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CONSENSUS PROBLEMS FOR HIGH-ORDER LTI SYSTEMS: A DECENTRALIZED STATIC OUTPUT FEEDBACK METHOD

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Abstract. *This paper deals with the consensus problem in multi-agent systems (MASs) with fixed and switching interaction topologies. All agents are modeled by identical linear high-order dynamical systems. Through some coordinate transformation and the augment of the output matrix, some consensus protocols are provided based on the output measurements of the adjacent agents. Sufficient conditions for consensus were presented in terms of coupled linear matrix inequalities (LMIs), the size of which will not increase according to the number of agents. Some numerical examples are presented to illustrate the effectiveness of the proposed method.*
Keywords: Multi-agent system, Synchronization, Static output feedback, LMI

1. **Introduction.** In the past several years, consensus problems of multi-agent systems have been developing very fast and several research topics have been addressed, such as wireless unmanned system networks (UMSN) [1, 2], swarms and flolcks [3], multi-vehicle systems [4], distributed sensor networks [5]. Since Vicsek et al. [6] proposed a discrete-time mode for multi-agent systems, the theoretical framework with fixed or switching topologies was provided [7], and some of the consensus conditions were further relaxed in [8]. Readers are referred to the surveys [9, 10] for a relatively complete coverage of the literature. Recently, the synchronization of complex dynamical networks, such as small world and scale-free networks, has been widely studied [11-18].

One well-known problem in most existing works is that the agents' dynamics is often restricted to be single or double integrators [19]. In some applications, agents of higher dynamical order are required if consensus of more than two variables is aimed at. Ren et al. [20] studied a special high-order consensus model, which can be regarded as a special controllability canonical form. Wang et al. considered a high-order model with fewer structural limitations, and a sufficient condition for consensus was presented under the assumption that interaction topologies are undirected [21]. A general high-order model is studied in Xiao and Wang [22] with a time-invariant consensus function. Generally, for consensus problems considering general high-order systems, the relevant results are very limited [4, 23-26].

Another common problem is that most proposed distributed consensus protocols are based on the state information of the neighboring agents, which is not always available in practice. Usually, each agent has only access to its adjacent agents' output measurements. The consensus problem based on the output information of the agents is called the "output feedback consensus" problem, which is difficult to be implemented compared with that of state feedback [27]. The difficulty lies in the fact that the system output matrix is always singular, and the conventional robust control theory will no longer apply. The consensus algorithm based on the static output feedback has started gaining attention in the literature [24], but the approach always relies on the initial system states, which will be difficult for large networks. This motivated the present research.

In this paper, we are interested in the synchronization problem for high-order multi-agent systems with fixed and switching interaction topology. The consensus protocol is "designed" based on the output measurements of the adjacent agents. Toward this end, the idea of augmenting the output matrix to be square and invertible is adopted for solving the consensus problem [28, 29]. Through some coordination transformation, the consensus problem is converted into the stability issue, and sufficient conditions are given in terms of a set of LMIs. In contrast to the conditions presented, these conditions are scalable to large undirected networks because the size of the LMI will not increase according to the number of agents. Moreover, it can provide more flexibility in circumventing constraint when compared with a solution that is based on the Riccati equation.

The remainder of this paper is organized as follows: we recall some preliminary results on the graph theory and the consensus problem of MASs in Section 2. In Section 3, the main theorems and the approaches are proposed for the fixed and switching interaction topology. Section 4 is devoted to examples illustrating the efficiency of our proposed approach. Some conclusions and open research topics are presented in Section 5.

A. Notation and Preliminaries. The following notations will be used throughout this paper. A^{-1} and A^T denote the inverse and the transpose of matrix A. $A \geq 0 (> 0)$ denotes that matrix A is positive semi-definite (positive definite). $\lambda(A)$ denotes the eigenvalue set of matrix A; $\mathrm{Re}(\lambda(A))$ and $\mathrm{Im}(\lambda(A))$ denote the real part and imaginary part of the matrix eigenvalue. Matrix A is a Hurwitz (or stable) matrix if all its eigenvalues have strictly negative real parts. $\otimes$ denotes the Kronecker product. $\mathbb{1}_n$ refers to an n-dimensional column vector with the same components 1, I_n is the $n \times n$ identity matrix.

B. Graph Theory. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a diagraph (directed graph) with the set of vertices $\mathcal{V} = \{1, 2, \cdots, N\}$ and the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The set of neighbors of the ith agent is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$. $\mathcal{A} = (a_{ij}) \in R^{N \times N}$ is called the adjacency matrix with non-negative elements, $a_{ij} > 0 \Leftrightarrow j \in N_i$ and $a_{ii} = 0$. The Laplacian matrix of the weighted graph is defined as $L_{\mathcal{G}} = [l_{ij}]_{N \times N}$, where $l_{ii} = \sum_j a_{ij}$ and $l_{ij} = -a_{ij}$, $i \neq j$. A diagraph is called strongly connected if there is a directed path from every node to every other node. An undirected graph is called connected if there is a path between any distinct pair of nodes.

2. **Problem Statement.** Assuming a set of agents indexed by $i = 1, 2, \cdots, N$, suppose that the dynamical representation of each agent be governed by

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t) \\ y_i(t) &= Cx_i(t), \quad i = 1, 2, \cdots, N \end{aligned} \tag{1}$$

where $x_i(t) \in R^n$, $u_i(t) \in R^m$ and $y_i(t) \in R^p$ denote the state vector, the input vector and the output vector of agents, respectively. $A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{p \times n}$ are constant matrices, where C is assumed to be full-row rank.

With regarding the above N agents as vertices, the topology relationships among them can be conveniently described by a diagraph (undirected graph) with $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ and $\mathcal{A} = [a_{ij}] \in R^{N \times N}$. We assume that (A, B) is stabilizable and (A, C) is detectable. The consensus protocol for each agent in MASs is distributed and only relies on the information of the agent itself and its neighbors, since each agent has limited capability of collecting information [3].

It should be pointed out that the agents' dynamics (1) is excluded from having poles in the open right-half plane, otherwise, the consensus value achieved by the agents will tend to infinity exponentially. Therefore, it is assumed that matrix A has no eigenvalues with positive real parts. Typical examples of this case include the single and double integrators considered in the existing literature [4, 8, 10, 11].

The consensus protocol adopted in this paper is

$$u_i(t) = K \sum_{j \in \mathcal{N}_i} a_{ij}(y_j(t) - y_i(t)), \quad i = 1, 2, \cdots, N \tag{2}$$

where $\mathcal{N}_i$ denotes the set of neighbors for agent i, a_{ij} is an adjacent element of $\mathcal{A}$ in the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, and K is the consensus gain of the protocol.

Let $x(t) \in R^{nN}$, $u(t) \in R^{mN}$, $y(t) \in R^{pN}$ be the vectors which collect the states and inputs of the N systems at time t, together with (2), System (1) can be written in a more compact form

$$\dot{x}(t) = A_1 x(t) \tag{3}$$

where

$$A_1 = I_N \otimes A - L_{\mathcal{G}} \otimes BKC.$$

The synchronization is achieved if there is a feedback gain K such that

$$\lim_{t \to \infty} \|x_j(t) - x_i(t)\| = 0, \quad i, j = 1, 2, \cdots, N \tag{4}$$

for any initial value $x(0)$. Let

$$\delta_i(t) = x_1(t) - x_i(t), \quad i = 2, 3, \cdots, N \tag{5}$$

synchronization is achieved if and only if $\delta_i(t) \to 0$ as $t \to 0$ $(i = 2, 3, \cdots, N)$.

Consider an affine transformation, $x(t) \longmapsto (P \otimes I_n) = \begin{bmatrix} x_1(t) \\ \delta(t) \end{bmatrix}$, where

$$P = \begin{bmatrix} 1 & 0 \\ \mathbb{1}_{N-1} & -I_{N-1} \end{bmatrix}, \quad \delta = \left[\delta_2^T, \cdots, \delta_N^N\right]^T$$

Suppose that the Laplacian matrix $L_{\mathcal{G}}$ is partitioned as

$$L_{\mathcal{G}} = \begin{bmatrix} l_{11} & \alpha^T \\ \beta & L_{22} \end{bmatrix} \tag{6}$$

where

$$\alpha = \begin{bmatrix} -a_{12} & -a_{13} & \cdots & -a_{1N} \end{bmatrix}^T$$

and

$$\beta = \begin{bmatrix} l_{21} & l_{31} & \cdots & l_{N1} \end{bmatrix}^T$$

We have

$$PL_{\mathcal{G}}P^{-1} = \begin{bmatrix} 0 & \alpha^T \\ 0 & L_{22} + \mathbb{1}\alpha^T \end{bmatrix} \tag{7}$$

In the new coordinates, System (3) is transformed into

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{\delta}(t) \end{bmatrix} = \begin{pmatrix} I_N \otimes A & -PL_{\mathcal{G}}P^{-1} \otimes BKC \end{pmatrix} x(t) \tag{8}$$

which yields

$$\dot{\delta}(t) = \hat{A}x(t) \tag{9}$$

where

$$\hat{A} = \left[I_{N-1} \otimes A - (L_{22} + \mathbb{1}\alpha^T) \otimes BKC\right]$$

3. Main Results.

3.1. Network with fixed topology. In this section, we will design the consensus protocol based on the output measurements of the adjacent agents. Sufficient conditions are established in terms of LMIs with the complex eigenvalues of the Laplacian matrix as the coefficients.

Lemma 3.1. [8] *The Laplacian matrix $L_{\mathcal{G}}$ of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ has at least one zero eigenvalue and all the nonzero eigenvalues are in open right half plane. Moreover, $L_{\mathcal{G}}$ has a simple zero eigenvalue with eigenvector $\mathbb{1}_n$ if and only if $\mathcal{G}$ has a spanning tree.*

Lemma 3.2. [23] *If System (1) is consensable, then (A, B, C) is stabilizable and detectable, and the topology $\mathcal{G}$ has a spanning tree.*

The following lemma presents necessary and sufficient conditions for the consensus problem under the static output feedback consensus protocol (2).

Lemma 3.3. [23] *For a directed (or undirected) network of agents with communication topology $\mathcal{G}$ that has a direct spanning tree. Protocol (2) solves the consensus problem of System (1) if and only if the following equivalent statements hold:*
(i) There exists a static output feedback gain K such that $\hat{A}$ is Hurwitz.
(ii) There exists a static output feedback gain K, such that

$$A_2 = I_{N-1} \otimes A - J \otimes BKC \tag{10}$$

is Hurwitz, where J is the Jordan canonical matrix of $L_{22}+\mathbb{1}_{N-1}\alpha^T$.
(iii) Static output feedback gain K simultaneously stabilizes the $N-1$ matrices

$$A_3 = A + \lambda_i BKC, \quad i = 2, 3, \cdots, N \tag{11}$$

where λ_i $(i = 2, 3, \cdots, N)$ are the nonzero eigenvalues of matrix $L = -L_{\mathcal{G}}$.

Proof:
$(i) \Leftrightarrow (ii)$ This can be concluded from [23] directly.
$(ii) \Leftrightarrow (iii)$ It is noted from (10) that A_2 is either block diagonal or block upper triangular, hence, A_2 and A_3 have the same pole, e.g., A_2 is Hurwitz if and only if the $N-1$ subsystems

$$\dot{x}_i(t) = (A + \lambda_i BKC)x_i(t), \quad i = 2, 3, \cdots, N$$

are asymptotically stable along the trajectories. This ends the proof of Lemma 3.3.

The purpose of this paper is to design the static output feedback consensus protocol (2) such that System (1) is consensable. Generally, the static output feedback control problem is difficult in the control community. In what follows, we will take a simpler LTI system

$$\begin{aligned} \dot{z}_1(t) &= Az_1(t) + Bu(t) \\ z_2(t) &= Cz_1(t) \end{aligned} \tag{12}$$

as a breakthrough point to address this issue, i.e., design the static output feedback protocol

$$u(t) = Kz_2(t) \tag{13}$$

such that the resulting closed-loop system

$$\dot{z}_1(t) = (A + BKC)z_1(t) \tag{14}$$

is asymptotically stable.

Lemma 3.4. [28] *System (12) is asymptotically stable under the static output feedback protocol (13) if there exists a positive definite matrix* $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \in R^{n\times n}$, $M \in R^{m\times m}$ *such that*

$$\tilde{A}Q + Q\tilde{A}^T + \tilde{B}MS + S^T M^T \tilde{B}^T < 0 \tag{15}$$

where

$$\begin{array}{ll} M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, & S = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \\ \tilde{A} = TAT^{-1}, & \tilde{B} = TB \end{array}$$

and $T = \begin{bmatrix} C^T & * \end{bmatrix}^T$ *is square and full rank. In this case, the feedback gain is given by*

$$K = MSQ^{-1} \begin{bmatrix} I_p & 0 \end{bmatrix}^T \tag{16}$$

where I_p *denotes the identity matrix of order* p*.*

Proof: First, let us augment C by adding some rows. In such a way, T is square and full rank, i.e., choosing a matrix $H \in R^{(n-p)\times n}$ such that

$$T = \begin{bmatrix} C^T & H^T \end{bmatrix}^T \tag{17}$$

It is simple to see that there are an infinite number of matrices satisfying this condition.

In order to solve the problem, we introduce a change of coordinate $z_1 \longmapsto Tz_1 = \tilde{z}_1$. Following the change of coordinates, $(A, B, K) \longmapsto (\tilde{A}, \tilde{B}, \tilde{K})$, where $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, $\tilde{K} = KCT^{-1} = \begin{bmatrix} K & 0 \end{bmatrix}$. System (12) can be further written as

$$\begin{array}{l} \dot{\tilde{z}}_1(t) = \tilde{A}\tilde{z}_1(t) + \tilde{B}\tilde{K}\tilde{z}_1(t) \\ z_2(t) = [I_p \; 0]\tilde{z}_1(t) \end{array} \tag{18}$$

Observe that (18) has the same form with the system that has the state feedback, if the structure constraint on the feedback gain $\tilde{K}$ is neglected.

As we can see, System (18) is asymptotically stable if there exists positive definite matrix $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$, such that

$$(\tilde{A} + \tilde{B}\tilde{K})^T P + P(\tilde{A} + \tilde{B}\tilde{K}) < 0 \tag{19}$$

Denoting P^{-1} by $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$, we obtain (15). Moreover, the static output feedback gain K can be obtained via the equation

$$\tilde{K}Q = MS \tag{20}$$

This completes the proof of Lemma 3.4.

Remark 3.1. *In Lemma 3.4, we have some degree of freedom to choose matrix* H*. This could be exploited to improve the effectiveness of the proposed output feedback stabilization method. The problem left for future discussion is the development of an effective procedure to find an optimal matrix* H*.*

Following the previous considerations, we are interested in evaluating the synchronization problem of System (1) according to the different structure of the Laplacian matrix $L_{\mathcal{G}}$*.*

Theorem 3.1. *Assuming the undirected communication topology $\mathcal{G}$ has a spanning tree. Consensus is achieved under protocol (2) if there exist a positive definite matrix* $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \in R^{m\times m}$, $M \in R^{n\times n}$, *such that*

$$\tilde{A}Q + Q^T\tilde{A}^T + \lambda_i\tilde{B}MS + \lambda_i S^T M^T \tilde{B}^T < 0, \quad i = 2, N \tag{21}$$

where $\tilde{A}$, $\tilde{B}$, M, S *are defined in Lemma 3.4.* $\lambda_2 \leq \cdots \leq \lambda_N$ *are the nonzero eigenvalues of* L. $H \in R^{(n-p)\times n}$ *is chosen such that* T *is invertible. In this case, the feedback gain* K *is given by*

$$K = MSQ^{-1}\begin{bmatrix} I_p & 0 \end{bmatrix}^T \tag{22}$$

Proof: Following from Lemma 3.3, consensus is achieved if and only if $A + \lambda_i BKC$ $(i = 2, 3, \cdots, N)$ are Hurwitz. Namely, there exists matrix $P = P^T = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} > 0$ such that

$$A^TP + PA + \lambda_i PBK + \lambda_i(BK)^TP < 0, \quad i = 2, 3, \cdots, N$$

This, together with Lemma 3.4 yields

$$\tilde{A}^TQ + Q\tilde{A} + \lambda_i\tilde{B}MS + \lambda_i(MS)^T\tilde{B}^T < 0, \quad i = 2, 3, \cdots, N$$

where $Q = Q^T = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$. It is noted that the solution space of the above inequality is convex with respect to λ_i, which yields (21). Moreover, the structure of the matrix M, S and Q guarantees that $\tilde{K}$ has the form

$$\begin{aligned} \tilde{K} &= MSQ^{-1} \\ &= M\begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & Q_2^{-1} \end{bmatrix} \\ &= \begin{bmatrix} K & 0 \end{bmatrix} \end{aligned}$$

This concludes the proof of Theorem 3.1.

Remark 3.2. *The condition that $\mathcal{G}$ has a spanning tree is quite general and weak, as it is intuitively clear that consensus is impossible to reach if $\mathcal{G}$ has disconnected components.*

Remark 3.3. *Theorem 3.1 generalizes the existing results on the consensus problem in at least two aspects. First, the agents' dynamics is extended to be of general LTI, which was not limited to single-integrator as usually assumed in most existing papers [7-9]. Second, the consensus problem with static output feedback protocol is considered, which can be converted into solving two LMIs with two eigenvalues of the Laplacian matrix as the coefficients, thereby significantly reduce the computational complexity. In addition, the effect of the communication topology on the consensus problem is characterized by the nonzero eigenvalues* λ_i $(i = 2, 3, \cdots, N)$ *of the corresponding matrix* L*, which will be real in the case of undirected graph, or may be complex in the general case of directed network. As we know, the LMIs with the complex coefficients cannot be solved directly in Matlab. The following theorem tries to solve this problem.*

Lemma 3.5. *Assuming that* $\lambda_i = u_i \pm v_i \cdot \mathrm{j}$ *are a pair of conjugate eigenvalues of matrix* L*. The following propositions are equivalent:*
(i) $\tilde{A} + \lambda_i\tilde{B}\tilde{K}$ *is Hurwitz, with eigenvalue* s_j *satisfying* $\mathrm{Re}(s_j) < -v$, $j = 1, 2, \ldots, n$, *where* v *is a constant.*
(ii) There exists $P = P^T \in R^{n\times n} > 0$, *such that*

$$\tilde{A}^TP + P\tilde{A}^T + \lambda_i P\tilde{B}\tilde{K} + \lambda_i\tilde{K}^T\tilde{B}^TP < -2vP \tag{23}$$

for the above $v > 0$.
(iii) There exists $Q = Q^T \in R^{n\times n} > 0$, $M \in R^{n\times n}$, *such that*

$$C_0 + \mathrm{Re}(\lambda_i(L))C_R + \mathrm{Im}(\lambda_i(L))C_I < 0 \tag{24}$$

for the above $v > 0$, *where*

$$C_0 = \begin{bmatrix} \tilde{A}Q + Q\tilde{A}^T + 2vQ & 0 \\ * & \tilde{A}Q + Q\tilde{A}^T + 2vQ \end{bmatrix}$$
$$C_R = \begin{bmatrix} \tilde{B}MS + (\tilde{B}MS)^T & 0 \\ * & \tilde{B}MS + (\tilde{B}MS)^T \end{bmatrix}$$
$$C_I = \begin{bmatrix} 0 & (\tilde{B}MS)^T - \tilde{B}MS \\ * & 0 \end{bmatrix}$$

Proof: Let $\lambda_i^1 = u_i + v_i \cdot \mathrm{j}$, $\lambda_i^2 = u_i - v_i \cdot \mathrm{j}$, where $u_i \neq 0$.
(i) $\Leftrightarrow$ *(ii)*
It is observed that

$$\begin{aligned} &\left|\lambda I - (\tilde{A} + \lambda_i^1\tilde{B}\tilde{K})\right| \cdot \left|\lambda I - (\tilde{A} + \lambda_i^2\tilde{B}\tilde{K})\right| \\ &= \begin{vmatrix} \lambda I - (\tilde{A} + u_i\tilde{B}\tilde{K}) & v_i\tilde{B}\tilde{K} \\ -v_i\tilde{B}\tilde{K} & \lambda I - (\tilde{A} + u_i\tilde{B}\tilde{K}) \end{vmatrix} \end{aligned} \tag{25}$$

Hence, if there exists a matrix $\tilde{K}$, such that $\tilde{A} + \lambda_i^1\tilde{B}\tilde{K}$ and $\tilde{A} + \lambda_i^2\tilde{B}\tilde{K}$ are Hurwitz simultaneously, then

$$\begin{bmatrix} \tilde{A} + u_i\tilde{B}\tilde{K} & -v_i\tilde{B}\tilde{K} \\ v_i\tilde{B}\tilde{K} & \tilde{A} + u_i\tilde{B}\tilde{K} \end{bmatrix} \tag{26}$$

are Hurwitz. The principle also works in reverse. Moreover, (26) can also be rewritten as

$$\underbrace{\begin{bmatrix} \tilde{A} & 0 \\ 0 & \tilde{A} \end{bmatrix}}_{\doteq\bar{\bar{A}}} + \underbrace{\begin{bmatrix} \tilde{B} & 0 \\ 0 & \tilde{B} \end{bmatrix}}_{\doteq\bar{\bar{B}}} \underbrace{\begin{bmatrix} u_i\tilde{K} & -v_i\tilde{K} \\ v_i\tilde{K} & u_i\tilde{K} \end{bmatrix}}_{\doteq\bar{\bar{K}}}$$

which are Hurwitz if and only if there is a matrix $\tilde{P} = \tilde{P}^T = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} > 0$, such that

$$\bar{\bar{A}}^T\tilde{P} + \tilde{P}\bar{\bar{A}} + \tilde{P}\bar{\bar{B}}\bar{\bar{K}} + \bar{\bar{K}}^T\bar{\bar{B}}^T\tilde{P} < -2v\tilde{P} \tag{27}$$

This, after simple computation yields (24). Assuming that the characteristic polynomial of (27) is given by $P_i^2(s)$ (if $\lambda_i = u_i$, $v_i = 0$) or $P_{i+}(s) \cdot P_{i-}(s)$ (if $\lambda_i = u_i \pm v_i \cdot \mathrm{j}$, $v_j \neq 0$). If there exist a constant $v > 0$ and $\tilde{P} = \tilde{P}^T$ such that (27) is satisfied, polynomials $P_i^2(s)$ as well as $P_{i+}(s) \cdot P_{i-}(s)$ are both Hurwitz, with roots s_j satisfying $\mathrm{Re}(s_j) < -v$, $j = 1, 2, \cdots, n$.
(ii) $\Leftrightarrow$ *(iii)*
The equivalence of (ii) and (iii) is easily obtained if we let $\tilde{Q} = \tilde{P}^{-1}$ and $\tilde{K}Q = MS$. This ends the proof of Lemma 3.5.

Theorem 3.2. *Assuming the Laplacian matrix* $L_{\mathcal{G}}$ *of System (1) has* P *nonzero real eigenvalues and* $(N - P)/2$ *conjugate complex eigenvalues. Consensus is achieved under the distributed protocol (2) if the directed graph has a spanning tree and there exists a positive definite matrices* $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \in R^{n\times n}$, $M \in R^{m\times m}$, *such that*

$$\tilde{A}Q + Q^T\tilde{A}^T + \lambda_i\tilde{B}MS + \lambda_i S^T M^T \tilde{B}^T < 0, \quad i = 2, P \tag{28}$$

and

$$C_0 + \mathrm{Re}(\lambda_i(L))C_R + \mathrm{Im}(\lambda_i(L))C_I < 0, \quad i = P+1, P+2, \cdots, (N-P)/2 \tag{29}$$

where C_0, C_R, C_I are defined in Lemma 3.5. In this case, the feedback gain is given by

$$K = MSQ^{-1}\begin{bmatrix} I_p & 0 \end{bmatrix}^T$$

Proof: The proof of Theorem 3.2 is easily obtained based on Lemma 3.5 and Theorem 3.1.

Remark 3.4. *It is noted that Theorem 3.2 degenerates to Theorem 3.1 when all the eigenvalues of the Laplacian Matrix $L_{\mathcal{G}}$ are real, i.e., $v_i = 0$ $(i = P+1, P+2, \cdots, (N-P)/2)$.*

3.2. **Network with switching topology.** Consider a network of mobile agents that communicate with each other and need to agree upon performing synchronization. Since the nodes of the network are moving, some of the existing communication links may fail due to the existence of an obstacle between two agents, and some new links, on the other hand may create in terms of the network topology. Here, we are interest in designing of the feedback gain K for the network with the switching topology and considering the possibility of reaching a consensus.

Suppose the continuous-state of the switching system evolve according to the following dynamics

$$\dot{x}(t) = (I_N \otimes A + L_s \otimes BKC)x(t) \tag{30}$$

where $x(t) \in R^n$. The discrete-state $\{s : s = \sigma(t) \in \psi_0\}$ belongs to a finite collection of directed graphs, $\psi_0 \subset Z$ is a finite index set, $\sigma(t)$ is a switching signal that determines the network topology, $L_s = -L_{\mathcal{G}s}$. The topology of $L_{\mathcal{G}s}$ is assumed to be connected for each $s \in \varphi_0$.

Theorem 3.3. *Assuming the undirected switching topology $\mathcal{G}$ of System (30) has a spanning tree. Consensus is achieved if there exists a positive definite matrices $Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \in R^{n\times n}$, $M \in R^{m\times m}$ such that*

$$\tilde{A}Q + Q^T\tilde{A}^T + \lambda_{is}\tilde{B}MS + \lambda_{is}S^TM^T\tilde{B}^T < 0, \quad i = 2, P \tag{31}$$

and

$$C_0 + \mathrm{Re}(\lambda_{is}(L))C_R + \mathrm{Im}(\lambda_{is}(L))C_I < 0, \quad i = P+1, P+2, \cdots, (N-P)/2 \tag{32}$$

the consensus gain is given by

$$K = MSQ^{-1}\begin{bmatrix} I_p & 0 \end{bmatrix}^T$$

where C_0, C_R and C_I are defined in Lemma 3.5, λ_{is} are the non zero real $(i = 2, \cdots, P)$ and conjugate complex $(i = P+1, \cdots, (N-P)/2)$ eigenvalues of matrix L_s $(s \in \psi_0)$, respectively.

Proof: Let

$$V(t) = x^T(t)Px(t)$$

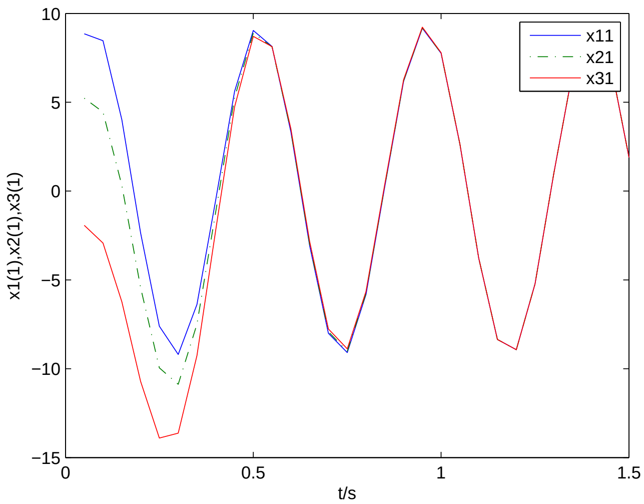
be a common Lyapunov function. Assume that the sth subsystem is achieved at time t, i.e., $\sigma(t) = s$. Taking the derivative of $V(t)$ along the trajectories of (30) implies that there exist $\beta_s > 0$, such that $\dot{V}(t) \le -\beta_s\|x(t)\|^2$.

Let $\beta = \min\{\beta_s : s \in \varphi_0\}$, we have $\dot{V}(t) \le -\beta\|x(t)\|^2$ for any switching signal $\sigma(t)$, and hence the zero solution of (30) is asymptotically stable for any switching signal. The proof of Theorem 3.3 is completed.

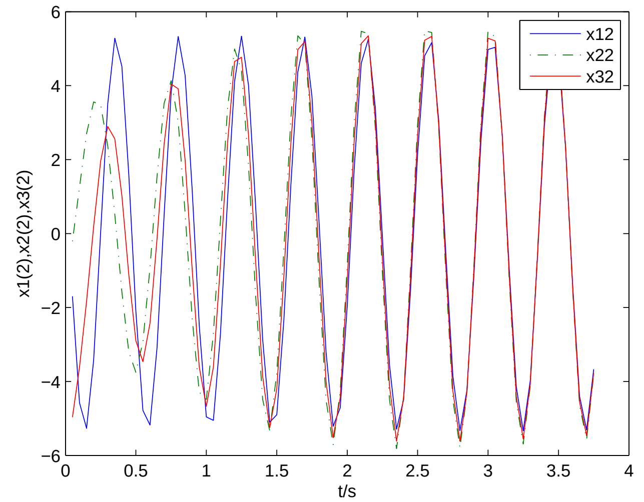
4. **Numerical Examples.**
In this section, the distributed protocol is designed using the approaches sketched above.

Example 4.1. *For an earthquake damage-preventing building, when an earthquake occurs, each agent in this system tries to keep the building horizontal. After the earthquake, the system should recover to be static and horizontal. It is clear that the consensus function of an earthquake damage-preventing building is time-invariant, that is, the velocity and position of each agent converge to zero and a constant respectively. In this and the following examples, we always assume that the dynamic of each agent is LTI and be depicted by*

$$\dot{x}_i(t) = \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix} x_i(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u_i(t)$$
$$y_i(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} x_i(t)$$



(a)

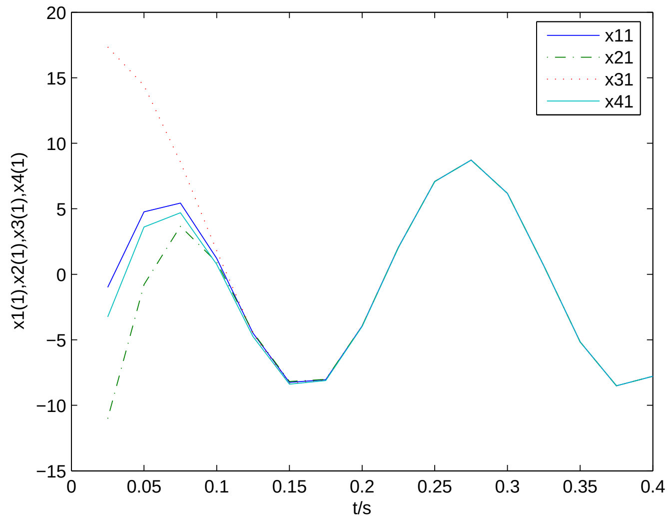


(b)

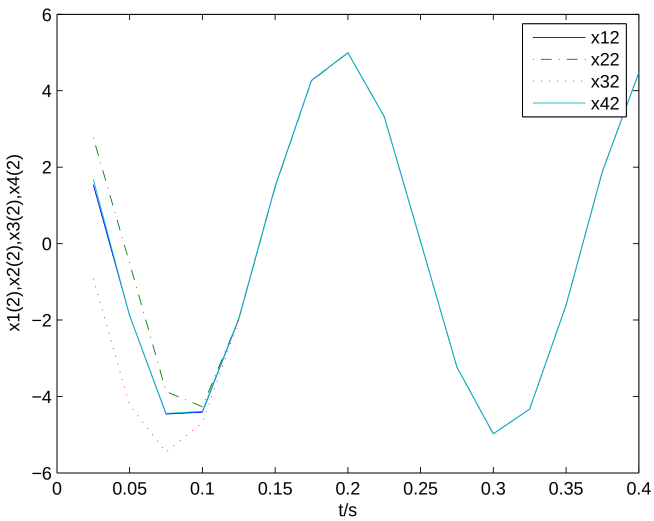
FIGURE 1. (a) x-component of the position profiles, (b) y-component of the velocity profiles

It is easy to check that (A, B) is completely controllable. Here x_i represents the states of the ith agent, which consists of two components, i.e., x (position) and y (velocity). As a first step, a matrix H is chosen to be $H = \begin{bmatrix} -0.1 & 10 \end{bmatrix}$. It is always possible to obtain T so that the transformation can be employed. In the transformed coordinates, $(\tilde{A}, \tilde{B}, \tilde{K})$ is created which is exploited in the second step of the design procedure.

A cyclic nearest neighbor interconnection is assumed among the three agents, e.g., the edge set is given by $\{\varepsilon_{12}, \varepsilon_{21}, \varepsilon_{23}\}$. Obviously, the graph has a spanning tree. A distributed control law as in (2) is designed using the LMIs provided in (21), and the control gain is obtained as $K = 0.4696$. The initial values of the agents are selected as $x_1(0) = [5\ 2]^T$, $x_2(0) = [3\ \ -1]^T$, $x_3(0) = [-2\ \ -6]^T$. In Figure 1(a) and Figure 1(b), we have confirmed that the designed distributed protocol performs very well. One can see that the MASs can recover to be static and horizontal asymptotically.



(a)



(b)

FIGURE 2. (a) x-component corresponding to the solution of LMIs, (b) y-component corresponding to the solution of LMIs

Example 4.2. *Consider the network with the switching topology* $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$*, which is described by the Laplacian matrix*

$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & -1 & -2 & -3 \\ -3 & 4 & -1 & 0 \\ 0 & 0 & 2 & -2 \\ -1 & -2 & -3 & -6 \end{bmatrix},$$

$$\begin{bmatrix} 1.3590 & 0 & 0 & -1.3590 \\ -1.8243 & 1.8243 & 0 & 0 \\ 0 & -1.1319 & 1.9736 & -0.8417 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In this case, some of the existing communication links fail and some of them are created due to the moving of the agents. We can easily see that topologies $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$ are all connected, the state of such network starts at $\mathcal{G}_1$ and then switches every 0.01s to the next state $\mathcal{G}_2$ and then $\mathcal{G}_3$. By using the LMI Toolbox in Matlab, it can be solved that exist S and M such that the (31) and (32) holds. Figure 2(a) and Figure 2(b) demonstrate the convergence of the states to the consensus when the network is running the consensus protocol with different rates of convergence.

5. Conclusions.

This paper investigates the consensus problem of the multi-agent system under the fixed and switching interaction topology. A distributed protocol based on relative output measurements of adjacent agents has been proposed and analyzed. The consensus problem has been converted to the feasibility of a set of LMIs. Finally, numerical examples are presented to illustrate the effectiveness of the proposed methods. Further research includes extensions to MIMO heterogeneous multi-agent systems, and takes the uncertainties into account.

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