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DELAY-DEPENDENT ROBUST STABILITY ANALYSIS FOR RECURRENT NEURAL NETWORKS WITH TIME-VARYING DELAY

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Abstract. *In this paper, the global asymptotic stability problem is dealt with for a class of recurrent neural networks (RNNs) with time-varying delays. The time delays are not necessarily differentiable and the uncertainties are assumed to be time-varying but norm-bounded. The activation functions are assumed to be neither monotonic, nor differentiable, nor bounded. By constructing the Lyapunov-Krasovskii functional and integral inequality approach, an improved delay-dependent stability criterion for delay RNNs is established in terms of linear matrix inequalities (LMIs). It is shown that the obtained criterion can provide less conservative results than some existing ones. Numerical examples are given to demonstrate the applicability of the proposed approach.*

Keywords: Recurrent neural networks (RNNs), Time-varying delay systems, Linear matrix inequality (LMI), Integral inequality approach (IIA)

1. **Introduction.** In recent years, neural networks (NNs) have attracted much attention in research and have found successful applications in many areas such as pattern recognition, image processing, association, optimization problems [7,18]. One of the important research topics is the globally asymptotic stability of the neural network models. However, in the implementation of artificial NNs, time delays are unavoidable due to the finite switching speed of amplifiers. It has been shown that the existence of time delays in NNs may lead to oscillation, divergence or instability. Recently, the stability issue of NNs with time delays has been extensively studied [1,3-23,25,27-51].

Recently, there has been increasing interest in the study of recurrent neural networks (RNNs) since RNNs have found extensive applications in solving some optimization problems, associative memory, and classification of patterns, reconstruction of moving images, and other areas [7]. In the implementation of artificial NNs, time delays are unavoidable due to the finite switching speed of amplifiers. It has been shown that the existence of time delays in RNNs and NNs may lead to oscillation, divergence or instability. Therefore, the stability issue of neural networks with time delays has recently drawn particular research interests [3,5,15,19,27,28,31,38,43,45,47]. Depending on whether the stability criterion itself contains the size of delay, criteria for RNNs can be classified into two categories, namely delay-independent criteria [33,38,39,48,51] and delay-dependent criteria [6,8,9,11,20,22,29,44,46,49]. Usually the latter is less conservative when the value of delay is small. In the Lyapunov-based delay-dependent results, the slow-varying constraints $\dot{h}(t) < 1$ are usually imposed on the time-varying delays [1,7,31,41,48]. These constraints will be relaxed and delay-dependent results will be proposed in this paper.

It is well known that a suitable form of Lyapunov-Krasovskii functional may lead to less conservative delay-dependent stability criterion for recurrent neural networks with time delays. Among various stability methods, a notable one is the free-weighting matrix method in [9-12], which is very effective to tackle the delay-dependent stability problem for time delay NNs since neither bounding techniques on some cross-product terms nor model transformations are involved. However, free-weighting matrix method is too complicated and it needs heavy computational burden. In order to overcome this limitation, recently, new convex combination conditions for the stability of the systems with time-varying delays are established without free-weighting matrices [25]. One natural question is how to simplify existing stability results using matrix variables as less as possible while maintaining the effectiveness of the stability conditions. Very recently, an integral inequality matrix approach (IIA) derived in [24,25] has been employed to derive some less conservative stability criteria. As a matter of fact, there exists conservativeness inevitably. This motivates us to establish a new delay-dependent condition and further reduce the conservatism.

However, as far as we know, in most existing literature, the above analyses have been treated separately. Up to now, the robust stability analysis for uncertain recurrent neural networks with time-varying delays has not been fully studied, which is still open. In this paper, we construct a new differential equation model for the uncertain recurrent neural networks with time-varying delays. By employing Lyapunov-Krasovskii functional and integral inequality approach, some less conservative delay-dependent stability criteria have been derived. Because we have carefully considered the ranges for the time-varying delays, our criteria are applicable to both fast and slow time-varying delays. The stability criteria derived turn out to be less conservative with fewer matrix variables than some recently reported ones. Our stability criteria are in LMI forms and can be easily checked in practice. Finally, numerical examples are also given to demonstrate the effectiveness and advantages of our analysis.

2. **Stability Analysis.** Consider the following recurrent neural network with time-varying delays and parameter uncertainties:

$$\dot{u}(t) = -(C + \Delta C(t))u(t) + (A + \Delta A(t))f(u(t)) + (B + \Delta B(t))f(u(t - h(t))) + J \quad (1)$$

where $u(t) = [u_1(t), \ \cdots, \ u_n(t)]^T \in R^n$ is the state vector with the n neurons; $f(u(t)) = [f_1(u_1(t)), \cdots, f_n(u_n(t))]^T \in R^n$ is called an activation function indicating how the j-th neuron responses to its input; $C = \mathrm{diag}(c_1, \ldots, c_n)$ is a diagonal matrix with each $c_i > 0$ controlling the rate with which the i-th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs; $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$, $A = (a_{ij})_{n\times n}$, and $B = (b_{ij})_{n\times n}$ are the feedback and the delayed feedback matrix, respectively; $J = [J_1, \cdots, J_n]^T \in R^n$ is a constant input vector, $\Delta A(t)$, $\Delta B(t)$, and $\Delta C(t)$ are unknown matrices that represent the time-varying parameter uncertainties and $h(t)$ is the time delay of the system satisfies

$$0 \le h(t) \le h, \quad \dot{h}(t) \le h_d, \quad (2)$$

where h and h_d are some positive constants.

In this paper, the neuron activation functions are assumed to be bounded and satisfy the following assumption.

Assumption 2.1. It is assumed that each of the activation functions f_j $(j = 1, 2, \ldots, n)$ possesses the following condition

$$\gamma_i \le \frac{f_i(\varsigma_1) - f_i(\varsigma_2)}{\varsigma_1 - \varsigma_2} \le k_i, \quad \varsigma_1 \ne \varsigma_2 \in R, \quad i = 1, 2, \ldots, n, \quad (3)$$

where γ_i and k_i are known constant scalars.

Remark 2.1. *If the neuron activation functions satisfy Assumption 2.1, then they satisfy*

$$|f_i(\varsigma_1) - f_i(\varsigma_2)| \leq \max\{|\gamma_i|, |k_i|\} |\varsigma_1 - \varsigma_2| = \rho_i |\varsigma_1 - \varsigma_2|, \quad i = 1, 2, \ldots, n. \tag{4}$$

It is noted that the assumption condition (4) has been investigated in many research papers [20,44]. However, we shall point out that this assumption is much *strong* and may lead to conservative conditions for the delay-dependent stability analysis of delayed neural networks. For example, if $\gamma_i < k_i < 0$, then the delay-dependent stability result obtained by using (3) is generally less conservative than the one obtained by using (4). This will be shown via numerical examples in Section 4 in this paper.

Next, the equilibrium point $u^* = [u_1^*, \ \cdots, \ u_n^*]^T$ of system (1) is shifted to the origin through the transformation $x(t) = u(t) - u^*$, then system (1) can be equivalently written as the following system

$$\dot{x}(t) = -(C + \Delta C(t))x(t) + (A + \Delta A(t))g(x(t)) + (B + \Delta B(t))g(x(t - h(t))) \tag{5}$$

where $x(\cdot) = [x_1(\cdot), \ \cdots, \ x_n(\cdot)]^T, g(x(\cdot)) = [g_1(x_1(\cdot)), \cdots, g_n(x_n(\cdot))]^T$, and $g_i(x_i(\cdot)) = f_i(x_i(\cdot) + u_i^*) - f_i(u_i^*)$, $i = 1, 2, \ldots, n$.

The matrices $\Delta C(t)$, $\Delta A(t)$ and $\Delta B(t)$ are the uncertainties of the system and have the form

$$\left[\begin{array}{ccc} \Delta C(t) & \Delta A(t) & \Delta B(t) \end{array}\right] = DF(t) \left[\begin{array}{ccc} E_c & E_a & E_b \end{array}\right] \tag{6}$$

where D, E_c, E_a, and E_b are known constant real matrices with appropriate dimensions and $F(t)$ is an unknown matrix function with Lebesgue-measurable elements bounded by

$$F^T(t)F(t) \leq I, \quad \forall t, \tag{7}$$

where I is an appropriately dimensioned identity matrix.

In the following, we will develop some practically computable stability criteria for the system described (5). The following lemmas are useful in deriving the criteria.

First, we introduce the following integral inequality approach (IIA), which be used in the proof of ours.

Lemma 2.1. [24]. *For any positive semi-definite matrices*

$$X = \left[\begin{array}{ccc} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{array}\right] \geq 0 \tag{8a}$$

the following integral inequality holds

$$\begin{aligned} &-\int_{t-h(t)}^{t} \dot{x}^T(s)\, X_{33}\, \dot{x}(s) ds \leq \\ &\int_{t-h(t)}^{t} \left[\begin{array}{ccc} x^T(t) & x^T(t-h(t)) & \dot{x}^T(s) \end{array}\right] \left[\begin{array}{ccc} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & 0 \end{array}\right] \left[\begin{array}{c} x(t) \\ x(t-h(t)) \\ \dot{x}(s) \end{array}\right] ds \end{aligned} \tag{8b}$$

Secondary, the following Schur complement result, which is essential in the proofs of Theorem 3.1, is introduced.

Lemma 2.2. [2]. *The following matrix inequality*

$$\left[\begin{array}{cc} Q(x) & S(x) \\ S^T(x) & R(x) \end{array}\right] < 0 \tag{9a}$$

where $Q(x) = Q^T(x)$, $R(x) = R^T(x)$ and $S(x)$ depend affine on x, is equivalent to

$$R(x) < 0 \tag{9b}$$

$$Q(x) < 0 \tag{9c}$$

and

$$Q(x) - S(x)\, R^{-1}(x)\, S^T(x) < 0 \tag{9d}$$

Finally, the following Lemma 2.3 will be used to handle the parametrical perturbation.

Lemma 2.3. [2]. *Given matrices Ω, D, and E of appropriate dimensions*

$$\Omega + DF(t)E + E^T\, F^T(t)\, D^T < 0 \tag{10a}$$

for all $F(t)$ satisfying $F^T(t)F(t) \le I$, if and only if there exists some $\varepsilon > 0$ such that

$$\Omega + \varepsilon D\, D^T + \varepsilon^{-1}\, E^T\, E < 0 \tag{10b}$$

3. **Main Results.** In this section, we use the integral inequality approach (IIA) to obtain stability criterion for recurrent neural network with time-varying delays. First, we take up the case where $\Delta C(t) = 0$, $\Delta A(t) = 0$ and $\Delta B(t) = 0$ in system (5) as follows:

$$\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t - h(t))) \tag{11a}$$

$$x(t) = \phi(t), \quad t \in [-h, 0] \tag{11b}$$

Based on the Lyapunov-Krasovskii stability theorem and integral inequality approach (IIA), the following result is obtained.

Theorem 3.1. *For given positive scalars h and h_d, the nominal neural network system with time-varying delay (11) is asymptotically stable if there exist symmetry positive-definite matrices $P = P^T > 0$, $Q = Q^T > 0$, $R = R^T > 0$, $Z = Z^T > 0$, diagonal matrices $S \ge 0$, $U \ge 0$, $V \ge 0$, and $X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{bmatrix} \ge 0$, $\quad Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{bmatrix} \ge 0$, such that the following LMIs hold for*

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & 0 & \Omega_{16} \\ \Omega_{12}^T & \Omega_{22} & \Omega_{23} & 0 & 0 & \Omega_{26} \\ \Omega_{13}^T & \Omega_{23}^T & \Omega_{33} & \Omega_{34} & 0 & \Omega_{36} \\ \Omega_{14}^T & 0 & \Omega_{34}^T & \Omega_{44} & \Omega_{45} & 0 \\ 0 & 0 & 0 & \Omega_{45}^T & \Omega_{55} & 0 \\ \Omega_{16}^T & \Omega_{26}^T & \Omega_{36}^T & 0 & 0 & \Omega_{66} \end{bmatrix} < 0 \tag{12a}$$

and

$$Z - X_{33} \ge 0 \tag{12b}$$

$$Z - Y_{33} \ge 0 \tag{12c}$$

where

$K = \mathrm{diag}\{k_1, k_2, \dots, k_n\}$, $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \dots, \gamma_n\}$,
$\Omega_{11} = -C^T P + PC + Q + R + \Gamma SC + C^T S^T \Gamma - \Gamma UK + h\, Y_{11} + Y_{13} + Y_{13}^T$,
$\Omega_{12} = PA - \Gamma SA - C^T S + \frac{U}{2}(\Gamma + K)$, $\Omega_{13} = PB - \Gamma SB$, $\Omega_{14} = h\, Y_{12} - Y_{13} + Y_{23}^T$,
$\Omega_{16} = -h\, C^T Z$, $\Omega_{22} = A^T S + SA - U$, $\Omega_{23} = SB$,
$\Omega_{26} = h\, A^T Z$, $\Omega_{33} = -V$, $\Omega_{34} = \frac{V}{2}(\Gamma + K)$, $\Omega_{36} = h\, B^T Z$,
$\Omega_{44} = -(1 - h_d)Q - \Gamma VK + h\, X_{11} + X_{13} + X_{13}^T + h\, Y_{22} - Y_{23} - Y_{23}^T$,
$\Omega_{45} = h\, X_{12} - X_{13} + X_{23}^T$, $\Omega_{55} = -R + h\, X_{22} - X_{23} - X_{23}^T$, $\Omega_{66} = -hZ$.

Proof: Choose the following Lyapunov-Kravoskii functional candidate to be

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) \tag{13}$$

where

$$\begin{aligned}
&V_1(t)) = x^T(t)Px(t) \\
&V_2(t) = 2\sum_{i=1}^{n} s_i \int_0^{x_i(t)} (g_i(s) - \gamma_i s)ds \\
&V_3(t) = \int_{t-h(t)}^{t} x^T(s)Qx(s)ds + \int_{t-h}^{t} x^T(s)Rx(s)ds \\
&V_4(t) = \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)Z\dot{x}(s)dsd\theta
\end{aligned}$$

Then, taking the time derivative of $V(t)$ with respect to t along the system (11) yield

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t) \tag{14}$$

where

$$\begin{aligned}
\dot{V}_1(x_t) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) \\
&= x^T(t)(-C^T P - PC)x(t) + 2\,x^T(t)P[Ag(x(t)) + Bg(x(t-h(t)))]
\end{aligned} \tag{15}$$

$$\begin{aligned}
\dot{V}_2(t) =& 2[g^T(x(t)) - x^T(t)\Gamma]S\dot{x}(t) \\
=& 2[g^T(x(t)) - x^T(t)\Gamma]S[-Cx(t) + Ag(x(t)) + Bg(x(t-h(t)))] \\
=& -g^T(x(t))SCx(t) + g^T(x(t))SAg(x(t)) + g^T(x(t))SBg(x(t-h(t))) \\
&+ x^T(t)\Gamma SCx(t) - x^T(t)\Gamma SAg(x(t)) - x^T(t)\Gamma SBg(x(t-h(t))) \\
&- x^T(t)\,C^T\,Sg(x(t)) + x^T(t)\,C^T\,S^T\,\Gamma x(t) \\
&+ g^T(x(t))\,A^T\,Sg(x(t)) - g^T(x(t))\,A^T\,S\Gamma x(t) \\
&+ g^T(x(t-h(t)))\,B^T\,Sg(x(t)) - g^T(x(t-h(t)))\,B^T\,S\Gamma x(t)
\end{aligned} \tag{16}$$

$$\begin{aligned}
\dot{V}_3(t) &= x^T(t)Qx(t) - (1-\dot{h}(t))\,x^T(t-h(t))Qx(t-h(t)) \\
&\quad + x^T(t)Rx(t) - x^T(t-h)Rx(t-h) \\
&\le x^T(t)Qx(t) - (1-h_d)\,x^T(t-h(t))Qx(t-h(t)) \\
&\quad + x^T(t)Rx(t) - x^T(t-h)Rx(t-h)
\end{aligned} \tag{17}$$

and

$$\dot{V}_4(t) = \dot{x}^T(t)hZ\dot{x}(t) - \int_{t-h}^{t} \dot{x}^T(s)Z\dot{x}(s)ds \tag{18}$$

Alternatively, the following equations are true:

$$\begin{aligned}
-\int_{t-h}^{t} \dot{x}^T(s)Z\dot{x}(s)ds =& -\int_{t-h}^{t-h(t)} \dot{x}^T(s)Z\dot{x}(s)ds - \int_{t-h(t)}^{t} \dot{x}^T(s)Z\dot{x}(s)ds \\
=& -\int_{t-h}^{t-h(t)} \dot{x}^T(s)(Z - X_{33})\dot{x}(s)ds - \int_{t-h}^{t-h(t)} \dot{x}^T(s)\,X_{33}\,\dot{x}(s)ds \\
&- \int_{t-h(t)}^{t} \dot{x}^T(s)(Z - Y_{33})\dot{x}(s)ds - \int_{t-h(t)}^{t} \dot{x}^T(s)\,Y_{33}\,\dot{x}(s)ds
\end{aligned} \tag{19}$$

Using Lemma 2.1, the term $-\int_{t-h(t)}^{t} \dot{x}^T(s)\,X_{33}\,\dot{x}(s)ds$ can be written that

$$-\int_{t-h}^{t-h(t)} \dot{x}^T(s)\,X_{33}\,\dot{x}(s)ds$$

$$\leq \int_{t-h}^{t-h(t)} \left[\begin{array}{ccc} x^T(t-h(t)) & x^T(t-h)) & \dot{x}^T(s) \end{array}\right] \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & 0 \end{bmatrix} \begin{bmatrix} x(t-h(t)) \\ x(t-h) \\ \dot{x}(s) \end{bmatrix} ds$$

$$\begin{aligned}
&\leq x^T(t-h(t))h\,X_{11}\,x(t-h(t)) + x^T(t-h(t))h\,X_{12}\,x(t-h) + x^T(t)\,X_{13}\int_{t-h}^{t-h(t)} \dot{x}(s)ds \\
&\quad + x^T(t-h)\,hX_{12}^T\,x(t-h(t)) + x^T(t-h)h\,X_{22}\,x(t-h) + x^T(t-h)\,X_{23}\int_{t-h}^{t-h(t)} \dot{x}(s)ds \\
&\quad + \int_{t-h)}^{t-h(t)} \dot{x}^T(s)ds\,X_{13}^T\,x(t-h(t)) + \int_{t-h}^{t-h(t)} \dot{x}^T(s)ds\,X_{23}^T\,x(t-h) \\
&= x^T(t-h(t))[h\,X_{11} + X_{13}^T + X_{13}]x(t-h(t)) + x^T(t-h(t))[h\,X_{12} - X_{13} + X_{23}^T]x(t-h) \\
&\quad + x^T(t-h)[hX_{12}^T - X_{13}^T + X_{23}]x(t-h(t)) + x^T(t-h)[h\,X_{22} - X_{23} - X_{23}^T]x(t-h)
\end{aligned} \tag{20}$$

Similarly, we have

$$\begin{aligned}
&-\int_{t-h(t)}^{t} \dot{x}^T(s)\,Y_{33}\,\dot{x}(s)ds \\
&\leq x^T(t)[h\,Y_{11} + Y_{13}^T + Y_{13}]x(t) + x^T(t)[h\,Y_{12} - Y_{13} + Y_{23}^T]x(t-h(t)) \\
&\quad + x^T(t-h(t))[hY_{12}^T - Y_{13}^T + Y_{23}]x(t) + x^T(t-h(t))[h\,Y_{22} - Y_{23} - Y_{23}^T]x(t-h(t))
\end{aligned} \tag{21}$$

Evaluating $\dot{x}^T(t)hZ\dot{x}(t)$ along solution to (11), gives as follows:

$$\begin{aligned}
&\dot{x}^T(t)hZ\dot{x}(t) \\
&= [-Cx(t) + Ag(x(t)) + Bg(x(t-h(t)))]^T\,(hZ) \\
&\quad [-Cx(t) + Ag(x(t)) + Bg(x(t-h(t)))] \\
&= x^T(t)h\,C^T\,ZCx(t) - x^T(t)\,hC^T\,ZAg(x(t)) \\
&\quad - x^T(t)\,hC^T\,ZBg(x(t-h(t))) - g^T(x(t))h\,A^T\,ZCx(t) \\
&\quad + g^T(x(t))h\,A^T\,ZAg(x(t)) + g^T(x(t))h\,A^T\,ZBg(x(t-h(t))) \\
&\quad - g^T(x(t-h(t)))h\,A^T\,ZCx(t) + g^T(x(t-h(t)))h\,A^T\,ZAg(x(t)) \\
&\quad + g^T(x(t-h(t)))h\,A^T\,ZBg(x(t-h(t)))
\end{aligned} \tag{22}$$

Applying (3), it can be verified that

$$\begin{aligned}
0 &= g^T(x(t))Ug(x(t)) - g^T(x(t))Ug(x(t)) \\
&= -x^T(t)\Gamma UKx(t) + x^T(t)U(\Gamma + K)g(x(t)) - g^T(x(t))Ug(x(t))
\end{aligned} \tag{23}$$

Similarly, there holds

$$\begin{aligned}
0 &= g^T(x(t-h(t)))Vg(x(t-h(t))) - g^T(x(t-h(t)))Vg(x(t-h(t))) \\
&= -x^T(t-h(t))\Gamma VKx(t-h(t)) + x^T(t-h(t))V(\Gamma + K)g(x(t-h(t))) \\
&\quad - g^T(x(t-h(t)))Vg(x(t-h(t)))
\end{aligned} \tag{24}$$

Substituting the above Equations (15)-(24) into (14), we obtain

$$\dot{V}(t) \leq \xi^T(t)\Xi\xi(t) - \int_{t-h}^{t-h(t)} \dot{x}^T(s)(Z - X_{33})\dot{x}(s)ds - \int_{t-h(t)}^{t} \dot{x}^T(s)(Z - Y_{33})\dot{x}(s)ds \tag{25}$$

where $\xi^T(t) = \left[\begin{array}{ccccc} x^T(t) & g^T(x(t)) & g^T(x(t-h(t))) & x^T(t-h(t) & x^T(t-h) \end{array}\right]$ and

$$\Xi = \left[\begin{array}{ccccc} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & 0 \\ \Xi_{12}^T & \Xi_{22} & \Xi_{23} & 0 & 0 \\ \Xi_{13}^T & \Xi_{23}^T & \Xi_{33} & \Xi_{34} & 0 \\ \Xi_{14}^T & 0 & \Xi_{34}^T & \Xi_{44} & \Xi_{45} \\ 0 & 0 & 0 & \Xi_{45}^T & \Xi_{55} \end{array}\right] < 0, \text{ with}$$

$K = \mathrm{diag}\{k_1, k_2, \ldots, k_n\}, \Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$,
$\Xi_{11} = -C^T P + PC + Q + R + \Gamma SC + C^T S^T \Gamma - \Gamma UK + h\,Y_{11} + Y_{13} + Y_{13}^T + h\,C^T ZC$,
$\Xi_{12} = PA - \Gamma SA - C^T S + \frac{U}{2}(\Gamma + K) - h\,C^T ZA$,
$\Xi_{13} = PB - \Gamma SB,\ \Xi_{14} = h\,Y_{12} - Y_{13} + Y_{23}^T$,
$\Xi_{22} = A^T S + SA - U + h\,A^T ZA,\ \Xi_{23} = SB + h\,A^T ZB,\ \Xi_{33} = -V$,
$\Xi_{34} = \frac{V}{2}(\Gamma + K),\ \Xi_{44} = -(1 - h_d)Q - \Gamma VK + h\,X_{11} + X_{13} + X_{13}^T + h\,Y_{22} - Y_{23} - Y_{23}^T$,
$\Xi_{45} = h\,X_{12} - X_{13} + X_{23}^T,\ \Xi_{55} = -R + h\,X_{22} - X_{23} - X_{23}^T$.

Finally, using the Schur complements of Lemma 2.2, with some effort we can show that (25) guarantees of $\dot{V}(t) < -\delta\,\|x(t)\|^2$ for a sufficiently small $\delta > 0$. It is clear that if $\Xi < 0$, $Z - X_{33} \geq 0$, and $Z - Y_{33} \geq 0$. Furthermore, (12) implies $\Omega < 0$, which is equivalent to (25). Therefore, if LMIs (12) are feasible, the system (11) is asymptotically stable. This completes the proof.

Based on Theorem 3.1, we have the following result for uncertain recurrent neural networks with time-varying delay (5).

Theorem 3.2. *For given positive scalars h and h_d, the uncertain recurrent neural networks with time-varying delay (5) is asymptotically stable if there exist symmetric positive-definite matrices $P = P^T > 0$, $Q = Q^T > 0$, $R = R^T > 0$, $Z = Z^T > 0$, diagonal matrices $S \geq 0$, $U \geq 0$, $V \geq 0$, a scalar $\varepsilon > 0$ and positive semi-definite matrices $X = \left[\begin{array}{ccc} X_{11} & X_{12} & X_{13} \\ X_{12}^T & X_{22} & X_{23} \\ X_{13}^T & X_{23}^T & X_{33} \end{array}\right] \geq 0$, $Y = \left[\begin{array}{ccc} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{array}\right] \geq 0$, such that the following LMIs are true*

$$\overline{\Omega} = \left[\begin{array}{ccccccc} \Omega_{11} + \varepsilon\,E_c^T E_c & \Omega_{12} - \varepsilon\,E_c^T E_a & \Omega_{13} - \varepsilon\,E_c^T E_b & \Omega_{14} & 0 & \Omega_{16} & PD \\ \Omega_{12}^T - \varepsilon\,E_a^T E_a & \Omega_{22} + \varepsilon\,E_a^T E_a & \Omega_{23} + \varepsilon\,E_a^T E_b & 0 & 0 & \Omega_{26} & 0 \\ \Omega_{13}^T - \varepsilon\,E_b^T E_a & \Omega_{23}^T + \varepsilon\,E_b^T E_a & \Omega_{33} + \varepsilon\,E_b^T E_b & \Omega_{34} & 0 & \Omega_{36} & 0 \\ \Omega_{14}^T & 0 & \Omega_{34}^T & \Omega_{44} & \Omega_{45} & 0 & 0 \\ 0 & 0 & 0 & \Omega_{45}^T & \Omega_{55} & 0 & 0 \\ \Omega_{16}^T & \Omega_{26}^T & \Omega_{36}^T & 0 & 0 & \Omega_{66} & hZD \\ D^T P & 0 & 0 & 0 & 0 & h\,D^T Z & -\varepsilon I \end{array}\right] < 0 \tag{26a}$$

and

$$Z - X_{33} \geq 0 \tag{26b}$$

$$Z - Y_{33} \geq 0 \tag{26c}$$

where Ω_{ij}, $(i, j = 1, \ldots, 6; i < j \leq 6)$ are defined in (12).

It is, incidentally, worth noting that the uncertain recurrent neural networks with time-varying delay (5) is asymptotically stable, that is, the uncertain parts of the nominal system can be tolerated within allowable time delay h.

Proof: Replacing A, B, and C in (12) with $A + DF(t)\,E_a$, $B + DF(t)\,E_b$, and $C + DF(t)\,E_c$, respectively, we apply Lemma 2.2 for system (12) is equivalent to the following condition:

$$\Omega + \Gamma_d\,F(t)\,\Gamma_e + \Gamma_e^T\,F(t)\,\Gamma_d^T < 0 \tag{27}$$

where $\Gamma_d = \begin{bmatrix} PD & 0 & 0 & 0 & 0 & hZD \end{bmatrix}^T$ and $\Gamma_e = \begin{bmatrix} E_c & E_a & E_b & 0 & 0 & 0 \end{bmatrix}$.

According to Lemma 2.3, (27) is true if there exist a scalar $\varepsilon > 0$ such that the following inequality holds

$$\Omega + \varepsilon^{-1} \Gamma_d^T \Gamma_d + \varepsilon \Gamma_e^T \Gamma_e < 0 \tag{28}$$

Applying the Schur complement shows that (28) is equivalent to (26a). This completes the proof.

If the upper bound of the derivative of time-varying delay h_d is unknown, Theorem 3.2 can be reduced to the result with $Q = 0$ and $X = 0$, we have the following Corollary 3.1.

Corollary 3.1. *Consider system (5) with constant delay. For given a positive scalar h, the system is asymptotically stable if there exist symmetric positive-definite matrices $P = P^T > 0$, $R = R^T > 0$, $Z = Z^T > 0$, diagonal matrices $S \geq 0$, $U \geq 0$, $V \geq 0$, $\varepsilon > 0$ and $Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12}^T & Y_{22} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{bmatrix} \geq 0$ such that the following LMIs are true*

$$\overline{\Psi} = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & \Psi_{14} & -h\,C^T Z & PD \\ \Psi_{12}^T & \Psi_{22} & \Psi_{23} & 0 & h\,A^T Z & 0 \\ \Psi_{13}^T & \Psi_{23}^T & \Psi_{33} & \Psi_{34} & h\,B^T Z & 0 \\ \Psi_{14}^T & 0 & \Psi_{34}^T & \Psi_{44} & 0 & 0 \\ -hZC & hZA & hZB & 0 & -hZ & hZD \\ D^T P & 0 & 0 & 0 & h\,D^T Z & -\varepsilon I \end{bmatrix} < 0 \tag{29a}$$

and

$$Z - X_{33} \geq 0 \tag{29b}$$

where

$K = \mathrm{diag}\{k_1, k_2, \ldots, k_n\}$, $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$,
$\Psi_{11} = -C^T P + PC + R + \Gamma SC + C^T S^T \Gamma - \Gamma UK + h\,Y_{11} + Y_{13} + Y_{13}^T + \varepsilon E_c^T E_c$,
$\Psi_{12} = PA - \Gamma SA - C^T S + \frac{U}{2}(\Gamma + K) - \varepsilon E_c^T E_a$, $\Psi_{13} = PB - \Gamma SB + \varepsilon E_c^T E_b$,
$\Psi_{14} = h\,Y_{12} - Y_{13} + Y_{23}^T$, $\Psi_{22} = A^T S + SA - U + \varepsilon E_a^T E_a$, $\Psi_{23} = SB + \varepsilon E_a^T E_b$,
$\Psi_{33} = -V + \varepsilon E_b^T E_b$, $\Psi_{34} = \frac{V}{2}(\Gamma + K)$, $\Psi_{44} = -R - \Gamma VK + h\,Y_{22} - Y_{23} - Y_{23}^T$.

Proof: If the matrix $Q = 0$ is selected in (13). This proof can be completed in a similar formulation to Theorems 3.1 and 3.2.

Remark 3.1. *Theorem 3.2 provides delay-dependent robust asymptotic stability criterion for the uncertain recurrent neural networks with time-varying delay (5) in terms of solvability of LMIs* [2]. *Based on them, we can obtain the maximum allowable delay bound (MADB) $\overline{h}$ such that (5) is stable by solving the following convex optimization problem*

$$\begin{cases} \text{Maximize} & \overline{h} \\ \text{Subject to} & (26) \end{cases} \tag{30}$$

Inequality (30) is a convex optimization problem and can be obtained efficiently using the MATLAB LMI Toolbox.

4. **Numerical Examples.**

In this section, we provide four numerical examples to demonstrate the effectiveness and less conservatism of our delay-dependent stability criteria.

Example 4.1. *Consider an uncertain delayed recurrent neural network with parameters as follows:*

$$\dot{x}(t) = -(C + \Delta C(t))x(t) + (A + \Delta A(t))g(x(t)) + (B + \Delta B(t))g(x(t - h(t))) \tag{31}$$

where

$$C = \begin{bmatrix} 6.5618 & 0 & 0 \\ 0 & 5.5784 & 0 \\ 0 & 0 & 7.3269 \end{bmatrix}, \quad A = \begin{bmatrix} 0.3526 & -0.1904 & 0.3322 \\ -0.1564 & 0.2446 & 0.3674 \\ -0.1753 & 0.2956 & -0.3115 \end{bmatrix},$$
$$B = \begin{bmatrix} 0.1981 & -0.1313 & 0.1185 \\ 0.1645 & 0.0901 & 0.1013 \\ 0.0274 & -0.1518 & 0.0742 \end{bmatrix}, \quad D = \alpha \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$E_a = E_b = E_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution: To calculate the maximum allowable delay bound (MADB) $\overline{h}$ by Theorem 3.2 in this paper, we consider the following four cases. The neuron activation functions satisfy Assumption 2.1 with

Case (I): $K = \mathrm{diag}(1.2051, 0.2562, 1.3593)$ and $\Gamma = \mathrm{diag}(0.7051, 0.0342, 0.5593)$.

Case (II): $K = \mathrm{diag}(-1.1147, -0.3251, -1.4962)$.
and $\Gamma = \mathrm{diag}(-2.5720, -1.0134, -2.5271)$.

Case (III): $K = \mathrm{diag}(0.8053, 0.7956, 0.9332)$ and $\Gamma = \mathrm{diag}(-0.8053, -0.7956, -0.9332)$.

Case (IV): $K = \mathrm{diag}(-0.6324, 1.4257, 1.7671)$ and $\Gamma = \mathrm{diag}(-1.5553, 0.5000, -1.4020)$.

For these cases, the maximum allowable bounds on time delays for different value of α are given in Table 1, respectively.

For Case I, by taking the parameter $\alpha = 0.1$ and $h_d = 0.5$, we get Theorem 3.2 remains feasible for any delay time $h \leq 4.0082$. In case of $\overline{h} = 4.0082$, solving Theorem 3.2 yields the following set of feasible solutions

$$P = \begin{bmatrix} 44.0466 & 1.3716 & 0.7087 \\ 1.3716 & 37.8747 & 1.6734 \\ 0.7087 & 1.6734 & 46.3251 \end{bmatrix}, \quad Q = \begin{bmatrix} 34.5449 & -0.1520 & 2.1352 \\ -0.1520 & 22.0457 & -0.4715 \\ 2.1352 & -0.4715 & 33.4641 \end{bmatrix},$$
$$R = \begin{bmatrix} 9.9987 & 0.1010 & -0.0064 \\ 0.1010 & 9.9888 & 0.0610 \\ -0.0064 & 0.0610 & 9.9537 \end{bmatrix}, \quad Z = \begin{bmatrix} 2.1499 & -0.0297 & 0.0020 \\ -0.0297 & 2.1906 & -0.0184 \\ 0.0020 & -0.0184 & 2.1442 \end{bmatrix},$$
$$X_{11} = \begin{bmatrix} 1.1029 & 0.0005 & -0.0027 \\ 0.0005 & 1.1512 & 0.0010 \\ 0.0027 & 0.0010 & 1.1048 \end{bmatrix}, \quad X_{12} = \begin{bmatrix} -0.0065 & 0.0014 & -0.0001 \\ 0.0014 & -0.0083 & 0.0009 \\ -0.0001 & 0.0009 & -0.0062 \end{bmatrix},$$
$$X_{13} = \begin{bmatrix} -0.0180 & 0.0039 & -0.0003 \\ 0.0039 & -0.0230 & 0.0024 \\ -0.0003 & 0.0024 & -0.0172 \end{bmatrix}, \quad X_{22} = \begin{bmatrix} 1.6180 & 0.0107 & -0.0007 \\ 0.0107 & 1.6186 & 0.0064 \\ -0.0007 & 0.0064 & 1.6124 \end{bmatrix},$$
$$X_{23} = \begin{bmatrix} 0.0173 & -0.0037 & 0.0003 \\ -0.0037 & 0.0222 & -0.0023 \\ 0.0003 & -0.0023 & 0.0166 \end{bmatrix}, \quad X_{33} = \begin{bmatrix} 1.0768 & -0.0156 & 0.0012 \\ -0.0156 & 1.0974 & -0.0097 \\ 0.0012 & -0.0097 & 1.0737 \end{bmatrix},$$
$$Y_{11} = \begin{bmatrix} 4.0052 & 0.9246 & 0.3912 \\ 0.9246 & 3.7491 & 0.2378 \\ 0.3912 & 0.2378 & 3.9254 \end{bmatrix}, \quad Y_{12} = \begin{bmatrix} 0.4075 & -0.0460 & 0.1188 \\ 0.3571 & 0.0640 & 0.0355 \\ 0.0820 & -0.1030 & 0.3842 \end{bmatrix},$$
$$Y_{13} = \begin{bmatrix} -0.0099 & 0.0067 & 0.0007 \\ 0.0052 & -0.0132 & 0.0022 \\ 0.0011 & 0.0026 & -0.0094 \end{bmatrix}, \quad Y_{22} = \begin{bmatrix} 1.1805 & 0.0011 & 0.0161 \\ 0.0011 & 1.1580 & -0.0137 \\ 0.0161 & -0.0137 & 1.1569 \end{bmatrix},$$
$$Y_{23} = \begin{bmatrix} 0.0190 & -0.0026 & 0.0004 \\ -0.0040 & 0.0233 & -0.0027 \\ 0.0007 & -0.0024 & 0.0184 \end{bmatrix}, \quad Y_{33} = \begin{bmatrix} 1.0769 & -0.0154 & 0.0012 \\ -0.0154 & 1.0972 & -0.0098 \\ 0.0012 & -0.0098 & 1.0740 \end{bmatrix},$$

$$S = \begin{bmatrix} 1.4701 & 0 & 0 \\ 0 & 1.4132 & 0 \\ 0 & 0 & 3.2164 \end{bmatrix}, \quad U = \begin{bmatrix} 13.7953 & 0 & 0 \\ 0 & 16.0365 & 0 \\ 0 & 0 & 13.3682 \end{bmatrix},$$
$$V = \begin{bmatrix} 14.6677 & 0 & 0 \\ 0 & 10.2710 & 0 \\ 0 & 0 & 10.8685 \end{bmatrix}, \quad \varepsilon = 1.6995.$$

Moreover, for various α, the computed maximum allowable delay bound (MADB), $\overline{h}$, which guarantee the stability of system (31), are listed in Table 1, which also illustrates the merits of the method proposed in this paper. It clearly shows the superiority of our results to those from [44].

TABLE 1. MADB $\overline{h}$ for various $h_d = 0.5$ in Example 4.1

Case	α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Case (I)	[44]	2.3015	1.5484	1.1078	0.8197	0.6393	0.5190	0.4330	0.3686
	Theorem 3.2	4.0082	3.9871	3.9510	3.8995	3.8420	3.7782	3.7117	3.6450
Case (II)	[44]	1.1197	0.7362	0.4960	0.3254	0.2224	0.1536	0.1044	0.0675
	Theorem 3.2	2.8982	2.1560	1.6510	1.1250	0.8366	0.7782	0.7091	0.5046
Case (III)	[44]	2.6785	1.9373	1.4481	1.1120	0.8696	0.6904	0.5601	0.4621
	Theorem 3.2	3.7865	3.1566	2.6790	2.5250	2.4659	2.1782	2.0991	1.9846
Case (IV)	[44]	1.4744	1.0581	0.7778	0.5823	0.4494	0.3546	0.2842	0.2291
	Theorem 3.2	3.4955	3.0966	2.5788	2.3250	2.2659	2.1622	1.9998	1.8835

It is noted that all the elements of K and Γ in Case (I) are positive, while in Case (II) they are negative. In Case (III), it is easy to see that $K = -\Gamma$, which just the case is where the activation functions satisfying (4). In Case (IV), neither K nor Γ is required to be positive definite or negative definite, and the two matrices are only required to satisfy $K \geq \Gamma$. We say that the elements of the two matrices in Case (IV) are very free. The numerical results in such tables illustrate the effectiveness of the delay-dependent stability criteria proposed in Theorem 3.2. It also can be seen from Table 1 that maximum allowable delay bound (MADB) $\overline{h}$ decreases as α increases. This shows that the uncertainty affects the maximum time delay for stability.

Example 4.2. *Consider a delayed recurrent neural network with parameters as follows:*

$$\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t - h(t))) \tag{32}$$

where

$$C = \begin{bmatrix} 1.3 & 0 & 0 \\ 0 & 1.1 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}, \quad A = \begin{bmatrix} 1.2 & 0.6 & 1.1 \\ 0.4 & 0.8 & 1.2 \\ 0.5 & 0.9 & 1.0 \end{bmatrix}, \quad B = \begin{bmatrix} 1.7 & 1.2 & 1.3 \\ 1.4 & 0.9 & 1.1 \\ 0.2 & 1.5 & 0.7 \end{bmatrix}.$$

Solution: The neuron activation functions are assumed to satisfy Assumption 2.1 with $k_1 = k_2 = k_3 = k$, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$, where k and γ are some scalars satisfying $k \geq \gamma$.

For different choices of k and γ. Table 2 gives the maximum allowable bounds on time delays based on Theorem 3.1. For this case, it can be verified that the stability conditions in [34] are not applicable when $h_d \geq 0.4$ and the stability conditions in [1,7,31,41,48] are not applicable when $h_d > 1$. This implies that for this example the stability condition in Theorem 3.1 in this paper is less conservative than those in [6,8,20,22,34,35,46]. If the upper bound of the derivative of time-varying delay h_d is unknown and $k_1 = k_2 = k_3 = 0.3, \gamma_1 = \gamma_2 = \gamma_3 = -0.3$, by Corollary 3.1 in this paper, we have that the delayed recurrent neural network under consideration is asymptotically stable for any time varying delay $h(t)$ satisfying $0 < h(t) \leq 1.1132$. By using the Matlab LMI toolbox, we solve LMIs (29) for the case $\overline{h} = 1.1132$, and obtain:

$$P = \begin{bmatrix} 0.5289 & -0.4131 & -0.5876 \\ -0.4131 & 0.4795 & 0.4913 \\ -0.5876 & 0.4913 & 0.9563 \end{bmatrix}, \quad R = \begin{bmatrix} 0.4817 & -0.2705 & -0.3968 \\ -0.2705 & 0.3872 & 0.1917 \\ -0.3968 & 0.1917 & 0.6262 \end{bmatrix},$$
$$Z = \begin{bmatrix} 0.2857 & -0.2237 & -0.3081 \\ -0.2237 & 0.2756 & 0.2625 \\ -0.3081 & 0.2625 & 0.5001 \end{bmatrix}, \quad Y_{11} = \begin{bmatrix} 0.5763 & -0.1598 & -0.2481 \\ -0.1598 & 0.4577 & 0.2557 \\ -0.2481 & 0.2557 & 0.7376 \end{bmatrix},$$
$$Y_{12} = \begin{bmatrix} -0.1053 & 0.0591 & 0.0701 \\ 0.0631 & -0.1266 & -0.0528 \\ 0.0673 & -0.0465 & -0.1722 \end{bmatrix}, \quad Y_{13} = \begin{bmatrix} -0.0465 & 0.0176 & 0.0131 \\ 0.0228 & -0.0683 & -0.0058 \\ 0.0105 & 0.0018 & -0.0668 \end{bmatrix},$$
$$Y_{22} = \begin{bmatrix} 0.1619 & -0.1174 & -0.1557 \\ -0.1174 & 0.1641 & 0.1268 \\ -0.1557 & 0.1268 & 0.2765 \end{bmatrix}, \quad Y_{23} = \begin{bmatrix} 0.1411 & -0.0955 & -0.1274 \\ -0.0960 & 0.1460 & 0.1050 \\ -0.1294 & 0.1062 & 0.2484 \end{bmatrix},$$
$$Y_{33} = \begin{bmatrix} 0.2075 & -0.1527 & -0.2100 \\ -0.1527 & 0.2066 & 0.1783 \\ -0.2100 & 0.1783 & 0.3698 \end{bmatrix}, \quad S = \begin{bmatrix} 0.0302 & 0 & 0 \\ 0 & 0.0362 & 0 \\ 0 & 0 & 0.0366 \end{bmatrix},$$
$$U = \begin{bmatrix} 9.7400 & 0 & 0 \\ 0 & 7.1528 & 0 \\ 0 & 0 & 6.2620 \end{bmatrix}, \quad V = \begin{bmatrix} 0.3779 & 0 & 0 \\ 0 & 0.5283 & 0 \\ 0 & 0 & 0.8284 \end{bmatrix}.$$

TABLE 2. MADB $\overline{h}$ for various $h_d = 0.5$ in Example 4.2

Case	$k = 0.3$, $\gamma = -0.3$	$k = 0.3$, $\gamma = -0.1$	$k = 0.3$, $\gamma = 0$	$k = 0.3$, $\gamma = 0.2$	$k = 0$, $\gamma = -0.3$	$k = -0.1$, $\gamma = -0.3$
[44]	0.5952	0.8636	0.9348	1.0665	1.2455	1.2896
Theorem 3.1	0.8337	2.5223	3.6734	4.4561	6.4379	6.5265

Figure 1 shows the state response of Example 4.2 with constant delay $\overline{h} = 1.1132$, when the initial value is $\begin{bmatrix} -1 & 1 & -1 \end{bmatrix}^T$.

Example 4.3. *Consider a delayed recurrent neural network with parameters as follows:*

$$\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t - h(t))) \tag{33}$$

where

$$C = \begin{bmatrix} 1.2769 & 0 & 0 & 0 \\ 0 & 0.6321 & 0 & 0 \\ 0 & 0 & 0.9230 & 0 \\ 0 & 0 & 0 & 0.4480 \end{bmatrix},$$
$$A = \begin{bmatrix} -0.0370 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{bmatrix},$$
$$B = \begin{bmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{bmatrix},$$
$$K = \mathrm{diag}(0.1137, 0.1279, 0.7994, 0.2368),$$
$$\Gamma = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3, \gamma_4).$$

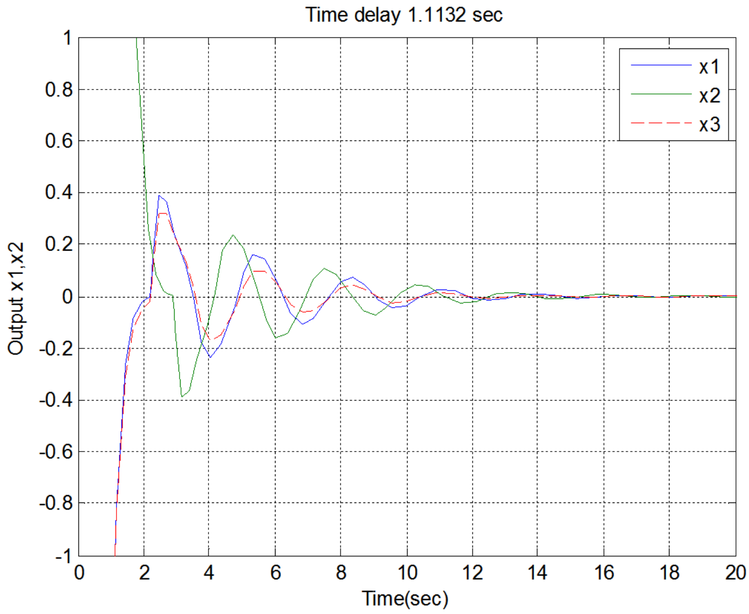

FIGURE 1. The simulation of Example 4.2 for $h = 1.1132$ sec

Solution: For $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$, Table 3 provides some comparisons of the maximum allowable delay bounds in the results of this paper and [6,8,20,22,35,46]. From Table 3, the RNN with time-varying delay in this example is asymptotically stable, which shows that the delay-dependent stability condition in this paper is satisfied. This implies that for this example the delay dependent stability condition in Theorem 3.1 is less conservative than those in [6,8,20,22,35,46].

TABLE 3. Comparisons of the MADB $\overline{h}$ for Example 4.3

Method	[35]	[6,29]	[8]	[22]	[46]	[20]	Theorem 3.1
$\overline{h}$	1.4224	1.9321	3.5841	3.5891	3.6156	4.0120	7.2154

For $\gamma_1 = 0.1$, $\gamma_2 = \gamma_3 = 0$, $\gamma_4 = -0.2$, the MADB $\overline{h}$ that guarantees the delayed NNs to be asymptotically stable is calculated to be 7.1056. If we use Theorem 1 in [20], we can calculate the maximum allowable delay bound (MADB) $\overline{h} = 3.3668$, which is smaller than the result obtained by our methods. Therefore, our method is less conservative in some degree than that in [20].

Example 4.4. *Consider a delayed recurrent neural network with parameters as follows:*

$$\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t - h(t))) \tag{34}$$

where $C = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}$.

The neuron activation functions are assumed to satisfy Assumption 2.1 with

$$K = \mathrm{diag}\{0.4, 0.8\}, \quad \Gamma = \mathrm{diag}\{\gamma_1, \gamma_2\}.$$

Solution: For $\gamma_1 = \gamma_2 = 0$, and $h_d = 0$, by applying the results in [9,11,14,20,22] to this example, the MADB $\overline{h}$ are listed in Table 4. For $\gamma_1 = 0.2$, $\gamma_2 = -0.1$, the MADB $\overline{h}$ that guarantees the system (34) to be asymptotically stable is calculated to be $\overline{h} = 1.1380$ in [20], which is $\overline{h} = 9.6185$ by using Theorem 3.1 in this paper. It is seen that our results improve the existing results [9,11,14,20,22].

Table 4. Comparisons of the MADB $\overline{h}$ for Example 4.4

Method	[9,14]	[11,22]	[20]	Theorem 3.1
$\overline{h}$	0.8298	1.0880	1.1345	12.9895

5. **Conclusions.** In this paper, we present an improved robust delay-dependent stability criterion for RNNs with time-varying delays. A modified Lyapunov-Krasovskii functional which is represented by a convex combination is provided for the stability analysis. Then, a novel criterion for the stability of the RNNs is derived in terms of LMIs. The criterion without free-weighting matrices is less conservative than ones in the literature. Finally, several examples are given to show the superiority of our proposed stability conditions.

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