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A SWITCHING CONTROLLER FOR THE STABILIZATION OF THE DAMPING INVERTED PENDULUM CART SYSTEM

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Abstract. *We present a switching approach for the stabilization of the strongly damped inverted pendulum cart system, provided that the pendulum is initialized inside of the upper-half plane, and the linear viscous force is known. The control strategy uses two stabilizing controllers. The first is a nonlinear controller, devoted to bringing the pendulum very close to its unstable equilibrium point. Then, this controller switches to a second linear controller in charge of asymptotically and exponentially renders the pendulum to the origin. The stability analysis and the estimation of the attraction domain were made using Lyapunov related tools. Convincing numerical experiments were included, where some comparison among others well-known strategies were performed.*

Keywords: Inverted pendulum system, Direct Lyapunov method, Flatness approach, Switched system

1. **Introduction.** One of the most important mechanical systems studied in Control Theory is the inverted pendulum mounted on a cart. This system is formed by a cart that moves backwards and forwards horizontally along a track, and a pendulum hinged to the cart at the bottom of its length, such that it can freely rotate in the same plane as the cart. This system is inherently unstable and it is almost impossible to keep the pendulum balanced in its inverted position without using an external force applied to the cart. As the external force is only applied to the cart, we do not have any control of the pendulum angular acceleration. That is why the inverted pendulum is an underactuated system and a well-established challenge in Control Theory. As this system is not input-output linearizable by means of static state feedback [1, 2], many control strategies developed for fully actuated systems are useless to control this class of system. Moreover, when the pendulum passes through the horizontal plane, it cannot be controlled and loses some geometric properties [3, 4]. On the other hand, the inverted pendulum is locally controllable around the unstable equilibrium point, and by using the direct pole placement procedure it can be stabilized [2, 5].

Related to the inverted pendulum there are two important issues. One is devoted to swinging the pendulum from the hanging position to the upright position. To this end, the pendulum is brought to a homoclinic orbit; then, once the system is close enough to the desired upright position, with a conveniently slow velocity, a simple change from a non-linear to a linear controller allows keeping the pendulum at this position [6, 7, 8, 9, 10, 11, 12, 13, 14]. The second issue consists of stabilizing the system around its unstable equilibrium point, assuming that the pendulum is initially above the horizontal plane, or lies inside an open vicinity of zero, which is related to the attraction region of the closed-loop system [10, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22]. This work is devoted to solving the upward pendulum stabilization problem with the cart at the origin, tacking into account the undesirable effects of the damping forces presented in both coordinates, the actuated and non actuated. The proposed control strategy, which is based on two control laws, assumes that the pendulum is initialized in the upper-half plane and that contact friction is available. The first control law is nonlinear and is devoted to bringing the position pendulum, the linear cart velocity and the angular pendulum velocity close enough to the origin. Then, the strategy switches to a linear controller, which renders all the system variables to the origin. The nonlinear control law is based on the direct Lyapunov function [23] in conjunction with a suitable change of coordinate, while the linear controller uses a simple linear feedback stabilization controller. Note that when the friction compensation is omitted, the pendulum could not reach the top rest position and the steady-state error in the horizontal cart position can appear. We underscore that the stabilization of the damped inverted pendulum system (**DIPC**) cannot be completely solved by using shaping energy control as pointed out in [24, 25]. For this reason, the undesired damping effect has been neglected in most of the previous works related to controlling the inverted pendulum. As far as we know this problem has only been partially solved in two different manners. The first uses the sliding mode method and the robust linear control method, with the disadvantage of being impossible to compute the attraction domain [26, 27, 28, 29, 30]. The second partial solution is based on the Linear Control Theory, and uses a linearized version of the system.

The following sections are organized as follows. The nonlinear model of the system is presented in Section 2. Our proposed control strategy is developed in Section 3. The numerical simulations to show the effectiveness of our control strategy, by means of some numerical comparisons among three well-known control methods, are shown in Section 4, and the conclusions are in Section 5.

2. The Damped Inverted Pendulum Cart System.

Consider the inverted pendulum mounted on a cart (see Figure 1). This system is described by a set of normalized differential equations [2]:

$$\begin{aligned} &\cos\theta\ddot{x} + \ddot{\theta} - \sin\theta + \lambda_1\dot{\theta} = 0, \\ &(1+\delta)\ddot{x} + \cos\theta\ddot{\theta} - \dot{\theta}^2\sin\theta + \lambda_2\dot{x} = f; \end{aligned} \tag{1}$$

where x is the normalized cart displacement, θ is the angle that the pendulum forms with the vertical, and f the normalized force applied to the cart, which is the input of the system, and $\delta > 0$ is a constant which directly depends on both the cart and pendulum masses. The cart and pendulum viscous frictions are $\lambda_1\dot{\theta}$ and $\lambda_2\dot{q}$ respectively, with their corresponding viscous friction coefficients, $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. Now, if the following feedback:

$$f = u(\delta + \sin^2\theta) + \lambda_2\dot{x} - \lambda_1\cos\theta\dot{\theta} - \dot{\theta}^2\sin\theta + \cos\theta\sin\theta, \tag{2}$$

is introduced into the second equation of (1), we have:

$$\begin{aligned} \ddot{\theta} &= \sin\theta - \cos\theta u - \lambda\dot{\theta} \\ \ddot{x} &= u; \end{aligned} \tag{3}$$

where $\lambda = \lambda_1$.

Motivation: It is well-known that damping in the unactuated directions does tend to enhance stability. However, damping in the controlled directions must be "reversed" through feedback ([24]). That is to say, damping in the actuated coordinated can always

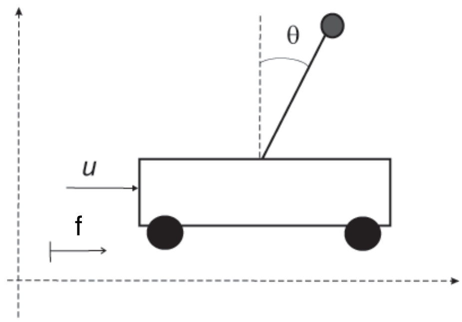

FIGURE 1. The inverted pendulum mounted on a cart

be compensated, which is not the case in the non-actuated coordinate. Besides, damping in the non-actuated coordinated can destroy the stability of the whole system as happened in the Furuta pendulum, the cart pendulum system and the ball and the beam system, to mention a few. It is worth mentioning that all the stability schemes based on shaping energy function, like Lagrange or Hamiltonian approaches, cannot in general assure stability. Roughly speaking these methods consist of shaping a locally strictly positive and proper energy function, with a minimum at the origin. Simultaneously, the controller is proposed such that, the time derivative of the energy function is semi-definite negative. Particularly, the obtained time derivative is semi-definite negative with respect to a manifold (which depends on the velocities and the positions). Afterwards, the stability analysis based on the theorem of LaSalle, has to be applied to analyze the mentioned manifold, and assure that the largest invariant set inside of the manifold is the origin. Unfortunately, these methods are useless when applied to the Furuta Pendulum and the Inverted Pendulum Cart system, because it is not possible to guarantee that the corresponding energy function is a non-increasing function. Consequently, we cannot use the theorem of LaSalle, nor assure convergence. A trick often used to assure asymptotic stability, consists of linearizing the system in closed-loop and proposing the gains of the energy based controller, such that this is locally and exponentially stable. Also, the methods based on saturation function, and bounded control ([17, 20]) neither assures asymptotic stability; only, as already mentioned, we can only assure local exponential stability by using simple linearized models. There are several works where this problem is tackled using the linear quadratic regulator method applied to the linearized control model. As far as is known, this problem has been partially solved using the second order sliding modes method, having the inconvenient of show chattering and discontinuities in the control. The main motivation of this work is introducing a novel control strategy, which allows controlling the Damped Inverted Cart System (**DICS**) by means of an almost continuous and smooth controller, with a conveniently large attraction domain. In our opinion, the innovation of our control strategy consists in a virtual compensation of the dissipative force using a coordinates change in the non-actuated coordinate. This coordinates change permits managing the system as if it were a traditional, non-linear cart pendulum system. This transformed model, allows us to design a control law, based on the Direct Lyapunov method, obtaining a continuous and smooth controller, which renders the pendulum to its upper-right position while both the pendulum and the cart almost stop. Afterwards, a switch to a linear controller brings the cart to the origin while keeping the pendulum in its upper-right position.

Remark 2.1. *To simplify the stabilization problem, the unmatched physical damping term, $\lambda\dot{\theta}$, is not considered in the model. However, the damping force can make the*

DIPC *system unstable, specially when the pendulum is moving inside the upper-half-plane* [24, 31].

Problem formulation: *Consider the dampened inverted pendulum on the cart system (DIPC), described in (3). The goal is to stabilize the pendulum around its upright position, while simultaneously rendering the cart to the origin, provided that the pendulum is initially located inside the upper-half plane and the damping viscous force is presented.*

The following comment is for clarification: The problem will be solved assuming that all the state variables and the system parameters are given. Also, we assume that the pendulum is initialized in the upper-half plane. In order to simplify the development of this work, we assume that λ_2 is given; if not, it can be estimated using an adaptive control low.

3. Nonlinear Control Strategy.

The control strategy presented here consists of two control laws. The first law, which is nonlinear, brings the pendulum to its upright and its velocity and the cart's are taken to zero. Then, this law is switched to a linear law, which is in charge of setting the whole state to the origin.

3.1. Bringing the states θ, $\dot{\theta}$ and $\dot{x}$ to some close vicinity at the origin.

The first step consists of designing a nonlinear law to bring the system very close to its unstable equilibrium point. To this end, we propose a change of variables in the following manner.

We introduce the following coordinates.

$$\begin{aligned} y &= x + 2l_c \int \arctan h\left(\tan\left(\tfrac{\theta}{2}\right)\right); \quad l_c > 0 \\ \dot{y} &= \dot{x} + 2l_c \arctan h\left(\tan\left(\tfrac{\theta}{2}\right)\right) \end{aligned} \tag{4}$$

For simplicity, we denote the quantity, $\int_0^t \phi(\theta(s))ds$, by $\int \phi(\theta)$. Then we split the control action, u, as:

$$u = u_s - l_c \sec\theta\dot{\theta} \tag{5}$$

Thus, after substituting the new coordinates and the new controller, (4) and (5), into the system (3), we obtain:

$$\begin{aligned} \ddot{\theta} &= \sin\theta - \cos\theta u_s + (l_c - \lambda)\dot{}\theta, \\ \ddot{y} &= u_s. \end{aligned} \tag{6}$$

From the above equation, it can be seen that, if the vector positions variables, $q = (\theta, y)^T$, and the vector velocity, $p = (\dot{\theta}, \dot{y})^T$, are brought close enough to the origin, we can assure from (4) that variables, $\{\theta, \dot{\theta}, \dot{x}\}$, are also close to zero with q bounded. Hence, our control strategy consists of bringing variables, q and p, to zero. Then we switch the proposed control law (5) to a linear controller, u_L, as shown below.

To bring variables, q and p, to the origin, we use the Direct Lyapunov Method. To do this, we propose the following Lyapunov function:

$$E(q, p) = \Phi(q) + \frac{1}{2}p^T K p; \tag{7}$$

where the constant matrix, K, fulfills $K = K^T > 0$, and Φ is selected in such a way that $\nabla_q \Phi(0)\big|_{q=0} = 0$ and $\nabla_q^2 \Phi(0)\big|_{q=0} > 0$; therefore, this function is strictly convex locally with a local minimum at the origin.

Now we will find the unknown variables, K, Φ and u_s. To this end, we split the controller u_s, as:

$$u_s(q, p) = u_p(z) + u_d(\theta, p), \tag{8}$$

where u_p is used to simultaneously shape the potential and kinetic energies, while u_d introduces the necessary damping to the closed-loop system. Hence, from (6) and (8), we have that:

$$\begin{aligned} \dot{q} &= p, \\ \dot{p} &= G_d(q, u_p) + F(\theta)u_d + Dp; \end{aligned} \tag{9}$$

where

$$G_d(q, u_p) = \begin{bmatrix} s_1 - c_1 u_p \\ u_p \end{bmatrix}; \quad F(\theta) = \begin{bmatrix} -c_1 \\ 1 \end{bmatrix}; \quad D = \begin{bmatrix} l_c - \lambda & 0 \\ 0 & 0 \end{bmatrix} \tag{10}$$

Now, computing the time derivative of E along the system trajectories (9), we easily have:

$$\dot{E} = p^T \left(\nabla_q \Phi(q) + KG_d(q, u_p)\right) + p^T KF(q_1)u_d + p^T KDp. \tag{11}$$

We should remark that, if the following conditions are satisfied:
A) K, $\Phi(q)$ and $u_p(q)$ satisfy the following equation:

$$\nabla_q \Phi(q) + KG_d(q, u_p) = 0, \tag{12}$$

and,

$$u_d = -\gamma F^T(q_1)Kp; \quad \gamma > 0. \tag{13}$$

B) For some $\gamma > 0$ and $l_c > 0$, there is a symmetric matrix, $M(q_1) < 0$, $q_1 \in I_\gamma$, such that:

$$p^T M(q_1)p = -\gamma p^T KF(q_1)F^T(q_1)Kp + p^T KDp. \tag{14}$$

Then, $\dot{E} \leq 0$. Summarizing we present the following proposition.

Proposition 3.1. *Consider the nonlinear system* (9)*, with $q_1 = \theta \in I_0 \subset (-\pi/2, \pi/2)$; and, K and Φ, as already described. If the set of unknown variables, $\{K, \Phi, u_d, u_p\}$, are selected according to the above conditions,* ***A*** *and* ***B****, then the system* (9) *is locally stable.*

Proof: After substituting (12) and (14) in (11), we have:

$$\dot{E} = p^T M(q_1)p \leq 0; \quad \forall q_1 \in I_0. \tag{15}$$

Since $E(q, p)$ is locally defined strictly positive and a non increasing function for the above relation, then the closed-loop system (9) is at least stable in the Lyapunov sense. Besides, from (15), we can easily assure that, $p \to 0$, as long as, $t \to \infty$. Remember that the proposed Lyapunov function has a locally minimum at the origin.

To apply the last proposition, we assure condition **A**, and then we assure condition **B**.
Solving the condition A: Fixing matrix, K, as:

$$K = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}, \tag{16}$$

where $k_1 > 0$ and $\Delta_K = k_1 k_3 - k_2^2 > 0$. Substituting G_d and K, given in (10) and (16), into (12), we have:

$$\begin{aligned} k_1 (s_1 - c_1 u_p) + k_2 u_p + \frac{\partial \Phi}{\partial q_1} &= 0, \\ k_2 (s_1 - c_1 u_p) + k_3 u_p + \frac{\partial \Phi}{\partial q_2} &= 0. \end{aligned} \tag{17}$$

From the first equation of (17), the control parameter, u_p, is given by:

$$u_p = \frac{\left(k_1 s_1 + \frac{\partial \Phi}{\partial q_1}\right)}{-k_2 + k_1 c_1}, \tag{18}$$

where $-k_2+k_1c_1$ must be different from zero (see below). Substituting u_p into the second equation of (17), we have:

$$\frac{\partial \Phi}{\partial q_2}+\frac{\partial \Phi}{\partial q_1}\frac{\psi(q_1)}{\chi(q_1)}=\frac{\Delta_K s_1}{\chi(q_1)}; \tag{19}$$

where:

$$\chi(q_1)=k_2-k_1c_1; \quad \psi(q_1)=-k_3+k_2c_1. \tag{20}$$

In order to assure that Equations (18) and (19) are well defined, we select k_1, k_2 and k_3 provided that $\chi(q_1)\neq 0$ and $\psi(q_1)\neq 0$; for all, $q_1\in I_0\subset(-\pi/2,\pi/2)$, where I_0 will be determined below. Now, after using the mathematical program to solve (19), we have that Φ can be expressed, as:

$$\Phi(q)=-\beta_K\log\left(-\psi(q_1)\right)+S\left[q_2+\Upsilon(q_1)\right]; \tag{21}$$

where

$$\Upsilon(q_1)=\frac{2\Delta_K}{k_2\delta_K}\arctan\left(\frac{\eta_K}{\delta_K}\tan\left(\frac{q_1}{2}\right)\right)+\frac{k_1}{k_2}q_1, \tag{22}$$

and the remaining constants, η_K, δ_K and β_K, are given by:

$$\delta_K=\sqrt{k_2^2-k_3^2}; \quad \eta_K=k_2+k_3; \quad \beta_K=\tfrac{k_1k_3-k_2^2}{k_2}>0, \tag{23}$$

where function $S(q)$ is selected such that:

$$\Phi(0)=0; \quad \nabla_q\Phi(q)\big|_{q=0}=0; \quad \nabla_q^2\Phi(q)\big|_{q=0}=0. \tag{24}$$

Therefore, from (24), we have that the potential energy $\Phi(q)$ [1] can be proposed, as:

$$\Phi(q)=\beta_K\log\left(\frac{\psi(0)}{\psi(q_1)}\right)+\frac{k_p}{2}\left(q_2+\Upsilon(q_1)\right)^2, \tag{25}$$

where $k_p>0$. The above discussion is summarized by the following remarks.

Remark 3.1. *Defining the potential energy, as:*

$$\Phi(q)=\beta_K\log\left(\frac{\psi(0)}{\psi(q_1)}\right)+\frac{k_p}{2}\rho_2^2 \tag{26}$$

where

$$\rho_2=q_2+\frac{2\Delta_K}{k_2\delta_K}\arctan h\left(\frac{\eta_K}{\delta_K}\tan\left(\frac{q_1}{2}\right)\right)+\frac{k_1}{k_2}q_1, \tag{27}$$

then the function Φ is locally strictly positive with a minimum at the origin, if the positive constants, k_1, k_2 and k_3, are selected, such that:

$$k_2-k_1<0; \quad k_2-k_3>0; \quad k_2^2-k_3^2>0. \tag{28}$$

Based on the previous remark, we introduce the following important lemma, which allows us to estimate the region where Φ is strictly positive and convex.

Lemma 3.1. *Defining the constant:*

$$\bar{c}<\beta_k\log\left(\frac{\psi(0)}{\psi(\bar{\theta})}\right), \tag{29}$$

where

$$\bar{\theta}=\min\left\{\cos^{-1}\left(\frac{k_2}{k_1}\right),\cos^{-1}\left(\frac{k_2\varepsilon}{4\gamma\Delta_K}+\frac{k_3}{k_2}\right)\right\} \tag{30}$$

[1] Notice that $\nabla_q^2\Phi(q)\big|_{q=0}=\frac{k_p\Delta_K}{\psi(0)}$ which implies that $k_p>0$ and $\psi(0)=k_2-k_3>0$.

being $\varepsilon = l_c - \lambda \geq 0$ small as we desired. Then, under the assumption in Remark 3.1, we have that the set:

$$W_c = \{q \in (-\overline{\theta}, \overline{\theta}) = I_0 \times R : \Phi(q) \leq c < \overline{c}\}, \tag{31}$$

is bounded and convex. That is, $\Phi(q)$ is proper on its sub-level, W_c. The proof of this Lemma can be found in [32].

Checking the condition (14): From the restriction **B**, we have that the symmetric matrix $M(q_1) = \{m_{ij}\}$, is given by:

$$\{m_{ij}\} = \begin{bmatrix} -\gamma\chi^2(q_1) + k_1\varepsilon & \gamma\chi(q_1)\psi(q_1) + k_2\varepsilon/2 \\ \gamma\chi(q_1)\psi(q_1) + k_2\varepsilon/2 & -\gamma\psi^2(q_1) \end{bmatrix}$$

where, evidently, from Lemma 3.1, we have, $m_{22} < 0$; for all, $q_1 \in I_0$, and:

$$\det M = \frac{1}{4}(l_c - \lambda)(-k_2^2(l_c - \lambda) + 4\gamma\Delta_K(k_2 c_1 - k_3)).$$

Therefore, selecting $k_2 > k$ and $\gamma >> \frac{k_2\varepsilon}{4\Delta_K}$, we assure that $M(q_1) > 0$ for all $q_1 \in (-\theta_\gamma, \theta_\gamma) \subset I_0$, where θ_γ is given by:

$$\theta_\gamma = \cos^{-1}\left(\frac{k_2\varepsilon}{4\gamma\Delta_K} + \frac{k_3}{k_2}\right).$$

Notice that, from the definition of $\overline{\theta}$, we have $\overline{\theta} \leq \theta_\gamma$.

Convergence analysis of the states, q and p: Now we show that, $q \to 0$, and, $p \to 0$, as long as, $t \to \infty$, with an attraction domain contained inside of, $I_0 \times R^3$. For this end, we note that, according to Lemma 3.1, $\Phi(q)$, is a strictly positive definite and convex function, for all, $q \in I_0 \times R$. Hence, the total energy:

$$E(q,p) = \frac{1}{2}\dot{p}^T K \dot{p} + \beta_K \log\left(\frac{\psi(0)}{\psi(q_1)}\right) + \frac{k_p}{2}(q_2 + \Upsilon(q_1))^2, \tag{32}$$

is strictly positive definite and proper, through $\Omega_{\tilde{c}} \subset I_0 \times R^3$, where:

$$\Omega_{\tilde{c}} = \{(q,p) \in I_\gamma \times R^3 : V(q,p) \leq \overline{c}\}. \tag{33}$$

Then, for Equation (15), we have that, $\dot{V} \leq 0$, implying that, $V(q,p) \leq V(q_0,p_0)$. In order to avoid the singular points of the proposed, V, it is enough that the initial condition satisfies the inequity, $V(q,p_0) \leq \overline{c}$, because, $V(q,p) \leq \overline{c}$, assuring that, $\Phi(q) \leq \overline{c}$, with, $q(t) \in I_0 \times R$. That is, all the solutions starting in, $\Omega_{\tilde{c}}$, remain inside of, $\Omega_{\overline{c}}$, implying that, q, and, p, are bounded, with, $q_1(t) \in I_0$. Notice that, $q_1(t) \subset I_0$, assures, $\psi(q_1) \neq 0$, and, $\chi(q_1) \neq 0$; for all, $q_1 \in I_0$. This fact also assures that, u_p, (18) is well defined.

In order to assure that the closed-loop system converges asymptotically to zero, we most invoke the LaSalle theorem [33]. So, we define the set:

$$S = \left\{(q,p) \in \Omega_{\tilde{c}} : p^T M(q_1) p = 0\right\} \tag{34}$$

where, $M(q_1) > 0$; because, $q_1(t) \in I_0$. From the above we have, $p = 0$, in S, which means that, $\dot{p} = 0$, and, $q = \overline{q}$ (where, $\overline{q}$, is a constant). Now, from (9) and (10), we easily have, $\sin\overline{q}_1 = 0$, implying that, $\overline{q}_1 = 0$, because, $\overline{q}_1 \in I_0$. Similarly, we can show that, $\overline{q}_2 = 0$. Consequently, the largest invariant set, S_0, contained in S, is given by, $S_0 = (q = 0, p = 0)$. According to the theorem of LaSalle, all the closed-loop solutions starting in, $\Omega_{\tilde{c}}$, asymptotically converge toward the largest invariant set, $S_0 = 0$.

3.2. **A linear controller.** The nonlinear control law introduced in the previous section brings the state variables to the origin, except the cart position, which is bounded. Thus, we need to propose a new controller to render the whole system to the origin. It is well known that the pendulum system is a flat system when the pendulum is located inside of an small vicinity of the unstable equilibrium point (see [2]). Therefore, the local regulation problem around the rest upright position can be solved by the pole placement procedure.

Therefore, in the case when θ, $\dot{\theta}$ and $\dot{x}$ are inside of an small vicinity of the origin, for instance:

$$\sqrt{\theta^2 + \dot{\theta}^2 + \dot{x}^2} \leq \theta_P;$$

where $\theta_P < 0.2$. Now, according to this condition, we can define the following set:

$$U_0 = \left\{ z = (\theta, x, \dot{\theta}, \dot{x})^T : \sqrt{\theta^2 + \dot{\theta}^2 + \dot{x}^2} \leq \theta_P \right\}. \tag{35}$$

In this case, the solution of the normalized system (3) can be approximately written as:

$$\begin{aligned} \ddot{\theta} &= \theta - u_L - \lambda\dot{\theta}, \\ \ddot{x} &= u_L \end{aligned} \tag{36}$$

Thus, by inspection we have that the flat output is given by:

$$y = \left(\lambda^2 + 1\right)\theta + \lambda\dot{\theta} + x + \lambda\dot{x}.$$

In fact, the whole system variables can be expressed in terms of, F, and a finite number of its time derivative, as follows:

$$z = \begin{bmatrix} \theta \\ x \\ \dot{\theta} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & -\lambda & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -\lambda & -1 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \\ y^{(3)} \end{bmatrix} = \Xi Y$$

where $Y = [y, \dot{y}, \ddot{y}, y^{(3)}]^T$. Hence, the state dependent input-coordinate transformation:

$$u_L = \ddot{y} - \lambda y^{(3)} - y^{(4)}, \tag{37}$$

shows that the system (36), is an equivalent approximation of the following chain of integrations:

$$y^{(4)} = -u_L + \ddot{y} - \lambda y^{(3)} = v. \tag{38}$$

A stabilizing feedback controller may be readily obtained by setting:

$$v = -\left(k_{0L}y + k_{1L}\dot{y} + k_{2L}\ddot{y} + k_{3L}y^{(3)}\right); \tag{39}$$

where the set of coefficients, $\{k_{0L}, k_{1L}, k_{2L}, k_{3L}\}$, is chosen such that the closed loop characteristic polynomial of the linearized system, defined as:

$$p(s) = s^4 + k_{3L}s^3 + k_{2L}s^2 + k_{1L}s + k_{0L}$$

is a Hurwitz polynomial. Observe that, $[y, \dot{y}, \ddot{y}, y^{(3)}]$, can be expressed in terms of actual coordinates, by, $Y = \Xi^{-1}z$, because, $\det(\Xi) \neq 0$. Therefore, if $z(t_i) = (x_i, \dot{x}_i, \theta_i, \dot{\theta}_i) \in U_0$, with $t_i > 0$, then we can always propose a simple linear control, which asymptotically brings the whole system variables to zero:

$$v = -k_{0L}(y - y_r) - k_{1L}(\dot{y} - \dot{y}_r) - k_{2L}(\ddot{y} - \ddot{y}_r) - k_{3L}(y^{(3)} - y_r^{(3)}) + y_r^{(4)}; \tag{40}$$

where y_r is an auxiliary reference proposed such that the cart asymptotically and exponentially goes to zero, from some position and velocity, x_i and $\dot{x}_i$, given by:

$$y_r(t) = -\left(\frac{\alpha_2 x_i + \dot{x}_i}{\alpha_1 - \alpha_2}\right) e^{-\alpha_1(t-t_i)} + \left(\frac{\alpha_1 x_i + \dot{x}_i}{\alpha_1 - \alpha_2}\right) e^{-\alpha_2(t-t_i)}.$$

Notice that $y_r(t_i) = x_i$ and $\dot{y}_r(t_i) = \dot{x}_i$, where $\alpha_1 > 0$ and $\alpha_2 > 0$, selected small enough to make the second, third and fourth time derivatives almost zero.

Remark 3.2. *Notice that once, $z(t_i)$, is inside the set, U_0, the system is practically converted to a linear system, with the respective pendulum angular position and angular velocity being almost zero. Then, it is always possible to accomplish a rest-to-rest maneuver. Moreover, this maneuver can be done in a desired finite time by using an eight grade spline ([2]). Note that, for practical purposes, it is useful fixing, $\theta_p \leq 0.2$, because the functional error is in the order of:*

$$\int_0^{\theta_p} |\sin s - s|\, ds \simeq 7 \times 10^{-5}.$$

3.3. **A switching based control strategy.** In the following proposition, we introduce our control strategy, based on the two already defined control laws.

Proposition 3.2. *Considers the damped DIPC system* (3)*, under the assumption that the system is initialized inside the attraction region* (33) *in closed-loop with:*

$$u = \begin{cases} u_L, & \text{if} \sqrt{\theta^2 + \dot{\theta}^2 + \dot{x}^2} < \theta_p; \\ u_p(z) + u_d(\theta, p) - l_c \dot{\theta} \sec\theta, & \text{otherwise} \end{cases} \tag{41}$$

then the whole state asymptotically converges to the origin. Where, u_L, is defined by Equations (37) *to* (40)*, while, u, $_p(z)$ and, $u_d(\theta, p)$, are previously defined in* (18) *and* (13)*, respectively.*

Proof: Let us suppose that the state variables, $z_0 = (\theta_0, x_0, \dot{\theta}_0, \dot{x}_0)$, are initialized, such that, $E(z_0) < \bar{c}$. Then, $E(z(t)) < \bar{c}$, with $\theta(t) \in I_0$. In other words, $z(t) \in \Omega_{\widetilde{c}}$. Under these circumstances, we have two possibilities: $z(t) \in U_0$, or, $z(t) \in U_0^c \cap \Omega_{\widetilde{c}}$. In the first possibility, the system behaves as a linear system, and the solution, as expected, converges exponentially almost always to zero. In the second possibility, the state variables, $(\dot{\theta}, \theta, \dot{x})$, asymptotically converge to zero, while state variable, x, converges to a constant, not necessarily the origin. That is, in the second case, for some finite time, we expect that z enters into U_0, where the closed-loop system behaves as a linear system. Evidently, the solution may be trapped inside U_0, making that z converges to zero. Now, in the case that the solution escapes from U_0, we have, once again, that $z \in U_0^c \cap \Omega_{\widetilde{c}}$ and u_N lead the solution to come into the set U_0, where z is forced to converge exponentially and asymptotically to zero. This procedure will occur until x reaches the origin.

4. **Numerical Simulations.** To show the effectiveness of the proposed nonlinear control strategy, we carried out some numerical simulations. To this end the following setup was used for the nonlinear part of our controller:

$$k_1 = 7.8; \quad k_2 = 2.1; \quad k_3 = 0.7; \quad k_p = 0.35; \quad \gamma = 0.9; \quad \delta = 1; \quad l_c = 0.6;$$

while the corresponding setup of the linear part was fixed, such that the corresponding characteristic polynomial were:

$$p(s) = 1 + 3.8s + 5.7s^2 + 12.16s^3 + s^4; \tag{42}$$

with, $\theta_p = 0.2$. The parameters of the exponential reference trajectory were set such that $\alpha_1 = 0.75$ and $\alpha_2 = 0.8$. The hypothesized initial condition was: $z = (\theta = 1.1, x = 0.1, \dot{\theta} = 0, \dot{x} = 0)$. We pointed out that z belongs to the already computed attraction domain, given by $\widetilde{c} = 1.12$.

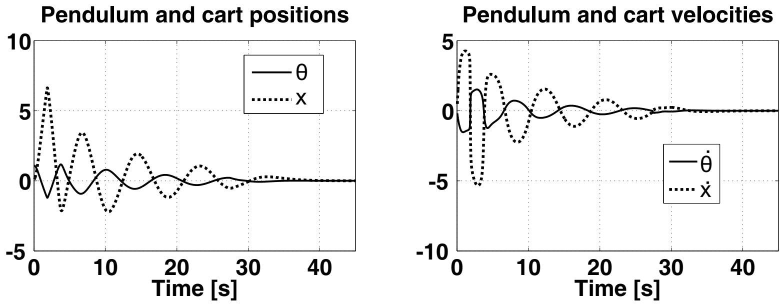

Figure 2. Closed-loop response of the **DIPC** to the proposed control strategy

Figure 2 shows the corresponding response of the system (3) in closed-loop with the switching control strategy (41). From this figure, we can see that our control strategy effectively renders the system to the origin in about 40 time units.

In order to provide an intuitive idea of how good our nonlinear control strategy (**OCL**) is, we ran two numerical experiments. The first experiment comperes (**OCL**) with the control techniques proposed by Woolsey *et al.* in [24] and by Olfati-Saber in [17], here respectively referred to as (**WCL**) and (**OSCL**). The experiment setup was the same as in the previous experiment, with the new initial condition, $z = (\theta = -0.9, x = 0.2, \dot{\theta} = 0, \dot{x} = 0)$. The corresponding control law parameters of the linearized **WCL** and **OSCL**, were tuned such that the obtained characteristic polynomial coincides with (42). The simulation results are shown in Figure 3. There, we can see that our control strategy is able to render asymptotically the system to the origin, while the others two need more time. In fact, it seems that these strategies maintain the system oscillating close to the origin. The other experiment comperes once again (**OCL**) with a control law introduced by Riachy *et al.* in [30] (**RCL**), based on sliding modes, by whose form of construction is well known that is robust. This control law consists of proposing a sliding surface, as:

$$s = \tan(\theta) + \lambda_1(x + \phi(\theta)) + \lambda_2(\dot{x} + \phi'(\theta)\dot{\theta}),$$
$$\phi(\theta) = \log\left(\frac{1 + \tan(\theta/2)}{1 - \tan(\theta/2)}\right);$$

where, $\lambda_1 > 0$ and $\lambda_2 > 0$, and from which a control law is proposed, such that the surfaces, $(s, \dot{s}, \ddot{s})$, are brought to zero in a finite time, and having the characteristic of set the whole state to zero, when $s = 0$.

For this second experiment, the initial conditions were $z = (\theta = -0.9, x = 0.2, \dot{\theta} = 0, \dot{x} = 0)$; the control parameters for (**RCL**) were $\lambda_1 = 0.1$ and $\lambda_2 = 0.2$; for our control strategy we used the same control parameters as in the previous experiments. The initial conditions for the results of this second experiment are shown in Figure 4, where we can see that our control strategy, as before, renders the pendulum and the cart to the origin, while the **RCL** presents chattering in the cart position coordinate, because the sliding mode based control action acts directly to it.

A formal comparative study between our control strategy and others found in literature is beyond the scope of this work.

5. **Conclusions.** In this work we presented a control strategy for the stabilization of the strongly damped inverted pendulum cart system, provided that the pendulum is initialized inside the upper-half plane and the viscosity coefficients are known. The proposed strategy consists of switching from a nonlinear to a linear controller. The first controller

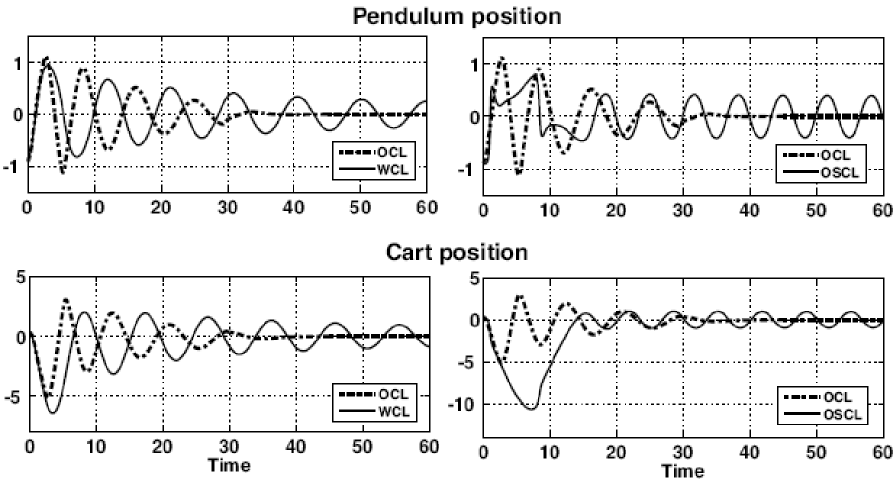

FIGURE 3. Comparison between our control strategy (**OCL**) and the strategies **WCL** and **OSCL**

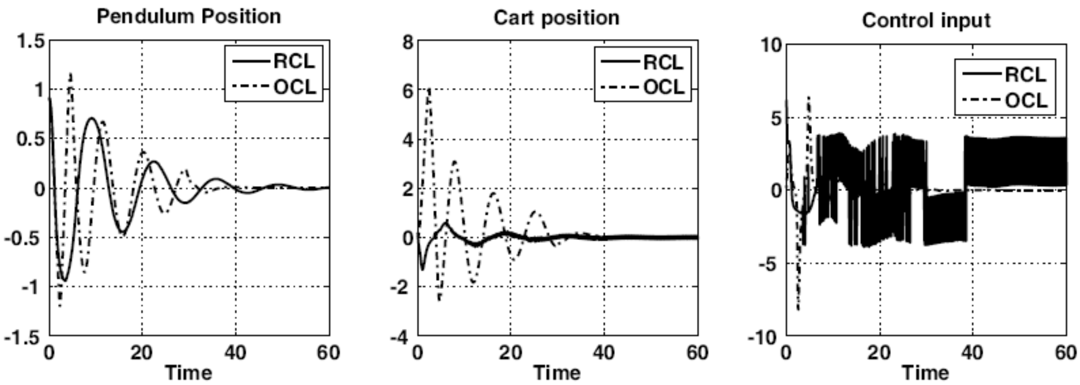

FIGURE 4. Comparison between our control strategy (**OCL**) and the strategy **RCL**

brings the pendulum position, its angular velocity and the cart velocity very close to zero. Afterwards, the second controller exponentially renders the whole state to the origin. The first controller included a nonlinear integrator of the pendulum angular position, which allowed us to translate the pendulum damping destabilize force to the force that directly acts over the cart movement controller. This means that the cart displacements were increased. The first controller is in charge of bringing the system variables inside the attraction domain of the second linear controller; then, the second controller renders the system to the origin. Numerical simulations were presented to assess the performance of the designed control law. Finally, we believe that our control law can also be used to stabilize the Furuta pendulum.

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