International Journal of Innovative Computing, Information and Control
Volume **9**, Number **8**, August **2013**
ICIC International ©2013 ISSN 1349-4198
pp. 3091–3113

UTILIZING PERMUTATIONAL SYMMETRIES IN DYNAMIC PROGRAMMING – WITH AN APPLICATION TO THE OPTIMAL CONTROL OF WATER DISTRIBUTION SYSTEMS UNDER WATER DEMAND UNCERTAINTIES

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Received June 2012; revised October 2012

Abstract. *This paper develops a sub-optimal control tool for high dimensional nonlinear stochastic systems which are subject to permutational invariance. Using an open-loop feedback control scheme, a management policy is obtained by exploiting the solutions of consecutive high dimensional stochastic nonlinear mixed-integer programs on a rolling horizon. The concept of permutational invariance as system property is introduced and used to define a pseudo state space over the control domain. Applying state aggregation over the pseudo state space, a one dimensional equivalent problem is obtained and solved by stochastic dynamic programming. An application of the proposed method to the water distribution system of Sopron (Hungary) is presented.*
Keywords: Process control, Pump scheduling, Water supply

1. **Introduction.** The aim of water network operational optimization is to satisfy the residential and industrial water demands while minimizing the operation costs under various operational constraints related to safety and capacity. In conventional water supply systems, pumping of treated water represents the main fraction of the total operation cost [1]. Consequently, cost efficient operation can be achieved by a management policy implemented as operation rules for the active hydraulic elements (pumps, valves) of the network. Policy is to be based on all available information on the system's state, to compensate for the disturbances of the system, such as uncertainty in water demand. Typically, human operators use rules of thumb to provide "cost efficient" operation.

In systems subject to uncertainties, an optimal management policy can be obtained as the solution of an optimal control problem by means of Stochastic Dynamic Programming (SDP) (closed-loop optimal control). However, SDP usually provides a feasible alternative only when the problem dimension is small enough. As the problem dimension increases, SDP encounters serious computational issues, including excessive demand of computing time and storage requirements. More precisely, the computational and storage requirements grow exponentially with respect to the state, control and distrurbance dimension. This fact is well known as the "curse of dimensionality" and represents a severe limiting factor for the application of SDP.

To resolve the computational difficulties, predictive control strategies have recently gained much interest. Under predictive control, an optimal open-loop control problem is defined and solved on a finite horizon, using nonlinear programming techniques which are (usually) not subject to dimensionality issues. The open-loop optimization results in an optimal operating plan. As management policy, only the first stage of this operating plan is actually implemented, the remaining components are discarded, and at the next control instant new problem is formulated and solved by shifting the time horizon (receding horizon principle).

In the context of water systems, the predictive control strategy is often utilized in the form of Certainty Equivalent Control (CEC). In CEC, the randomness is handled implicitly replacing the stochastic variables by their nominal values [2]. This creates a deterministic problem, which is then solved by means of a deterministic optimization model such as Dynamic Programming (DP) [3], Linear Programming (LP) [4], Nonlinear Programming (NLP) [5, 6, 7] or Mixed Integer Nonlinear Programming (MINLP) [8]. The CEC approach often performs well and yields a near optimal solution [9].

Contrary to CEC, the Open-Loop Feedback Control (OLFC) handles the randomness explicitly (see [2]) by taking into account the extra information in probability distributions describing uncertainties. The OLFC usually performs better than CEC. More precisely, OLFC performs at least as well as an optimal open loop policy that uses a sequence of controls that is independent of the values of the measurements received. By contrast CEC does not have this property [2]. However, OLFC involves a heavier computational burden, since the expectations with respect to uncertain quantities must be computed to determine the operating plan. In certain cases, when the stochastic finite horizon optimization problem exhibits a special property (e.g., linearity, convexity, continuous search space), powerful optimization models can be applied such as nested benders decomposition [10].

On the other hand, when a general non-linear function must be optimized over a search space including discrete as well as continuous decisions, SDP is still the most flexible and robust alternative to solve the open-loop optimal control problem, which makes it attractive for researchers and scientists. To overcome the difficulties caused by the dimensionality, great efforts have been devoted to manipulating the problems in order to make them suitable for SDP. This continuing research has resulted in numerous proposals establishing the field of Approximate Dynamic Programming (ADP) (for overview in the context of water systems see [11, 12, 13]).

The present paper uses ADP utilizing aggregation/disaggregation techniques. The idea of aggregation is to construct a simpler more traceable problem based on reducing the number of states by "combining" them together [2]. The resulting aggregated problem has a reduced dimension which gives the possibility to solve it by SDP.

In the context of water systems, the aggregation/disaggregation method was first proposed by [14] and applied to multireservoir power systems. In [14] the whole network was aggregated into a single virtual reservoir unit (often called as equivalent reservoir) and the problem was solved by dynamic programming. The technique of state aggregation has been utilized by several researchers for the optimal control of water distribution networks [15] and large-scale hydropower systems under uncertainty [16, 17, 18, 19, 20] often combined with spatial decomposition [21, 22]. The key idea is to perform aggregation in state space by adding the potential energy of the reservoirs together.

Today it is understood that the state aggregation approaches encounter the following problems. (1) State aggregation introduces information loss; therefore, the local constraints on reservoirs cannot be taken into account and the obtained solution may not be feasible [16, 17, 21]. (2) Disaggregation is required to utilize the solution of the aggregated system, to generate the individual policies for each control input of the original system.

The disaggregation becomes difficult when nonlinear relations among state variables exist [18, 19]. (3) Aggregation methods are practically useless when the state space includes system components other than reservoirs (e.g., correlated disturbance models, forecasts) [23]. To resolve the first two issues, advanced disaggregation techniques have been developed using neural networks [18, 19]; however, to the best of the author's knowledge the third problem still remains unsolved.

This paper is devoted to decision making under uncertainty. A sub-optimal (closed-loop) control method is developed and applied to the operational optimization of the water distribution network of Sopron (Hungary) where the optimality criteria are to minimize the pumping costs while satisfying the system requirements under uncertain water demand. The proposed method exploits the open-loop feedback control scheme (OLFC). The main contribution of this paper is the introduction and utilization of permutational symmetries with state aggregation. The proposed approach allows the application of sequential decision making by SDP on high dimensional stochastic mixed-integer problems by resolving the aforementioned difficulties of aggregation/disaggregation methods. Under OLFC, the solutions of these problems are required in order to derive a sub-optimal management policy.

The utilization of permutational symmetries for the optimal control of water distribution networks was first hinted by the authors of this study and published in [24]. The concept called "permutational invariance" was introduced and utilized for optimal control in a deterministic combinatorial framework. This study extends and generalizes the results presented in [24] for the optimal control of stochastic nonlinear systems.

The paper is organized as follows. The stochastic optimal control problem is presented in Section 2. Section 3 details the solution strategy by OLFC and introduces the corresponding finite horizon (open-loop optimization) problems. The main contributions of this paper – the utilization of permutational symmetries for the solution of the finite horizon problems – are presented in Section 3.2. The interpretation of the introduced technique for water distribution systems is presented in Section 4 including a simple clarifying example. A case study is provided in Section 5. Section 6 highlights the feasibility of hydraulic modeling for real-time control, while Section 7 provides a clear overview on the benefits. Finally Section 8 gives a short summary, draws conclusions, and points at the future research directions.

2. **Problem Statement.**

The core problem is to achieve cost efficient long term operation of a water distribution system. This requires the development of an optimal control strategy for the active hydraulic elements. The control strategy needs to fulfill the water demand of users and minimize the energy consumed by water pump operations [1, 5, 25, 26].

In general, the model of the water network of interest is represented by nonlinear dynamics

$$\mathbf{x}(k+1) = \mathbf{f}(\mathbf{x}(k),\, \mathbf{u}(k),\, \mathbf{w}(k),\, k), \quad k = 0, \ldots, K-1 \tag{1}$$

which describes the evolution of the system at discrete time instants k, where $\mathbf{x}(k) = (x_1(k), \ldots, x_n(k))^T \in X$ is the state vector, $\mathbf{u}(k) = (u_1(k), \ldots, u_m(k))^T \in U$ denotes the control vector and $\mathbf{w}(k) = (w_1(k), \ldots, w_z(k))^T \in W$ is the disturbance representing the stochastic inputs of the system. The disturbance is expressed by a vector of random variables which are characterized by known probability density functions ($w_i(k) \sim \mathcal{F}_i(., k)$, $i = 1, \ldots, z$).

The aim is to find an optimal control law (policy) π^* providing the control decision(s) based on the system's state. This policy consists of a sequence of functions $\boldsymbol{\pi}(., k) : X \rightarrow$

U

$$\pi^* = \{\boldsymbol{\pi}(\mathbf{x}, 0), \ldots, \boldsymbol{\pi}(\mathbf{x}, K-1)\} \tag{2}$$

which maps the states into feasible controls $\mathbf{u}(k) = \boldsymbol{\pi}(\mathbf{x}(k), k)$ for all $\mathbf{x}(k) \in X$, and minimizes the associated cost

$$J(\pi) = \lim_{K\to\infty} \frac{1}{K}\left(\underset{\mathbf{w}(0),\ldots,\mathbf{w}(K-1)}{E} \left\{ \sum_{k=0}^{K-1} c(\mathbf{x}(k),\ \boldsymbol{\pi}(\mathbf{x}(k), k),\ \mathbf{w}(k),\ k) \right\}\right). \tag{3}$$

The cost function is defined over an infinite horizon requiring the minimization of the average expected cost per stage, where $c(\mathbf{x}(k),\ \mathbf{u}(k),\ \mathbf{w}(k),\ k)$ is the state transition cost (step cost). Equations (1)-(3) define an infinite horizon non-stationary stochastic optimal control problem.

The water demand pattern $\mathcal{F}_i(., k)$, $i = 1, \ldots, z$ (representing the stochastic inputs of the system) and the step cost $c(., k)$ can be modeled as periodic with period one year and one day respectively. Likewise, the state transition function $\mathbf{f}(., k)$ can be considered as periodic with a period one year. Taking into account the periodicity of the system, the optimal policy is a periodic sequence of control laws

$$\boldsymbol{\pi}(\mathbf{x}, k) = \boldsymbol{\pi}(\mathbf{x}, k + lT_{\mathrm{p}}), \quad l \in \mathbb{N}. \tag{4}$$

Assuming that the expected step cost is bounded

$$0 \le E\{c(\mathbf{x}(k),\ \mathbf{u}(k),\ \mathbf{w}(k),\ k)\} \le c_{\max}, \quad c_{\max} < \infty \tag{5}$$

for all $(\mathbf{x}(k), \mathbf{u}(k), \mathbf{w}(k)) \in (X, U, W)$ and $k = 0, \ldots, K-1$, the average cost $J(\pi)$ becomes well defined over an infinite number of stages, and it can be meaningfully minimized.

3. Solution Strategy.

Conceptually, the detailed optimal control problem can be solved by Stochastic Dynamic Programming (SDP). However, the determination of the optimal policy by SDP is often impossible, as the method encounters serious computational issues when the problem dimension increases. In large scale systems, the computational requirements usually grow far beyond the capabilities of existing digital hardware. In practise, the application of SDP is restricted to relatively small networks where the number of reservoirs is not greater than few units [12]. Consequently, it is usually possible to derive only a sub optimal control law for large scale problems through manipulation of the original problem. In the next section an approximate solution is examined via a reformulation of the original problem.

3.1. Finite horizon problem.

To resolve the computational difficulties, the Open-Loop Feedback Control (OLFC) provides a good alternative [2]. Under OLFC the decision making proceeds as follows: At each time instant k an optimal open-loop control problem is stated over a finite time horizon $[k, k+N_{\mathrm{t}}]$ formulating a stochastic nonlinear programming problem. The control decisions on the defined horizon constitute the search space rather than the control policy. Once the solution is obtained, only the first decision is implemented as a policy. The decision making then proceeds by formulating and solving a new open-loop control problem over $[k+1, k+N_{\mathrm{t}}+1]$ horizon.

The optimal open-loop control problem is defined as follows: given an initial state $\mathbf{x}(k)$ find a control sequence $\{\mathbf{u}(k), \ldots, \mathbf{u}(k+N_{\mathrm{t}}-1)\}$ that minimizes

$$\underset{\mathbf{w}(k),\ldots,\mathbf{w}(k+N_{\mathrm{t}}-1)}{E} \left\{ c(\mathbf{x}(k+N_{\mathrm{t}}), k+N_{\mathrm{t}}) + \sum_{t=0}^{N_{\mathrm{t}}-1} c(\mathbf{x}(k+t),\ \mathbf{u}(k+t),\ \mathbf{w}(k+t),\ k+t) \right\}, \tag{6a}$$

on a finite optimization horizon $[k, k+N_{\mathrm{t}}]$ subject to constraints on state and control:

$$\mathbf{x}(k+t+1) = \mathbf{f}(\mathbf{x}(k+t),\, \mathbf{u}(k+t),\, \mathbf{w}(k+t),\, k+t), \quad t = 0, \ldots, N_{\mathrm{t}}-1 \tag{6b}$$

$$\mathbf{g}(\mathbf{x}(k+t), \mathbf{u}(k+t)) \leq 0. \tag{6c}$$

Perfect state information is assumed, i.e., the exact value of the initial state $\mathbf{x}(k)$ is available at each time step k. Once the optimal control sequence $\{\mathbf{u}^*(k), \ldots, \mathbf{u}^*(k+N_{\mathrm{t}}-1)\}$ is obtained, the first control decision is implemented for the system as a sub-optimal control law at time k, that is,

$$\boldsymbol{\pi}(\mathbf{x}(k), k) := \mathbf{u}^*(k). \tag{7}$$

The formulated optimization problem (6) is a stochastic nonlinear program of dimension mN_{t}. The solution must be found over the free search space $S(N_{\mathrm{t}}-1) = U^{N_{\mathrm{t}}}$. The term free search space was introduced in [27] to describe the entire search domain of interest. A solution within a free search space has nothing to do with constraints; instead, it is merely the specification of the domains of the variables. That is, $S(N_{\mathrm{t}}-1)$ is spanned by feasible (satisfying all constraints) as well as non feasible (violating at least one constraint) candidate solutions.

The present paper focuses on the presented stochastic nonlinear programming problem (6) with a mixed-integer search domain $S(N_{\mathrm{t}}-1)$, i.e., some components of the control vector are restricted to be integers and defined over discrete domains while others are represented by continuous variables defined over continuous domains. In water distribution systems the constant speed pumps running in parallel [5, 25, 28] and on/off type valves [25] are simple examples of discrete decision spaces, where the task is to decide when and which pumps/valves to switch on and off. By contrast, variable speed pumps use frequency converters to allow the setting of arbitrary speed within a given range, thereby representing continuous decision variables.

3.2. **Solving the finite horizon problem.** An approximate dynamic programming approach is proposed in this section to solve the finite horizon control problem. The most attractive feature of the dynamic programming (which makes it superior to other nonlinear programming models) is that, it uses sequential decision making in time. Consequently, the optimization over the set of control sequences $S(N_{\mathrm{t}}-1)$ is reduced to a sequence of optimal decisions in time over the domain of single control vectors U. In dynamic programming an optimal solution is obtained recursively by solving the Bellman equation

$$V(\mathbf{x}(t),\, t) = \min_{\mathbf{u}(k+t)} \; \mathop{E}_{\mathbf{w}(k+t)} \left\{ c(\mathbf{x}(t), \mathbf{u}(k+t), \mathbf{w}(k+t), k+t) + V\left(\mathbf{x}(t+1), t+1\right) \right\} \tag{8}$$

where $t = 0, \ldots, N_{\mathrm{t}}-1$ and $V(\mathbf{x}(t),\, t)$ is the cost-to-go function (or value function) over the state space X. In large scale problems the true value function must be approximated to render (8) computationally traceable. A feasible approximation can be achieved by state manipulation, where the manipulated state $\mathbf{y}(t) \in Y$ is constructed subject to the following criteria: $\mathbf{y}(t)$ must have as few dimensions as possible; the use of $\mathbf{y}(t)$ must enable to find a sub-optimal solution close to the global optima. The next section presents a novel approach that can be used to construct such a manipulated state in practice.

3.2.1. *Dynamic programming over the control domain.* The essential idea is to construct a one dimensional manipulated state variable by utilizing symmetries and state aggregation. The symmetries are introduced by the invariance of the controllable domain of the state space under the permutations of control sequences. This property is referred to as *permutational invariance* throughout this paper.

Technically speaking, let $\Gamma_j(t)$ denote the set of all possible permutations of the control sequence $u_j(k:k+t) := \{u_j(k),\, u_j(k+1), \ldots, u_j(k+t)\}$ by time t and let $\Upsilon(t) =$

$\Gamma_1(t) \times \cdots \times \Gamma_m(t)$ be the set of all possible control actions over the permutation sub-sets by time t, that is,

$$\Upsilon(t) = \{(u_1(k:k+t), \ldots, u_m(k:k+t)) \mid u_j(k:k+t) \in \Gamma_j(t), j = 1, \ldots, m\}, \tag{9}$$

A permutation set $\Upsilon(t) \subseteq S(t)$ is a partition of the free search space $S(t) = U^{(t+1)}$ at time t, and originated simply by changing the order of the values in control sequences $u_j(k:k+t)$ of a given control trajectory $\mathbf{u}(k:k+t)$. By determining all possible permutation sets on $S(t)$ the free search space becomes partitioned, where $S(t) = \bigcup_{j=1}^{N_{\mathrm{p}}(t)} \Upsilon_j(t)$ and $N_{\mathrm{p}}(t)$ denotes the number of permutation sets at time t. For illustration see Figure 1.

Following the definition, the number of permutation sets at time t is proportional to the number of possible control sequences by time t. Consequently, $N_{\mathrm{p}}(t)$ is finite only if the number of possible control trajectories on the time horizon $[k, k+t]$ is finite; thus, the problem is of integer type. As soon as the decision domain has at least one continuous component (mixed-integer and continuous problems) $N_{\mathrm{p}}(t) = \infty$ for all $t = 0, \ldots, N_{\mathrm{t}} - 1$.

Let the state of the system be represented by two components $\mathbf{x}(k) = (\hat{\mathbf{x}}(k), \breve{\mathbf{x}}(k))^T$. The evolution of the main component $\hat{\mathbf{x}}(k) \in \hat{X}$ is determined by $\mathbf{u}(k)$ while the other component of the state vector $\breve{\mathbf{x}}(k) \in \breve{X}$ cannot be affected by control. The reason for splitting is to emphasize that only part of the state space is actually controllable. Using this, the general dynamics (1) are reformulated as follows:

$$\hat{\mathbf{x}}(k+t+1) = \hat{\mathbf{f}}(\hat{\mathbf{x}}(k+t), \breve{\mathbf{x}}(k+t), \mathbf{u}(k+t), \mathbf{w}(k+t), k+t) \tag{10a}$$

$$\breve{\mathbf{x}}(k+t+1) = \breve{\mathbf{f}}(\breve{\mathbf{x}}(k+t), \mathbf{w}(k+t), k+t), \quad t = 0, \ldots, N_{\mathrm{t}} - 1 \tag{10b}$$

Applying two controls, $\mathbf{u}^{(1)}(k:k+t)$ and $\mathbf{u}^{(2)}(k:k+t)$, under an anticipated disturbance scenario $\overline{\mathbf{w}}(k:k+t) := \{\overline{\mathbf{w}}(k), \ldots, \overline{\mathbf{w}}(k+t)\}$, for a system represented by (10), let $\hat{\mathbf{x}}^{(1)}(k+t+1)$ and $\hat{\mathbf{x}}^{(2)}(k+t+1)$ be the obtained controllable state components. The underlying system is called permutationally invariant if

$$\hat{\mathbf{x}}^{(1)}(k+t+1) \equiv \hat{\mathbf{x}}^{(2)}(k+t+1), \tag{11}$$

(a) for all pairs $\mathbf{u}^{(1)}(k:k+t), \mathbf{u}^{(2)}(k:k+t) \in \Upsilon_j(t)$ $(\forall j \in \{1, \ldots, N_{\mathrm{p}}(k)\})$, (b) for all possible disturbance scenarios $\overline{\mathbf{w}}(k:k+t)$, (c) for all initial conditions $\mathbf{x}(k) \in X$ and (d) for all $t \in \{0, \ldots, N_{\mathrm{t}} - 1\}$.

The nice property of the permutational invariance is that, each permutation set $\Upsilon_j(t)$ represents a unique state $\hat{\mathbf{x}} \in \hat{X}$ regardless of the disturbance profile. In other words, the elements (control sequences) of a particular permutation set map the system into one and only one state $\hat{\mathbf{x}} \in \hat{X}$. Note that going forward in time, the value of this state is ambiguous due to the stochastic inputs. The particular state is resolved after the disturbance scenario becomes available. However, the value of the corresponding state is irrelevant, as only the equivalence is utilized in finding a solution to the finite horizon problem.

In order to "quantify" the permutation sets, let us define the pseudo state $\boldsymbol{\xi}(t+1) \in \Phi(t+1)$ over the free search space $S(t)$ by the "scalarization" of control sequences

$$\boldsymbol{\xi}(t+1) := \mathbf{s}_t\Big(\mathbf{u}(k), \ldots, \mathbf{u}(k+t)\Big) \equiv \begin{pmatrix} s_{t,1}\Big(u_1(k), \ldots, u_1(k+t)\Big) \\ \vdots \\ s_{t,m}\Big(u_m(k), \ldots, u_m(k+t)\Big) \end{pmatrix}, \tag{12}$$

using symmetric functions where $\mathbf{s}_t : S(t) \to \Phi(t+1)$. These functions are obtained by the recursive application of commutative binary functions ϕ_j such that

$$\boldsymbol{\xi}(t+1) = \boldsymbol{\phi}\Big(\boldsymbol{\xi}(t), \mathbf{u}(k+t)\Big) \equiv \begin{pmatrix} \xi_1(t+1) \\ \vdots \\ \xi_m(t+1) \end{pmatrix} = \begin{pmatrix} \phi_1(\xi_1(t), u_1(k+t)) \\ \vdots \\ \phi_m(\xi_m(t), u_m(k+t)) \end{pmatrix}, \quad (13)$$

$\boldsymbol{\phi} : \Phi(t) \times U \to \Phi(t+1)$, $t = 0, \ldots, N_{\mathrm{t}} - 1$ subject to $\boldsymbol{\xi}(0) = \mathbf{I}$ where $\mathbf{I} = (I_1, \ldots, I_m)^T$ denotes the vector of identity elements, that is, $\boldsymbol{\phi}(\mathbf{I}, \mathbf{u}) = \mathbf{u}$ for any $\mathbf{u} \in U$.

Definition (12) ensures the permutational invariance on the domain of pseudo states, that is, $\boldsymbol{\xi}(t+1) = \boldsymbol{\xi}_j^*$ for all $\mathbf{u}(k : k+t) \in \Upsilon_j(t)$ and $j = 1, \ldots, N_{\mathrm{p}}(t)$. In what follows, each permutation set $\Upsilon_j(t) \subseteq S(t)$ is mapped into one and only one pseudo state $\boldsymbol{\xi}(t+1) \in \Phi(t+1)$. Furthermore, (13) assures that the defined variable can be utilized as a (pseudo) state over the control domain.

Under permutational invariance, the set of pseudo states $\Phi(t)$ and the controllable domain $\hat{X}$ of the state space are connected by (in general) a surjective map $\mathbf{h} : \Phi(t) \to \hat{X}$. Using the pseudo state, the evolution of the main component becomes

$$\hat{\mathbf{x}}(k+t+1) = \mathbf{h}(\hat{\mathbf{x}}(k), \boldsymbol{\xi}(t+1), \check{\mathbf{x}}(k+t), \mathbf{w}(k+t), k+t), \quad t = 0, \ldots, N_{\mathrm{t}} - 1 \quad (14)$$

Figure 1(b) depicts the map $\mathbf{s}_t : S(t) \to \Phi(t+1)$ indicating the permutation sets considering a simple example. In general $\mathbf{s}_t$ is a surjection (more to one correspondence) by simply considering the fact that two control actions from different permutation sets may lead the system into the same state (see the example in Section 4.1).

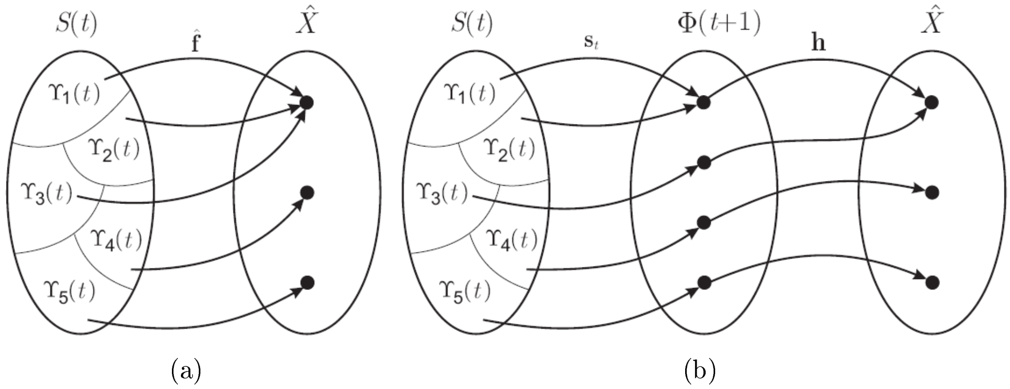

FIGURE 1. The construction of pseudo states assuming 5 permutation sets on free search space at time t. (a) Equivalence (elements of a particular permutation set map the system into one and only one state), (b) the utilization of permutation sets as (pseudo) states.

The aim of the use of pseudo states is to establish means to perform dynamic programming over the control domain. Using the pseudo state space $\Phi(t)$ the evolution of the system can be tracked by the pseudo dynamics (13) which is a system of independent equations coupled by the constraint system. Under tracking, we mean that, using (13) one is able to compute the state (and obtain the step cost information) utilizing the disturbance statistics. A nice property of (13) is that, if the control vector does not include stochastic component (which is usually the case), the pseudo dynamics is deterministic.

Due to permutational invariance, (13) replaces the controllable sub-system's evolution when the disturbance pattern becomes available. Under replacement, we mean that for every $\boldsymbol{\xi}(t) \in \Phi(t)$ there is a unique $\hat{\mathbf{x}}(k+t) \in \hat{X}$ for all $t = 0, \ldots, N_{\mathrm{t}} - 1$. This is simply due to the fact that when the disturbance values are known the system dynamics becomes deterministic and a specific control trajectory generates a unique solution.

The global optimum for the finite horizon problem (6) can be obtained by recursively computing the iterative functional equation over the domain of pseudo states

$$\begin{aligned} V(\boldsymbol{\xi}(t),t) = \min_{\mathbf{u}(k+t)} \underset{\mathbf{w}(k+t)}{E} \Big\{ & c\big(\hat{\mathbf{x}}(k),\, \boldsymbol{\xi}(t),\, \mathbf{u}(k+t),\, \check{\mathbf{x}}(k+t),\, \mathbf{w}(k+t),\, k+t\big) \\ & + V\left(\boldsymbol{\xi}(t+1), t+1\right) \Big\} \end{aligned} \tag{15}$$

subject to (13) and (10). If the control vector does not include stochastic components (thus the evolution in pseudo state space is deterministic) the forward iteration of the corresponding Bellman equation (i.e., forward generation of pseudo states) is preferred.

Unfortunately, the curse of dimensionality is not resolved using the pseudo states as the manipulated state, $\mathbf{y}(t) = \boldsymbol{\xi}(t)$. The computational and storage requirements still grow exponentially with the dimension of the control domain. Equation (15) may provide some improvement for the state space dimensionality problem, since in many applications the dimension of the control domain is smaller than the dimension of the state space.

3.2.2. *Pseudo state aggregation.* The reduction achieved by pseudo states is not usually sufficient in practise. However, due to the "weak coupling" structure of the pseudo dynamics, it is straightforward to propose a solution to the problem by state aggregation. The idea of aggregation is to construct a simpler more traceable problem by reducing the number of states by "combining" them together (see [2]). The resulting aggregated problem has a reduced dimension which gives the possibility to solve it by stochastic dynamic programming. A one dimensional manipulated state variable can be constructed by state aggregation over the pseudo state space $\Phi(t)$:

$$y(t) = \cup\big(\boldsymbol{\xi}(t)\big) \tag{16}$$

where $\cup : \Phi(t) \to Y$ is a suitable aggregation function. Using the one dimensional manipulated state variable, the solution of the finite horizon problem is obtained by solving the following Bellman equation:

$$\begin{aligned} V(y(t),t) = \min_{\mathbf{u}(k+t)} \underset{\mathbf{w}(k+t)}{E} \Big\{ & c\big(\hat{\mathbf{x}}(k),\, y(t),\, \mathbf{u}(k+t),\, \check{\mathbf{x}}(k+t),\, \mathbf{w}(k+t),\, k+t\big) \\ & + V\left(y(t+1), t+1\right) \Big\} \end{aligned} \tag{17}$$

$t = 0, \dots, N_{\mathrm{t}} - 1$. Although the problem is reduced into just one dimension, the solution of (17) still involves a substantial computational effort. The next section gives a detailed overview on the solution of the one dimensional Bellman equation, and the interpretation of the presented approach.

4. **Interpretation.** In practise, the volume or level of water in the reservoirs is chosen to form the controllable domain of the state space for water resource management problems [11, 13, 29, 30, 31, 32]. Using such representation, one important observation can be made: a storage tank acts as a simple integrator. The net inflow into the tank is proportional to the rate of change of the water volume stored in the tank. Using the law of mass conservation, the integrator dynamics are represented as follows:

$$\mathbf{x}(k+t+1) = \mathbf{x}(k) + \sum_{\tau=0}^{t} \mathbf{B}\mathbf{u}(k+\tau) + \sum_{\tau=0}^{t} \mathbf{D}\mathbf{d}(k+\tau) \tag{18}$$

where the state vector $\mathbf{x}(k)$ (m^3) describes the water volumes in the reservoirs, the control variable $\mathbf{u}(k+\tau)$ (m^3) represents the delivered water between tanks (by pumping) and

$\mathbf{d}(k+\tau)$ (m^3) denotes the water inflow or demand by sources/consumers between time $k+\tau$ and $k+\tau+1$.

The integrator dynamics highlight the fact, that, the actual amount of water in the reservoirs at time $k+t+1$ depends on the total delivered water (cumulative control variable) rather than the schedule itself. In other words, given a sequence of delivered fluid volumes, one is able to freely permute the order in the sequence without changing the resulting state. This introduces permutational invariance with respect to delivered water.

In general, the dynamics of a water distribution system are represented by

$$\hat{\mathbf{x}}(k+t+1) = \hat{\mathbf{x}}(k+t) + \Delta t \mathbf{B}\mathbf{q}_{\mathrm{r}}(k+t) + \mathbf{D}\breve{\mathbf{f}}(\breve{\mathbf{x}}(k+t), \mathbf{w}(k+t), k+t) \tag{19a}$$

$$\breve{\mathbf{x}}(k+t+1) = \breve{\mathbf{f}}(\breve{\mathbf{x}}(k+t), \mathbf{w}(k+t), k+t) \tag{19b}$$

subject to

$$\mathbf{q}_{\mathrm{r}}(k+t) = \mathbf{f}_{\mathrm{q}}\big(\hat{\mathbf{x}}(k+t), \mathbf{u}(k+t), \breve{\mathbf{f}}(\breve{\mathbf{x}}(k+t), \mathbf{w}(k+t), k+t)\big) \tag{19c}$$

where $\mathbf{q}_{\mathrm{r}}(k+t) \in \mathbf{R}^{m_{\mathrm{r}}}$ (m^3/h) denotes the vector of reservoir inflows/outflows and Δt is the sampling time. The controlled state domain $\hat{\mathbf{x}}(k+t) \in \hat{X}$ represents the mass conservation law, quantifying the amount of stored water in reservoirs. The uncontrolled component $\breve{\mathbf{x}}(k+t) \in \breve{X}$ includes a nonlinear model of uncertain water demand. Finally, Equation (19c) is the hydraulic model of the network, which describes the dependence of reservoir inflows/outflows on network head, control variable (including pump speeds, valve openings, pump and valve switches) and consumer demand.

Recall that, due to the nonlinear characteristics of the hydraulic model, the system might not have the property of permutational invariance under the control variable $\mathbf{u}$ which is composed by pump speeds, valve openings, etc. To ensure permutational invariance, the construction of the pseudo state space relies on a "dummy" control variable instead. For this purpose, the "dummy" control

$$\mathbf{u}_{\mathrm{r}}(k+t) = \Delta t \mathbf{q}_{\mathrm{r}}(k+t) \tag{20}$$

is defined by the vector of reservoir inflows/outflows which, without doubt, guarantees permutational invariance through the mass conservation law. (It is easy to see, that, using $\mathbf{u}_{\mathrm{r}}(k+t)$, (19a) can be represented in the integrator form (18)).

The pseudo state represents the total water discharged into/out of reservoirs on the time horizon $[k, k+t]$, that is,

$$\boldsymbol{\xi}(t+1) := \mathbf{s}_t(\mathbf{u}_{\mathrm{r}}(k), \dots, \mathbf{u}_{\mathrm{r}}(k+t)) \equiv \begin{pmatrix} \sum_{\tau=0}^{t} u_{\mathrm{r},1}(k+\tau) \\ \vdots \\ \sum_{\tau=0}^{t} u_{\mathrm{r},m_{\mathrm{r}}}(k+\tau) \end{pmatrix} \tag{21}$$

quantifying the cumulative control variable. The pseudo dynamics is composed by a system of independent equations,

$$\boldsymbol{\xi}(t+1) = \boldsymbol{\phi}(\boldsymbol{\xi}(t), \mathbf{u}_{\mathrm{r}}(k+t)) \equiv \begin{pmatrix} \xi_1(t+1) \\ \vdots \\ \xi_{m_{\mathrm{r}}}(t+1) \end{pmatrix} = \begin{pmatrix} \xi_1(t) + u_{\mathrm{r},1}(k+t) \\ \vdots \\ \xi_{m_{\mathrm{r}}}(t) + u_{\mathrm{r},m_{\mathrm{r}}}(k+t) \end{pmatrix} \tag{22}$$

where $\boldsymbol{\xi}(0) = (0, \dots, 0)^T$. The manipulated state variable $y(t)$ is defined by the aggregation of the components of the pseudo state vector,

$$y(t) := \cup(\boldsymbol{\xi}(t)) \equiv \sum_{i=1}^{m_{\mathrm{r}}} \xi_i(t). \tag{23}$$

Putting it together, a sub-optimal control law for water distribution system is derived by solving the Bellman Equation (17) subject to (19) and

$$y(t+1) = y(t) + \sum_{i=1}^{m_{\mathrm{r}}} f_{\mathrm{q},i}(\hat{\mathbf{x}}(k+t), \mathbf{u}(k+t), \breve{\mathbf{f}}(\breve{\mathbf{x}}(k+t), \mathbf{w}(k+t), k+t)) \tag{24a}$$

$$\mathbf{g}(\hat{\mathbf{x}}(k+t), \mathbf{u}(k+t)) \leq 0. \tag{24b}$$

Equation (24b) represents the constraint system (reservoir bounds, control constraints etc.), $y(0) = 0$ and $t = 0, \ldots, N_{\mathrm{t}} - 1$.

4.1. **Simple example.** To ensure understanding, a simple tank filling example is considered in this section. The hydraulic system of interest is depicted in Figure 2. The evolution of the system is described by the following dynamics:

$$x(k+t+1) = x(k+t) + \Delta t q_{\mathrm{r}}(k+t) - w(k+t). \tag{25}$$

The "network" hydraulics are simply

$$q_{\mathrm{r}}(k+t) = q_{\mathrm{p}}(k+t) \tag{26}$$

where $q_{\mathrm{p}}(k+t)$ denotes the pump discharge. In this simple case, the pump discharge can be considered as manipulated variable, since it is uniquely determined by the control action (e.g., pump speed) and vice versa. If the pump discharge is known, the corresponding control action can be derived using the characteristic curves of the pump and the pipe.

Let the initial state of the system be $x(0) = 8\,(\mathrm{m}^3)$. Assume that, the pump has three possible operating states $q_{\mathrm{p}}(k+t) \in \{0,\,1,\,2\}\,(\mathrm{m}^3/\mathrm{h})$ and the state of the system is observed hourly $\Delta t = 1\,(\mathrm{h})$.

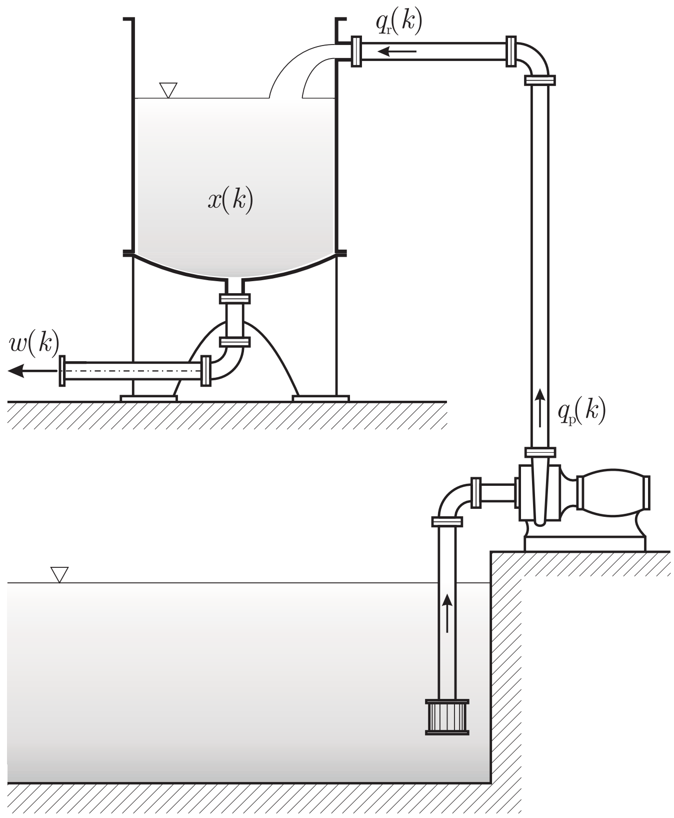

FIGURE 2. Tank filling hydraulic system

Applying the schema detailed in Section 4 to this particular example, the control variable $u(k+t) \equiv u_{\mathrm{r}}(k+t) = \Delta t q_r(k+t)$ denotes the water delivered by the pump between time $k+t$ and $k+t+1$. The pseudo state represents the total delivered water on the time horizon $[k, k+t]$ (i.e., water cumulative),

$$\xi(t+1) = \sum_{\tau=0}^{t} u(k+\tau). \tag{27}$$

The integrator dynamics of the tank is revealed by substituting $\xi(t)$ into (25).

$$x(k+t+1) = x(k) + \xi(t+1) - \sum_{\tau=0}^{t} w(k+\tau) \tag{28}$$

Using a three step lookahead, $N_{\mathrm{t}} = 3$, the free search space – state space mapping chain indicating permutation sets is shown in Figure 3 at time $k = 0$ and $t = 2$.

At this particular time instant, there are $|S(2)| = 3^3 \equiv 27$ possible control actions forming the free search space which is partitioned into $N_{\mathrm{p}}(2) = 10$ different permutation sets. Then, the permutation sets are mapped into 7 pseudo states which are stochastically mapped onto X according to the probability distribution function of $\sum_{\tau=0}^{2} w(0+\tau)$ (Figure 3, top). Recall that each permutation set "represents" one and only one system state. The

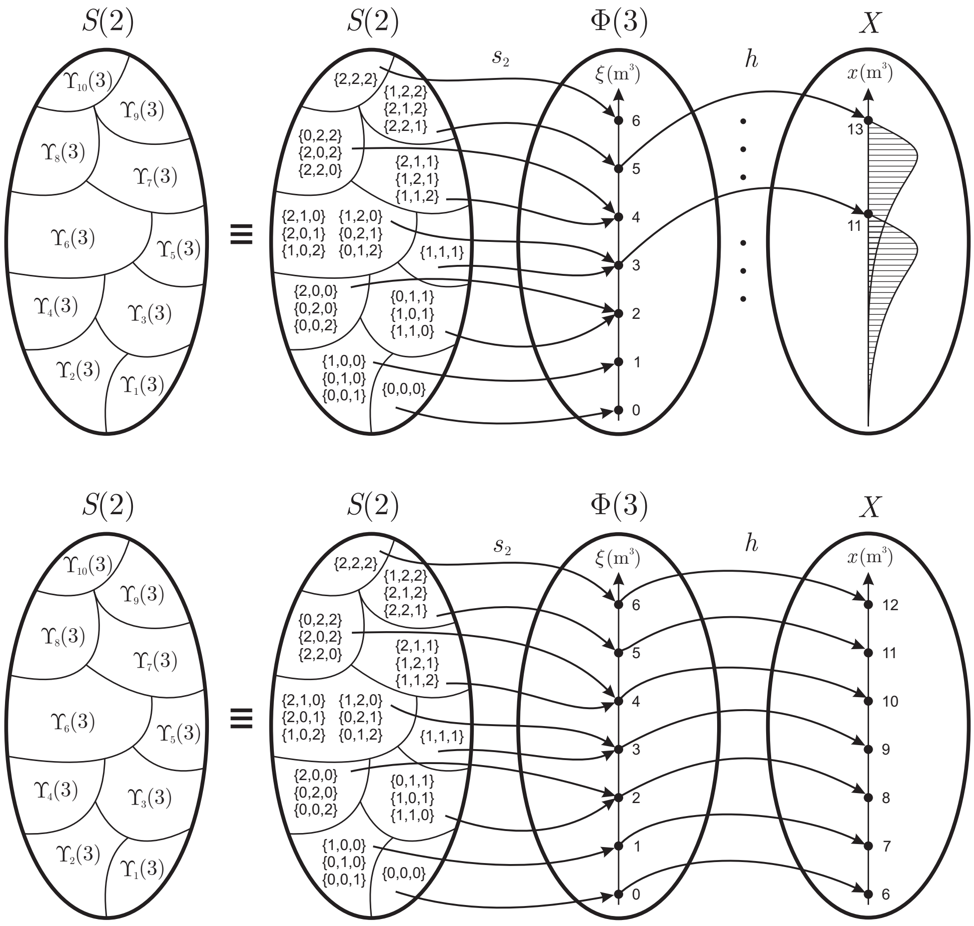

FIGURE 3. Free search space – state space mapping chain at time $k = 0$ and $t = 2$. All possible control sequences $u(0:2)$ forming the search space $S(2)$ are indicated within square braces. The search space is partitioned according to permutation sets $\Upsilon_j(3)$, $j = 1, \ldots, 10$.

corresponding systems states $\mathbf{x}(3)$ are determined unambiguously once the water demand pattern $w(0:2) = \{1, 0, 1\}\,(\mathrm{m}^3)$ becomes available at time $k = 3$ (Figure 3, bottom).

5. **Case Study.** The outlined approach was implemented at the regional water distribution network of Sopron, Hungary. The networks serves the city of Sopron and its surroundings, with a total population of about 120,000. The topology of the water distribution system is shown in Figure 4. The network includes 11 pumping stations, 8 reservoirs and 5 main consumer demands allocated to the corresponding service reservoirs. Each pumping station in the model represents a group of individual pumps running in parallel indicated as "Pump" units in Figure 4.

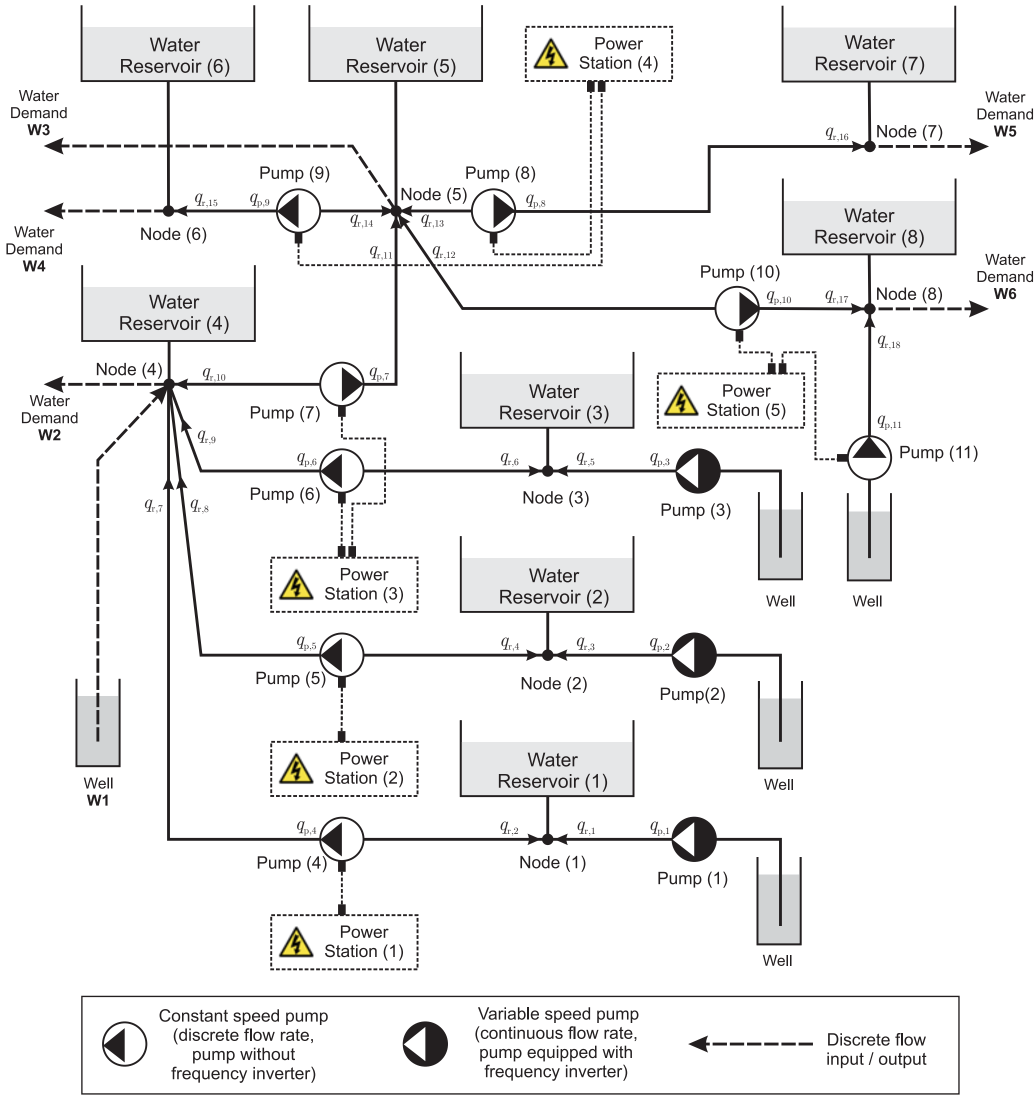

FIGURE 4. The topology of the regional water distribution network of Sopron, Hungary (Schematic)

5.1. **Optimization model.** The hydraulic characteristic of the presented network allows the use of mass-balance modeling. For this particular case, a mass-balance description provides a computationally traceable model for real time control with sufficient accuracy. Mass balance models rely on the following assumptions [4, 33]:

- The water distribution network is a well designed network, where internal network pressure remains within acceptable bounds for allowable service reservoir storage fluctuations;
- The head lift for each pumping station is large compared with the network nodal head changes induced by pump/valve switchings elsewhere in the system;
- The flows a given pumping station will deliver depend on zonal consumer demands, and not on the changes in the network head/flow pattern caused by pump/valve switchings elsewhere in the system.

The outlined assumptions were well confirmed by a full-hydraulic simulator of the network. This allows the pumping stations to be represented by a set of flow rates and corresponding energy consumptions. Hence, there is a unique control action for each discharge of the pumping station, including operation rules (pump speeds, pump switchings) for the individual pumps within the group.

Using the vector of reservoir inflows $\mathbf{q}_{\mathrm{r}}(k) = (q_{\mathrm{r},1}(k), \ldots, q_{\mathrm{r},18}(k))^T$ as indicated in Figure 4 the mass conservation law can be written as follows:

$$\hat{\mathbf{x}}(k+1) = \hat{\mathbf{x}}(k) + \Delta t \mathbf{B} \mathbf{q}_{\mathrm{r}}(k) + \mathbf{D}\mathbf{w}(k) \tag{29}$$

where $\hat{\mathbf{x}}(k) = (\hat{x}_1(k), \ldots, \hat{x}_8(k))^T$ and $\mathbf{w}(k) = (w_1(k), \ldots, w_6(k))^T$. The hydraulic model of the network becomes

$$\mathbf{q}_{\mathrm{r}}(k) = \mathbf{F}\mathbf{q}_{\mathrm{p}}(k) \tag{30}$$

where $\mathbf{q}_{\mathrm{p}}(k) = (q_{\mathrm{p},1}(k), \ldots, q_{\mathrm{p},11}(k))^T$ denotes the pump discharge. The water demand $\mathbf{w}(k)$ was implemented as a random truncated Gaussian noise,

$$w_i(k) \sim \mathcal{N}(\mu_i(k), \sigma_i(k)) \text{ and } w_i^{\min}(k) \leq w_i(k) \leq w_i^{\max}(k) \quad i = 1, \ldots, 6. \tag{31}$$

The parameters of the distributions $\mu_i(k)$, $\sigma_i(k)$, $w_i^{\min}(k)$ and $w_i^{\max}(k)$ were obtained using empirical values (mean, standard deviation, min and max demand) computed from historical records of the water demand. The parameters were considered as periodic with a period of one year.

Finally, the model of the system is formulated as follows

$$\mathbf{x}(k+1) = \mathbf{x}(k) + \Delta t \overbrace{\mathbf{B}\mathbf{F}}^{\widetilde{\mathbf{B}}} \mathbf{q}_{\mathrm{p}}(k) + \mathbf{D}\mathbf{w}(k). \tag{32}$$

Since demand is represented as state noise the state space X is totally controllable (i.e., $X \equiv \hat{X}$). The control variable is defined by pump discharges $\mathbf{u}(k) = \Delta t \mathbf{q}_{\mathrm{p}}(k)$. The system matrices $\widetilde{\mathbf{B}}$ and $\mathbf{D}$ are given in the Appendix. The state of the system is observed on hourly basis $\Delta t = 1\,\mathrm{h}$ which is a good compromise between computational complexity of the model and flexibility of the operation.

The goal is to find an optimal water pump operation policy which minimizes the cost of the electric energy required by pumping while satisfying the water demand subject to reservoir constraints. The cost of energy has the following form:

$$J(\mathbf{u}(k)) = \lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} \left(\sum_{i=1}^{3} Q_i(u_i(k))) + \mathbf{a}_1^T \mathbf{u}_{4:11}(k) + \mathbf{a}_2^T \mathbf{u}_{4:11}^2(k) + \mathbf{a}_3^T \mathbf{u}_{4:11}^3(k) \right) e_{\mathrm{c}}(k), \tag{33}$$

which must be minimized, subject to reservoir constraints $\mathbf{x}^{\min} \le \mathbf{x}(k) \le \mathbf{x}^{\max}$ and control constraints

$$\mathbf{u}(k) = \Delta t \begin{pmatrix} q_{\mathrm{p},1}(k) \in [0,\ 270] \\ q_{\mathrm{p},2}(k) \in [0,\ 250] \\ q_{\mathrm{p},3}(k) \in [0,\ 460] \\ q_{\mathrm{p},4}(k) \in \{0, 150, 360\} \\ q_{\mathrm{p},5}(k) \in \{0, 110\} \\ q_{\mathrm{p},6}(k) \in \{0, 270, 500\} \\ q_{\mathrm{p},7}(k) \in \{0, 550\} \\ q_{\mathrm{p},8}(k) \in \{0, 66, 116\} \\ q_{\mathrm{p},9}(k) \in \{0, 66, 118, 148\} \\ q_{\mathrm{p},10}(k) \in \{0, 90, 114\} \\ q_{\mathrm{p},11}(k) \in \{0, 72, 130\} \end{pmatrix}. \tag{34}$$

where $\mathbf{u}^p_{4:11}(k) = (u^p_4(k), \ldots, u^p_{11}(k))^T$, $(p = 2, 3)$ and $Q_i(.)$ are non-polynomial nonlinear functions.

The energy tariff $e_{\mathrm{c}}(k)$ varies during the day, involving peak (price of electric energy is high) and off peak periods (price of electric energy is low). The tariff has the following pattern: 1 (Unit) {[0h-7h), [13h-17h), [20h-24h)} and 1.25 (Unit) {[7h-13h), [17h-20h)}. Unit denotes the price of the electric energy in terms of a given currency (e.g., EUR/kWh, USD/kWh).

The complete problem definition (including water demand data, cost function coefficients, etc.) can be downloaded from [34].

5.2. **Solution strategy.** The control variable of the outlined optimization model is not subject to uncertainties. Consequently, the state transition in the space of manipulated states (24a) is deterministic, and the forward iteration of the corresponding Bellman Equation (17) is used.

The continuous domain of the manipulated state variable was approximated by a finite set of states by discretization of the finite interval $[0, y_{\max}(t)]$ into N_{y} levels. Furthermore, the continuous probability distribution functions of the random inputs were approximated by bounded finite-sample distributions containing N_{s} samples $(\mathbf{w}_1(k+t), \ldots, \mathbf{w}_{N_{\mathrm{s}}}(k+t))$ for all $t = 0, \ldots, N_{\mathrm{t}} - 1$.

5.2.1. *Numerical approximation of the step cost.* Any local constraints on the control variable $\mathbf{u}(k+t)$ can be considered at the moment of choice of a particular control action. On the other hand, the handling of local constraints on the controllable state component (reservoirs) is incorporated within the transition cost by penalty functions, due to the stochastic nature of the problem. Consequently, applying a control action $\mathbf{u}(k+t)$ generates two different costs: $c_u(.)$ is the energy cost of the particular action and $c_p(.)$ is the cost of constraint violation (penalty) on the controllable state component. While the energy cost is obvious (see (33)) one has to pay attention to how the constraint handling is expressed in terms of penalty.

In this case study, the relative constraint violation was used to express the penalty, that is, the percent of overshoot on each reservoir is determined and aggregated to form the total violation cost. More precisely, the penalty on reservoir j is derived as follows:

$$c_p^j = \begin{cases} 0 & \text{if} \quad x_j^{\min} \le x_j(k+t) \le x_j^{\max} \\ (x_j^{\min} - x_j(k+t))/x_j^{\min} & \text{if} \quad x_j(k+t) < x_j^{\min} \\ (x_j(k+t) - x_j^{\max})/x_j^{\max} & \text{if} \quad x_j(k+t) > x_j^{\max} \end{cases}, \quad j = 1, \ldots, 8 \tag{35}$$

The energy and penalty costs representing two different, conflicting, non-commeasurable objectives (objectives are defined on different units) can be calculated for each realization of the stochastic inputs $(\mathbf{w}_1(k+t), \ldots, \mathbf{w}_{N_s}(k+t))$ at time t. The cost vector for a particular control action becomes

$$\begin{pmatrix} E_{\mathbf{w}(k+t)}\{c_u(\ldots, \mathbf{w}(k+t))\} \\ E_{\mathbf{w}(k+t)}\{c_p(\ldots, \mathbf{w}(k+t))\} \end{pmatrix} \approx \frac{1}{N_s} \begin{pmatrix} \sum_{i=1}^{N_s} c_{u,i}(\ldots, \mathbf{w}_i(k+t)) \\ \frac{1}{8}\sum_{j=1}^{8}\sum_{i=1}^{N_s} c_{p,i}^{j}(\ldots, \mathbf{w}_i(k+t)) \end{pmatrix} \tag{36}$$

To derive the (scalar) step cost $c(.)$, the non-commeasurability issue must be resolved, requiring the objectives to be defined on the same units. This needs the identification of a set of (Pareto-optimal) control actions which all map into a particular node. Once this set is obtained, the costs corresponding to two different objectives are normalized to interval $[0, 1]$ and the transition cost of a particular control action is computed based on linear weighting,

$$E\{c(.)\} = ||E\{c_u(.)\}|| + \omega||E\{c_p(.)\}|| \tag{37}$$

where $||E\{c_u(.)\}||$ and $||E\{c_p(.)\}||$ are normalized expected costs on interval $[0, 1]$. The weighting factor $\omega \in [0, \infty)$ reflects the relative importance of the constraint satisfaction against energy cost. The higher the weighting factor, the more important it is to satisfy the constraints on the controllable state component.

5.2.2. *Algorithm implementation.* At this point, the optimization problem is well defined, and the introduced dynamic programming framework can be implemented to derive a solution. The importance of the implementation in practise is unquestionable. Therefore, the authors' implementation is described in detail in this section.

The decision making at each stage is performed in two steps. At first, a control action is derived which performs the state transition between two particular nodes $y_a(t) \to y_b(t+1)$ with minimal cost (filtering step). The corresponding problem is formulated as follows: find a control action $\mathbf{u}(k+t) \in U$ which minimizes

$$\underset{\mathbf{w}(k+t)}{E}\left\{c\Big(\mathbf{x}(k),\, y_a(t),\, \mathbf{u}(k+t),\, \mathbf{w}(k+t),\, k+t\Big)\right\} \tag{38a}$$

subject to (32) and

$$\sum_{i=1}^{m} u_i(k+t) = y_b(t+1) - y_a(t). \tag{38b}$$

Secondly, using the outcome of the filtering step, the optimal control decision is derived by which a particular node $y_b(t+1)$ is "optimally" reached from the domain of manipulated states $y(t) \in Y$ at time t. The decision making phases are illustrated in Figure 5.

At the filtering stage, the solution for (38) is obtained by dynamic programming. The underlying problem is decomposed spatially rather than temporally. Decomposition requires that each component of the control vector contributes individually to the step cost.

The energy cost can be computed individually in each dimension. However, the computation of penalties requires two or more control components simultaneously according to the connectivity matrix $\widetilde{\mathbf{B}}$. This prevents the application of dynamic programming on the entire control domain U since it cannot be spatially decomposed. Taking into account the structure of the connectivity matrix, the decomposition is possible only on the continuous domain since each continuous pump is connected to only one reservoir; thus, the state penalties can be individually computed when the discrete control actions are known.

This gives the inspiration to integrate enumeration with dynamic programming in order to compute the step cost. First, the discrete domain of the decision variable is enumerated, calculating partial expected costs for all possible control actions. Then, the decisions on

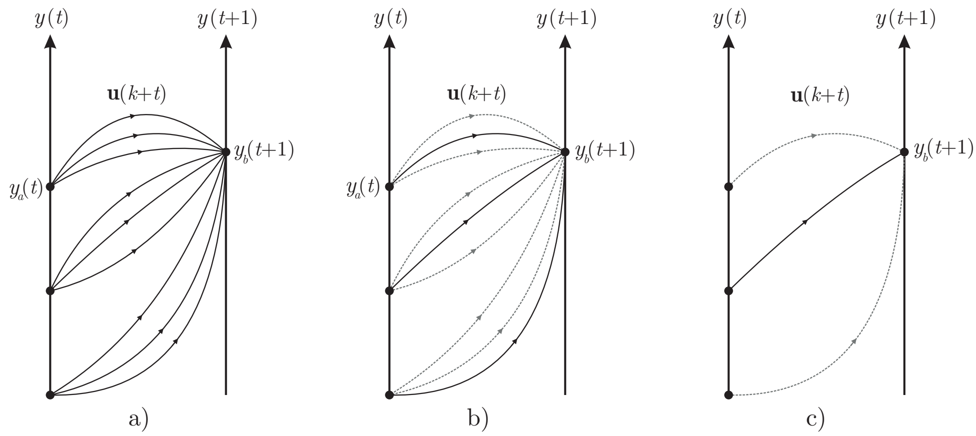

FIGURE 5. The solution stages of the Bellman Equation (17). a) Initialization of the state transition. There exists many different control alternatives between particular initial and target state pairs. b) Filtering step. Internal optimization derives the unique control actions (highlighted by solid black lines) for state transition between particular state pairs. c) Identification of the optimal control action (solid black line) for a particular target state.

the continuous pump deliveries are obtained by dynamic programming utilizing permutational invariance.

Let the control variable be represented by $\mathbf{u}(k+t) = (\mathbf{u}_{1:3}(k+t), \mathbf{u}_{4:11}(k+t))^T$ where $\mathbf{u}_{1:3}(k+t) = (u_1(k+t), \ldots, u_3(k+t))^T$ is the domain of continuous decisions and $\mathbf{u}_{4:11}(k+t) = (u_4(k+t), \ldots, u_{11}(k+t))^T$ is the domain of discrete decisions respectively. Let U_{d} denote the set of all possible control candidates on the discrete domain; thus, U_{d} is simply the cartesian product of the discrete control sub-domains $U_{\mathrm{d}} = U_4 \times U_5 \times, \ldots, \times U_{11}$. In this particular problem the discrete domain includes $3^5 \cdot 2^2 \cdot 4 = 3888$ decisions (see (34)).

The solution for (38) is obtained by solving the Bellman equation

$$V(\eta(i+1), i+1) = \min_{u_{i+1}(k+t)} \left\{ c\Big(\mathbf{x}(k),\, y(t),\, \eta(i),\, u_{i+1}(k+t),\, k+t\Big) + V(\eta(i), i) \right\} \tag{39a}$$

subject to

$$\eta(i+1) = \eta(i) + u_{i+1}(k+t), \quad i = 0, \ldots, 2. \tag{39b}$$

The state $\eta(i)$ is initialized by the enumeration of the discrete domain as follows

$$\eta(0) = \left\{ \sum_{i=4}^{11} u_i(k+t) \mid \forall\, \mathbf{u}_{4:11}(k+t) \in U_{\mathrm{d}} \right\}. \tag{39c}$$

Note that (39) is deterministic since the continuous control domain affects only the first three states which are not subject to stochastic events (see disturbance matrix $\mathbf{D}$).

5.3. **Results.** The presented method was implemented under MATLAB R2011b and executed on a computer equipped with Intel Core i7 CPU (2.93 GHz) using parallelization (the computation tasks were distributed to 8 cores). In order to ensure the reproducibility of the presented results, the implemented algorithm is available to download from [34].

Table 1 summarizes the algorithm's parameter setup which was derived by experimentation in order to meet the time limit on the determination of numerical solution of the finite horizon problem. This limit was set to 6 minutes considering that the system's state is observed on an hourly basis.

TABLE 1. Parameter settings

Parameter name	Notation	Value
Manipulated state $y(t)$ discretization level	N_{y}	150
Internal state $\eta(i)$ discretization level	N_{η}	100
Water demand finite-sample distribution, sample size	N_{s}	1000
Number of lookahead steps (prediction horizon)	N_{t}	10

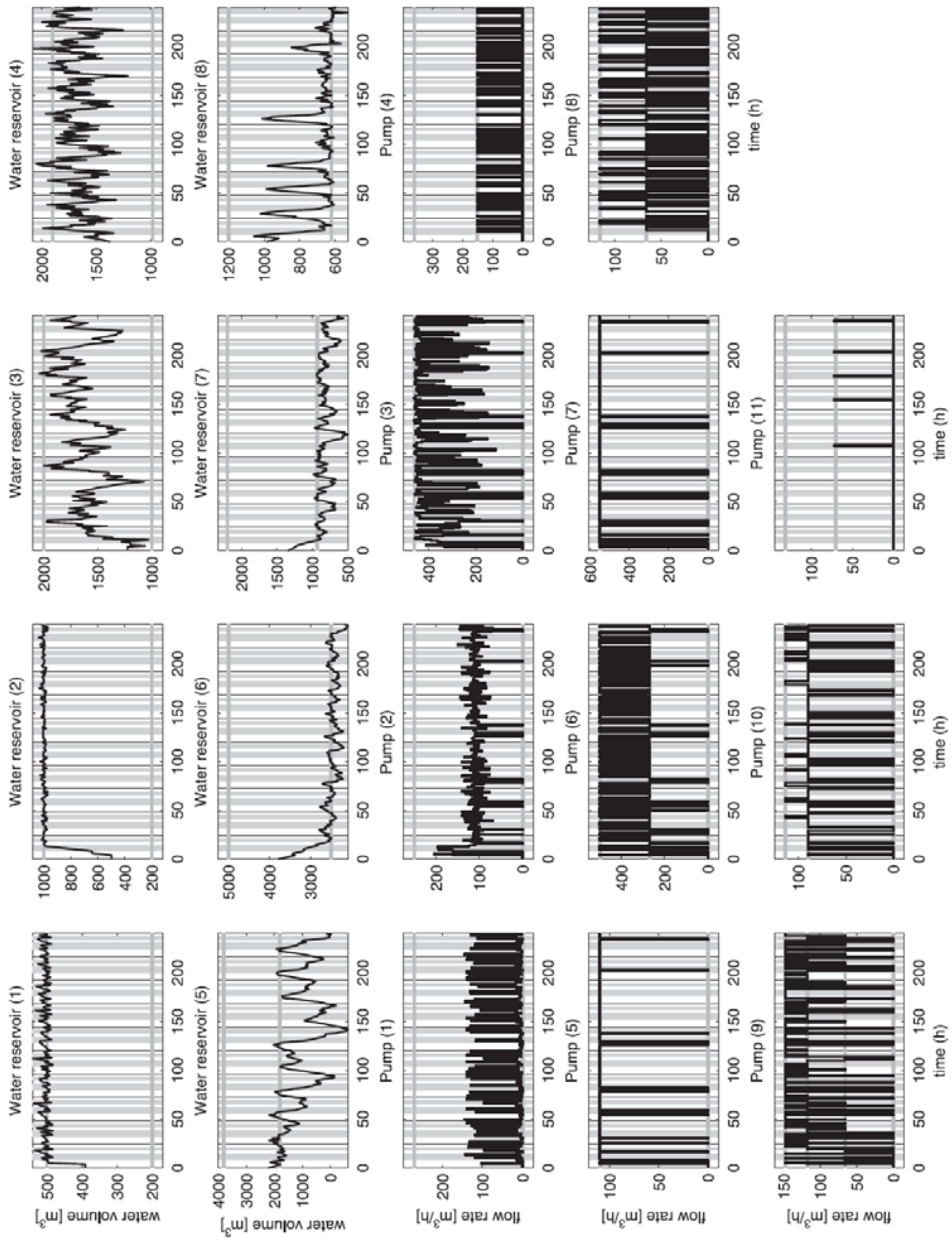

FIGURE 6. 10 day sub-optimal pump control policy ($\omega = 1$). Peak charging period is gray shaded while off-peak periods are uncolored. Reservoir upper and lower bounds are indicated by solid gray lines.

Using the parameter settings presented in Table 1 the solution of the finite horizon problem takes 352.08 (sec) on average in MATLAB. However, the authors strongly believe that using a lower level programming language (C++ for example) more efficient implementation of the numerical solver can be constructed and the computation time can be significantly reduced.

Some experiments are presented to illustrate the performance of the control system with respect to the objective weighting factor ω. The weighting factor is used in the calculation of the step cost (37); thus, it has a great influence on the choice of the optimal control.

Two experiments were carried out setting $\omega = 1$ (pumping cost and constraint satisfaction have the same importance) and $\omega = 10$ (high priority on constraint satisfaction). The

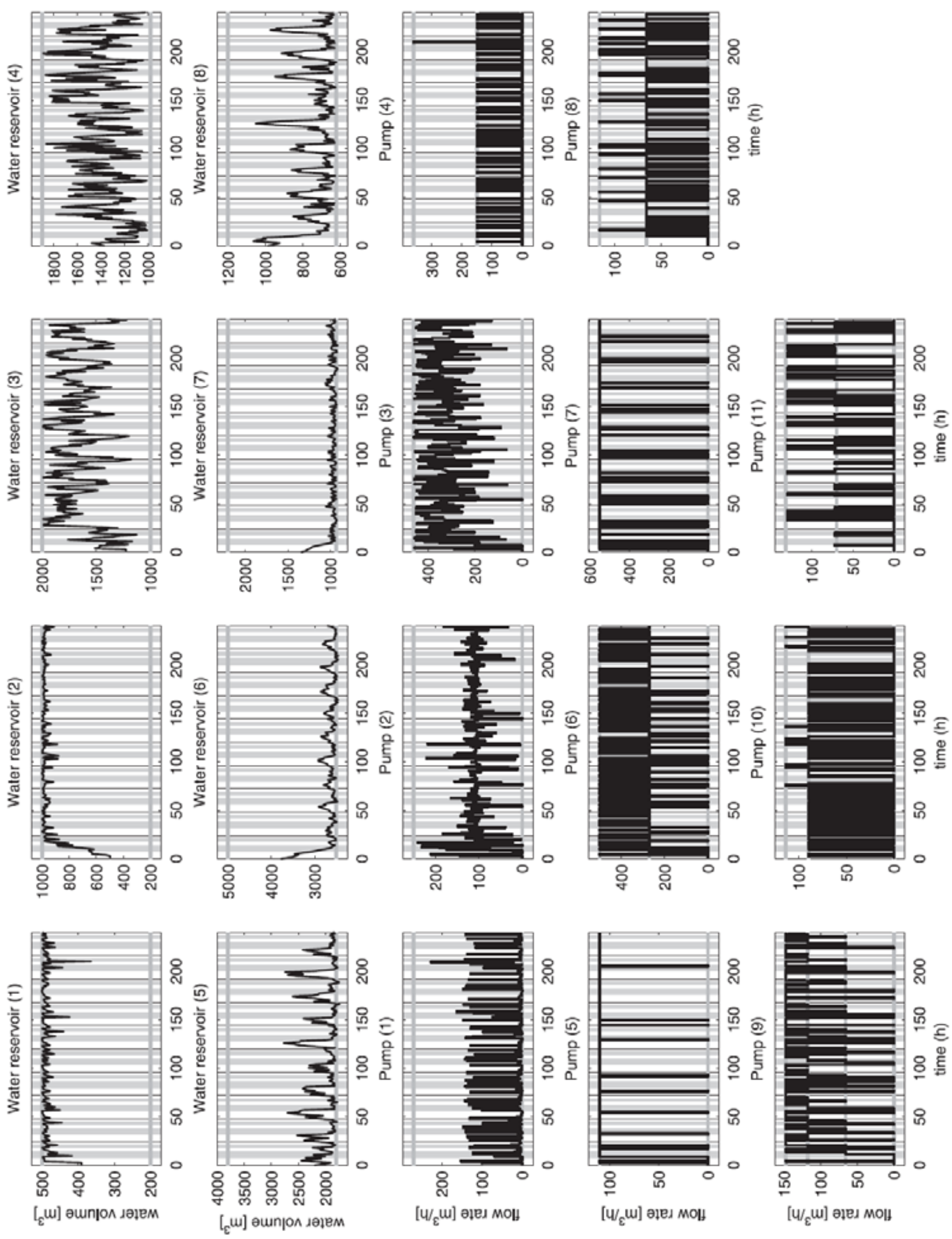

FIGURE 7. 10 day sub-optimal pump control policy ($\omega = 10$). Peak charging period is gray shaded while off-peak periods are uncolored. Reservoir upper and lower bounds are indicated by solid gray lines.

obtained results are presented for 10 days of operation using a randomly chosen feasible state $\mathbf{x}(0) = (391, 498, 1226, 1370, 2210, 3783, 1345, 989)^T$ (m^3) as initial condition. Figure 6 shows the control strategy and the evolution of reservoirs for $\omega = 1$. With this weighting, the constraints on reservoirs (especially on reservoirs 5-7) are heavily violated. Figure 7 shows the control strategy and the evolution of reservoirs when constraint satisfaction is emphasized ($\omega = 10$). In this case, the water demand is satisfied without significant constraint violations.

However, under the two different weighings, the characteristics of the reservoir trajectories exhibit similarities. As expected, in both cases the reservoir filling is performed when the electrical tariff is low (off peak period is uncolored in figures). This confirms intuitively the cost efficiency of the control.

The corresponding average cost per stage (including only energy costs) is 319.82 units ($\omega = 1$) and 319.19 units ($\omega = 10$) respectively. The results show that the choice on the objective weighting factor does not convey any particular advantage for the control strategy with respect to pumping costs under the detailed system model. It seems that a sensible weighting ($\omega \in [0, \infty)$) can improve the control objective with respect to constraints while having no effect on the pumping costs. As a consequence, the operational goals of the water distribution network can be achieved (satisfying the water demand subject to system requirements with minimal cost) without experimental parameter tuning on ω.

6. **Discussion.** In the presented case study, a simple mass-balance model was used to capture the network dynamics. In this particular network, mass balance equations provide a computationally traceable model for real time control with sufficient accuracy.

It must be emphasized that the introduced method is not restricted to simple, linear mass balance equations. In the case of nonlinearity, the state transition (24a) is processed numerically. However, the introduced method is computationally intensive. It relies on the network's hydraulic model which is evaluated frequently. If a single evaluation of the hydraulic model is "expensive", the associated computational requirements become overwhelming, making the underlying model unsuitable for real time control when used in association with optimization. For example, coupled hydraulic (or full hydraulic) models require the solution of a set of nonlinear quasi-steady-state hydraulic equations which is processed iteratively. Since most urban water distribution networks comprise thousands of interconnected pipes, the numerical solution for such system induces a very heavy computational burden under the open loop feedback control framework.

One possibility for addressing this problem is to render the hydraulic model in a computationally efficient form by means of function approximation, such as simplified hydraulics (if applicable) [4, 35], or artificial neural networks [36, 37]. Nevertheless, the authors promote the utilization of parallel computing architecture in order to keep the computing time within reasonable limits. The authors strongly believe that parallelization can make accurate coupled hydraulic modeling suitable for real-time operational control.

7. **Comparison.** Operational optimization of water distribution systems is highly challenging task. Difficulties are induced by (1) nonlinearity (cost function, water system dynamics), (2) high dimensionality (numerous water tanks, large number of consumption nodes, pumps of different types etc.) and (3) uncertainty (water demand).

The methods available in the literature can be classified into three main categories:

1) Handling nonlinearity and dimensionality, ignoring uncertainty
2) Handling nonlinearity and uncertainty, ignoring dimensionality
3) Handling dimensionality and uncertainty, ignoring nonlinearity

The first category is the largest by far, confirming the well-known fact that demand uncertainty is almost completely ignored in the corresponding literature. The vast majority of published methods rely on the perfect foreknowledge of the future water demand profile. This profile is provided by some forecasting method describing stochastic events by their nominal values. The continuous research in this direction has resulted in numerous mathematical tools for optimal control resolving nonlinearity and dimensionality issues. The most significant research projects in this field (POWADIMA and WATERNET) which have proven results on real networks utilize soft computing methods [36, 38, 37, 39, 40] and nonlinear programming [25]. By ignoring uncertainty, it is implicitly assumed that the optimal decision obtained by a deterministic model is feasible under a range of uncertainties, and it is not sensitive to unknown future conditions (i.e., water demand perturbations). Correspondingly, the robustness of operations has been a neglected area in the publications [41]. In fact, robust operation cannot be guaranteed by deterministic planning, due to the fact that future water demand trajectory realizations cannot be predicted with high accuracy by a sequence of nominal values if the water demand is subject to very high degree of uncertainty.

The second category relies on Stochastic Programming utilizing the framework of Controlled Finite Markov Chains (CFMC). The field of CFMC and the related stage-wise optimization techniques are commonly referred to as Markov decision processes (MDP). The MDP approach can be seen as an extension of model predictive control or model-based (optimal) process control, providing an excellent framework for the analysis of nonlinear stochastic processes. With the breakthrough of increased and reasonably priced computing power and memory resources, the application of CFMC/MDP techniques is becoming feasible. However, like SDP, CFMC is subject to the curse of dimensionality which prevents direct application on large scale networks. Consequently, the feasibility of CFMC based process control design is restricted to small scale (hypothetical) water distribution systems [26, 42, 43, 44].

The third category is the field of Linear Programming applied for water distribution network operational optimization. It works under the assumption that the optimal control problem can be formulated as a linear program. While the linearity of system dynamics (i.e., mass-balance modeling) is feasible under reasonable assumptions [4], the assumption that the cost function is a single-objective linear function (or can be well approximated by a single linear function) is unrealistic in most of the cases (e.g., cost is represented by multiple conflicting objectives). If applicable linear programming is extremely powerful in handling dimensionality and uncertainty [10, 33]. However, full linearity is a very strong restriction for real life applications.

The presented method in this study has clear advantages over the outlined techniques because it takes nonlinearity, dimensionality as well as uncertainty fully into account. According to the best of the author's knowledge, this has not been successfully addressed by any of the previous publications.

A direct comparison with methods from the outlined categories would require very precise planning because the outcome is strongly affected by the definition of the underlying problem. If this problem is defined as a high dimensional model, involving significant nonlinearities and uncertainties the presented approach would be favorable, since deterministic planning would encounter serious issues with robustness and feasibility and MDP is simply unsuitable as is linear programming. Nevertheless, the dominance of MDP approaches is unquestionable in the case of low dimensionality and the presented method would be clearly inferior to linear programming in a linear framework.

On the other hand, handling of uncertainty results in the increase of the optimal value as compared with the deterministic case. This is well supported by a comparison with the authors' previous study. In [35] a genetic algorithm was introduced and benchmarked on the Sopron network ignoring uncertainty. With a perfect knowledge on future demand, the genetic algorithm resulted in $7,216.96/24 = 300.7$ units cost on average (see Table 9 in [35]), in contrast to 319.19 units achieved by the OLFC method handling uncertainty. Although, the results are not directly comparable due to the differences in problem statement, it is worth to point out that the smaller costs were achieved by deterministic planning under a stricter constraint system.

The presented OLFC approach is suitable for deterministic planning ignoring uncertainty. Indeed, such framework would provide a basis for a fair comparison with the methods of the first category. However, deterministic problems are not in our interest.

8. **Summary and Conclusions.** The development and application of a sub-optimal (closed loop) control tool for nonlinear stochastic systems is presented and applied for full scale water distribution networks under demand uncertainties. The optimization goal was to minimize the pumping costs while satisfying the water demand and other system requirements. To achieve this goal, an open-loop feedback control scheme was implemented utilizing stochastic dynamic programming.

The contribution of this paper is to provide solution to the curse of dimensionality on the underlying finite horizon problems. Exploiting permutational symmetries, a pseudo state space is defined over the control domain, and a one dimensional problem is formulated applying state aggregation in the pseudo state space. Consequently, the optimal control solution of a high dimensional stochastic dynamic system was obtained by the application of SDP on the associated one dimensional problem. Since SDP is executed in a pseudo state space, the system states need not be manipulated and can be composed by dynamical components other than reservoirs (e.g., disturbance model). Consequently, the proposed optimal control tool is capable of dealing with very large systems, allowing the enlargement of the state space by external dynamical components. For example, it allows the use of forecasting models of any order.

The presented method in this study has clear advantages over the existing techniques because it takes nonlinearity, dimensionality as well as uncertainty fully into account. According to the best of the author's knowledge, this has not been successfully addressed by any of the previous publications.

The presented approach has been tested and applied to real time management of the regional water distribution network of the city of Sopron in Hungary. Although the presented case study is preliminary, it clearly highlights the good potential of the proposed approach and suggest further studies.

Acknowledgment. The presented work has been carried out within the OPUS project (Project ID: 138349) funded by the Academy of Finland. This work was partially supported by the scientific program of the 'Development of quality-oriented and harmonized R+D+I strategy and functional model at BME' project, New Hungary Development Plan (Project ID: TÁMOP-4.2.1/B-09/1/KMR-2010-0002).

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Appendix (System matrices).

$$\widetilde{\mathbf{B}} = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$