldsymbol{z}(k) = [z_1(k), z_2(k), \ldots, z_v(k)]^T$ is the premise variable vector whose elements may be states or measurable external variables, $\boldsymbol{x}(k) \in \Re^n$ is the state vector, $\boldsymbol{u}(k) \in \Re^p$ is the input vector, $\boldsymbol{y}(k) \in \Re^q$ is the output vector, r is the number of inference rules, F_j^l are the fuzzy sets, and $(\boldsymbol{A}_l, \boldsymbol{B}_l, \boldsymbol{C}_l)$ are the matrices of the l-th local model.

By the singleton fuzzifier, the product inference engine and center average defuzzification, the final output of (1) is inferred as:

$$
\begin{aligned}
\boldsymbol{x}(k+1) &= \boldsymbol{A}(\mu)\boldsymbol{x}(k) + \boldsymbol{B}(\mu)\boldsymbol{u}(k) \\
\boldsymbol{y}(k) &= \boldsymbol{C}(\mu)\boldsymbol{x}(k)
\end{aligned}
\tag{2}
$$

where

$$
\boldsymbol{A}(\mu) = \sum_{l=1}^{r} \mu_l \boldsymbol{A}_l, \quad \boldsymbol{B}(\mu) = \sum_{l=1}^{r} \mu_l \boldsymbol{B}_l, \quad \boldsymbol{C}(\mu) = \sum_{l=1}^{r} \mu_l \boldsymbol{C}_l.
\tag{3}
$$

μ_l is the normalized membership function satisfying:

$$
\varepsilon_l = \prod_{i=1}^{v} F_i^l(z_i), \quad \mu_l(z) = \varepsilon_l \bigg/ \sum_{i=1}^{r} \varepsilon_i,
\tag{4}
$$

and $F_i^l(z_i)$ is the grade of membership of z_i in the fuzzy set F_i^l. It is simple to show that $\sum_{l=1}^{r} \mu_l = 1$.

2.2. Basic Lyapunov stability theories.
Consider the discrete time dynamic system:

$$
\boldsymbol{x}(k+1) = f(\boldsymbol{x}(k)),
\tag{5}
$$

where the function $f : \Re^n \to \Re^n$ can be in general nonlinear. Without loss of generality, stability of origin is studied. If the equilibrium point is at any other point, one can simply transfer the coordinates so that in the new coordinates the origin is the equilibrium point. The formal stability definitions are presented as follows:

Definition 2.1. *The origin is a globally asymptotically stable equilibrium of (5) if:*

$$
\begin{aligned}
& \forall \varepsilon > 0 \ \ \exists \delta > 0 \ \textit{such that} \ \|\boldsymbol{x}(0)\| < \delta \ \Rightarrow \ \|\boldsymbol{x}(k)\| < \varepsilon \ \ \forall k \quad (a) \\
& Lim_{k \to \infty} \boldsymbol{x}(k) = 0 \quad \forall \boldsymbol{x}(0) \in \Re^n \qquad\qquad\qquad\qquad\qquad (b)
\end{aligned}
\tag{6}
$$

Currently, the primary tool for establishing stability of discrete nonlinear systems is the well known Lyapunov's direct method. A sufficient condition to check the stability of a discrete control system based on Lyapunov's direct method can be expressed as:

Lemma 2.1. *Consider a discrete system described by (5), where $\boldsymbol{x}(k) \in \Re^n$, $f : \Re^n \to \Re^n$ and satisfies $f(0) = 0$. If there exists a continuous scalar function $V(\boldsymbol{x}(k))$ satisfying*
1) $V(\boldsymbol{x}(k))$ is a positive definite function (pdf),
2) $V(\boldsymbol{x}(k)) \to \infty$ as $\boldsymbol{x}(k) \to \infty$,
3) $V(\boldsymbol{x}(k+1)) - V(\boldsymbol{x}(k)) < 0$ for $\boldsymbol{x}(k) \neq 0$,
then the equilibrium state $\boldsymbol{x}(k) = 0$ is globally asymptotically stable, and $V(\boldsymbol{x}(k))$ is a Lyapunov function [35].

Stability analysis of T-S fuzzy systems has been pursued mainly based on Lyapunov stability theory but with different Lyapunov function. The following stability condition proposed by Sugeno and Tanaka is well known based on a common Lyapunov function $V(\boldsymbol{x}(k)) = \boldsymbol{x}(k)^T \boldsymbol{P} \boldsymbol{x}(k)$.

Lemma 2.2. *The origin of discrete time T-S fuzzy system described by (2) with $\boldsymbol{u}(k) = 0$ is a globally asymptotically stable if there exists a common positive matrix $\boldsymbol{P}$ satisfying*

$$\boldsymbol{A}_l^T \boldsymbol{P} \boldsymbol{A}_l - \boldsymbol{P} < 0 \quad l = 1, 2, \ldots, r. \tag{7}$$

Lemma 2.2 [1,2] shows that a common positive definite matrix $\boldsymbol{P}$ must satisfy r inequalities (7) in order to guarantee the stability of T-S fuzzy systems. If the number of rules r is large, it might be difficult to find the common matrix $\boldsymbol{P}$. The difficulty stems from the fact that the stability condition of T-S fuzzy systems is conservative. In order to relax this conservatism, stability condition based on a fuzzy Lyapunov function [12,13] described by $V(\boldsymbol{x}(k)) = \sum_{l=1}^{r} \mu_l \boldsymbol{x}(k)^T \boldsymbol{P}_l \boldsymbol{x}(k)$ is proposed to replace the common Lyapunov function as follows.

Lemma 2.3. *The origin of discrete time T-S fuzzy system described by (2) with $\boldsymbol{u}(k) = 0$ is a globally asymptotically stable if there exist r local positive definite matrices $\boldsymbol{P_1}$, $\boldsymbol{P_2}$, $\ldots$, $\boldsymbol{P_r}$ satisfying*

$$\boldsymbol{A}_l^T \boldsymbol{P}_i \boldsymbol{A}_l - \boldsymbol{P}_l < 0 \quad i, l \in \{1, 2, \ldots, r\}, \tag{8}$$

r^2 Lyapunov inequalities must be checked for Lemma 2.3. This result is also the same as the stability criterions of switched (piecewise) hybrid systems derived in [36] based on the switched Lyapunov function and in [37] based on the piecewise quadratic Lyapunov function.

3. **Main Results.**

3.1. **Non-monotonic Lyapunov stability criteria.** Since in case of $V \to 0$ as $k \to \infty$, it is not required that V decreases monotonically, the monotonicity requirement of Lyapunov's theorem is extended to enlarge the class of functions that can provide stability. In order to relax monotonicity requirement, it is necessary to answer the following question [34]:
1) Is it possible to replace inequality $V(\boldsymbol{x}(k+1)) < V(\boldsymbol{x}(k))$ by a condition that allows Lyapunov functions to increase locally but yet guarantee their convergence to zero in the limit?
2) Can the search for a Lyapunov function with the new condition be cast as a convex program, so that earlier techniques can be readily applied?

In this section, affirmative answers to these questions are presented. Before presenting the main theorem, the globally Lipschitz function needs to be defined.

Definition 3.1. *$f(x)$ is globally Lipschitz function if there exists $L > 0$ such that $\forall \boldsymbol{x}, \boldsymbol{y} \in \Re^n$, $|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq L |\boldsymbol{x} - \boldsymbol{y}|$. L is known as a Lipschitz constant.*

Lipschitz condition is used to replace the monotonicity requirement of Lyapunov's theorem with the more relaxed condition which is summarized in the following theorem.

Theorem 3.1. *Consider a discrete time system described by (5), where $\boldsymbol{x}(k) \in \Re^n$, $f : \Re^n \to \Re^n$ is globally Lipschitz function with Lipschitz constant L that satisfies $f(0) = 0$. If there exists a continuous scalar function $V(\boldsymbol{x}(k))$ satisfying*
1) $V(\boldsymbol{x}(k))$ as a positive definite function (pdf),
2) $V(\boldsymbol{x}(k)) \to \infty$ as $\boldsymbol{x}(k) \to \infty$,
3) $V(\boldsymbol{x}(k+2)) - V(\boldsymbol{x}(k)) < 0$ for $\boldsymbol{x}(k) \neq 0$,
then the origin is globally asymptotically stable.

Proof: Since V is a pdf, there exists a function α of class K such that $\alpha(\|\boldsymbol{x}\|) \leq V(\boldsymbol{x})$. To show that the origin is a globally asymptotically stable equilibrium, we must show that conditions (6-a) and (6-b) in Definition 2.1 are satisfied.

Let $\varepsilon > 0$. We can choose γ where $0 < \gamma < \varepsilon$. Since V is a continuous and positive definite function on the boundary of sphere $B_\gamma(0)$ where $\|\boldsymbol{x}\| = \gamma$, we have $\alpha(\gamma) \leq V(\boldsymbol{x})$. Let $0 < \beta < \alpha(\gamma)$, and define the set $K_\beta := \{\boldsymbol{x} \in B_\gamma(0) : V(\boldsymbol{x}) < \beta\}$. The continuity of V at the origin implies that there exists $\lambda > 0$ such $\lambda < \gamma$, and $\|\boldsymbol{x}\| < \lambda \Rightarrow V(\boldsymbol{x}) < \beta$, therefore we have $B_\lambda(0) \subset K_\beta \subseteq B_\gamma(0) \subset B_\varepsilon(0)$. We can define ρ as $\rho = \max(1, L)$ and pick δ such that $\delta = \lambda/\rho$.

To show that the above choice of δ satisfies (6-a), suppose $\boldsymbol{x}(0) \in B_\delta(0)$. Since f is a globally Lipschitz function with Lipschitz constant L, $\boldsymbol{x}(1) = f(\boldsymbol{x}(0)) \in B_{L\delta}(0)$. On the other hand, $\rho = \max(1, L)$ leads to $\lambda \geq \delta$. Consequently $\boldsymbol{x}(1) \in B_{L\delta}(0) \subseteq B_{\rho\delta}(0) = B_\lambda(0)$ and $B_\delta(0) \subseteq B_\lambda(0) \subset K_\beta \subseteq B_\gamma(0) \subset B_\varepsilon(0)$ are achieved. Therefore, $\boldsymbol{x}(0) \in B_\delta(0)$ results in $\boldsymbol{x}(1) \in B_\lambda(0)$, $V(\boldsymbol{x}(0)) < \beta$, and $V(\boldsymbol{x}(1)) < \beta$.

On the other hand, let $\boldsymbol{x}(k)$ be a solution with initial condition $\boldsymbol{x}(0) \in B_\delta(0)$. Consider the subsequences $\{V(\boldsymbol{x}(2k))\}$ and $\{V(\boldsymbol{x}(2k+1))\}$ of sequence $\{V(\boldsymbol{x}(k))\}$. Since $V(\boldsymbol{x}(k+2)) - V(\boldsymbol{x}(k)) < 0$ for $\boldsymbol{x}(k) \neq 0$, then subsequences $\{V(\boldsymbol{x}(2k))\}$ and $\{V(\boldsymbol{x}(2k+1))\}$ are monotonically decreasing, therefore for all k, we have $V(\boldsymbol{x}(2k)) < V(\boldsymbol{x}(0))$ and $V(\boldsymbol{x}(2k+1)) < V(\boldsymbol{x}(1))$. Hence, from $V(\boldsymbol{x}(0)) < \beta$ and $V(\boldsymbol{x}(1)) < \beta$ we can imply that $V(\boldsymbol{x}(k)) < \beta < \alpha(\gamma)$, and because $\alpha(.)$ is strictly increasing, we have $\|\boldsymbol{x}(k)\| < \gamma < \varepsilon$.

To show (6-b), we must show that given any $\eta > 0$ and $\boldsymbol{x}(0) \in \Re^n$, we can find a $\bar{k}$ such that $\|\boldsymbol{x}(k)\| < \eta$, $\forall k \geq \bar{k}$. Let $\eta > 0$, and let $\boldsymbol{x}(k)$ be a solution with initial condition $\boldsymbol{x}(0) \in \Re^n$. Since V is continuous and positive definite function on the boundary sphere where $\|x\| = \eta$, we have $\alpha(\eta) \leq V(\boldsymbol{x}(k))$ for some $k > 0$. Therefore, we must show that we can find a $\bar{k}$ such that $V(\boldsymbol{x}(k)) < \alpha(\eta)$, $\forall k \geq \bar{k}$.

Because $V(\boldsymbol{x}(k+2)) < V(\boldsymbol{x}(k))$ for $\boldsymbol{x}(k) \neq 0$, subsequences $\{V(\boldsymbol{x}(2k))\}$ and $\{V(\boldsymbol{x}(2k+1))\}$ are monotonically decreasing. Since the subsequences are lower bounded by zero, they must converge to some $c > 0$. It can be shown (e.g., by contradiction) that because of continuity of $V(\boldsymbol{x}(k))$, c must be zero. This part of the proof is similar to the proof of standard Lyapunov theory [35]. Now we can find k_1 and k_2 such that $V(\boldsymbol{x}(2k+1)) < \alpha(\eta)$, $\forall k \geq k_1$ and $V(\boldsymbol{x}(2k)) < \alpha(\eta)$, $\forall k \geq k_2$. We can pick $\bar{k}$ such that $\bar{k} = \max(2k_1 + 1, 2k_2)$.

To show that the above choice of $\bar{k}$ satisfies (6-b), we must show that $\|\boldsymbol{x}\| < \eta$, $\forall k \geq \bar{k}$. Since for $k \geq \bar{k}$, $V(\boldsymbol{x}(k)) < \alpha(\eta)$ then since $\alpha(.)$ is strictly increasing, then $\|\boldsymbol{x}\| < \eta$. Therefore, $\|\boldsymbol{x}\| \to 0$, which implies $x(k) \to 0$.

The proof is now completed.

Next, we generalize Theorem 3.1 to m-step differences.

Corollary 3.1. *Consider a discrete time system described by (5), where $\boldsymbol{x}(k) \in \Re^n$, $f : \Re^n \to \Re^n$ is globally Lipschitz function with Lipschitz constant L and satisfies $f(0) = 0$. If there exists a continuous scalar function $V(\boldsymbol{x}(k))$ satisfying*
1) $V(\boldsymbol{x}(k))$ as a positive definite function (pdf),
2) $V(\boldsymbol{x}(k)) \to \infty$ as $\boldsymbol{x}(k) \to \infty$,
3) $V(\boldsymbol{x}(k+m)) - V(\boldsymbol{x}(k)) < 0$ for $\boldsymbol{x}(k) \neq 0$,
then the origin is globally asymptotically stable.

3.2. **Non-monotonicity fuzzy T-S stability criteria.** The non-monotonicity stability conditions presented in the previous part can be applied to T-S fuzzy systems. However, Lemma 3.1 [9] needs to be stated before the main result can be evolved. In this section, the analysis of this problem is presented.

Lemma 3.1. *If $\boldsymbol{P_i}$ and $\boldsymbol{P_j}$ are positive definite matrices and matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are of appropriate dimensions such that $\boldsymbol{A^T P_i A} - \boldsymbol{P_j} < 0$ and $\boldsymbol{B^T P_i B} - \boldsymbol{P_j} < 0$, then*

$$\boldsymbol{A^T P_i B} + \boldsymbol{B^T P_i A} - 2\boldsymbol{P_j} < 0. \tag{9}$$

Hence, Theorem 3.1 and its corollary can be applied to present new sufficient conditions for the development of global asymptotic stability of discrete time T-S fuzzy systems.

Theorem 3.2. *The origin of discrete time T-S fuzzy system described by (2) with $\boldsymbol{u}(k) = 0$ is globally asymptotically stable if*
i. There exists a positive definite matrix $\boldsymbol{P}$ such that the following LMIs are satisfied:

$$\boldsymbol{A_j^T A_i^T P A_i A_j} - \boldsymbol{P} < 0 \quad i, j \in L : \{1, 2, \ldots, r\} \tag{10}$$

or, equivalently;
ii. There exists a positive definite matrix $\boldsymbol{X}$ such that the following LMIs are satisfied:

$$\begin{bmatrix} -\boldsymbol{X} & \boldsymbol{X A_j^T A_i^T} \\ \boldsymbol{A_i A_j X} & -\boldsymbol{X} \end{bmatrix} < 0 \quad i, j \in L \tag{11}$$

Proof: Consider the following Lyapunov candidate $V(\boldsymbol{x}(k))$:

$$V(\boldsymbol{x}(k)) = \boldsymbol{x}(k)^T \boldsymbol{P} \boldsymbol{x}(k) \tag{12}$$

since $\boldsymbol{P}$ is a positive definite matrix, $V(\boldsymbol{x}(k))$ satisfies [35]:
1) $V(\boldsymbol{x}(k))$ is a positive definite function (pdf),
2) $V(\boldsymbol{x}(k)) \to \infty$ as $\boldsymbol{x}(k) \to \infty$.
Due to the fact that T-S fuzzy system is a combination of local linear models and each fuzzy membership is assumed to be bounded and satisfy the Lipschitz condition, it can be concluded that the function $f(\boldsymbol{x}(k)) = \boldsymbol{A}(\eta(k))\boldsymbol{x}(k)$ satisfies Lipschitz condition [38]. Based on Theorem 3.1, if the following inequality holds

$$V(\boldsymbol{x}(k+2)) - V(\boldsymbol{x}(k)) < 0, \tag{13}$$

then the origin of discrete time T-S fuzzy system described by (2) with $\boldsymbol{u}(k) = 0$ is globally asymptotically stable. From (2) and (3), we have

$$V(\boldsymbol{x}(k+2)) - V(\boldsymbol{x}(k)) = \boldsymbol{x}^T(k+2)\boldsymbol{P}\boldsymbol{x}(k+2) - \boldsymbol{x}^T(k)\boldsymbol{P}\boldsymbol{x}(k) \tag{14}$$

$$= [\boldsymbol{A}(\mu(k+1))\boldsymbol{x}(k+1)]^T \boldsymbol{P}\left[\boldsymbol{A}(\mu(k+1))\boldsymbol{x}(k+1)\right] - \boldsymbol{x}^T(k)\boldsymbol{P}\boldsymbol{x}(k) \tag{15}$$

$$= [\boldsymbol{A}(\mu(k+1))\boldsymbol{A}(\mu(k))\boldsymbol{x}(k)]^T \boldsymbol{P}\left[\boldsymbol{A}(\mu(k+1))\boldsymbol{A}(\mu(k))\boldsymbol{x}(k)\right] - \boldsymbol{x}^T(k)\boldsymbol{P}\boldsymbol{x}(k) \tag{16}$$

$$= \left[\sum_{i=1}^{r}\mu_i(k+1)\boldsymbol{A}_i\sum_{i=1}^{r}\mu_i(k)\boldsymbol{A}_i\boldsymbol{x}(k)\right]^T \boldsymbol{P}\left[\sum_{i=1}^{r}\mu_i(k+1)\boldsymbol{A}_i\sum_{i=1}^{r}\mu_i(k)\boldsymbol{A}_i\boldsymbol{x}(k)\right]$$
$$-\boldsymbol{x}^T(k)\boldsymbol{P}\boldsymbol{x}(k) \tag{17}$$

$$= \left[\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(k+1)\mu_j(k)\boldsymbol{A}_i\boldsymbol{A}_j\boldsymbol{x}(k)\right]^T \boldsymbol{P}\left[\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(k+1)\mu_j(k)\boldsymbol{A}_i\boldsymbol{A}_j\boldsymbol{x}(k)\right]$$
$$-\boldsymbol{x}^T(k)\boldsymbol{P}\boldsymbol{x}(k) \tag{18}$$

$$= \boldsymbol{x}(k)^T\left(\left[\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(k+1)\mu_j(k)\boldsymbol{A}_j^T\boldsymbol{A}_i^T\right]\boldsymbol{P}\left[\sum_{m=1}^{r}\sum_{l=1}^{r}\mu_m(k+1)\mu_l(k)\boldsymbol{A}_m\boldsymbol{A}_l\right]-\boldsymbol{P}\right)\boldsymbol{x}(k) \tag{19}$$

Because of $\sum\limits_{i=1}^{r}\sum\limits_{j=1}^{r}\sum\limits_{k=1}^{r}\sum\limits_{l=1}^{r}\mu_i(k+1)\mu_j(k)\mu_k(k+1)\mu_l(k) = 1$ and $\sum\limits_{i=1}^{r}\sum\limits_{j=1}^{r}\mu_i(k+1)\mu_j(k) = 1$, we have

$$V(\boldsymbol{x}(k+2)) - V(\boldsymbol{x}(k))$$
$$= \boldsymbol{x}(k)^T\left(\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{l=1}^{r}\mu_m(k+1)\mu_l(k)\mu_i(k+1)\mu_j(k)(\boldsymbol{A}_j^T\boldsymbol{A}_i^T\boldsymbol{P}\boldsymbol{A}_m\boldsymbol{A}_l - \boldsymbol{P})\right)\boldsymbol{x}(k)$$
$$= \boldsymbol{x}(k)^T(\boldsymbol{\Omega} + \boldsymbol{\Lambda})\boldsymbol{x}(k) \tag{20}$$

where

$$\boldsymbol{\Omega} = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i^{\;2}(k+1)\mu_j^{\;2}(k)(\boldsymbol{A}_j^T\boldsymbol{A}_i^T\boldsymbol{P}\boldsymbol{A}_i\boldsymbol{A}_j - \boldsymbol{P}) \tag{21}$$

$$\boldsymbol{\Lambda} = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m\geq i}^{r}\sum_{l\geq j}^{r}\mu_i(k+1)\mu_m(k+1)\mu_j(k)\mu_l(k)\boldsymbol{\Gamma}_{jiml} \tag{22}$$

and

$$\boldsymbol{\Gamma}_{jiml} = \boldsymbol{A}_j^T\boldsymbol{A}_i^T\boldsymbol{P}\boldsymbol{A}_m\boldsymbol{A}_l + \boldsymbol{A}_l^T\boldsymbol{A}_m^T\boldsymbol{P}\boldsymbol{A}_i\boldsymbol{A}_j - 2\boldsymbol{P} \tag{23}$$

Because of $\sum\limits_{i=1}^{r}\sum\limits_{j=1}^{r}\sum\limits_{k=1}^{r}\sum\limits_{l=1}^{r}\mu_i(k+1)\mu_j(k)\mu_k(k+1)\mu_l(k) = 1$, $\mu_i \geq 0$, $\mu_j \geq 0$, $\mu_l \geq 0$ and $\mu_m \geq 0$, if $\boldsymbol{\Gamma}_{jiml} < 0$ and $\boldsymbol{A}_j^T\boldsymbol{A}_i^T\boldsymbol{P}\boldsymbol{A}_i\boldsymbol{A}_j - \boldsymbol{P} < 0$ for every $i, j, m, l \in L$ then $V(\boldsymbol{x}(k+2)) - V(\boldsymbol{x}(k)) < 0$ for every $\boldsymbol{x}(k) \neq 0$.

From Lemma 3.1, if $\boldsymbol{A}_j^T\boldsymbol{A}_i^T\boldsymbol{P}\boldsymbol{A}_i\boldsymbol{A}_j - \boldsymbol{P} < 0$ and $\boldsymbol{A}_l^T\boldsymbol{A}_m^T\boldsymbol{P}\boldsymbol{A}_m\boldsymbol{A}_l - \boldsymbol{P} < 0$ for every $i, j, m, l \in L$, then $\boldsymbol{\Gamma}_{jiml} < 0$. Therefore, if $\boldsymbol{A}_j^T\boldsymbol{A}_i^T\boldsymbol{P}\boldsymbol{A}_i\boldsymbol{A}_j - \boldsymbol{P} < 0$ for every $i, j \in L$, then $V(\boldsymbol{x}(k+2)) - V(\boldsymbol{x}(k)) < 0$ for $\boldsymbol{x}(k) \neq 0$ and the origin is globally asymptotically stable.

The proof is now completed.

The equivalence of (10) and (11) can be easily established by using Schur complement together with $\boldsymbol{x} = \boldsymbol{P}^{-1}$ [39].

Next, we generalize Theorem 3.2 to m-step differences.

Corollary 3.2. *The origin of discrete time T-S fuzzy system described by (2) with $\boldsymbol{u}(k) = 0$ is globally asymptotically stable if*
i. There exists a positive definite matrix $\boldsymbol{P}$ such that the following LMIs are satisfied:

$$\underbrace{\boldsymbol{A}_o^T \dots \boldsymbol{A}_j^T \boldsymbol{A}_i^T}_{m} \boldsymbol{P} \boldsymbol{A}_i \boldsymbol{A}_j \dots \boldsymbol{A}_o - \boldsymbol{P} < 0 \quad i,j,\dots,o \in L \tag{24}$$

or, equivalently;
ii. There exists a positive definite matrix X such that the following LMIs are satisfied:

$$\begin{bmatrix} -\boldsymbol{X} & \boldsymbol{X}\boldsymbol{A}_o^T \dots \boldsymbol{A}_j^T \boldsymbol{A}_i^T \\ \boldsymbol{A}_i \boldsymbol{A}_j \dots \boldsymbol{A}_o \boldsymbol{X} & -\boldsymbol{X} \end{bmatrix} < 0 \quad i,j,\dots,o \in L \tag{25}$$

Proof: Similar to the proof of Theorem 3.2, it is straightforward to show that:

$$V(\boldsymbol{x}(k+m)) - V(\boldsymbol{x}(k)) < 0 \text{ for } \boldsymbol{x}(k) \neq 0. \tag{26}$$

Remark 3.1. *Based on stability condition derived from Corollary 3.2, it is necessary to find a positive definite matrix $\boldsymbol{P}$ which satisfies r^m inequalities.*

In order to prove the global asymptotic stability of T-S fuzzy systems, Theorem 3.2 and its corollary can be applied to obtain new sufficient conditions which are more suitable for controller synthesis.

Theorem 3.3. *The origin of discrete time T-S fuzzy system described by (2) with $\boldsymbol{u}(k) = 0$ is a globally asymptotically stable if*
i. There exists a set of positive definite matrices $\boldsymbol{P}$ and $\boldsymbol{P}_{ij}$ for every $i,j \in L$ such that the following LMIs are satisfied:

$$\boldsymbol{A}_i^T \boldsymbol{P} \boldsymbol{A}_i - \boldsymbol{P}_{ij} < 0 \tag{27}$$

$$\boldsymbol{A}_j^T \boldsymbol{P}_{ij} \boldsymbol{A}_j - \boldsymbol{P} < 0 \tag{28}$$

or, equivalently;
ii. There exists a set of positive definite matrices $\boldsymbol{P}$ and $\boldsymbol{P}_{ij}$ for every $i,j \in L$ such that the following inequalities are satisfied:

$$\begin{bmatrix} -\boldsymbol{P}_{ij} & \boldsymbol{A}_i^T \\ \boldsymbol{A}_i & -\boldsymbol{P}^{-1} \end{bmatrix} < 0 \tag{29}$$

$$\begin{bmatrix} -\boldsymbol{P} & \boldsymbol{A}_j^T \\ \boldsymbol{A}_j & -\boldsymbol{P}_{ij}^{-1} \end{bmatrix} < 0 \tag{30}$$

or, equivalently;
iii. There exists a set of positive definite matrices $\boldsymbol{X}$ and $\boldsymbol{X}_{ij}$ for every $i,j \in L$ such that the following LMIs are satisfied:

$$\begin{bmatrix} -\boldsymbol{X}_{ij} & \boldsymbol{X}_{ij}\boldsymbol{A}_i^T \\ \boldsymbol{A}_i \boldsymbol{X}_{ij} & -\boldsymbol{X} \end{bmatrix} < 0 \tag{31}$$

$$\begin{bmatrix} -\boldsymbol{X} & \boldsymbol{X}\boldsymbol{A}_j^T \\ \boldsymbol{A}_j \boldsymbol{X} & -\boldsymbol{X}_{ij} \end{bmatrix} < 0 \tag{32}$$

Proof: Based on Theorem 3.2, in order to show that the origin is a globally asymptotically stable equilibrium, it is necessary to show that the inequality (10) holds for every $i,j \in L$. In this regard we must show that for all $\boldsymbol{x} \in \Re^n$ and $\boldsymbol{x} \neq 0$, $\Phi < 0$, where Φ is defined as:

$$\Phi = \boldsymbol{x}^T \left[\boldsymbol{A}_j^T \boldsymbol{A}_i^T \boldsymbol{P} \boldsymbol{A}_i \boldsymbol{A}_j - \boldsymbol{P} \right] \boldsymbol{x} \tag{33}$$

(33) can be written as follows:

$$\Phi = \boldsymbol{x}^T \left[\boldsymbol{A}_j^T \boldsymbol{A}_i^T \boldsymbol{P} \boldsymbol{A}_i \boldsymbol{A}_j - \boldsymbol{A}_j^T \boldsymbol{P}_{ij} \boldsymbol{A}_j + \boldsymbol{A}_j^T \boldsymbol{P}_{ij} \boldsymbol{A}_j - \boldsymbol{P} \right] \boldsymbol{x} \tag{34}$$

$$= \boldsymbol{x}^T \left[\boldsymbol{A}_j^T (\boldsymbol{A}_i^T \boldsymbol{P} \boldsymbol{A}_i - \boldsymbol{P}_{ij}) \boldsymbol{A}_j + (\boldsymbol{A}_j^T \boldsymbol{P}_{ij} \boldsymbol{A}_j - \boldsymbol{P}) \right] \boldsymbol{x} \tag{35}$$

$$= \boldsymbol{x}^T \boldsymbol{A}_j^T \left[(\boldsymbol{A}_i^T \boldsymbol{P} \boldsymbol{A}_i - \boldsymbol{P}_{ij}) \right] \boldsymbol{A}_j \boldsymbol{x} + \boldsymbol{x}^T \left[\boldsymbol{A}_j^T \boldsymbol{P}_{ij} \boldsymbol{A}_j - \boldsymbol{P} \right] \boldsymbol{x} \tag{36}$$

we define $\boldsymbol{y} = \boldsymbol{A}_j \boldsymbol{x}$. Therefore, (36) can be rewritten as:

$$\Phi = \boldsymbol{y}^T \left[(\boldsymbol{A}_i^T \boldsymbol{P} \boldsymbol{A}_i - \boldsymbol{P}_{ij}) \right] \boldsymbol{y} + \boldsymbol{x}^T \left[\boldsymbol{A}_j^T \boldsymbol{P}_{ij} \boldsymbol{A}_j - \boldsymbol{P} \right] \boldsymbol{x}. \tag{37}$$

If $\boldsymbol{A}_i^T \boldsymbol{P} \boldsymbol{A}_i - \boldsymbol{P}_{ij} < 0$ and $\boldsymbol{A}_j^T \boldsymbol{P}_{ij} \boldsymbol{A}_j - \boldsymbol{P} < 0$ for every $i, j \in L$ then $\Phi < 0$ and the origin is globally asymptotically stable.

The proof is now completed.

The equivalence of (27-28), (29-30) and (31-32) can be easily established by using Schur complement together with $\boldsymbol{X} = \boldsymbol{P}^{-1}$, $\boldsymbol{X}_{ij} = \boldsymbol{P}_{ij}^{-1}$ [39].

Next, we generalize Theorem 3.3 to m-step differences.

Corollary 3.3. *The origin of discrete time T-S fuzzy system described by (2) with $\boldsymbol{u}(k) = 0$ is globally asymptotically stable if*
i. There exists a set of positive definite matrices $\boldsymbol{P}$, $\boldsymbol{P}_{ij}$, $\boldsymbol{P}_{ijk}$, ..., and $\boldsymbol{P}_{ijk...o}$ for every $i, j, k, \ldots, o \in L$ such that the following LMIs are satisfied:

$$\begin{aligned} \boldsymbol{A}_i^T \boldsymbol{P} \boldsymbol{A}_i - \boldsymbol{P}_{ij} &< 0 \\ \boldsymbol{A}_j^T \boldsymbol{P}_{ij} \boldsymbol{A}_j - \boldsymbol{P}_{ijk} &< 0 \\ &\vdots \\ \boldsymbol{A}_o^T \boldsymbol{P}_{ijk...o} \boldsymbol{A}_o - \boldsymbol{P} &< 0 \end{aligned} \tag{38}$$

or, equivalently;
ii. There exists a set of positive definite matrices $\boldsymbol{P}$, $\boldsymbol{P}_{ij}$, $\boldsymbol{P}_{ijk}$, ..., and $\boldsymbol{P}_{ijk...o}$ for every $i, j, k, \ldots, o \in L$ such that the following inequalities are satisfied:

$$\begin{bmatrix} -\boldsymbol{P}_{ij} & \boldsymbol{A}_i^T \\ \boldsymbol{A}_i & -\boldsymbol{P}^{-1} \end{bmatrix} < 0$$
$$\begin{bmatrix} -\boldsymbol{P}_{ijk} & \boldsymbol{A}_j^T \\ \boldsymbol{A}_j & -\boldsymbol{P}_{ij}^{-1} \end{bmatrix} < 0$$
$$\vdots$$
$$\begin{bmatrix} -\boldsymbol{P} & \boldsymbol{A}_o^T \\ \boldsymbol{A}_o & -\boldsymbol{P}_{ijk...o}^{-1} \end{bmatrix} < 0 \tag{39}$$

iii. There exists a set of positive definite matrices $\boldsymbol{X}$, $\boldsymbol{X}_{ij}$, $\boldsymbol{X}_{ijk}$, ..., and $\boldsymbol{X}_{ijk...o}$ for every $i, j, k, \ldots, o \in L$ such that the following LMIs are satisfied:

$$\begin{bmatrix} -\boldsymbol{X}_{ij} & \boldsymbol{X}_{ij} \boldsymbol{A}_i^T \\ \boldsymbol{A}_i \boldsymbol{X}_{ij} & -\boldsymbol{X} \end{bmatrix} < 0$$
$$\begin{bmatrix} -\boldsymbol{X}_{ijk} & \boldsymbol{X}_{ij} \boldsymbol{A}_j^T \\ \boldsymbol{A}_j \boldsymbol{X}_{ij} & -\boldsymbol{X}_{ij} \end{bmatrix} < 0$$
$$\vdots$$
$$\begin{bmatrix} -\boldsymbol{X} & \boldsymbol{X}_{ijk...o} \boldsymbol{A}_o^T \\ \boldsymbol{A}_o \boldsymbol{X}_{ijk...o} & -\boldsymbol{X}_{ijk...o} \end{bmatrix} < 0 \tag{40}$$

Proof: Similar to the proof of Theorem 3.3, it can be shown that the inequality (24) holds for every $i, j, k, \ldots, o \in L$.

Remark 3.2. *It is noted that when the positive definite matrices in (38) and (39) (or equivalently (40)) are chosen as a common one, $\boldsymbol{P} = \boldsymbol{P}_{ij} = \boldsymbol{P}_{ijk} = \ldots = \boldsymbol{P}_{ijk\ldots o}$ (or equivalently, $\boldsymbol{X} = \boldsymbol{X}_{ij} = \boldsymbol{X}_{ijk} = \ldots = \boldsymbol{X}_{ijk\ldots o}$), then the results of Theorem 3.3 and its corollary reduce to that of Lemma 2.2. So it can be easily seen that common quadratic Lyapunov functions are a special case of Theorem 3.3 and its corollary. It is obvious that the new results are less conservative; however, the computational cost of them would be higher in general.*

Remark 3.3. *For the resulting stability condition derived from this corollary, it is necessary to find $((r^{m+1} - 1)/(r - 1)) - r$ positive definite matrices $\boldsymbol{P}$, $\boldsymbol{P}_{ij}$, $\boldsymbol{P}_{ijk}$, $\ldots$, and $\boldsymbol{P}_{ijk\ldots o}$ satisfying $(((r^{m+1} - 1)/(r - 1)) - r - 1 + r^m)$ inequalities.*

4. **Stabilization Based on Non-Monotonic Lyapunov Functions.** Consider the discrete time T-S fuzzy system described by (2). In order to stabilize this fuzzy system, we consider $\boldsymbol{u}(k) = \boldsymbol{F}\boldsymbol{x}(k)$ as a state feedback controller, so the closed loop fuzzy system is

$$\boldsymbol{x}(k + 1) = \sum_{i=1}^{r} \mu_i (\boldsymbol{A}_i + \boldsymbol{B}_i \boldsymbol{F})\boldsymbol{x}(k) \tag{41}$$

In order to achieve a suitable form for controller synthesis, the inequalities (29-30) from Theorem 3.3 and (39) from Corollary 3.3 are used to propose a new method to design a state feedback controller which is presented as follows:

Theorem 4.1. *The closed loop fuzzy system (41) is globally asymptotically stable, if there exists a set of positive definite matrices $\boldsymbol{P}$ and $\boldsymbol{P}_{ij}$ for every $i, j \in L$ and a matrix F such that the following inequalities are satisfied:*

$$\begin{aligned} \begin{bmatrix} -\boldsymbol{P}_{ij} & \boldsymbol{A}_i^T + \boldsymbol{F}^T \boldsymbol{B}_i^T \\ \boldsymbol{A}_i + \boldsymbol{B}_i \boldsymbol{F} & -\boldsymbol{P}^{-1} \end{bmatrix} &< 0 \\ \begin{bmatrix} -\boldsymbol{P} & \boldsymbol{A}_j^T + \boldsymbol{F}^T \boldsymbol{B}_j^T \\ \boldsymbol{A}_j + \boldsymbol{B}_j \boldsymbol{F} & -\boldsymbol{P}_{ij}^{-1} \end{bmatrix} &< 0 \end{aligned} \tag{42}$$

Proof: Theorem 4.1 directly follows from Theorem 3.3 and closed loop fuzzy system described by (41).

Next, we generalize Theorem 4.1 to m-step differences.

Corollary 4.1. *The closed loop fuzzy system (41) is globally asymptotically stable, if there exists a set of positive definite matrices $\boldsymbol{P}$, $\boldsymbol{P}_{ij}$, $\boldsymbol{P}_{ijk}$, $\ldots$, and $\boldsymbol{P}_{ijk\ldots o}$ for every $i, j, k, \ldots, o \in L$ and a matrix $\boldsymbol{F}$ such that the following inequalities are satisfied:*

$$\begin{aligned} \begin{bmatrix} -\boldsymbol{P}_{ij} & \boldsymbol{A}_i^T + \boldsymbol{F}^T \boldsymbol{B}_i^T \\ \boldsymbol{A}_i + \boldsymbol{B}_i \boldsymbol{F} & -\boldsymbol{P}^{-1} \end{bmatrix} &< 0 \\ \begin{bmatrix} -\boldsymbol{P}_{ijk} & \boldsymbol{A}_j^T + \boldsymbol{F}^T \boldsymbol{B}_j^T \\ \boldsymbol{A}_j + \boldsymbol{B}_j \boldsymbol{F} & -\boldsymbol{P}_{ij}^{-1} \end{bmatrix} &< 0 \\ &\vdots \\ \begin{bmatrix} -\boldsymbol{P} & \boldsymbol{A}_o^T + \boldsymbol{F}^T \boldsymbol{B}_o^T \\ \boldsymbol{A}_o + \boldsymbol{B}_o \boldsymbol{F} & -\boldsymbol{P}_{ijk\ldots o}^{-1} \end{bmatrix} &< 0 \end{aligned} \tag{43}$$

Proof: Similar to the proof of Theorem 4.1, the result directly follows from Corollary 4.1.

It is noted that the resulting conditions for controller synthesis problem in Theorem 4.1 and its corollary are not LMI conditions because of the terms $\boldsymbol{P}^{-1}$, $\boldsymbol{P}_{ij}^{-1}$, $\boldsymbol{P}_{ijk}^{-1}$, $\ldots$, and $\boldsymbol{P}_{ijk\ldots o}^{-1}$ in (42) and (43). To obtain a solution for these inequalities, the iterative algorithm presented in [40] can be employed.

First, we define new variables $\boldsymbol{X}$, $\boldsymbol{X}_{ij}$, ..., and $\boldsymbol{X}_{ijk...o}$ and replace the condition (42) with (44) and condition (43) with (45).

$$\left.\begin{array}{l}\begin{bmatrix} -\boldsymbol{P}_{ij} & \boldsymbol{A}_i^T + \boldsymbol{F}^T\boldsymbol{B}_i^T \\ \boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{F} & -\boldsymbol{X} \end{bmatrix} < 0 \\ \begin{bmatrix} -\boldsymbol{P} & \boldsymbol{A}_j^T + \boldsymbol{F}^T\boldsymbol{B}_j^T \\ \boldsymbol{A}_j + \boldsymbol{B}_j\boldsymbol{F} & -\boldsymbol{X}_{ij} \end{bmatrix} < 0 \end{array}\right\} \tag{44}$$

$$\left.\begin{array}{l}\begin{bmatrix} -\boldsymbol{P}_{ij} & \boldsymbol{A}_i^T + \boldsymbol{F}^T\boldsymbol{B}_i^T \\ \boldsymbol{A}_i + \boldsymbol{B}_i\boldsymbol{F} & -\boldsymbol{X} \end{bmatrix} < 0 \\ \begin{bmatrix} -\boldsymbol{P}_{ijk} & \boldsymbol{A}_j^T + \boldsymbol{F}^T\boldsymbol{B}_j^T \\ \boldsymbol{A}_j + \boldsymbol{B}_j\boldsymbol{F} & -\boldsymbol{X}_{ij} \end{bmatrix} < 0 \\ \vdots \\ \begin{bmatrix} -\boldsymbol{P} & \boldsymbol{A}_o^T + \boldsymbol{F}^T\boldsymbol{B}_o^T \\ \boldsymbol{A}_o + \boldsymbol{B}_o\boldsymbol{F} & -\boldsymbol{X}_{ijk...o} \end{bmatrix} < 0 \end{array}\right\} \tag{45}$$

Now, using a cone complementarity problem [41], we suggest the following nonlinear minimization problem involving LMI conditions instead of the original non convex feasibility problem of Corollary 4.1,

Minimize $Tr\left(\boldsymbol{PX} + \boldsymbol{P}_{ij}\boldsymbol{X}_{ij} + \ldots + \boldsymbol{P}_{ijk...o}\boldsymbol{X}_{ijk...o}\right)$,

subject to (45) and

$$\left.\begin{array}{l}\begin{bmatrix} \boldsymbol{P} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{X} \end{bmatrix} \geq 0 \\ \begin{bmatrix} \boldsymbol{P}_{ij} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{X}_{ij} \end{bmatrix} \geq 0 \\ \vdots \\ \begin{bmatrix} \boldsymbol{P}_{ij...o} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{X}_{ij...o} \end{bmatrix} \geq 0 \end{array}\right\} \tag{46}$$

If the solution of the above minimization problem exists, we can say from Corollary 4.1 that the closed loop fuzzy system (41) is globally asymptotically stable. Actually, utilizing the linearization method [41], we can use an iterative algorithm presented as follows, where h and n denote the number of iterations and state variables, respectively.

Algorithm 4.1:

Step 1: *Find a feasible set $(\boldsymbol{X}^0, \boldsymbol{X}_{ij}^0, \ldots, \boldsymbol{X}_{ijk...o}^0, \boldsymbol{P}^0, \boldsymbol{P}_{ij}^0, \ldots, \boldsymbol{P}_{ijk...o}^0)$ satisfying (46). Set $h = 1$.*

Step 2: *Solve the following LMI problem for the variables $(\boldsymbol{X}, \boldsymbol{X}_{ij}, \ldots, \boldsymbol{X}_{ijk...o}, \boldsymbol{P}, \boldsymbol{P}_{ij}, \ldots, \boldsymbol{P}_{ijk...o}$ and $\boldsymbol{F})$:*
Minimize $Tr(\boldsymbol{PX}^{h-1} + \boldsymbol{P}_{ij}\boldsymbol{X}_{ij}^{h-1} + \ldots + \boldsymbol{P}_{ijk...o}\boldsymbol{X}_{ijk...o}^{h-1} + \boldsymbol{P}^{h-1}\boldsymbol{X} + \boldsymbol{P}_{ij}^{h-1}\boldsymbol{X}_{ij} + \ldots + \boldsymbol{P}_{ijk...o}^{h-1}\boldsymbol{X}_{ijk...o})$
Subject to (45) and (46).
Set $\boldsymbol{X}^h = \boldsymbol{X}$, $\boldsymbol{X}_{ij}^h = \boldsymbol{X}_{ij}, \ldots, \boldsymbol{X}_{ijk...o}^h = \boldsymbol{X}_{ijk...o}$, $\boldsymbol{P}^h = \boldsymbol{P}$, $\boldsymbol{P}_{ij}^h = \boldsymbol{P}_{ij}, \ldots$, and $\boldsymbol{P}_{ijk...o}^h = \boldsymbol{P}_{ijk...o}$.

Step 3: *If the solution of the above minimization problem is approximately equal to $2n((r^{m+1}-1)/(r-1))-r$, then $(\boldsymbol{X}, \boldsymbol{X}_{ij}, \ldots, \boldsymbol{X}_{ijk...o}, \boldsymbol{P}, \boldsymbol{P}_{ij}, \ldots, \boldsymbol{P}_{ijk...o}$, and $\boldsymbol{F})$ are a feasible solution for the ILMI of Theorem 4.1 and exit.*

Step 4: *If $h \geq q$, where q is a predetermined large value, then the given ILMI may not be feasible and exit. Otherwise, set $h = h + 1$, and go to Step 2.*

5. **Numerical Examples.** In this section, three different examples are provided to demonstrate the effectiveness, validity and applicability of our proposed stability analysis and synthesis methods. In Example 5.1 [44] a comparison of our proposed stability methods based on Theorems 3.2 and 3.3 and the stability analysis based on quadratic Lyapunov function is given for T-S discrete fuzzy system. However, in the second example, our proposed theorem (Theorem 4.1) is compared with Theorem (5-1) in [42] which is utilized to design a state feedback controller. Also Example 5.3 is provided as a comparison of our proposed method (Theorem 4.1) with Theorem (5-1) and (6-2) in [42] and Theorem 4 in [43], which are based on common quadratic, piecewise and fuzzy Lyapunov functions, respectively. From the examples, it is apparent that our proposed methods outperform the available schemes in the literature. It thus can be easily seen that in our method conservativeness is reduced; however, the computational complexity is increased.

Example 5.1. *Consider a T-S fuzzy free system with two state variables that switches between two rules,*

R^1 :   *If* $\|x_1\| - \|x_2\|$ *is about negative*
    Then $\boldsymbol{x}\,(k+1) = \boldsymbol{A}_1\boldsymbol{x}\,(k)\,.$
R^2 :   *If* $\|x_1\| - \|x_2\|$ *is about positive*
    Then $\boldsymbol{x}\,(k+1) = \boldsymbol{A}_2\boldsymbol{x}\,(k)\,.$
Let the corresponding characteristic matrices $\boldsymbol{A}_i$*s be*

$$\boldsymbol{A}_1 = \begin{bmatrix} 0.4 & 0 \\ -\sqrt{7/8} & 0 \end{bmatrix}, \quad \boldsymbol{A}_2 = \begin{bmatrix} 0 & \sqrt{7/8} \\ 0 & 0.4 \end{bmatrix}.$$

Using Matlab LMI toolbox, one can easily verify that there exists no positive define matrix $\boldsymbol{P}$ *for the fuzzy system to show its stability. In other words, the fuzzy system does not admit a global quadratic Lyapunov function.*

According to Theorem 3.3 we should find 10 symmetric positive definite matrices $\boldsymbol{P}$, $\boldsymbol{X}$, $\boldsymbol{P}_{11}$, $\boldsymbol{X}_{11}$, $\boldsymbol{P}_{12}$, $\boldsymbol{X}_{12}$, $\boldsymbol{P}_{21}$, $\boldsymbol{X}_{21}$, $\boldsymbol{P}_{22}$, *and* $\boldsymbol{X}_{22}$ *to satisfy (10-11) and (27-32). Using Matlab LMI toolbox, the following solutions are obtained for the aforementioned LMIs:*

$$\boldsymbol{P} = \begin{bmatrix} 1.0357 & 0 \\ 0 & 1.0357 \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} 0.9655 & 0 \\ 0 & 0.9655 \end{bmatrix},$$

$$\boldsymbol{P}_{11} = \begin{bmatrix} 1.4877 & 0.1345 \\ 0.1345 & 0.6947 \end{bmatrix}, \quad \boldsymbol{X}_{11} = \begin{bmatrix} 0.6842 & -0.1324 \\ -0.1324 & 1.4650 \end{bmatrix},$$

$$\boldsymbol{P}_{12} = \begin{bmatrix} 1.1215 & -0.1812 \\ -0.1812 & 0.9317 \end{bmatrix}, \quad \boldsymbol{X}_{12} = \begin{bmatrix} 0.9206 & 0.1790 \\ 0.1790 & 1.1081 \end{bmatrix},$$

$$\boldsymbol{P}_{21} = \begin{bmatrix} 0.9317 & 0.1812 \\ 0.1812 & 1.1215 \end{bmatrix}, \quad \boldsymbol{X}_{21} = \begin{bmatrix} 1.1081 & -0.1790 \\ -0.1790 & 0.9206 \end{bmatrix},$$

$$\boldsymbol{P}_{22} = \begin{bmatrix} 0.6947 & -0.1345 \\ -0.1345 & 1.4877 \end{bmatrix}, \quad \boldsymbol{X}_{22} = \begin{bmatrix} 1.4650 & 0.1324 \\ 0.1324 & 0.6842 \end{bmatrix}.$$

Consequently, the origin of the discrete time T-S fuzzy system is globally asymptotically stable.

Example 5.2. *Consider the discrete time fuzzy system that switches between the following three rules,*

R^1 :   If x_1 is about negative
        Then $\boldsymbol{x}(k+1) = \boldsymbol{A}_1\boldsymbol{x}(k) + \boldsymbol{B}_1\boldsymbol{u}(k)$.
R^2 :   If x_1 is about zero
        Then $\boldsymbol{x}(k+1) = \boldsymbol{A}_2\boldsymbol{x}(k) + \boldsymbol{B}_2\boldsymbol{u}(k)$.
R^3 :   If x_1 is about positive
        Then $\boldsymbol{x}(k+1) = \boldsymbol{A}_3\boldsymbol{x}(k) + \boldsymbol{B}_3\boldsymbol{u}(k)$.

   The membership functions for "about negative", "about zero" and "about positive" are shown in Figure 1.
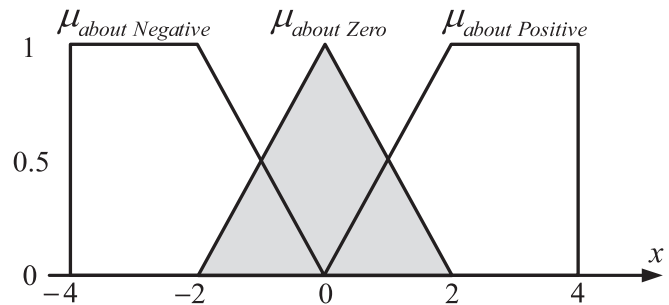
FIGURE 1. State space and membership functions for the illustrated system

The system matrices are given by

$$\boldsymbol{A}_1 = \begin{bmatrix} 1 & 0.1 \\ 1 & -0.5 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 0.5 & -0.6 \\ 0.6 & 0.5 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\boldsymbol{A}_3 = \begin{bmatrix} 1 & 0.5 \\ -0.1 & 1 \end{bmatrix}, \quad \boldsymbol{B}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

   It should be noted that the open loop fuzzy system is unstable and the stabilization cannot be provided by the common quadratic Lyapunov function [42, Th. 5.1] approach. However, by using the non-monotonic Lyapunov function in Theorem 4.1 and Algorithm 4.1, the stabilizing state feedback controller can be computed. In order to solve (42) with the initial values as $\boldsymbol{X}^0 = \boldsymbol{X}_{ij}^0 = \boldsymbol{P}^0 = \boldsymbol{P}_{ij}{}^0 = \boldsymbol{I}$, Matlab LMI Toolbox is utilized and after seven iterations, the minimum of $Tr(\boldsymbol{P}\boldsymbol{X}^k + \boldsymbol{P}_{ij}\boldsymbol{X}_{ij}^k + \boldsymbol{P}^k\boldsymbol{X} + \boldsymbol{P}_{ij}^k\boldsymbol{X}_{ij})$ is achieved as 40.0004.

The following feasible solutions are also obtained:

$$\boldsymbol{P} = \begin{bmatrix} 0.9968 & -0.0189 \\ -0.0189 & 1.0034 \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} 1.0036 & 0.0190 \\ 0.0190 & 0.9970 \end{bmatrix},$$

$$\boldsymbol{F} = \begin{bmatrix} -1.0113 & -0.4024 \end{bmatrix},$$

$$\boldsymbol{P}_{11} = \begin{bmatrix} 0.9984 & 0.1333 \\ 0.1333 & 1.0196 \end{bmatrix}, \quad \boldsymbol{X}_{11} = \begin{bmatrix} 1.0194 & -0.1333 \\ -0.1333 & 0.9982 \end{bmatrix},$$

$$\boldsymbol{P}_{12} = \begin{bmatrix} 0.9267 & 0.0298 \\ 0.0298 & 0.9864 \end{bmatrix}, \quad \boldsymbol{X}_{12} = \begin{bmatrix} 1.0801 & -0.0326 \\ -0.0326 & 1.0148 \end{bmatrix},$$

$$\boldsymbol{P}_{13} = \begin{bmatrix} 0.9176 & 0.0322 \\ 0.0322 & 1.0456 \end{bmatrix}, \quad \boldsymbol{X}_{13} = \begin{bmatrix} 1.0910 & -0.0336 \\ -0.0336 & 0.9574 \end{bmatrix},$$

$$\boldsymbol{P}_{21} = \begin{bmatrix} 1.0705 & 0.0361 \\ 0.0361 & 1.0194 \end{bmatrix}, \quad \boldsymbol{X}_{21} = \begin{bmatrix} 0.9352 & -0.0332 \\ -0.0322 & 0.9822 \end{bmatrix},$$

$$\boldsymbol{P}_{22} = \begin{bmatrix} 0.9998 & -0.0002 \\ -0.0002 & 1.0003 \end{bmatrix}, \quad \boldsymbol{X}_{22} = \begin{bmatrix} 1.0002 & 0.0002 \\ 0.0002 & 0.9997 \end{bmatrix},$$

$$\boldsymbol{P}_{23} = \begin{bmatrix} 0.9995 & -0.0022 \\ -0.0022 & 1.0235 \end{bmatrix}, \quad \boldsymbol{X}_{23} = \begin{bmatrix} 1.0005 & 0.0021 \\ 0.0021 & 0.9770 \end{bmatrix},$$

$$\boldsymbol{P}_{31} = \begin{bmatrix} 1.0831 & 0.0376 \\ 0.0376 & 0.9621 \end{bmatrix}, \quad \boldsymbol{X}_{31} = \begin{bmatrix} 0.9246 & -0.0362 \\ -0.0362 & 1.0408 \end{bmatrix},$$

$$\boldsymbol{P}_{32} = \begin{bmatrix} 1.0005 & -0.0017 \\ -0.0017 & 0.9837 \end{bmatrix}, \quad \boldsymbol{X}_{32} = \begin{bmatrix} 0.9995 & 0.0017 \\ 0.0017 & 1.0166 \end{bmatrix},$$

$$\boldsymbol{P}_{33} = \begin{bmatrix} 1.0152 & -0.1447 \\ -0.1447 & 1.0132 \end{bmatrix}, \quad \boldsymbol{X}_{33} = \begin{bmatrix} 1.0055 & 0.1436 \\ 0.1436 & 1.0075 \end{bmatrix}.$$

Based on Theorem 4.1, the closed loop state feedback control system is asymptotically stable. Figure 2 illustrates the trajectories of states for the closed loop feedback control system when the initial condition is $\boldsymbol{x}(0) = [-2 \quad -1]^T$. Under the same initial condition, Figure 3 shows the control signal and Lyapunov function trajectories against time and it is clear that this Lyapunov function does not monotonically decrease along trajectories; however, it decreases every two step.

Example 5.3. *Consider the discrete time fuzzy system with two state variables that switches between the following two rules,*

    R^1 : *If $z(k)$ is about zero*

        Then $\boldsymbol{x}(k+1) = \boldsymbol{A}_1\boldsymbol{x}(k) + \boldsymbol{B}_1\boldsymbol{u}(k)$

    R^1 : *If $z(k)$ is about 1 or –1*

        Then $\boldsymbol{x}(k+1) = \boldsymbol{A}_2\boldsymbol{x}(k) + \boldsymbol{B}_2\boldsymbol{u}(k)$

where $z(k) = x_2(k) - x_1(k)$ and the membership functions for "about zero", "about 1" and "about –1" are shown in Figure 4.

    The system matrices are

$$\boldsymbol{A}_1 = \begin{bmatrix} -1.9 & -a \\ 2 & 1 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 0.26 \\ 0.92 \end{bmatrix},$$

$$\boldsymbol{A}_2 = \begin{bmatrix} -0.08 & 0.8 \\ 0.58 & 0.69 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} -b \\ 0.21 \end{bmatrix}.$$

a and b are variable elements which are used to represent the feasible areas for stabilization. Suppose that a and b vary in the intervals $1 \le a \le 5.5$ and $0 \le b \le 2$, respectively.
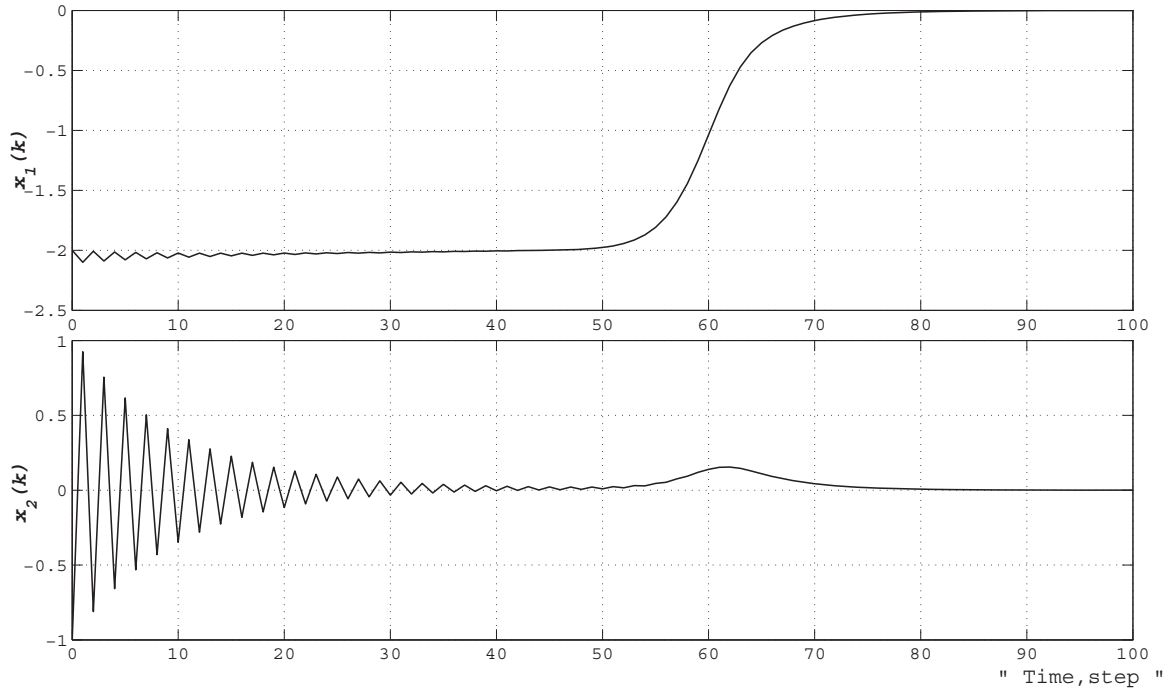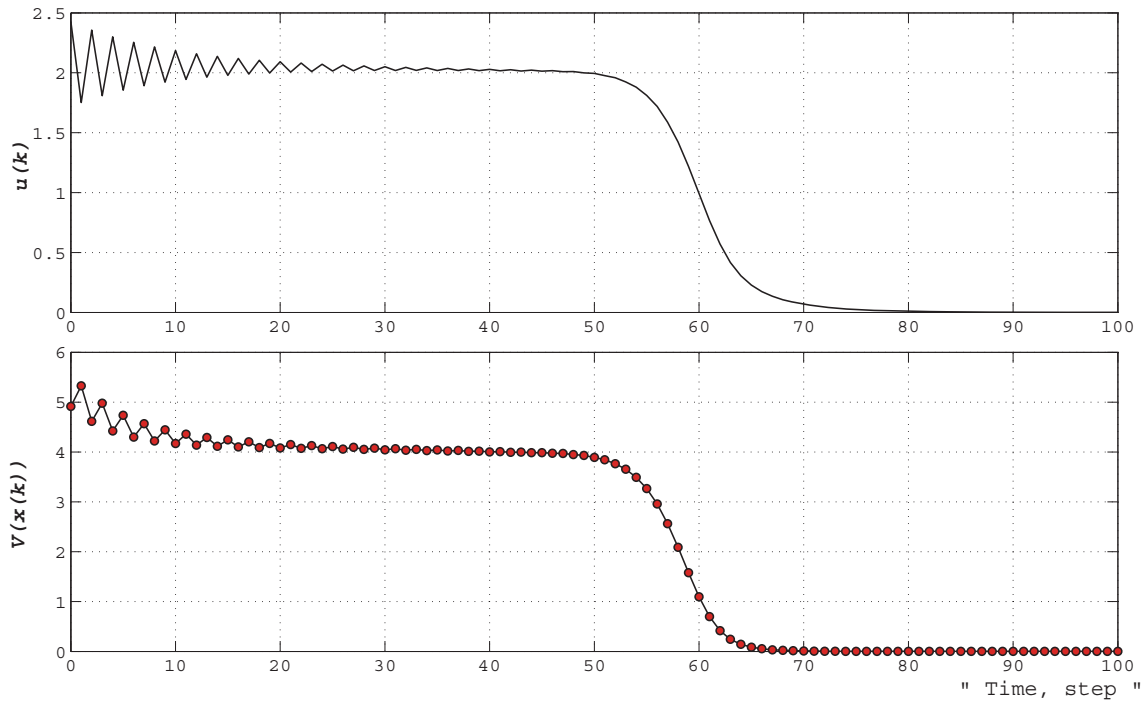
FIGURE 2. The trajectories of the state

FIGURE 3. The trajectories for control signal and Lyapunov function

Figures 5(a)-5(d) show the feasible area according to common quadratic Lyapunov function [42,Th. 5.1], piecewise quadratic Lyapunov function [42,Th. 6.2], fuzzy Lyapunov function [43,Th. 4] and non-monotonic Lyapunov function based on Theorem 4.1 of this paper, respectively. It is shown that the stabilization method based on common quadratic Lyapunov function [42,Th. 5.1], piecewise quadratic Lyapunov function [42,Th. 6.2] and
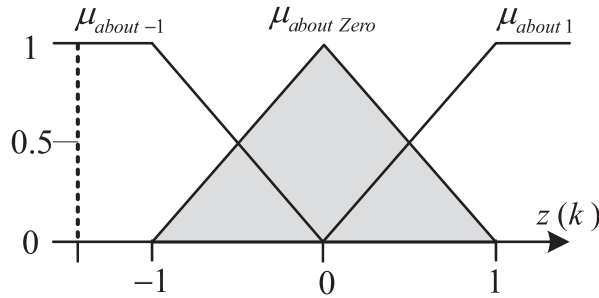
FIGURE 4. Membership functions for the illustrated system

fuzzy Lyapunov function [43,Th. 4] have the same conservative results. However, the stabilization method based on Theorem 4.1 which is proposed in this paper provides the most relaxed result.
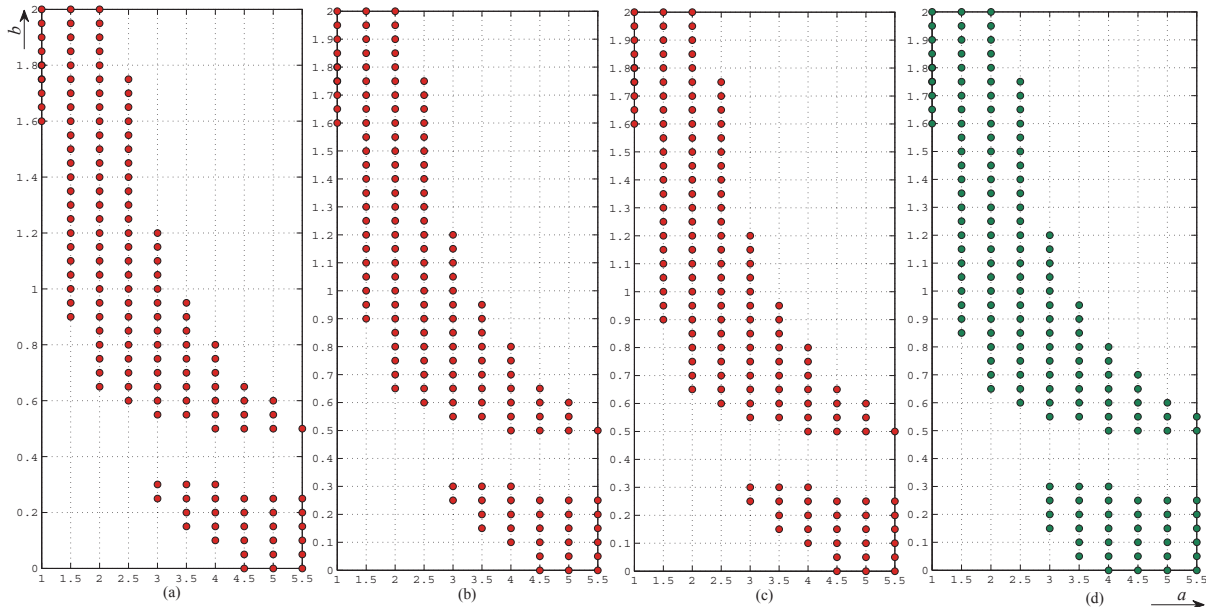

FIGURE 5. Feasible area for the stabilization based on (a) common quadratic Lyapunov function, (b) piecewise Lyapunov function, (c) fuzzy Lyapunov function, (d) non-monotonic Lyapunov function

A detailed comparison is given in Table 1, where the number of inequalities and the feasible areas are listed for each condition. From Table 1 it can be concluded that the stabilization method based on Theorem 4.1 has the most relaxed results; however, it has the largest number of inequalities which end in higher computational cost.

It should be noted that for $a = 1.5$ and $b = 0.85$, stability cannot be provided by the methods based on [42,Th. 5.1], [42,Th. 6.2], and [43,Th. 4]. However, using non-monotonic Lyapunov function based on Theorem 4.1 and Algorithm 4.1 of this paper, the stability can be provided. In order to solve (42), Matlab LMI Toolbox is utilized and after ten iterations, the minimum of $Tr(\boldsymbol{P}\boldsymbol{X}^k + \boldsymbol{P}_{ij}\boldsymbol{X}_{ij}^k + \boldsymbol{P}^k\boldsymbol{X} + \boldsymbol{P}_{ij}^k\boldsymbol{X}_{ij})$ is achieved as 20.0001. The initial values are assumed to be $\boldsymbol{X}^0 = \boldsymbol{X}_{ij}^0 = \boldsymbol{P}^0 = \boldsymbol{P}_{ij}^0 = \boldsymbol{I}$. The following feasible

TABLE 1. Numbers of inequalities and the feasible areas for each condition

	Number of Inequalities	Feasible Areas (Point)
Common Quadratic Lyapunov Function [42, Th.5.1]	4	151
Piecewise Quadratic Lyapunov Function [42, Th.6.2]	8	151
Fuzzy Lyapunov Function based on [43, Th.4]	8	151
Non-monotonic Lyapunov Function based on Theorem 4.1	13	160

solutions are also obtained:

$$\boldsymbol{P} = \begin{bmatrix} 1.4451 & 1.3780 \\ 1.3780 & 1.6507 \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} 3.3925 & -2.8320 \\ -2.8320 & 2.9699 \end{bmatrix},$$

$$\boldsymbol{F} = \begin{bmatrix} -0.0432 & 0.8286 \end{bmatrix},$$

$$\boldsymbol{P}_{11} = \begin{bmatrix} 1.3956 & 1.2789 \\ 1.2789 & 1.4696 \end{bmatrix}, \quad \boldsymbol{X}_{11} = \begin{bmatrix} 3.5372 & -3.0781 \\ -3.0781 & 3.3590 \end{bmatrix},$$

$$\boldsymbol{P}_{12} = \begin{bmatrix} 1.3371 & 1.4230 \\ 1.4230 & 1.8779 \end{bmatrix}, \quad \boldsymbol{X}_{12} = \begin{bmatrix} 3.8630 & -2.9272 \\ -2.9272 & 2.7506 \end{bmatrix},$$

$$\boldsymbol{P}_{21} = \begin{bmatrix} 2.3248 & 0.8851 \\ 0.8851 & 1.3368 \end{bmatrix}, \quad \boldsymbol{X}_{21} = \begin{bmatrix} 0.5751 & -0.3808 \\ -0.3808 & 1.0002 \end{bmatrix},$$

$$\boldsymbol{P}_{22} = \begin{bmatrix} 1.4268 & 0.5460 \\ 0.5460 & 1.5635 \end{bmatrix}, \quad \boldsymbol{X}_{22} = \begin{bmatrix} 0.8090 & -0.2825 \\ -0.2825 & 0.7382 \end{bmatrix}.$$

Figure 6 shows the evolution of the states and the control input of the closed loop system with the designed controller from the initial condition $\boldsymbol{x}(0) = \begin{bmatrix} -4 & 5 \end{bmatrix}^T$.

6. **Conclusions.** Stability analysis and synthesis methodology are the important issues in the study of fuzzy control systems. In this paper, we study the global asymptotic stability of the discrete time T-S fuzzy systems. First, the monotonic requirement of Lyapunov theorem is relaxed to achieve less conservative results in the stability analysis. Subsequently, based on non-monotonic Lyapunov Functions, a new stabilizability condition is proposed which is solved in an iterative manner. It should be noted that although this paper is safely concentrated on the stability analysis of fuzzy T-S models, the non-monotonic stability theorem presented, can still be employed for various other non-linear models. Numerical and simulation examples illustrate the validity and effectiveness of our proposed analysis and synthesis methodology in sense of being less conservative. Still, attention needs to be turned to establishing stability in the presence of uncertainty, performance study in the context of closed-loop control system, disturbance attenuation and decreasing computational complexity.
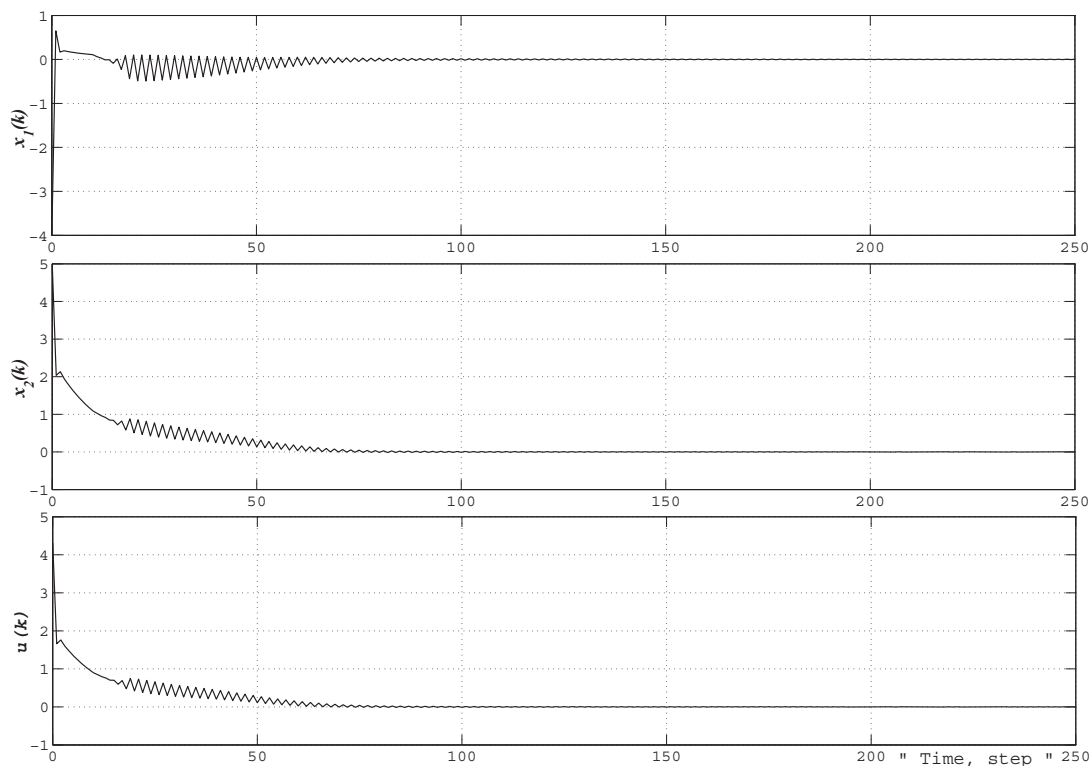
FIGURE 6. Simulation results for Example 5.3

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