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LINEAR MATRIX INEQUALITY-BASED REPETITIVE CONTROLLER DESIGN FOR LINEAR SYSTEMS WITH TIME-VARYING UNCERTAINTIES TO REJECT POSITION-DEPENDENT DISTURBANCES

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Abstract. *This paper addresses the problem of designing an optimal modified repetitive controller for a strictly proper plant with time-varying uncertainties to reject position-dependent disturbances. A modified repetitive controller with time-varying delay structure, inserted by a low-pass filter and an adjustable parameter, is developed for this class of system. Two linear matrix inequalities (LMIs)-based robust stability conditions of the closed-loop system with time-varying state delay are derived for fixed parameters. One is a delay-dependent robust stability condition that is derived based on the free-weight matrix. The other robust stability condition is obtained based on the H_∞ control problem by introducing a linear unitary operator. To obtain the desired controller, the design problems are converted to two LMI-constrained optimization problems by reformulating the LMIs given in the robust stability conditions. The validity of the proposed method is verified through a numerical example.*

Keywords: Modified repetitive controller, Low-pass filter, Position-dependent disturbance, Time-varying period-time, Linear matrix inequality (LMI), Uncertainty

1. **Introduction.** In practical applications, many control systems must deal with periodic reference and/or disturbance signals, for example industrial robots, computer disk drives, CD player tracking control, machine tool motion control, and vibration attenuation of engineering structures. One control system that can deal with periodic reference and/or disturbance signals is a repetitive control system, as proposed by Hara et al. [1]. This system is based on the internal model principle (IMP) proposed by Francis and Wonham [2], and has proved to be a very efficient scheme for tracking periodic reference signals or rejecting periodic disturbances. A disadvantage of typical repetitive controllers is that they are based on the constant period of the external signal. This means that in practical applications, either the period must be constant (±0.1%) or an accurate measurement of the periodicity is necessary.

However, in practice, rotary motion systems have found applications in various industry products. For most applications, the systems are required to operate at variable speeds while following repetitive trajectories and/or rejecting disturbances, such as the brushless DC electric motor in a typical laser printer described by Chen et al. [3]. In general, the periods of reference signals and/or disturbances are mostly time varying in such systems. For instance, consider the flat cam grinding system in Figure 1, which requires the control system to track a time-varying periodic reference signal. This system uses noncircular grinding and the cam is machined by utilizing a profile copier controlled by a linear servomotor. In the traditional grinding system, the cam rotates at a constant speed, which means the cam is machined at a varying tangent velocity. This leads to different

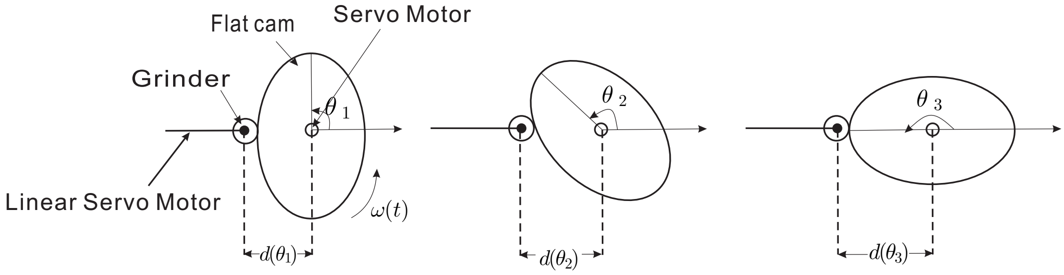

Figure 1. Flat cam grinding system

metal-removal rates and the cam may not meet the requirements. Therefore, to achieve the required machining conditions, the cam is controlled by a servomotor that is required to rotate at a varying speed $\omega(t)$, as shown in Figure 2, and this means that the reference input signal, the distance $d(\theta)$ between the circle centers of the grinder and flat cam, is a time-varying periodic signal, i.e., a position-dependent periodic signal. Hence, it is necessary to design a controller for the linear servomotor to track the position-dependent reference input signal $d(\theta)$. Because it is periodic with respect to angular position, but not necessarily with respect to time, the conventional repetitive control technique is not directly applicable in this case. A very common design method for this class of system is to transform a linear system from the time domain into a spatial domain.

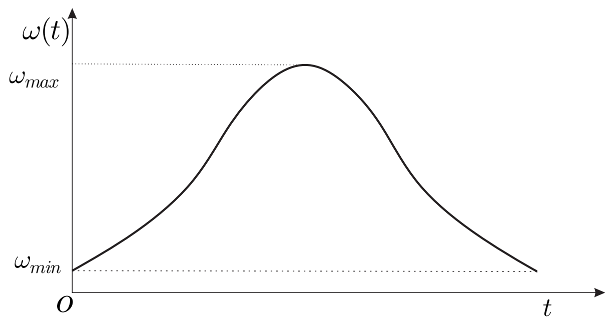

Figure 2. Rotation speed of servomotor

Recently, several studies have considered the problem of rejecting and/or tracking spatially periodic disturbances and/or reference inputs for rotary motion systems using a spatial-based repetitive controller [4, 5, 6, 7, 8, 9]. Nakano et al. [4] eliminated the angular position-dependent disturbances in constant-speed rotation control systems by transforming all signals defined in the time domain to the spatial domain, and obtained a stabilizing controller using coprime factorization. To track spatially periodic reference inputs, Mahawan and Luo [5] proposed a repetitive controller design method using operator-theoretic approaches. Sun [6] addressed the tracking or rejecting problem for position-dependent signals by converting the continuous-time system into a discrete spatial system. A more advanced design based on linearization using H_∞ robust control was proposed by Chen and Allebach [7]. Chen and Chiu [8] proved that the reformulated nonlinear plant model could be cast into a quasilinear parameter-varying system that can be used to address spatially periodic disturbances. In particular, a method of designing a spatial-based repetitive control system for rotary motion systems subject to position-dependent disturbances based on adaptive feedback linearization was presented by Chen and Yang [9].

With the domain transformation, the linear system in the time domain is cast into a nonlinear system in the spatial domain. Before designing the repetitive controller, it is necessary to linearize the nonlinear control system, which makes the design of the repetitive controller more complicated and difficult. In particular, for the control of plants with uncertainties or time-varying state delay, there exists a trade-off problem between robust stability and control performance in the design of repetitive control systems, and spatial-based design methods do not provide a satisfactory solution to this trade-off. Hence, there is a clear need to develop an efficient design method for repetitive control systems that track or reject the position-dependent signals.

In this paper, the position-dependent signal will be converted into a time-varying periodic signal. Inspired by the structure of the repetitive controller [1] and the structure of the optimal repetitive controller [10], we propose a new modified repetitive controller for position-dependent signals. Compared with the conventional modified repetitive controller, the constant time-delay element is replaced by a time-varying operator in our new controller. Moreover, an adjustable parameter is introduced in the new structure to adjust the convergence rate of the closed-loop system and improve the control precision. This controller is plugged into the closed-loop system for a strictly proper plant with uncertainties to reject position-dependent disturbances. The control performance of this repetitive control system then depends heavily on the cutoff frequency of the low-pass filter and the adjustable parameter that represent the trade-off between system robust stability and rejection performance. To achieve the optimal performance and guarantee robust stability, the design problem considered in this paper is converted into a robustly stabilizing problem based on linear matrix inequalities (LMIs). Two LMI-based robust stability conditions of the closed-loop system with time-varying state delay are derived for fixed parameters. One is a delay-dependent robust stability condition that is derived based on the free-weight matrix. The other robust stability condition is based on the H_∞ control approach and introduces a linear unitary operator. The optimal values of the cutoff frequency of the low-pass filter and the adjustable parameter can be obtained by solving the optimization problems with LMI-constrained conditions. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed design method.

Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{R}^{n\times n}$ is the set of all $n \times n$ real matrices; I is the identity matrix; a function $f(t) \in L_2[0, t_f]$ satisfies $\int_0^{t_f} f(t)f(t)dt < \infty$; $\|G(s)\|_\infty := sup_{0\le\omega\le\infty}|G(j\omega)|$ is the H_∞ norm of a transfer function $G(s)$; the superscript T stands for the transposition of a matrix and the symmetric terms in a symmetric matrix are denoted by $*$, for example: $\begin{bmatrix} A & B \\ * & C \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$.

2. **Problem Statement and Preliminaries.** In this paper, we convert the position-dependent reference into a time-varying periodic signal, in contrast with the conventional processing method, which transforms a linear system in the time domain into a nonlinear system in the spatial domain. The position-dependent disturbance $d(t)$ is given by

$$d(t) := \tilde{d}(\theta) = \tilde{d}(\theta - T_\theta), \tag{1}$$

where $\tilde{d}(\theta)$ is the position-dependent disturbance, T_θ is the period, and the rotational angle $\theta(t)$ is defined as:

$$\begin{cases} \theta(t) & := & f(t) = \int_0^t \omega(s)ds \\ \omega(t) & = & \dfrac{d\theta}{dt} > 0 \ \ \forall\, t > 0 \end{cases}, \tag{2}$$

where $\omega(t)$ is the rotational speed and guarantees that $\theta(t)$ is strictly monotonic such that $t = f^{-1}(\theta)$ exists. Thus, for a large enough t, there exist a $t_\theta > 0$ such that

$f(t_\theta) = f(t) - T_\theta$. We define a time-varying function $\tau(t)$ as

$$\tau(t) := \begin{cases} t_0 & 0 < t < t_0 \\ t - f^{-1}(f(t) - T_\theta) = t - t_\theta & t \geq t_0 \end{cases}, \tag{3}$$

where $t_0 = f^{-1}(T_\theta)$ satisfies $T_\theta = f(t_0) - f(0)$. Then by Lagrange's mean value theorem, there exists at least one point $\xi \in (t_\theta, t)$ such that

$$T_\theta = f(t) - f(t_\theta) = f'(\xi)\tau(t) = \omega(\xi)\tau(t). \tag{4}$$

Then

$$\tau(t) = \frac{T_\theta}{\omega(\xi)} \leq \frac{T_\theta}{\omega_{\min}}. \tag{5}$$

From the inverse function theorem, the derivative of function $\tau(t)$ is

$$\dot{\tau}(t) = \begin{cases} 0 & 0 < t < t_0 \\ 1 - \dfrac{\omega(t)}{\omega(t_\theta)} \leq 1 - \dfrac{\omega_{\min}}{\omega_{\max}} & t \geq t_0 \end{cases}. \tag{6}$$

From the Equations (5) and (6), there exist positive scalars $\bar{\tau}$ and μ such that $\tau(t)$ satisfies

$$0 < \tau(t) \leq \bar{\tau}, \quad \dot{\tau}(t) \leq \mu, \quad 0 \leq \mu < 1. \tag{7}$$

Then, the position-dependent disturbance signal can be transformed into a time-varying periodic signal as

$$d(t) = \begin{cases} \tilde{d}(\theta(t)) & 0 < t < t_0 \\ d(t - \tau(t)) & t \geq t_0 \end{cases}, \tag{8}$$

where $\tau(t)$ is the period defined in (3) and satisfying (7).

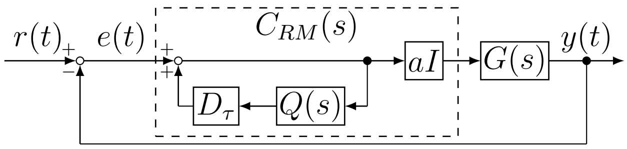

FIGURE 3. The new repetitive control system

Given the time-varying period and inspired by the structure of repetitive controllers [1] and optimal repetitive controllers [10], we establish a new repetitive controller, shown in Figure 3, for time-varying periodic signals. Compared with the conventional repetitive controller, the constant time-delay element is replaced by the time-varying operator D_τ defined as

$$D_\tau(v(t)) := v(t - \tau(t)), \tag{9}$$

where $\tau(t)$ is the period of the disturbance $d(t)$ in (8).

It is well known that the performance of a repetitive control system depends strongly on the cutoff frequency of the included low-pass filter, which represents the trade-off between system stability and control precision. However, it is hard to determine the optimal bandwidth in practice because of the plant uncertainty and system stability. To overcome this problem, we modify the system gain by introducing an adjustable parameter a into the repetitive controller. From the structure of the repetitive controller in Figure 3, the gain of $C_{RM}(s)$ is always proportional to the adjustable parameter a; thus, the performance of the repetitive control system is strongly dependent on both the cutoff frequency ω_c and the adjustable parameter a. Hence, it is clear that the cutoff frequency ω_c and the adjustable parameter a should be as high as possible to obtain good rejection.

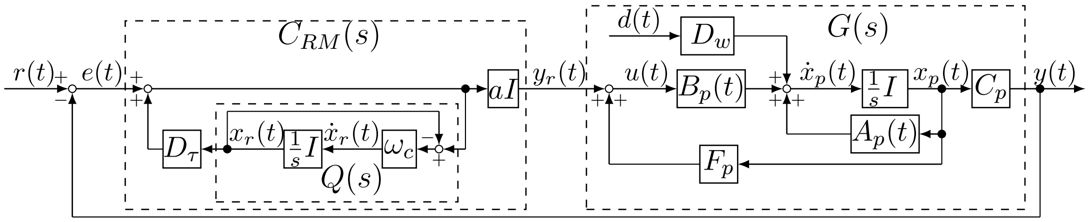

FIGURE 4. The repetitive control system with uncertainties

We consider the design problem of the modified repetitive control system shown in Figure 4 that rejects signals that are periodic in the spatial domain while the rotational speed varies in real-time. The strictly proper plant with uncertainties is described as

$$\begin{cases} \dot{x}_p(t) &= A_p(t)x_p(t) + B_p(t)u(t) + D_w d(t) \\ y(t) &= C_p x_p(t) \end{cases}, \tag{10}$$

where $x_p(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^m$ are the state, input, and output signals, respectively, $A_p(t) \in \mathbb{R}^{n\times n}$, $B_p(t) \in \mathbb{R}^{n\times m}$, $C_p \in \mathbb{R}^{m\times n}$, and $D_w \in \mathbb{R}^{n\times m}$. $d(t) \in \mathbb{R}^m$ is an input disturbance that is periodic in the spatial domain and belongs to $L_2[0, t_f]$. Assume that the uncertainties of the plant are given by

$$\begin{cases} \begin{bmatrix} A_p(t) & B_p(t) \end{bmatrix} &= \begin{bmatrix} A_p + \Delta A_p(t) & B_p + \Delta B_p(t) \end{bmatrix} \\ \begin{bmatrix} \Delta A_p(t) & \Delta B_p(t) \end{bmatrix} &= \Phi_p \Gamma(t) \begin{bmatrix} \Psi_A & \Psi_B \end{bmatrix} \end{cases}, \tag{11}$$

where $A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times m}$, Φ_p, Ψ_A, and Ψ_B are known constant matrices, and $\Gamma(t) \in \mathbb{R}^{n\times n}$ is an unknown real and possibly time-varying matrix with Lebesgue-measurable entries satisfying

$$\Gamma^T(t)\Gamma(t) \leq I, \quad \forall t \geq 0. \tag{12}$$

$Q(s)$, given by

$$Q(s) = \frac{\omega_c}{s + \omega_c} I \in \mathbb{R}^{m\times m}, \tag{13}$$

is the low-pass filter of the repetitive controller $C_{RM}(s)$, where ω_c is the cutoff frequency of the low-pass filter $Q(s)$, and a is an adjustable parameter.

The problem that should be addressed first is to design a feedback controller of the form

$$u(t) = F_p x_p(t) + e(t) \tag{14}$$

such that the closed-loop system, without the modified repetitive controller, is stabilized. Applying the control law (14) to (10) with $r(t) \equiv 0$ yields the closed-loop system

$$\begin{cases} \dot{x}(t) = \{A_p(t) - B_p(t)C_p + B_p(t)F_p\}x_p(t) + D_w d(t) \\ y(t) = C_p x_p(t) \end{cases}. \tag{15}$$

The following lemma presents a rate-dependent state-feedback controller to stabilize (15) robustly with a prescribed H_∞ norm-bound specification.

Lemma 2.1. [11] *For a prescribed scalar $\gamma > 0$, the closed-loop system* (15) *is robustly stable and satisfies $\|y(t)\|_2 < \gamma\|d(t)\|_2$, if there exists a matrix $P^T = P > 0$, a scalar $\lambda > 0$, and an arbitrary matrix W with appropriate dimensions satisfying*

$$\begin{bmatrix} \Lambda_1 & D_w & PC_p^T & \Lambda_2 & \lambda\Phi \\ * & -I & 0 & 0 & 0 \\ * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & -\lambda I & 0 \\ * & * & * & * & -\lambda I \end{bmatrix} < 0, \tag{16}$$

$$\Lambda_1 := (A_p - B_pC_p)P + B_pW + W^TB_p^T + P(A_p - B_pC_p)^T, \tag{17}$$

and

$$\Lambda_2 := P\Psi_A^T - PC_p^T\Psi_B^T + W^T\Psi_B^T. \tag{18}$$

Then the H_∞ state-feedback controller is given by $F_p = WP^{-1}$.

The scalar γ can be regarded as a disturbance performance index. The problem of robust stabilization, to find a state-feedback controller such that the closed-loop system is stable with disturbance attenuation γ, can easily be obtained by solving the above feasible problem for the given γ.

We next present an efficient method to find the optimal values of the cutoff frequency ω_c and the adjustable parameter a.

3. **Robust Stability Conditions.** In this section, we describe a design method to find the optimal values of the cutoff frequency of the low-pass filter and the adjustable parameter.

As shown in Figure 4, the state-space description of the repetitive controller is

$$\begin{cases} \dot{x}_r(t) = -\omega_c x_r(t) + \omega_c x_r(t - \tau(t)) + \omega_c e(t) \\ y_r(t) = ae(t) + ax_r(t - \tau(t)) \end{cases}. \tag{19}$$

By using the augmented state vector $x := [x_p^T, x_r^T]^T$, we combine (19) and (10) with $r(t) \equiv 0$, $d(t) \equiv 0$ and

$$u(t) = F_p x_p + y_r(t) \tag{20}$$

to yield the closed-loop system

$$\dot{x}(t) = (A + \Delta A(t))\, x(t) + (A_1 + \Delta A_1(t))\, x(t - \tau(t)), \tag{21}$$

where $A = \begin{bmatrix} A_p + B_pF_p - aB_pC_p & 0 \\ -\omega_c C_p & -\omega_c I \end{bmatrix}$, $A_1 = \begin{bmatrix} 0 & aB_p \\ 0 & \omega_c I \end{bmatrix}$, $\Delta A(t) = \Phi\Gamma(t)E_1$, $\Delta A_1(t) = \Phi\Gamma(t)E_2$, $\Phi = \begin{bmatrix} \Phi_p^T & 0 \end{bmatrix}^T$, $E_1 = \begin{bmatrix} \Psi_A + \Psi_B F_p - a\Psi_B C_p & 0 \end{bmatrix}$, and $E_2 = \begin{bmatrix} 0 & a\Psi_B \end{bmatrix}$.

To establish the design method, the following lemmas are required.

Lemma 3.1 (Schur complement [12])**.** *For a real matrix $\Sigma = \Sigma^T$, the following assertions are equivalent:*

1. $\Sigma := \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ * & \Sigma_{22} \end{bmatrix} > 0$.
2. $\Sigma_{11} > 0$, *and* $\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12} > 0$.
3. $\Sigma_{22} > 0$, *and* $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T > 0$.

Lemma 3.2 (BRL [13])**.** *For the system*

$$\begin{cases} \dot{x}(t) = Ax(t) + Bw(t) \\ z(t) = Cx(t) + Dw(t) \end{cases}, \tag{22}$$

the following assertions are equivalent:

1. *A is stable; and the H_∞ norm of the transfer function, $G_{zw}(s)$, from $w(t)$ to $z(t)$ satisfying $\|G_{zw}\|_\infty < 1$.*
2. *There exists a symmetric matrix $P > 0$ such that*

$$\begin{bmatrix} PA + A^TP & PB & C^T \\ * & -I & D^T \\ * & * & -I \end{bmatrix} < 0 \tag{23}$$

holds.

Lemma 3.3. [14] *Given the matrices $Q = Q^T$, H, E, and $R = R^T > 0$ of appropriate dimensions,*

$$Q + HFE + E^T F^T H^T < 0$$

for all F satisfying $F^T F \le R$, if and only if there exists some $\lambda > 0$ such that

$$Q + \lambda HH^T + \lambda^{-1} E^T RE < 0.$$

Lemma 3.4. [15] *Consider a nominal system with time-varying delay given by*

$$\begin{cases} \dot{x}(t) &= Ax(t) + A_1 x(t - \tau(t)), \qquad t > 0 \\ x(t) &= \phi(t), \qquad t \in [\bar{\tau}, 0] \end{cases}, \tag{24}$$

where the initial condition, $\phi(t)$, is a continuous vector-valued initial function of $t \in [\bar{\tau}, 0]$. Then, for given scalars $\bar{\tau}$ and μ, the system (24) *is globally asymptotically stable for any time delay satisfying* (7)*, if there exist symmetric positive definite matrices P, Q, and Z, symmetric matrices X_{11} and X_{22}, and arbitrary matrices X_{12}, Y, and T with appropriate dimensions such that the following LMIs are true.*

$$\begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix} \ge 0, \tag{25}$$

$$\begin{bmatrix} X_{11} & X_{12} & Y \\ * & X_{22} & T \\ * & * & Z \end{bmatrix} \ge 0, \tag{26}$$

and

$$\Sigma := \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \bar{\tau} A^T Z \\ * & \Sigma_{22} & \bar{\tau} A_1^T Z \\ * & * & -\bar{\tau} Z \end{bmatrix} < 0, \tag{27}$$

where

$$\Sigma_{11} = PA + A^T P + Y^T + Y + Q + \bar{\tau} X_{11},$$

$$\Sigma_{12} = PA_1 - Y + T^T + \bar{\tau} X_{12},$$

and

$$\Sigma_{22} = -T^T - T - (1 - \mu) Q + \bar{\tau} X_{22}.$$

Now, applying these lemmas to system (21) yields the following theorem.

Theorem 3.1. *For given scalars $\bar{\tau}$ and μ satisfying* (7)*, the system* (21) *is robustly stable if there exist symmetric positive definite matrices P, Q, and Z, symmetric matrices X_{11} and X_{22}, a positive scalar λ, and arbitrary matrices X_{12}, Y, and T with appropriate dimensions such that* (25) $\sim$ (26) *and the following LMI are true.*

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \bar{\tau} A^T Z & P\Phi & \lambda E_1^T \\ * & \Sigma_{22} & \bar{\tau} A_1^T Z & 0 & \lambda E_2^T \\ * & * & -\bar{\tau} Z & \bar{\tau} Z\Phi & 0 \\ * & * & * & -\lambda I & 0 \\ * & * & * & * & -\lambda I \end{bmatrix} < 0, \tag{28}$$

where Σ_{11}, Σ_{12}, and Σ_{22} are defined in (27).

Proof: The proof follows from Lemma 3.4. Let us reconsider the matrix inequality $\Sigma < 0$ defined in (27). We shall replace A and A_1 with $A(t) = A + \Phi\Gamma(t)E_1$ and $A_1(t) = A_1 + \Phi\Gamma(t)E_2$, respectively, in (27) and rewrite the resulting inequality in the form of nominal and uncertain parts as

$$\Sigma + \Sigma_u + \Sigma_u^T < 0, \tag{29}$$

where Σ is defined in (27) and

$$\Sigma_u := \begin{bmatrix} P\Delta A(t) & P\Delta A_1(t) & 0 \\ 0 & 0 & 0 \\ Z\Delta A(t) & \bar{\tau} Z\Delta A_1(t) & 0 \end{bmatrix}. \tag{30}$$

We can decompose Σ_u and express it as

$$\Sigma_u = H\Gamma(t)E, \tag{31}$$

where $H = \begin{bmatrix} \Phi^T P & 0 & \Phi^T Z \end{bmatrix}^T$ and $E = \begin{bmatrix} E_1 & E_2 & 0 \end{bmatrix}$. For $\lambda > 0$, applying Lemma 3.3 to (29) results in

$$\Sigma + \lambda^{-1} HH^T + \lambda E^T E = \Sigma + \lambda^{-1} HH^T + \lambda^{-1}(\lambda E^T)(\lambda E) < 0. \tag{32}$$

By employing the Schur complement Lemma 3.1, the LMI given in (27) is obtained. Thus, system (21) with admissible uncertainties (11) satisfying (12) is robustly asymptotically stable.

We have thus proved this theorem. □

Because the system matrices A and A_1 contain the design parameters ω_c and a, Theorem 3.1 cannot be used directly to obtain the optimal values of the cutoff frequency and adjustable parameter. However, as we now show, (28) can be converted into LMIs that can be used to calculate the optimal cutoff frequency for given a.

For convenience, we represent ω_c as the sum of $\hat{\omega}_c$ and $\delta\omega_c$ that is:

$$\omega_c = \hat{\omega}_c + \delta\omega_c, \tag{33}$$

where $\hat{\omega}_c$ is a roughly estimated value and $\delta\omega_c$ is an unknown value to be found. The matrices A and A_1 can then be represented in the following form:

$$A = \bar{A} + \hat{A} \times \delta\omega_c \tag{34}$$

and

$$A_1 = \bar{A}_1 + \hat{A}_1 \times \delta\omega_c, \tag{35}$$

where

$$\bar{A} = \begin{bmatrix} A_p + B_p F_p - aB_p C_p & 0 \\ -\hat{\omega}_c C_p & -\hat{\omega}_c I \end{bmatrix},$$

$$\hat{A} = \begin{bmatrix} 0 & 0 \\ -C_p & -I \end{bmatrix},$$

$$\bar{A}_1 = \begin{bmatrix} 0 & aB_p \\ 0 & \hat{\omega}_c I \end{bmatrix},$$

and

$$\hat{A}_1 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

Denote

$$Q := \bar{Q} - \hat{Q} \times \delta\omega_c > 0 \tag{36}$$

and

$$\lambda := \bar{\lambda} - \hat{\lambda} \times \delta\omega_c > 0, \tag{37}$$

where $\bar{Q}^T = \bar{Q}$, $\hat{Q}^T = \hat{Q}$, and $\bar{\lambda}$, $\hat{\lambda} \in R$. Then, the LMI (28) can be described by

$$\Xi + \hat{\Xi} \times \delta\omega_c < 0. \tag{38}$$

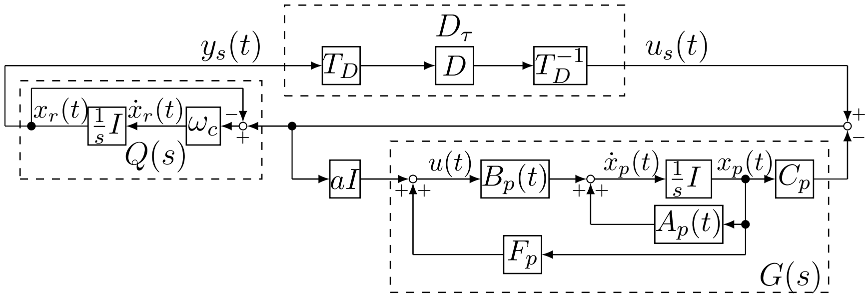

Figure 5. Equivalent diagram of Figure 4

Ξ and $\hat{\Xi}$ are represented as

$$\Xi := \begin{bmatrix} \Xi_{11} & \Xi_{12} & \bar{\tau}\bar{A}^T Z & P\Phi & \bar{\lambda}E_1^T \\ * & \Xi_{22} & \bar{\tau}\bar{A}_1^T Z & 0 & \bar{\lambda}E_2^T \\ * & * & -\bar{\tau}Z & \bar{\tau}Z\Phi & 0 \\ * & * & * & -\bar{\lambda}I & 0 \\ * & * & * & * & -\bar{\lambda}I \end{bmatrix} \tag{39}$$

and

$$\hat{\Xi} := \begin{bmatrix} \hat{\Xi}_{11} & P\hat{A}_1 & \bar{\tau}\hat{A}^T Z & 0 & -\hat{\lambda}E_1^T \\ * & (1-\mu)\hat{Q} & \bar{\tau}\hat{A}_1^T Z & 0 & -\hat{\lambda}E_2^T \\ * & * & 0 & 0 & 0 \\ * & * & * & \hat{\lambda}I & 0 \\ * & * & * & * & \hat{\lambda}I \end{bmatrix}, \tag{40}$$

where

$$\Xi_{11} = P\bar{A} + \bar{A}^T P + Y^T + Y + \bar{Q} + \bar{\tau}X_{11},$$

$$\Xi_{12} = P\bar{A}_1 - Y + T^T + \bar{\tau}X_{12},$$

$$\Xi_{22} = -T^T - T - (1-\mu)\bar{Q} + \bar{\tau}X_{22},$$

and

$$\hat{\Xi}_{11} = P\hat{A} + \hat{A}^T P - \hat{Q}.$$

By introducing a new variable $\sigma := 1/\delta\omega_c$, then (36) $\sim$ (38) can be rewritten as

$$\hat{\Xi} < -\sigma\Xi, \quad \hat{Q} < \sigma\bar{Q}, \quad \hat{\lambda} < \sigma\bar{\lambda}. \tag{41}$$

This gives the following result.

Theorem 3.2. *For given a, and scalars $\bar{\tau}$ and μ satisfying* (7)*, if there exist the symmetric positive definite matrices P and Z, symmetric matrices, $\bar{Q}$, $\hat{Q}$, X_{11}, and X_{22}, scalars $\bar{\lambda}$ and $\hat{\lambda}$, and arbitrary matrices X_{12}, Y, and T with appropriate dimensions such that* (25) $\sim$ (26) *and* (41) *are true, then the cutoff frequency given by* (33) *guarantees the robust stability of the repetitive control system* (21).

Proof: From Theorem 3.1 and Equations (33) $\sim$ (41), this theorem can be obtained directly. This completes the proof. □

Thus, for the given rough estimate $\hat{\omega}_c$, we can obtain the optimal cutoff frequency ω_c by solving the following LMI-constrained optimization problem

$$\min\ \sigma > 0\ \text{ subject to } (25), (26) \text{ and } (41). \tag{42}$$

On the other hand, Mahawan and Luo [5] proved that there exists a unitary operator T_D such that the control system shown in Figure 4 is equivalent to the control system shown in Figure 5 with $r(t) \equiv 0$ and $d(t) \equiv 0$. The unitary operator T_D satisfies

$$\left\| T_D^{-1} D T_D \right\|_\infty \le 1, \tag{43}$$

where the delay operator $D : L^2(0, \theta_f) \to L^2(0, \theta_f)$ is defined as

$$D\zeta(\theta) := \zeta(\theta - T_\theta) \tag{44}$$

and T_θ is the spatial period of the disturbances.

The transfer function $T_{y_s u_s}(s)$ from u_s to y_s is given by

$$T_{y_s u_s}(s) = Q(s) \left(I + aG(s) \right)^{-1}. \tag{45}$$

Then, from the small-gain theorem, the closed-loop system with the modified repetitive controller is asymptotically stable if

$$\|T_{y_s u_s}\|_\infty = \|Q(s)(I + aG(s))^{-1}\|_\infty < 1. \tag{46}$$

Hence, for the given $Q(s)$ and F_p, we can regulate the parameter a to the optimal value by using the H_∞ control method.

From Figure 5, the state space description of $T_{y_s u_s}$, in general, is given by

$$\begin{cases} \dot{x}(t) = (A_s + \Delta A_s(t))x(t) + (B_s + \Delta B_s(t))u_s(t) \\ y_s(t) = C_s x(t) \end{cases}, \tag{47}$$

where $x(t)$ is defined in (21) and $A_s = \begin{bmatrix} A_p + B_p F_p - aB_p C_p & 0 \\ -\omega_c C_p & -\omega_c I \end{bmatrix}$, $B_s = \begin{bmatrix} aB_p \\ \omega_c I \end{bmatrix}$, $C_s = \begin{bmatrix} I & 0 \end{bmatrix}$, $\Delta A_s(t) = \Phi_s \Gamma(t) E_s$, $\Delta B_s(t) = \Phi_s \Gamma(t) a \Psi_B$, $\Phi_s = \begin{bmatrix} \Phi_p^T & 0 \end{bmatrix}^T$ and $E_s = \begin{bmatrix} \Psi_A + \Psi_B F_p - a\Psi_B C_p & 0 \end{bmatrix}$.

Applying Lemmas 3.1 $\sim$ 3.3 to the above system yields the following result.

Theorem 3.3. *For the system* (47)*, if a symmetric matrix $P > 0$ and a positive scalar λ exist such that the LMI*

$$\begin{bmatrix} PA_s + A_s^T P & PB_s & C^T & P\Phi_s & \lambda E_s^T \\ * & -I & 0 & 0 & a\lambda \Psi_B^T \\ * & * & -I & 0 & 0 \\ * & * & * & -\lambda I & 0 \\ * & * & * & * & -\lambda I \end{bmatrix} < 0 \tag{48}$$

holds, then the closed-loop system in (47) *is robustly stable.*

Proof: According to Lemma 3.2, a necessary and sufficient condition that guarantees both that the closed-loop system in Figure 5 is robustly stable and also that (46) holds is that there exists a symmetric matrix $P > 0$ such that the following linear matrix inequality is feasible.

$$\Pi_n + \Pi_u + \Pi_u^T < 0, \tag{49}$$

where

$$\Pi_n := \begin{bmatrix} PA_s + A_s^T P & PB_s & C^T \\ * & -I & 0 \\ * & * & -I \end{bmatrix} \tag{50}$$

and

$$\Pi_u := \begin{bmatrix} P\Phi_s \\ 0 \\ 0 \end{bmatrix} \Gamma(t) \begin{bmatrix} E_s & a\Psi_B & 0 \end{bmatrix}. \tag{51}$$

For a positive scalar, $\lambda > 0$, employing Lemma 3.3, we obtain

$$\Pi_u + \Pi_u^T \leq \lambda^{-1} \begin{bmatrix} P\Phi_s \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} P\Phi_s & 0 & 0 \end{bmatrix} + \lambda^{-1} \begin{bmatrix} \lambda E_s^T \\ a\lambda\Psi_B^T \\ 0 \end{bmatrix} \begin{bmatrix} \lambda E_s & a\lambda\Psi_B & 0 \end{bmatrix}. \tag{52}$$

Substituting (52) into (49) appropriately and applying the Schur complement Lemma 3.1, the LMI given in (48) is obtained.

We have thus proved this theorem. □

Hence, the problem of regulating the parameter a satisfying (46) is converted into the problem of regulating the parameter a satisfying the LMI condition (48). We now find the largest parameter $a_{\max}$ to guarantee the system stability using the result of Theorem 3.3.

Without loss of generality, represent $a_{\max}$ as the sum of a_0 and $\bar{\delta}a$

$$a_{\max} = a_0 + \bar{\delta}a, \tag{53}$$

where a_0 is given in the Theorem 3.2 and $\bar{\delta}a$ is an unknown value to be decided. Then, A_s, B_s and E_s are affinities dependent on the free parameter $\bar{\delta}a$ and are represented as the following form:

$$A_s = \bar{A}_s + \hat{A}_s \times \bar{\delta}a, \tag{54}$$

$$B_s = \bar{B}_s + \hat{B}_s \times \bar{\delta}a \tag{55}$$

and

$$E_s = \bar{E}_s + \hat{E}_s \times \bar{\delta}a, \tag{56}$$

where

$$\bar{A}_s = \begin{bmatrix} A_p + B_pF_p - a_0B_pC_p & 0 \\ -\omega_cC_p & -\omega_cI \end{bmatrix},$$

$$\hat{A}_s = \begin{bmatrix} -B_pC_p & 0 \\ 0 & 0 \end{bmatrix},$$

$$\bar{B}_s = \begin{bmatrix} a_0B_p \\ \omega_cI \end{bmatrix},$$

$$\hat{B}_s = \begin{bmatrix} B_p \\ 0 \end{bmatrix},$$

$$\bar{E}_s = \begin{bmatrix} \Psi_A + \Psi_BF_p - a_0\Psi_BC_p & 0 \end{bmatrix}$$

and

$$\hat{E}_s = \begin{bmatrix} -\Psi_BC_p & 0 \end{bmatrix}.$$

In the following theorem, a modified stability condition is proposed, which is represented as an LMI.

Theorem 3.4. *For given ω_c and F_p, the adjustable parameter given by (53) guarantees the robust stability of the repetitive control system (47), if there is a symmetric positive definite matrix P, and positive scalars λ and $\rho := \bar{\delta}a^{-1}$ such that*

$$\hat{\Theta} < -\rho\Theta \tag{57}$$

holds with the shorthand

$$\Theta := \begin{bmatrix} P\bar{A}_s + \bar{A}_s^T P & P\bar{B}_s & C^T & P\Phi_s & \lambda \bar{E}_s^T \\ * & -I & 0 & 0 & a_0\lambda\Psi_B^T \\ * & * & -I & 0 & 0 \\ * & * & * & -\lambda I & 0 \\ * & * & * & * & -\lambda I \end{bmatrix} \tag{58}$$

and

$$\hat{\Theta} := \begin{bmatrix} P\hat{A}_s + \hat{A}_s^T P & P\hat{B}_s & 0 & 0 & \lambda \hat{E}_s^T \\ * & 0 & 0 & 0 & \lambda\Psi_B^T \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix}. \tag{59}$$

Proof: Replacing a, A_s, B_s and E_s by (54) $\sim$ (56) in (48), we have:

$$\Theta + \hat{\Theta} \times \bar{\delta}a < 0. \tag{60}$$

By introducing the new variable $\rho := \bar{\delta}a^{-1}$ and applying it to (60), the LMI condition (57) can be obtained.

We have thus proved this theorem. □

We observe that for a given optimal cutoff frequency and a, the maximum $\bar{\delta}a$ can be obtained by solving the optimization problem

$$\min \ \rho > 0 \ \text{ subject to (57)}. \tag{61}$$

The constraints in the optimization problems (42) and (61) have the standard forms of generalized eigenvalue minimization problems (GEVP) with semipositive conditions. Hence, they can be solved numerically using the bisection algorithm in YALMIP [16] or the GEVP solver in the LMI-toolbox [17].

4. Design Procedure.

In this section, we present a design procedure for a robust stabilizing modified repetitive controller with optimal performance for position-dependent disturbances.

<u>Procedure</u>

Step 1: Select a solution precision, ϵ, for the optimization problems and positive real scalars, γ, a, and $\hat{\omega}_c$ that are small enough.

Step 2: Solve the feasible problem (16) to obtain the state-feedback controller F_p with given γ for position-dependent disturbances without a repetitive controller.

Step 3: Check the feasibility of Theorem 3.1.

Step 4: If feasible, go to the next step. Otherwise, select new values for a and $\hat{\omega}_c$, and return to step 2.

Step 5: Solve the optimization problem (42) using a, $\hat{\omega}_c$, and F_p. If a solution exists, then set $\omega_c = \hat{\omega}_c + 1/\sigma$ and go to the next step. Otherwise, set $\omega_c = \hat{\omega}_c$ and go to the next step.

Step 6: Solve the optimization problem (61) using a, F_p and ω_c. If a solution exists, set $a_{\max} = a + 1/\rho$ and stop. Otherwise, set $a_{\max} = a$ and stop.

The design procedure proposed in this section is applicable for both single-input/single-output (SISO) linear systems and multiple-input/multiple-output (MIMO) linear systems by simply modifying the dimensions of some matrices.

5. **Numerical Example.** In this section, a numerical example is shown to illustrate the effectiveness of the proposed design method.

Consider the SISO system (10) with $A_p = \begin{bmatrix} -8 & -10 \\ 1 & 0 \end{bmatrix}$, $B_p = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $C_p = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $D_w = \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}$, $\Gamma(t) = \begin{bmatrix} \sin(0.1\pi t) & 0 \\ 0 & \cos(0.1\pi t) \end{bmatrix}$, $\Phi_p = \begin{bmatrix} 0 & 0 \\ 1 & 0.1 \end{bmatrix}$, $\Psi_A = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}$, and $\Psi_B = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$.

We set $\gamma = 0.1$. Then, the state-feedback controller F_p obtained by solving the feasible problem (16) is

$$F_p = \begin{bmatrix} -1.308 & -21.621 \end{bmatrix}. \tag{62}$$

Choose $\epsilon = 10^{-3}$, $a = 1$, $\hat{\omega}_c = 30$[rad/s] and suppose that the disturbance signal, as shown in Figure 6, is given by

$$d(t) = \sin\left(\frac{2\pi}{5}\theta\right) + \sin\left(\frac{4\pi}{5}\theta\right) \tag{63}$$

and

$$\frac{d\theta}{dt} = \omega(t) = 10 + 5\cos(t). \tag{64}$$

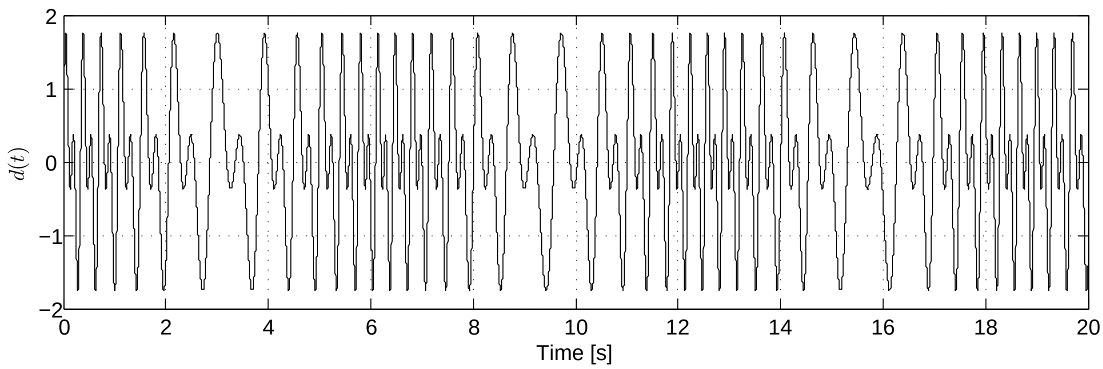

Figure 6. Disturbance signal used in simulations

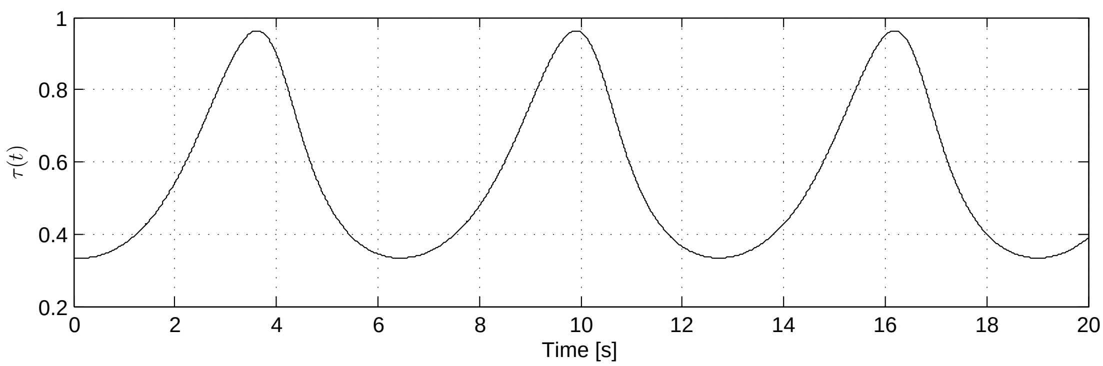

Figure 7. Time-varying period, $\tau(t)$

Then the position-dependent disturbance is converted into a time-varying periodic signal with period $\tau(t)$, shown in Figure 7 and its derivative is shown in Figure 8. From Figures 7 and 8, the time-varying period satisfies (7) and we set

$$\bar{\tau} = 1, \quad \mu = 0.4. \tag{65}$$

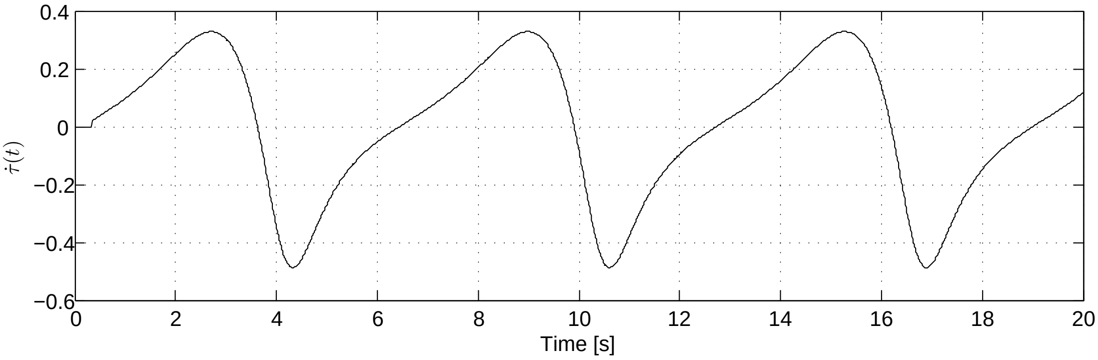

FIGURE 8. Derivative of $\tau(t)$, $\dot{\tau}(t)$

According to the design procedures in Section 3 and using the above parameters, the minimum σ is obtained by solving the optimization problem (42) as $\sigma = 7.755 \times 10^{-4}$. Therefore, the maximum cutoff frequency ω_c of the low-pass filter $Q(s)$ is

$$\omega_c = \hat{\omega}_c + \frac{1}{\sigma} = 1319.490 \text{ [rad/s]}. \tag{66}$$

After obtaining the optimal cutoff frequency ω_c, we solve the optimization problem (61) to obtain the largest adjustable parameter a_{max} as

$$a_{max} = 1 + \frac{1}{\rho} = 12.628 \tag{67}$$

with the minimum $\rho = 0.086$.

The simulation results in Figure 9 show that the system enters the steady state in the second period and that the output is 0.68% of the disturbance when considering the amplitude of the disturbance and the output after the application of the new repetitive controller. For comparison, we also simulated this control system without the repetitive controller. The simulation results in Figure 10 show that, without the repetitive controller, the disturbance is attenuated to about 4.00%. Clearly, better disturbance attenuation is obtained with the proposed repetitive control system than without the repetitive controller. This design procedure demonstrates that the control performance can be improved by optimizing the parameters of the new modified repetitive controller from a general disturbance attenuation control system and that robust stability can also be guaranteed. In contrast, the design methods proposed in [4, 9] achieve robust stability without considering the control precision and are required to deal with a nonlinear system in the spatial domain. Thus, an optimal modified repetitive controller can easily be designed as shown here for position-dependent disturbances.

6. **Conclusions.** In this paper, position-dependent disturbances are converted into time-varying periodic signals and a new modified repetitive controller structure is presented. To obtain good disturbance attenuation, we proposed a design method for the optimal modified repetitive control system based on LMIs, which can be applied to rotary motion systems. We also gave a complete proof of the theorems for the design method that were omitted previously [18]. By reformulating the LMI-constrained robust stability conditions, an optimal modified repetitive control system can be obtained by solving the resulting optimization problems. A numerical example was presented to demonstrate the effectiveness of the proposed design method. The results in this paper extend the application of the repetitive control technique to systems with time-varying uncertainties, and

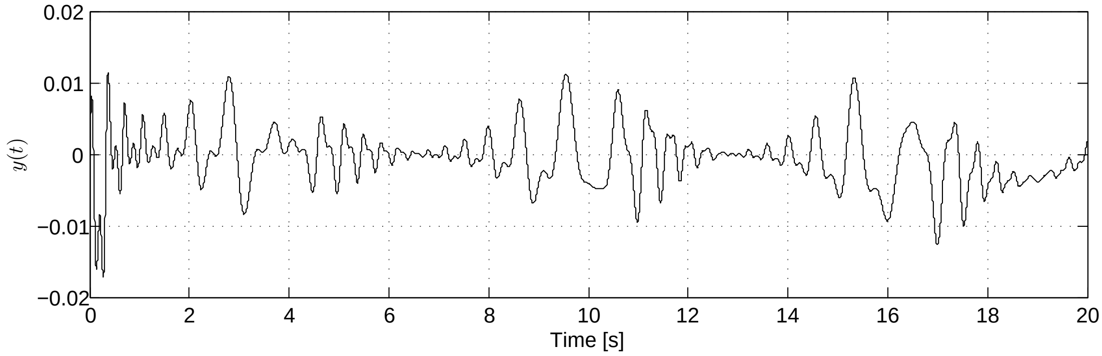

FIGURE 9. Response of the output $y(t)$ for the disturbance $d(t)$ with our repetitive controller

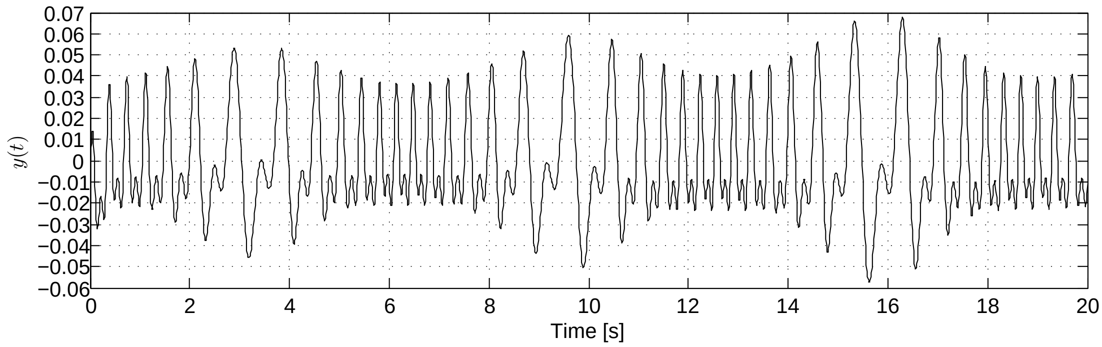

FIGURE 10. Response of the output $y(t)$ for the disturbance $d(t)$ without our repetitive controller

can also be potentially applied to the systems with time-varying state delay and input delay [19].

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