International Journal of Innovative Computing, Information and Control
Volume 9, Number 8, August 2013
 ISSN 1349-4198
pp. 3173–3183

A NEW APPROACH FOR RANKING NON-NORMAL TRAPEZOIDAL FUZZY NUMBER

Guixiang Wang[1], Jie Du[1], Baoping Wang[2] and Jiaxi Zhang[1]

[1]Institute of Operational Research and Cybernetics
Hangzhou Dianzi University
Xiasha Higher Education Zone, Hangzhou 310018, P. R. China
g.x.wang@hdu.edu.cn

[2]College of Mathematics and Information Science
Hebei Normal University
No. 20, Nanerhuan East Road, Shijiazhuang 050024, P. R. China

Received June 2012; revised October 2012

Abstract. *In this paper, the conception of the quasi-slope of fuzzy number is proposed, some properties of quasi-slope of fuzzy number are investigated, and a ranking index value is obtained based on the quasi-slope and geometrical distance. Then a novel approach to ranking non-normal trapezoidal fuzzy numbers is set up based on their ranking index values. This approach not only makes the calculation easy, but also improves some shortcomings of the previous method in ranking fuzzy numbers. At last, we present some examples to illustrate the advantage and practicality of the new approach.*
Keywords: Ranking fuzzy numbers, Quasi-slopes of fuzzy numbers, Non-normal trapezoidal fuzzy numbers, Distance

1. **Introduction.** The concept of general fuzzy number was introduced by Chang and Zadeh [12] in 1972. Since then, many workers studied the theory of fuzzy numbers, and achieved fruitful results [8,13,19]. On the other hand, ranking is a very important concept, and many methods for ranking have also been studied [15,22,28]. For ranking general fuzzy numbers, various approaches have been developed [5,14,16,21,25]. In recent years, people have proposed a number of methods to rank fuzzy numbers based on the distances in general fuzzy number space. In the existing research, the method commonly used is firstly to construct proper maps to transform fuzzy numbers into real numbers. Then these real numbers are compared. These approaches have their own advantages, and also have some disadvantages. For example, Yager [27] proposed a centroid index ranking approach with weighting function. Lee and Li [17], proposed a method for ranking fuzzy numbers, considered both the mean and dispersion of alternatives and gave two groups of indices based on the uniform and the proportional probability distributions. However, when the mean value and the spread are all higher, or the mean value and the spread are all lower, the method for ranking fuzzy numbers cannot work. In paper [7], Cheng proposed a centroid index ranking approach, where the distance of the centroid point of each fuzzy number and original point are calculated. However, Cheng's CV index and distance method cannot rank some fuzzy numbers. For example, for two different triangular fuzzy numbers u and v whose centroid values are respectively (x_u, y_u) and (x_v, y_v) with $x_u^2+y_u^2 = x_v^2+y_v^2$, Cheng's method cannot distinguish the two fuzzy numbers. Chu and Tsao [11] found these shortcomings and proposed an approach for ranking fuzzy numbers with the area between the centroid and original points. Recently, Wang and Lee [23] thought that the importance of the degree of representative location was higher

than average height and proposed a method to ranking fuzzy numbers by comparing the centroid horizontal and vertical coordinate values of fuzzy numbers. In paper [2], Abbasbandy and Asady proposed a distance based on an approach called sign distance and set up a method for ranking fuzzy numbers based on the distance. In paper [4], Asady and Zendehnam proposed an approach using minimum distance between the two fuzzy numbers, and presented a ranking method for fuzzy numbers by considering the nearest point.

However, these methods have still some drawbacks. For example, the proposed method in [23] cannot rank all normal symmetric triangular fuzzy numbers with the same normal point. In addition, although the proposed method in [23] can distinguish normal symmetric triangular fuzzy numbers without the same normal point, the ranking result may be counterintuitive. For example, for $u = (9, 10, 11)$ and $v = (0, 10.001, 20.002)$, we can consider u is better than v since their centroid points are almost the same, but the separation degree of v is much bigger than u. However, if we use the proposed method in [23], we will get a counterintuitive result that v is better than u. About the method proposed by authors in paper [2], for two triangular fuzzy numbers $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3)$ with $a_1 + a_3 = b_1 + b_3$ and $a_2 = b_2$, if the method is used, a counterintuitive result may be obtained. About the approach proposed by authors in [4], for all triangular fuzzy numbers such as $u = \left(\frac{\alpha-\beta}{4} - \alpha, \frac{\alpha-\beta}{4}, \frac{\alpha-\beta}{4} + \beta\right)$, it is obvious that these fuzzy numbers are different as α or β takes different values, but the approach proposed by authors in [4] cannot tell us the difference between them. Furthermore, for any two different trapezoidal fuzzy numbers $u = (a_1, b_1, c_1, d_1)$ and $v = (a_2, b_2, c_2, d_2)$ with $a_1 + b_1 + c_1 + d_1 = a_2 + b_2 + c_2 + d_2$, the ranking method of Asady and Zendehnam [4] also cannot distinguish the two fuzzy numbers. On the other hand, most of the former ranking methods mentioned above can only deal with normal triangular or trapezoidal fuzzy numbers, so these methods have also some limitations in applications (see Example 4.2).

In order to overcome these drawbacks and limitations, and to make computations more simple, in this paper, we give the concept of quasi-slope of fuzzy number, and propose a novel approach for ranking non-normal trapezoidal fuzzy numbers based on the quasi-slope and geometrical distance. In addition, we also illustrate the advantage (these drawbacks and limitations are overcome) and practicality of the novel approach by some numerical examples (see Example 4.1) and a specific example of application (see Example 4.2).

2. Preliminaries.

In this section, some basic concepts and definitions on fuzzy number space are reviewed from the literature.

Definition 2.1. [14] *A fuzzy number $u = (a, b, c, d; \omega)$ is described as any fuzzy subset of the real line R with the membership function u which has the following properties:*

(1) u is a continuous mapping from R to the closed interval $[0, \omega]$;
(2) $u(x) = 0$, for all $x \neq [a, d]$;
(3) u is strictly increasing on $[a, b]$;
(4) $u(x) = \omega$, for all $x \in [b, c]$, where ω is constant;
(5) u is strictly decreasing on $[c, d]$,

where a, b, c and d are real numbers with $a < b < c < d$ and $\omega \in (0, 1]$. Therefore, the membership function u can be expressed as:

$$u(x) = \begin{cases} u_L(x), & a \le x \le b, \\ \omega, & b \le x \le c, \\ u_R(x), & c \le x \le d, \\ 0, & \text{otherwise}, \end{cases}$$

where the mapping $u_L : [a,b] \to [0,\omega]$ *is continuous and strictly increasing, and* $u_R : [c,d] \to [0,\omega]$ *is continuous and strictly decreasing.*

The set of all fuzzy numbers is denoted as $F(R)$. For convenience, the fuzzy number in Definition 2.1 can be denoted by $u = (a,b,c,d;\omega)$. The image (opposite) of u can be given by $-u = (-d,-c,-b,-a;\omega)$.

Definition 2.2. [7] *A trapezoidal fuzzy number* $u = (a,b,c,d;\omega)$, *where* $a < b < c < d$ *and* $\omega \in (0,1]$, *is described as any fuzzy subset of the real line* R, *with the membership function* u *in* $F(R)$ *expressed as:*

$$u(x) = \begin{cases} \frac{\omega(x-a)}{b-a}, & a \le x \le b \\ \omega, & b \le x \le c \\ \frac{\omega(d-x)}{d-c}, & c \le x \le d \\ 0, & otherwise \end{cases}$$

The set of all trapezoidal fuzzy numbers, as defined in Definition 2.2, is denoted by $T(R)$. If $\omega = 1$, then u is called a normal trapezoidal fuzzy number and is denoted as $u = (a,b,c,d)$ (Figure 1). If $b = c$, then u is called a triangular fuzzy number and is denoted as $u = (a,b,d;\omega)$; furthermore, if $b - a = d - b$, we call $u = (a,b,d;\omega)$ a symmetric triangular fuzzy number. If $\omega = 1$ and $b = c$, then u is called a normal triangular fuzzy number and is denoted as $u = (a,b,d)$ (Figure 2). If $a = b = c = d$, then u is called a real number.

Definition 2.3. [9] *For any* $u = (a_1,b_1,c_1,d_1;\omega_u)$, $v = (a_2,b_2,c_2,d_2;\omega_v) \in T(R)$ *and* $k \in R$, *the addition and scalar multiplication of trapezoidal fuzzy numbers are defined as following:*

(1) $u + v = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; \min\{\omega_u, \omega_v\})$;

(2) when $k \ge 0$, $ku = (ka_1, kb_1, kc_1, kd_1; \omega_u)$; *when* $k \le 0$, $ku = (kd_1, kc_1, kb_1, ka_1; \omega_u)$.

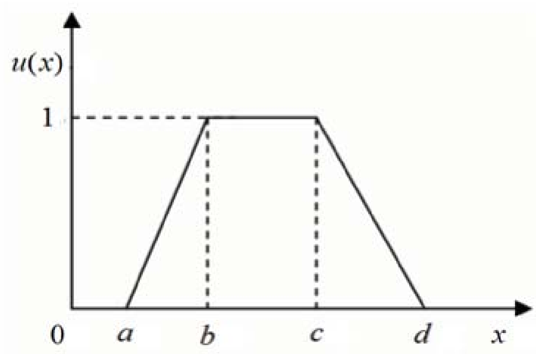

FIGURE 1. Normal trapezoidal fuzzy number

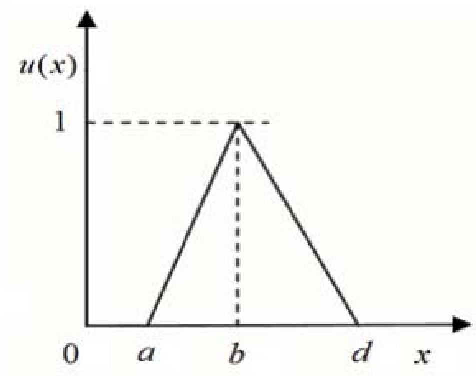

FIGURE 2. Normal triangular fuzzy number

3. The Ranking Index Based on the Quasi-Slope and Geometrical Distance.

Definition 3.1. *Any fuzzy number* $u = (a,b,c,d;\omega) \in T(R)$. $Lk_u = \frac{b-a}{\omega}$ *and* $Rk_u = \frac{c-d}{\omega}$ *are called L-quasi-slope and R-quasi-slope of u, respectively.*

Property 3.1. *Any fuzzy number* $u, v \in T(R)$, α *is an arbitrary real number, then*

(1) $Lk_{\alpha u} = \alpha Lk_u$, $Rk_{\alpha u} = \alpha Rk_u$, *when* $\alpha \ge 0$; $Rk_{\alpha u} = \alpha Lk_u$, $Lk_{\alpha u} = \alpha Rk_u$, *when* $\alpha \le 0$;

(2) $Lk_{(u+v)} \geq Lk_u + Lk_v$, $Rk_{(u+v)} \geq Rk_u + Rk_v$*; specially, when* $\omega_u = \omega_v$*, then* $Lk_{(u+v)} = Lk_u + Lk_v$, $Rk_{(u+v)R} = Rk_u + Rk_v$.

Proof: For $u \in T(R)$, we know that there exist $\omega_u \in (0,1]$ and $a_1, a_2, a_3, a_4 \in R$ with $a_1 < a_2 < a_3 < a_4$ such that $u = (a_1, a_2, a_3, a_4; \omega_u)$. When $\alpha \geq 0$, we can see that $\alpha u = (\alpha a_1, \alpha a_2, \alpha a_3, \alpha a_4; \omega_u)$. According to Definition 3.1, we can respectively get the L-quasi-slope and R-quasi-slope of u, i.e., $Lk_{\alpha u} = \frac{\alpha(a_2-a_1)}{\omega_u} = \alpha Lk_u$ and $Rk_{\alpha u} = \frac{\alpha(a_3-a_4)}{\omega_u} = \alpha Rk_u$. When $\alpha \leq 0$, $\alpha u = (\alpha a_4, \alpha a_3, \alpha a_2, \alpha a_1; \omega_u)$, so $Lk_{\alpha u} = \frac{\alpha(a_3-a_4)}{\omega_u} = \alpha Rk_u$, $Rk_{\alpha u} = \frac{\alpha(a_2-a_1)}{\omega_u} = \alpha Lk_u$. Therefore, the conclusion (1) holds.

From Definition 2.3, we know that $u+v = (a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4; \min\{\omega_u, \omega_v\})$, for any $u = (a_1, a_2, a_3, a_4; \omega_u)$, $v = (b_1, b_2, b_3, b_4; \omega_v) \in T(R)$. Then we have

$$\begin{aligned} Lk_{(u+v)} &= \frac{a_2 + b_2 - (a_1 + b_1)}{\min\{\omega_u, \omega_v\}} \\ &\geq \frac{a_2 - a_1}{\omega_u} + \frac{b_2 - b_1}{\omega_v} \\ &= Lk_u + Lk_v. \end{aligned}$$

We can prove that $Rk_{(u+v)} \geq Rk_u + Rk_v$ is established in the same way. When $\omega_u = \omega_v$, it is obvious that $Lk_{(u+v)} = Lk_u + Lk_v$, $Rk_{(u+v)} = Rk_{uR} + Rk_v$.

Definition 3.2. *The mapping* $d : T(R) \times T(R) \to [0, +\infty)$ *is defined as*

$$d(u,v) = (1-\alpha)\, d_1(u,v) + \alpha d_2(u,v)$$

for any $u = (a_1, a_2, a_3, a_4; \omega_u)$, $v = (b_1, b_2, b_3, b_4; \omega_v) \in T(R)$*, where* $\alpha \in (0,1)$,

$$\begin{aligned} d_1(u,v) &= \frac{1}{4}\left(|a_1 - b_1| + |a_4 - b_4|\right) + \frac{3}{4}\left(|a_2 - b_2| + |a_3 - b_3|\right), \\ d_2(u,v) &= \frac{3}{4}|\omega_u - \omega_v| + \frac{1}{4}\left(|Lk_u - Lk_v| + |Rk_u - Rk_v|\right), \end{aligned}$$

then d *is a distance on trapezoidal fuzzy number space, i.e., satisfies the following properties:*

(1) $d(u,v) \geq 0$*, for any* $u, v \in T(R)$*; and* $d(u,v) = 0$ *if and only if* $u = v$*;*
(2) $d(u,v) = d(v,u)$*, for any* $u, v \in T(R)$*;*
(3) $d(u,w) + d(w,v) \geq d(u,v)$*, for any* $u, v, w \in T(R)$.

Proof: By Definition 3.2, it is obvious that $d(u,v) \geq 0$ for any $u, v \in T(R)$, and $d(u,v) = 0$ if $u = v$. On the other hand, If $d(u,v) = 0$ for any $u = (a_1, a_2, a_3, a_4; \omega_u)$, $v = (b_1, b_2, b_3, b_4; \omega_v) \in T(R)$ then $d(u,v) = (1-\alpha)\, d_1(u,v) + \alpha d_2(u,v) = 0$, i.e.,

$$d_1(u,v) = \frac{1}{4}\left(|a_1 - b_1| + |a_4 - b_4|\right) + \frac{3}{4}\left(|a_2 - b_2| + |a_3 - b_3|\right) = 0$$

and

$$d_2(u,v) = \frac{3}{4}|\omega_u - \omega_v| + \frac{1}{4}\left(|Lk_u - Lk_v| + |Rk_u - Rk_v|\right) = 0.$$

It implies that $a_i = b_i$ $(i = 1, 2, 3, 4)$, $\omega_u = \omega_v$ and $Lk_u = Lk_v$, $Lk_u = Lk_v$. So we can see that $u = v$ is established. Therefore, the property (1) holds.

It is also obvious that the property (2) holds by Definition 3.2.

For any $u = (a_1, a_2, a_3, a_4; \omega_u)$, $v = (b_1, b_2, b_3, b_4; \omega_v)$, $w = (c_1, c_2, c_3, c_4; \omega_w) \in T(R)$, by Definition 3.2, we have $d(u,w) = (1-\alpha)\, d_1(u,w) + \alpha d_2(u,w)$, $d(w,v) = (1-\alpha)\, d_1(w,v) +$

$\alpha d_2(w, v)$ and

$$\begin{aligned}
d_1(u,w) + d_1(w,v) &= \frac{1}{4}\left(|a_1 - c_1| + |a_4 - c_4|\right) + \frac{3}{4}\left(|a_2 - c_2| + |a_3 - c_3|\right) \\
&\quad + \frac{1}{4}\left(|c_1 - b_1| + |c_4 - b_4|\right) + \frac{3}{4}\left(|c_2 - b_2| + |c_3 - b_3|\right) \\
&\geq \frac{1}{4}\left(|a_1 - b_1| + |a_4 - b_4|\right) + \frac{3}{4}\left(|a_2 - b_2| + |a_3 - b_3|\right) \\
&= d_1(u,v) \\
d_2(u,w) + d_2(w,v) &= \frac{3}{4}\left|\omega_u - \omega_w\right| + \frac{1}{4}\left(|Lk_u - Lk_w| + |Rk_u - Rk_w|\right) \\
&\quad + \frac{3}{4}\left|\omega_w - \omega_v\right| + \frac{1}{4}\left(|Lk_w - Lk_v| + |Rk_w - Rk_v|\right) \\
&\geq \frac{3}{4}\left(|\omega_u - \omega_w + \omega_w - \omega_v|\right) \\
&\quad + \frac{1}{4}\left(|Lk_u - Lk_w + Lk_w - Lk_v| + |Rk_u - Rk_w + Rk_w - Rk_v|\right) \\
&= \frac{3}{4}\left(|\omega_u - \omega_v|\right) + \frac{1}{4}\alpha\left(|Lk_u - Lk_v| + |Rk_u - Rk_v|\right) \\
&= d_2(u,v).
\end{aligned}$$

So we get $(1-\alpha)\, d_1(u,w) + \alpha d_2(u,w) + (1-\alpha)\, d_1(w,v) + \alpha d_2(w,v) \geq (1-\alpha)\, d_1(u,v) + \alpha d_2(u,v)$, i.e., $d(u,w) + d(w,v) \geq d(u,v)$. Therefore, d is a distance on trapezoidal fuzzy number space

Note: In the application, when measuring the difference between trapezoidal fuzzy numbers, the factor of difference between endpoints is more important than other factors to a trapezoidal fuzzy number, so we choose $\alpha \in (0, \frac{1}{2}]$ usually. In addition, since the bigger the membership degrees of points are, the greater the impacts on the metric of fuzzy numbers. We respectively choose the coefficients of $|a_2 - b_2| + |a_3 - b_3|$ and $|a_1 - b_1| + |a_4 - b_4|$ in the definition d_1 as $\frac{3}{4}$ and $\frac{1}{4}$.

Property 3.2. *Any three trapezoidal fuzzy numbers* $u = (a_1, a_2, a_3, a_4; \omega_u)$, $v = (b_1, b_2, b_3, b_4; \omega_v)$, $w = (c_1, c_2, c_3, c_4; \omega_w)$, *and* $\beta \in R$, when $\omega_u = \omega_v = \omega_w$, we have

(1) $d(\beta u, \beta v) = |\beta|\, d(u,v)$;
(2) $d(u + w, v + w) = d(u,v)$.

Proof: Let $\omega_u = \omega_v$. If $\beta \geq 0$, according to Definition 3.2, we can get

$$d(\beta u, \beta v) = (1-\alpha)\, d_1(\beta u, \beta v) + \alpha d_2(\beta u, \beta v),$$

and

$$\begin{aligned}
d_1(\beta u, \beta v) &= \frac{1}{4}\left(|\beta a_1 - \beta b_1| + |\beta a_4 - \beta b_4|\right) + \frac{3}{4}\left(|\beta a_2 - \beta b_2| + |\beta a_3 - \beta b_3|\right) \\
&= \frac{1}{4}\beta\left(|a_1 - b_1| + |a_4 - b_4|\right) + \frac{3}{4}\beta\left(|a_2 - b_2| + |a_3 - b_3|\right) \\
&= \beta d_1(u,v) \\
d_2(\beta u, \beta v) &= \frac{3}{4}\left|\omega_u - \omega_v\right| + \frac{1}{4}\left(|Lk_{\beta u} - Lk_{\beta v}| + |Rk_{\beta u} - Rk_{\beta v}|\right) \\
&= \frac{1}{4}\left(\beta\left|Lk_u - Lk_v\right| + \beta\left|Rk_u - Rk_v\right|\right) \\
&= \beta d_2(u,v)
\end{aligned}$$

Therefore, $d(\beta u, \beta v) = \beta(1-\alpha)\, d_1(u,v) + \beta\alpha d_2(u,v) = \beta d(u,v)$ is established. On the other hand, when $\beta \leq 0$, the result is $d(\beta u, \beta v) = -\beta d(u,v)$ in the same way. Therefore, the property (1) holds.

According to Definition 3.2, when $\omega_u = \omega_v = \omega_w$, we have

$$d(u+w, v+w) = (1-\alpha)d_1(u+w, v+w) + \alpha d_2(u+w, v+w)$$

and

$$\begin{aligned} d_1(u+w, v+w) &= \frac{1}{4}\left(|a_1 + c_1 - (b_1 + c_1)| + |a_4 + c_4 - (b_4 + c_4)|\right) \\ &\quad + \frac{3}{4}\left(|a_2 + c_2 - (b_2 + c_2)| + |a_4 + c_4 - (b_4 + c_4)|\right) \\ &= \frac{1}{4}\left(|a_1 - b_1| + |a_4 - b_4|\right) + + \frac{3}{4}\left(|a_2 - b_2| + |a_4 - b_4|\right) \\ &= d_1(u,v) \\ d_2(u+w, v+w) &= \frac{3}{4}|\omega_u - \omega_v| + \frac{1}{4}\left(\left|Lk_{(u+w)} - Lk_{(v+w)}\right| + \left|Rk_{(u+w)} - Rk_{(v+w)}\right|\right) \\ &= \frac{1}{4}\left(|Lk_u + Lk_w - (Lk_v + Lk_w)| + |Rk_u + Rk_w - (Rk_v + Rk_w)|\right) \\ &= \frac{1}{4}\left(|Lk_u - Lk_v| + |Rk_u - Rk_w|\right) \\ &= d_2(u,v) \end{aligned}$$

Therefore, $d(u+w, v+w) = (1-\alpha)\, d_1(u,v) + \alpha d_2(u,v) = d(u,v)$ is established.

From the Property 3.2, we can see when the maximums of membership function of trapezoidal fuzzy numbers are all the same, then d satisfies translation invariance and absolute homogeneity.

Denote $T(R^+) = \{u | u = (a,b,c,d;\omega) \in T(R), a,b,c,d \geq 0\}$, and $o = (0,0,0,0;1)$, then o is called zero fuzzy number.

Definition 3.3. *A binary relationship on $T(R^+)$, i.e., a subset of $T(R^+) \times T(R^+)$ as follows: $\prec = \{(u,v) \in T(R^+) \times T(R^+) : d(u,o) \leq d(v,o)\}$. Denote $u \prec v$ if and only if $(u,v) \in \prec$, and we say that u is smaller than v, or v is larger than u (respect to $\prec$).*

Property 3.3. *For any $u, v, w \in T(R^+)$, we have the following properties:*
(1) $u \prec u$ (Reflexivity);
(2) If $u \prec v$ and $v \prec w$, then $v \prec w$ (Transitivity);
(3) $u \prec v$ and $v \prec u$ at least one is established (Completeness).

Proof: By Definition 3.3, it is obvious that $u \prec u$. If $u \prec v$ and $v \prec w$, according to Definition 3.3, $d(u,o) \leq d(v,o)$ and $d(v,o) \leq d(w,o)$ are both established, then we can get $d(u,o) \leq d(w,o)$. Therefore, $u \prec w$ holds. By Definition 3.3, it is also obvious $u \prec v$ and $v \prec u$ at least one is established.

From the Property 3.3, we can see that the binary relationship "$\prec$" satisfies reflexivity, transitivity and completeness, but it does not satisfy the anti-symmetry. Although it is not a total order, it is a weak order.

4. Examples.

Example 4.1. *Literature [28] has shown the four sets as the following. See Figure 3.*
Set 1: $A = (0.4, 0.5, 1)$, $B = (0.4, 0.7, 1)$, $C = (0.4, 0.9, 1)$
Set 2: $A = (0.3, 0.4, 0.7, 0.9)$, $B = (0.3, 0.7, 0.9)$, $C = (0.5, 0.7, 0.9)$
Set 3: $A = (0.3, 0.5, 0.7)$, $B = (0.3, 0.5, 0.8, 0.9)$, $C = (0.3, 0.5, 0.9)$
Set 4: $A = (0, 0.4, 0.7, 0.8)$, $B = (0.2, 0.5, 0.9)$, $C = (0.1, 0.6, 0.8)$

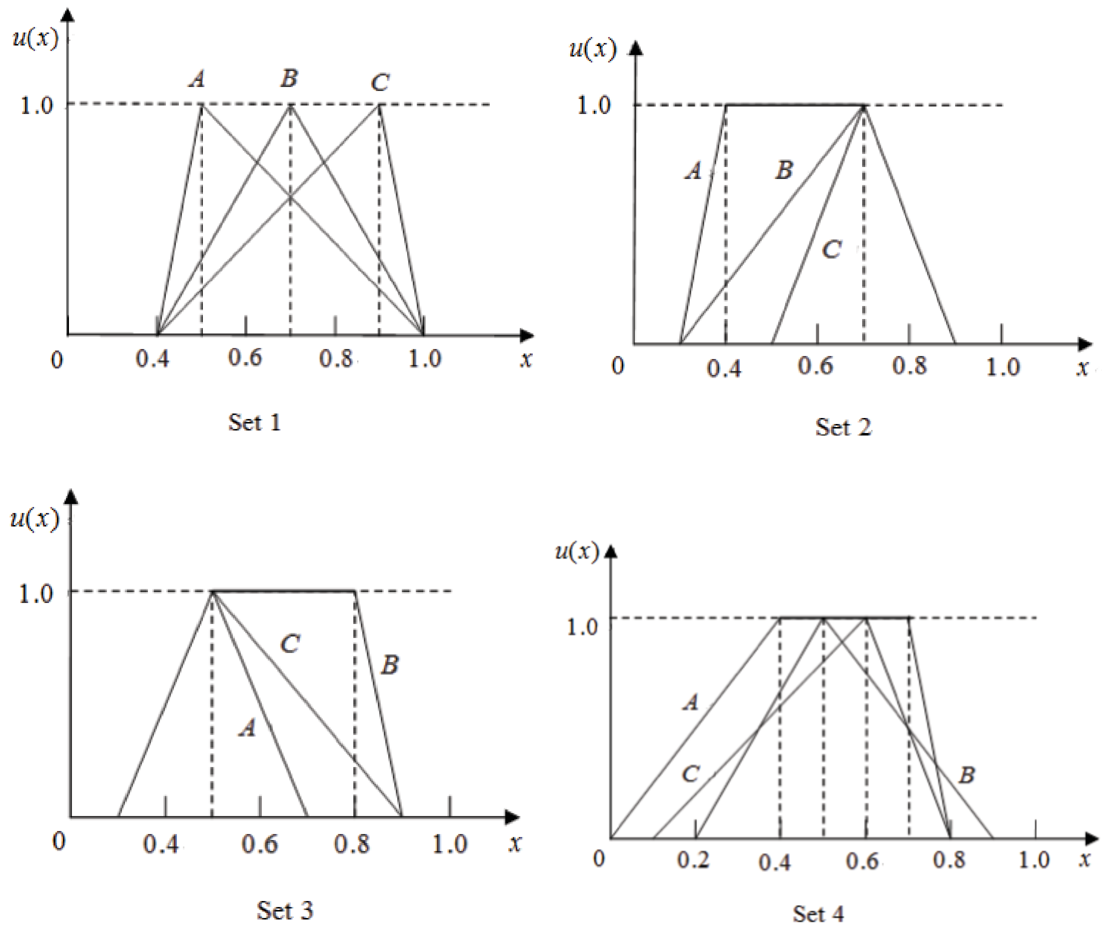

FIGURE 3. Four sets of fuzzy numbers

Through Table 1 we can see the following: By the approach proposed in this paper, the ranking index values of Set 1 can be obtained as $d(A, o) = 0.8625$, $d(B, o) = 1.0875$ and $d(C, o) = 1.3125$. Then the ranking order of fuzzy numbers is $A \prec B \prec C$. Although our result is the same with others, the approach proposed in this paper is easier in calculation. For the Set 2, the ranking index values are respectively $d(A, o) = 0.8625$, $d(B, o) = 1.0500$ and $d(C, o) = 1.0750$. The ranking result is also $A \prec B \prec C$. Then Baldwin and Guild [5] cannot rank A and B. And the result of Wang and Lee [23] is $B \prec C \prec A$, it is easy to see that the result is not consistent with human intuition. For the Set 3, the ranking order is $A \prec C \prec B$ by the approach proposed in this paper. The method of Choobineh and Li [7], Baldwin and Guild [5], Chen [6] and Yager [27] can obtain the result $A \prec B \prec C$. From Figure 3, obviously, our result is more reasonable. For the Set 4, Abbasbandy and Asady [3], Baldwin and Guild [5], Yao and Wu [28] can not give out the ranking order. The result of Chu and Tsao [11] and Cheng [7] is $A \prec C \prec B$. The result of Abbasbandy and Hajjari [3] is $B \prec A \prec C$, and the result of Wang and Lee [23] is $C \prec B \prec A$. By the approach proposed in this paper, the ranking index values are $d(A, o) = 0.8000$, $d(B, o) = 0.8125$ and $d(C, o) = 0.8875$ respectively. Then the ranking order is $A \prec B \prec C$.

The example illustrates that the novel approach proposed by us in this paper can overcome the drawbacks of the previous methods, and makes the calculation much easier than the previous methods.

Table 1. Comparison with previous method

Authors	Fuzzy numbers	Set 1	Set 2	Set 3	Set 4
Proposed method	A	0.8625	0.8625	0.7750	0.8000
	B	1.0875	1.0500	0.9500	0.8125
	C	1.3125	1.0750	0.8250	0.8875
results		$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \prec C$
Sign distance method $p=1$	A	1.2	1.15	1	0.95
	B	1.4	1.3	1.25	1.05
	C	1.6	1.4	1.1	1.05
Results		$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \sim C$
Sign distance method $p=2$	A	0.8669	0.8756	0.7257	0.7853
	B	1.0194	0.9522	0.9416	0.7958
	C	1.1605	1.0033	0.8165	0.8386
Results		$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \prec C$
Choobineh and Li	A	0.333	0.458	0.333	0.50
	B	0.50	0.583	0.4167	0.5833
	C	0.667	0.667	0.5417	0.6111
Results		$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$
Chu and Tsao	A	0.299	0.2847	0.25	0.24402
	B	0.350	0.32478	0.31526	0.26243
	C	0.3993	0.350	0.27475	0.2619
Results		$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec C \prec B$
Chen	A	0.3375	0.4135	0.375	0.52
	B	0.50	0.5625	0.425	0.57
	C	0.667	0.625	0.55	0.625
Results		$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$
Baldwin and Guild	A	0.30	0.27	0.27	0.40
	B	0.33	0.27	0.37	0.42
	C	0.44	0.37	0.45	0.42
Results		$A \prec B \prec C$	$A \sim B \prec C$	$A \prec B \prec C$	$A \prec B \sim C$
Yao and Wu	A	0.6	0.575	0.5	0.475
	B	0.7	0.65	0.625	0.525
	C	0.8	0.7	0.55	0.525
Results		$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \sim C$
Abbasbandy and Hajjari	A	0.5334	0.5584	0.5000	0.5250
	B	0.7000	0.6334	0.6416	0.5084
	C	0.8666	0.7000	0.5166	0.5750
Results		$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$B \prec A \prec C$
Cheng distance	A	0.79	0.7577	0.7071	0.7106
	B	0.8602	0.8149	0.8037	0.7256
	C	0.9268	0.8602	0.7458	0.7241
Results		$A \prec B \prec C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec C \prec B$
Wang centroid method	A	0.2111	0.2568	0.1778	0.1967
	B	0.2333	0.2111	0.2765	0.1778
	C	0.2555	0.2333	0.1889	0.1667
Results		$A \prec B \prec C$	$B \prec C \prec A$	$A \prec C \prec B$	$C \prec B \prec A$
Yager	A	0.60	0.575	0.50	0.45
	B	0.70	0.65	0.55	0.525
	C	0.80	0.70	0.625	0.55
Results		$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$	$A \prec B \prec C$

Example 4.2. *In general, the average personal income of residents per month and the city's harmony do not have a relationship that the higher income, the lower harmony or the lower income, the higher harmony, but have the direct relationship with the degree of polarization of personal income. In other words, the greater the degree of polarization of personal income is, the lower the degree of city's harmony is. How can we construct a one-dimensional element to represent the concept of "harmonious personal income" of a city? Apparently, it is a quantity (concept) with uncertain boundary, so using an exact real number to represent it is not suitable. In addition, it is well known that a city's "harmonious personal income" is also closely related to people's satisfaction degree to the city. Now we will put forward a method to construct a non-normal trapezoidal fuzzy number to represent the concept "harmonious personal income" of a city.*

We select n individuals randomly and record their income x_i, $i = 1, 2, \cdots, n$ and satisfaction ω_i, $i = 1, 2, \cdots, n$ to a city. Then calculate the mean value, L-deviation degree and R-deviation degree of these data respectively. It is shown as follows:

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad L\sigma = \frac{1}{n_i}\sum_{x_i<\overline{x}} (\overline{x} - x_i), \quad R\sigma = \frac{1}{n_i}\sum_{x_i>\overline{x}} (x_i - \overline{x})$$

Denote as $a = \bar{x} - \alpha L\sigma$, $b = \bar{x} - \beta L\sigma$, $c = \bar{x} + \beta R\sigma$, $d = \bar{x} + \alpha R\sigma$, $\omega = \frac{1}{n}\sum\limits_{i=1}^{n} \omega_i$, in which $\alpha \in [2, 4]$, $\beta \in [0, \frac{1}{2}]$, then $u = (a, b, c, d; \omega)$ is a non-normal trapezoidal fuzzy number and can represent the concept "harmonious personal income" of one city.

The follow data set (Table 2) are selected randomly in five cities about the personal income $x \in [0, +\infty)$ (Unit: Yuan) and the satisfaction $\omega \in [0, 1]$ to the corresponding city.

Now we will make a ranking to the five cities about the city's "harmonious personal income" by the new approach proposed in this paper. According to the data set, we can

TABLE 2. The personal income and satisfaction of five cities

	City 1		City 2		City 3		City 4		City 5	
	x_1	ω_1	x_2	ω_2	x_3	ω_3	x_4	ω_4	x_5	ω_5
1	3500	0.8	5000	0.7	13000	0.9	2700	0.9	2000	0.5
2	5500	0.9	1850	0.7	2000	0.9	5900	0.9	3900	0.8
3	4950	0.5	4530	0.8	5900	0.9	8400	0.8	4950	0.7
4	2080	0.3	9040	1.0	4800	0.7	8300	0.9	5800	0.7
5	4000	0.7	5840	0.9	3000	0.8	2000	0.7	1600	0.5
6	6210	0.8	2900	0.9	1300	0.4	3200	0.9	3800	0.7
7	10300	0.7	4300	0.7	9500	1.0	6300	0.7	7200	0.9
8	1800	0.4	1500	0.6	2600	0.7	5500	0.8	4600	0.6
9	31000	1.0	3210	0.6	8700	0.9	3000	0.7	13000	0.8
10	7800	0.8	2430	0.6	6000	0.9	1900	0.4	9500	0.8
11	3200	0.6	6000	0.9	5600	0.8	4500	0.8	2700	0.5
12	9000	0.9	6720	0.6	1900	0.6	20600	1.0	3650	0.6
13	5840	0.9	2600	0.4	3100	0.6	6780	0.8	18500	0.9
14	2400	0.6	3800	0.9	1600	0.5	15000	0.7	5500	0.7
15	3650	0.8	3500	0.7	8800	1.0	3800	0.5	4590	0.7
16	2560	0.7	8000	0.9	5000	0.7	4390	0.7	3520	0.6
17	4250	0.8	2500	0.6	2000	0.5	2850	0.6	9600	0.8
18	25000	0.6	3450	0.7	2200	0.4	14000	0.9	25000	1.0
19	3000	0.6	6300	0.8	4200	0.8	5000	0.8	3500	0.8
20	2700	0.5	2000	0.5	15000	0.9	6500	0.8	4000	0.7

construct five non-normal trapezoidal fuzzy numbers to represent the five cities' "harmonious personal income" respectively, as follows:

	*city*1	*city*2	*city*3	*city*4	*city*5
$\overline{x}$	6937	4273.5	5310	6531	6845.5
$L\sigma$	3227.7	15969.	2501.7	2421	2980.5
$R\sigma$	9683	1918.7	3752.5	5649	6954.5
ω	0.695	0.725	0.745	0.765	0.715

where $\alpha = 2$, $\beta = \frac{1}{2}$. The five non-normal trapezoidal fuzzy numbers are respectively $u_1 = (481.67, 5323.2, 11779, 26303; 0.695)$, $u_2 = (1133.8, 3488.6, 5232.9, 8110.9; 0.725)$, $u_3 = (306.67, 405.9, 7186.3, 12815; 0.745)$, $u_4 = (1689, 5320.5, 9355.5, 17829; 0.765)$, $u_5 = (884.5, 5355.3, 10323, 20755; 0.715)$. We can obtain the ranking index value of five cities respectively, they are $d(u_1, o) = 21608$, $d(u_2, o) = 8443.7$, $d(u_3, o) = 11934$, $d(u_4, o) = 15871$, $d(u_5, o) = 18087$. So according to the ranking index proposed in this paper, the order of five cities about "harmonious personal income" is $u_2 \prec u_3 \prec u_4 \prec u_5 \prec u_1$.

It is obvious that using normal fuzzy numbers to represent the quantity (concept) "harmonious personal income" with uncertain boundary is not suitable since people's satisfaction degree to the city is not certainly 1 (100%). So the previous methods are not suitable to deal with the specific example of application. It shows that the field of application of the novel approach proposed is wider than the previous methods.

5. **Conclusion.** In this paper, we first defined the concept of quasi-slope and discussed its properties. Then, we put forward to the distance on non-normal trapezoidal fuzzy number space based on the concept of quasi-slope, and proved that the distance satisfied translation invariance and absolute homogeneity when the maximum of membership function of non-normal trapezoidal fuzzy numbers are all the same. And then we set up a novel approach to rank non-normal trapezoidal fuzzy numbers based on the distance, and have given some numerical examples to illustrate the advantages of the proposed approach. At last, by a specific example of application, we introduced a method constructing a non-normal trapezoidal fuzzy number to represent a quantity (concept) with uncertain boundary, and illustrated the advantage and practicality of the main results proposed in this paper

Acknowledgment. This work is partially supported by Natural Science Foundations of China (No. 61273077) and the Nature Science Foundation of Zhejiang Province, China (No. LY12A01001).

REFERENCES

[1] B. Asady, The revised method of ranking LR fuzzy number based on deviation degree, *Expert Systems with Applications*, vol.37, no.7, pp.5056-5060, 2010.

[2] S. Abbasbandy and B. Asady, Ranking fuzzy numbers by sign distance, *Information Sciences*, vol.176, no.16, pp.2405-2416, 2006.

[3] S. Abbasbandy and T. Hajjari, A new approach for ranking of trapezoidal fuzzy numbers, *Computers and Mathematics with Applications*, vol.57, no.3, pp.413-419, 2009.

[4] B. Asady and A. Zendehnam, Ranking fuzzy numbers by distance minimization, *Appl. Math. Model.*, vol.31, no.11, pp.2589-2598, 2007.

[5] J. F. Baldwin and N. C. F. Guild, Comparison of fuzzy numbers on the same decision space, *Fuzzy Set and Systems*, vol.2, pp.213-233, 1979.

[6] S. Chen, Ranking fuzzy numbers with maximizing set and minimizing set, *Fuzzy Sets and Systems*, vol.17, no.2, pp.113-129, 1985.

[7] C. H. Cheng, A new approach for ranking fuzzy numbers by distance method, *Fuzzy Sets and Systems*, vol.95, no.3, pp.307-317, 1998.

[8] C. H. Chou, A fuzzy backorder inventory model and application to determining the optimal empty-container quantity at a port, *International Journal of Innovative Computing, Information and Control*, vol.5, no.12, pp.4825-4834, 2009.
[9] S. J. Chen and S. M. Chen, Fuzzy risk analysis based on similarity measures of generalized fuzzy numbers, *IEEE Transactions on Fuzzy Systems*, vol.11, no.1, pp.45-46, 2003.
[10] F. Choobineh and H. Li, An index for ordering fuzzy numbers, *Fuzzy Sets and Systems*, vol.54, no.3, pp.287-294, 1993.
[11] T. Chu and C. Tsao, Ranking fuzzy numbers with an area between the centroid point and original point, *Computers and Mathematics with Applications*, vol.43, no.1-2, pp.111-117, 2002.
[12] S. S. L. Chang and L. A. Zadeh, On fuzzy mapping and control, *IEEE Transactions on Systems, Man and Cybernetics*, vol.2, no.1, pp.30-34, 1972.
[13] J. F. Ding, Fuzzy MCDM approach for selecting strategic partner: An empirical study of a container shipping company in Taiwan, *International Journal of Innovative Computing, Information and Control*, vol.5, no.4, pp.1055-1068, 2009.
[14] D. Dubois and H. Prade, Operations on fuzzy numbers, *International Journal of Systems Science*, vol.9, no.6, pp.613-626, 1978.
[15] M. Detyniecki and R. Yager, Ranking fuzzy numbers using α-weighted valuations, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol.8 pp.573-591, 2000.
[16] H. L. Kwang and J. H. Lee, A method for ranking fuzzy numbers and its application to decision-making, *IEEE Transactions on Fuzzy Systems*, vol.7, no.6, pp.677-685, 1999.
[17] E. S. Lee and R. J. Li, Comparison of fuzzy numbers based on the probability measure of fuzzy events, *Computers and Mathematics with Applications*, vol.15, no.10, pp.887-896, 1988.
[18] A. M. Nejad and M. Mashinchi, Ranking fuzzy numbers based on the areas on the left and the right sides of fuzzy number, *Computers and Mathematics with Applications*, vol.61, no.2, pp.431-442, 2011.
[19] V. A. Niskanen, Application of fuzzy linguistic cognitive maps to prisoner's dilemma, *International Journal of Innovative Computing, Information and Control*, vol.2, no.1, pp.139-152, 2006.
[20] Y. Shi, P. Messenger and M. Mizumoto, Fuzzy inference modeling based on fuzzy singleton-type reasoning, *International Journal of Innovative Computing, Information and Control*, vol.3, no.1, pp.13-20, 2007.
[21] L. Tran and L. Duckein, Comparison of fuzzy numbers using a fuzzy distance measure, *Fuzzy Sets and Systems*, vol.35, no.3, pp.331-341, 2002.
[22] Y. L. P. Thorani and N. R. Shankar, Ordering generalized trapezoidal fuzzy numbers, *Int. J. Contemp. Math. Sciences*, vol.7, no.12, pp.555-573, 2012.
[23] Y. J. Wang and H. S. Lee, The revised method of ranking fuzzy numbers with an area between the cencroid and original points, *Computers and Mathematics with Applications*, vol.55, no.9, pp.2033-2042, 2008.
[24] Z. X. Wang and Y. J. Liu, Ranking L-R fuzzy number based on deviation degree, *Information Sciences*, vol.179, no.13, pp.2070-2077, 2009.
[25] Y. M. Wang and Y. Luo, Area ranking of fuzzy numbers based on positive and negative ideal points, *Computers and Mathematics with Applications*, vol.58, no.9, pp.1769-1779, 2009.
[26] Z. Y. Xu and S. C. Shang, A method for fuzzy risk analysis based on the new similarity of trapezoidal fuzzy numbers, *Expert Systems with Applications*, vol.37, no.3, pp.1920-1927, 2010.
[27] R. R. Yager, On a general class of fuzzy connectives, *Fuzzy Sets and Systems*, vol.4, no.3, pp.235-242, 1980.
[28] J. Yao and K. Wu, Ranking fuzzy numbers based on decomposition principle and signed distance, *Fuzzy Sets and Systems*, vol.116, no.2, pp.275-288, 2000.