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METHOD OF STRUCTURAL ANALYSIS, TAKING INTO ACCOUNT DEFORMATIONS BY FLEXURE, SHEAR AND AXIAL

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Abstract. *This paper proposes a full method for analysis of statically indeterminate structures, considering the deformations by flexure, shear and axial. This methodology can be used to analysis of any structure type to obtain optimal designs in terms of the dimensions of transverse section for those structures. The traditional method considers only deformations by flexure called the slope-deflection method, which is as usually done structural analysis of statically indeterminate rigid frames. Subsequently a method is developed by Luévanos-Rojas, which considers the deformations by flexure and shear for beams and also for rigid frames. It also makes a comparison between the proposed method and other two methods as can be seen in the results tables of the problems considered; in the traditional method the deference is greater, which the method developed by Luévanos-Rojas, with respect to the proposed method. Therefore, the general practice without considering the axial deformations will not be a recommended solution and it proposes the use of taking the deformations by flexure, shear and axial and also is related more with real conditions.*
Keywords: Flexure deformations, Shear deformations, Axial deformations, Poisson's ratio, Moment of inertia, Elasticity modulus, Shear modulus, Shear area

1. **Introduction.** In the structural systems analysis has been studied by diverse researchers in the past, making a brief historical review of progress in this subject.

In 1857, Benoit Paul Emile Clapeyron presented to the French Academy his "theorem of three moments" for analysis of continuous beams; in the same way Bertot had published two years ago in the Memories of the Society of Civil Engineers of France, but without giving some credit. It can be said that from this moment begins the development of a true "Theory of Structures" [1-4].

In 1854 the French Engineer Jacques Antoine Charles Bresse published his book *"Recherches Analytiques sur la Flexion et la Résistance de Pieces Courbés"* in which he presented practical methods for the analysis of curved beams and arcs [1-4].

In 1867 was introduced by the German Emil Winkler (1835-1888), the "Influence Line". He also made important contributions to the Resistance of materials, especially in the flexure theory of curved beams, flexure of beams, resting on elastic medium [1-4].

James Clerk Maxwell (1830-1879), from the University of Cambridge, published what might be called the first systematic method of analysis for statically indeterminate structures, based on the equality of the internal energy of deformation of a loaded structure and the external work done by applied loads, equality had been established by Clapeyron. In his analysis presented in the Theorem of the Reciprocal Deformations, which by in its brevity and lack of illustration, was not appreciated at the time. In another publication later presented his diagram of internal forces to trusses, which combine in one figure all the polygons of forces. The diagram was extended by Cremona, by what is known as the Maxwell-Cremona diagram [1-4].

The Italian Betti in 1872 published a generalized form of Maxwell's theorem, known as the reciprocal theorem of Maxwell-Betti [1-4].

The German Otto Mohr (1835-1918) made great contributions to the Structures Theory. He developed the method for determining the deflections in beams, known as the method of elastic loads or the conjugate beam. He also presented a derivation simpler and more extensive of the general method of Maxwell for analysis in indeterminate structures, using the principles of virtual work. He made contributions in the graphical analysis of deflections of trusses, complemented by Williot diagram, known as the Mohr-Williot diagram of great practical utility. He also earned his famous Mohr Circle for the graphical representation of the stresses in a stress biaxial state [1-4].

Alberto Castigliano (1847-1884) in 1873 introduced the principle of minimum work, which had been previously suggested by Menabrea, and is known as the First Theorem of Castigliano. Later, it presented the Theorem second Castigliano, to find deflections, as a corollary of the first. In 1879 his famous book published in Paris "Thèoreme de l'Equilibre de Systèmes Elastiques et ses Applications", remarkable by its originality and very important in the development of analysis of statically indeterminate structures [1-3].

Heinrich Müller-Breslau (1851-1925), published in 1886 a basic method for analysis of indeterminate structures, but was essentially a variation of those presented by Maxwell and Mohr. He gave great importance the Maxwell's Theorem of Reciprocal Deflections in the assessment of displacement. He discovered that the "influence line" for the reaction or an inner strength of a structure was, on some scale, the elastic produced by an action similar to that reaction, or inner strength. Known as the Müller-Breslau theorem is the basis for other indirect methods of structural analysis using models [1-3].

Hardy Cross (1885-1959) professor at the University of Illinois, published in 1930 his famous moments distribution method, can be said that revolutionized the analysis of structures of reinforced concrete by continuous frames and can be considered one of the greatest contributions to the analysis from indeterminate structures. This method of successive approximations evades solving systems of equations, as presented in the methods of Mohr and Maxwell. This method declined popularity with the availability of computers, with which the resolution of equations systems is no longer a problem. The general concepts of the method were later extended in the study on flow of pipes. Later became more popular the methods of Kani and Takabeya also of type iterative and today in disuse [1-7].

In the early 50's, Turner, Clough, Martin and Topp present what may be termed as the beginning of the application to structures of the matrix methods of stiffness, which have gained so much popularity today. Subsequently, it is developed the finite element methods, which have allowed the systematic analysis of large numbers of structures and obtain the forces and deformations in complex systems such as concrete dams used in hydroelectric plants. Among its promoters include: Clough, Wilson, Zienkiewics and Gallagher [1,2,8].

Luévanos-Rojas developed a method of structural analysis for statically indeterminate beams and rigid frames, in this method takes into account the flexure deformations and shear. That previously was considered only the flexure deformations [9,10].

Structural analysis is the study of structures such as discrete systems. The theory of the structures is essentially based on the fundamentals of mechanics with which are formulated the different structural elements. The laws or rules that define the balance and continuity of a structure can be expressed in different ways, including partial differential equations of continuous medium three-dimensional, ordinary differential equations that define a member or the theories various of beams, or simply, algebraic equations for a discrete structure. The more delves into the physics of problem, are developing theories that are most appropriate for solving certain types of structures and that prove more useful for practical calculations. However, in each new theory are made hypotheses about how the system behaves or element. Therefore, we must always be aware of these hypotheses when evaluating results, fruit of the theories that apply or develop [11-13].

Structural analysis can be addressed using three main approaches [14]: a) tensorial formulation (Newtonian mechanics and vectorial), b) formulation based on the principles of virtual work, c) formulation based on classical mechanics [15,16].

In the design of steel structures, reinforced concrete and prestressed, the study of structural analysis is a crucial stage into its design, since the axial forces, shear forces and moments are those that govern the design of rigid frames and for the case of beams only shear forces and moments, and the damage caused by such effects may become predominant among the various requests to consider for your design.

As regards the conventional techniques of structural analysis of beams and rigid frames, the common practice is considering the flexure deformations only, and to from of the work developed by Luévanos-Rojas, the structural analysis methods for beams and statically indeterminate rigid frames where is taken into account the flexure deformations and shear could be used.

This paper proposes a full method for analysis of statically indeterminate structures, considering the flexure deformations, shear and axial, since these effects are those which are present in all types of structures analyzed in two dimensions. Also a comparison between the proposed method with the traditional method and the method developed by Luévanos-Rojas is realized.

2. Mathematical Development of the Proposed Method.

2.1. Theoretical principles. In the scheme of deformation of a structure member is illustrated in Figure 1, shows the difference between the Timoshenko theory and Euler-Bernoulli theory: the first "θ_Z" and "dy/dx" not coincides necessarily, while in the second are equal [9,10,15-17].

The fundamental difference between the Euler-Bernoulli theory and Timoshenko's theory is that in the first the relative rotation of the section is approximated by the derivative of vertical displacement, this is an approximation valid only for long members in relation to the dimensions of cross section, and then it happens that due to shear deformations are negligible in comparison with the deformations caused by moment. On the Timoshenko theory, which considers the deformation due to shear, i.e., and is valid therefore for short members and long, the equation of the elastic curve is given by the complex system of equations:

$$G\left(\frac{dy}{dx} - \theta_Z\right) = \frac{V_y}{A_s} \tag{1}$$

$$E\left(\frac{d\theta_Z}{dx}\right) = \frac{M_z}{I_z} \tag{2}$$

where: G = shear modulus, dy/dx = total rotation around axis "Z", θ_Z = rotation around axis "Z", due to the flexure, V_y = shear force in direction "Y", A_s = shear area, $d\theta_Z/dx = d^2y/dx^2$, E = elasticity modulus, M_z = moment around axis "Z", I_z = moment of inertia around axis "Z".

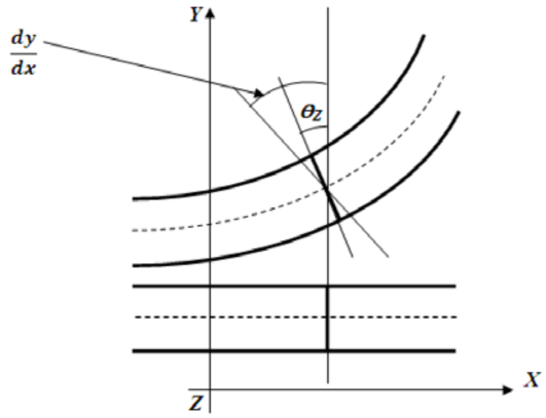

FIGURE 1. Deformation of a structure member

Deriving Equation (1) and substituting into Equation (2), it is arrived at the equation of the elastic curve including the effect of shear stress:

$$\frac{d^2y}{dx^2} = \frac{1}{GA_s}\frac{dV_y}{dx} + \frac{M_z}{EI_z} \tag{3}$$

by Equation (1) is obtained "dy/dx":

$$\frac{dy}{dx} = \frac{V_y}{GA_s} + \theta_Z \tag{4}$$

and Equation (2) is given "θ_Z":

$$\theta_Z = \int \frac{M_z}{EI_z} dx \tag{5}$$

now substituting Equation (5) into Equation (4) is:

$$\frac{dy}{dx} = \frac{V_y}{GA_s} + \int \frac{M_z}{EI_z} dx \tag{6}$$

2.2. **General conditions.** The method proposed can be used to analyze all type of beams and rigid frames statically indeterminate. In this method all joints are considered rigid; i.e., the angles between members at the joints are considered not to change in value, when the loads are applied. Thus the joints at the supports interior of statically indeterminate beams can be considered rigid joints of 180°; and usually the joints in rigid frames are rigid joints of 90°. When beams are deformed, the rigid joints are considered to rotate only as a whole; in other terms, the angles between the tangents to the various branches of the elastic curve in the joint remain the same as in the original undeformed structure.

In this method the rotations, displacements in direction perpendicular to the axis of the member (the end joints are subjected to unequal movements in direction perpendicular to the axis of the structural member) and displacements in direction parallel to the axis of the member (also these end joints are subjected to unequal movements in direction parallel to the axis of the structural member, i.e., the member is lengthened or is shortened), these deformations in the joints are treated as unknowns. Then the end moments are expressed in terms of the rotations and perpendicular displacements to the axis of the member. However, to satisfy the condition of equilibrium, the sum of the end moments which any joints exerted on the ends of union of the members must be zero, because the rigid joints in question are subject to the sum of these moments at the ends (Reversed only in the direction). Further, to satisfy the others two equilibrium conditions, i.e., sum of horizontal forces and the sum of vertical forces must be zero. This condition must apply, making a virtual cut in each joint of the structure, which provides the additional condition that corresponds to the unknown displacements.

This procedure solves the equations system of rotations and displacements for beams and rigid frames statically indeterminate. Therefore, it is important to remember the hypothesis under which, the equations are deduced: a) the material is homogeneous, isotropic and behaves linear elastic, i.e., the material is of the same nature, have identical physical properties in all directions and efforts that can withstand are directly proportional to the deformations that suffer and the factor of proportionality is called modulus elasticity, "E", i.e., $\sigma = E\varepsilon$ (Hooke's Law); b) the principle of the small deformations, which once the structure is loaded, the linear deformation and angular of the joints and each of the points of its members are rather small in such a way that in form do not change, nor are altered appreciably; c) the principle of superposition of effects, that supposes the totals displacements and internal forces total of the structure under a system of loads, these can be found separately by the sum of the effects of each one of the considered loads; d) you can only take into account the first order effects such as: internal deformations by flexure always, while the shear deformations can be taken into account or not.

2.3. **Equations for moments.** The equations for moments acting in the ends of the members are expressed in terms of the rotations, displacements perpendicular to the axis of the bar and the loads on the members. Then, the member "AB" is shown in Figure 2(a) can be expressed in terms of "θ_A", "θ_B", "Δ_{VA}" and "Δ_{VB}", also of the applied loads, "P_1" and "P_2". Counterclockwise end moments acting on the members are considered to be positive and clockwise end moments acting on the members are considered to be negative. Now, with the applied loading on the member, the fixed end moments, "M_{FAB}" and "M_{FBA}", these are moments required to hold the horizontal tangents at the ends fixed in Figure 2(b). Additionally at the fixed moments in the ends, "M'_{FAB}", and "M'_{FBA}", these are acting on the member in the fixed ends, when the perpendicular displacement to axis the member appears, according to be seen in Figure 2(c). Then the moments, "M'_{AB}" and "M'_{BA}", should be such as to cause rotations of "θ_A" and "θ_B". If "θ_{A1}" and "θ_{B1}" are the end rotations caused by "M'_{AB}", according to Figure 2(d), thus "θ_{A2}" and "θ_{B2}", due to "M'_{BA}", are observed in Figure 2(e).

The conditions required of geometry are [9,10,18-22]:

$$\theta_A = -\theta_{A1} + \theta_{A2} \tag{7a}$$

$$\theta_B = \theta_{B1} - \theta_{B2} \tag{7b}$$

by superposition:

$$M_{AB} = M_{FAB} + M'_{AB} + M'_{FAB} \tag{8a}$$

$$M_{BA} = M_{FBA} + M'_{BA} + M'_{FBA} \tag{8b}$$

Taking into account the member of Figure 2(c) and supposing that $M'_{FAB} = M'_{FBA}$ and, $V_A = V_B$, the sum of moments in the point "B" is realized to obtain "M'_{FAB}" in function "V_A":

$$M'_{FAB} = \frac{V_A L}{2} \tag{9}$$

Therefore, the shear forces and moments at a distance "x" are:

$$V_x = V_A \tag{10}$$

$$M_x = V_A\left(\frac{L}{2} - x\right) \tag{11}$$

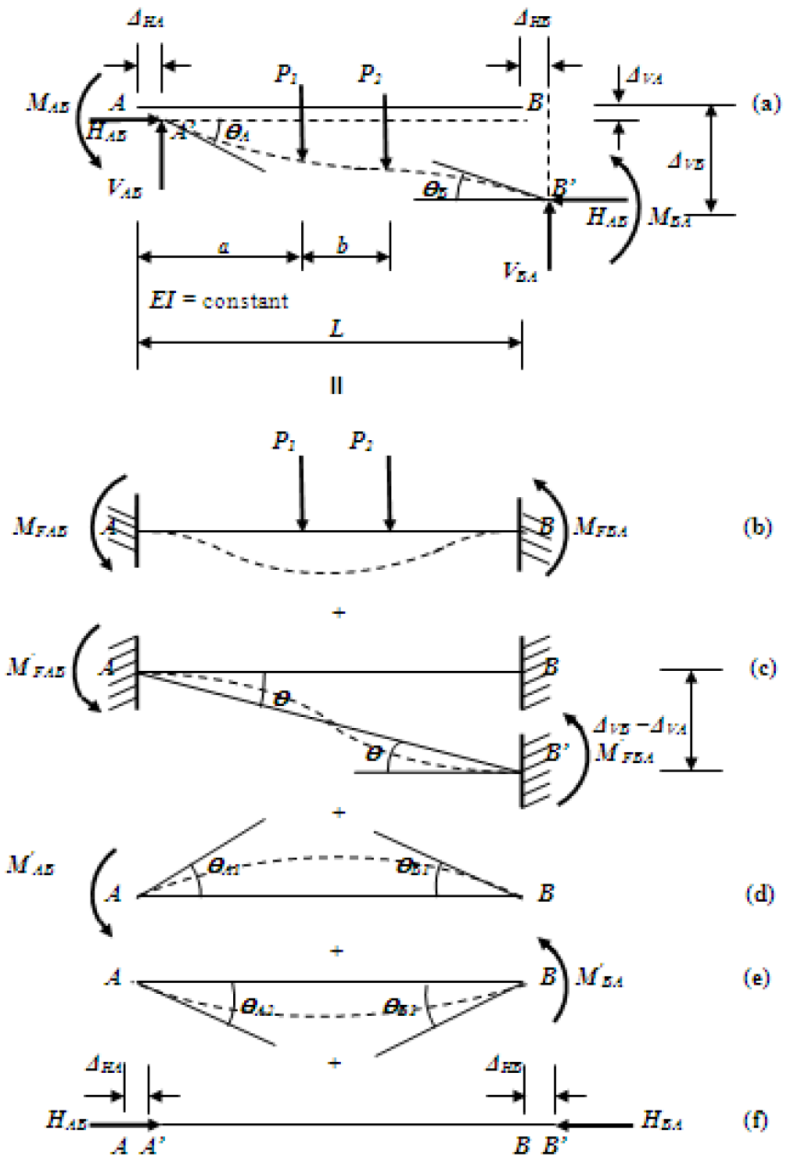

FIGURE 2. Derivation of equations

Substituting "M_x" and "V_x" in function of "V_A" into Equation (6), the shear deformation and flexure is separated to obtain the stiffness due to the displacement, it is presented as follows:

Shear deformation:

$$\frac{dy}{dx} = \frac{V_A}{GA_s} \tag{12}$$

Equation (12) is integrated to obtain:

$$y = \frac{V_A}{GA_s}x + C_1 \tag{13}$$

the conditions of border are considered, when $x = 0$; $y = 0$; then $C_1 = 0$.

$$y = \frac{V_A}{GA_s}x \tag{14}$$

Flexure deformation:

$$\frac{dy}{dx} = \frac{V_A}{EI_z}\int\left(\frac{L}{2} - x\right)dx \tag{15}$$

the integral of Equation (15) is developed, it is the expression follows:

$$\frac{dy}{dx} = \frac{V_A}{EI_z}\left(\frac{L}{2}x - \frac{x^2}{2} + C_2\right) \tag{16}$$

the conditions of border are taken into account, when $x = 0$; $dy/dx = 0$; it is obtained that $C_2 = 0$.

$$\frac{dy}{dx} = \frac{V_A}{EI_z}\left(\frac{L}{2}x - \frac{x^2}{2}\right) \tag{17}$$

Equation (17) is integrated to obtain:

$$y = \frac{V_A}{EI_z}\left(\frac{L}{4}x^2 - \frac{x^3}{6} + C_3\right) \tag{18}$$

the conditions of border are considered, when $x = 0$; $y = 0$; it is obtained that $C_3 = 0$.

$$y = \frac{V_A}{EI_z}\left(\frac{L}{4}x^2 - \frac{x^3}{6}\right) \tag{19}$$

the sum of Equation (14) due to shear deformation and Equation (19) due to flexure is developed as follows:

$$y = \frac{V_A}{GA_s}x + \frac{V_A}{EI_z}\left(\frac{L}{4}x^2 - \frac{x^3}{6}\right) \tag{20}$$

Substituting $x = L$ and $y = \Delta_{VB} - \Delta_{VA}$, into Equation (20), the displacement in the support "B" is found as follows:

$$\Delta_{VB} - \Delta_{VA} = \frac{V_A L^3}{12EI_z}\left(\frac{12EI_z}{GA_s L^2} + 1\right) \tag{21}$$

the substitution following is performed [9,10,23]:

$$\varnothing = \frac{12EI_z}{GA_s L^2} \tag{22}$$

it is obtained "G" as follows:

$$G = \frac{E}{2(1+\nu)} \tag{23}$$

where: $\varnothing$ = form factor, ν = Poisson's ratio.

Substituting Equation (22) into Equation (21) and the value "V_A" is:

$$V_A = \frac{12EI_z}{L^3(\varnothing + 1)}(\Delta_{VB} - \Delta_{VA}) \tag{24}$$

by replacing "V_A" into Equation (9) is:

$$M'_{FAB} = \frac{6EI_z}{L^2\,(\varnothing + 1)}(\Delta_{VB} - \Delta_{VA}) \tag{25}$$

Then, as $M'_{FAB} = M'_{FBA}$, the following equations are presented:

$$M'_{FAB} = \frac{6EI_z}{L^2\,(\varnothing + 1)}(\Delta_{VB} - \Delta_{VA}) \tag{26a}$$

$$M'_{FBA} = \frac{6EI_z}{L^2\,(\varnothing + 1)}(\Delta_{VB} - \Delta_{VA}) \tag{26b}$$

Now, the member of Figure 2(d) is analyzed to find "θ_{A1}" and "θ_{B1}" in function of "M'_{AB}", if is considered that $V_A = V_B$, the sum of moments in "B" is realized to obtain "M'_{AB}" in function of "V_A" is presented:

$$M'_{AB} = V_A L \tag{27}$$

Therefore, the shear forces and moments at a distance "x" are:

$$V_x = \frac{M'_{AB}}{L} \tag{28}$$

$$M_x = \frac{M'_{AB}}{L}(L - x) \tag{29}$$

Substituting Equations (28) and (29) into Equation (6), the shear deformation and flexure is separated to obtain stiffness as follows:

Shear deformation:

$$\frac{dy}{dx} = \frac{M'_{AB}}{GA_sL} \tag{30}$$

Equation (30) is integrated to obtain:

$$y = \frac{M'_{AB}}{GA_sL}x + C_1 \tag{31}$$

the conditions of border are considered, when $x = 0$; $y = 0$; it is obtained that $C_1 = 0$.

$$y = \frac{M'_{AB}}{GA_sL}x \tag{32}$$

Flexure deformation:

$$\frac{dy}{dx} = \frac{M'_{AB}}{EI_zL}\int(L - x)dx \tag{33}$$

the integral of Equation (33) is developed:

$$\frac{dy}{dx} = \frac{M'_{AB}}{EI_zL}\left(Lx - \frac{x^2}{2} + C_2\right) \tag{34}$$

now, the integral of Equation (34) is obtained:

$$y = \frac{M'_{AB}}{EI_zL}\left(\frac{L}{2}x^2 - \frac{x^3}{6} + C_2x + C_3\right) \tag{35}$$

the conditions of border are considered, when $x = 0$ and $y = 0$, it is $C_3 = 0$.

$$y = \frac{M'_{AB}}{EI_zL}\left(\frac{L}{2}x^2 - \frac{x^3}{6} + C_2x\right) \tag{36}$$

now, the conditions of border are considered, when $x = L$ and $y = 0$, it is obtained:

$$C_2 = -\frac{L^2}{3} \tag{37}$$

Substituting Equation (37) into Equations (34) and (36) is shown as follows:

$$\frac{dy}{dx} = \frac{M'_{AB}}{EI_zL}\left(Lx - \frac{x^2}{2} - \frac{L^2}{3}\right) \tag{38}$$

$$y = \frac{M'_{AB}}{EI_zL}\left(\frac{L}{2}x^2 - \frac{x^3}{6} - \frac{L^2}{3}x\right) \tag{39}$$

Substituting $x = 0$ into Equation (38) to find the rotation in support "A" due to the flexure deformation "θ_{A1F}", it is as follows:

$$\theta_{A1F} = -\frac{M'_{AB}L}{3EI_z} \tag{40}$$

Now, substituting $x = L$ into Equation (38) to find the rotation in support "B" due to the flexure deformation "θ_{B1F}" is obtained as follows:

$$\theta_{B1F} = \frac{M'_{AB}L}{6EI_z} \tag{41}$$

If the curvature radius is considered positive in the inferior part, the rotations are:

$$\theta_{A1F} = +\frac{M'_{AB}L}{3EI_z} \tag{42a}$$

$$\theta_{B1F} = +\frac{M'_{AB}L}{6EI_z} \tag{42b}$$

The rotation due to the shear deformation, "θ_{A1S}" and "θ_{B1S}" if the curvature radius is taken into account is:

$$\theta_{A1S} = \frac{dy}{dx} = \frac{M'_{AB}}{GA_sL} \tag{43a}$$

$$\theta_{B1S} = \frac{dy}{dx} = -\frac{M'_{AB}}{GA_sL} \tag{43b}$$

Shear deformation and flexure are added up in the joint "A" is obtained:

$$\theta_{A1} = \theta_{A1F} + \theta_{A1S} \tag{44}$$

substituting Equations (42a) and (43a) into Equation (44), it is as follows:

$$\theta_{A1} = \frac{M'_{AB}L}{3EI_z} + \frac{M'_{AB}}{GA_sL} \tag{45}$$

the common factor of Equation (45) for "M'_{AB}" is obtained:

$$\theta_{A1} = \frac{M'_{AB}L}{12EI_z}\left(4 + \frac{12EI_z}{GA_sL^2}\right) \tag{46}$$

substituting Equation (22) into Equation (46):

$$\theta_{A1} = \frac{M'_{AB}L}{12EI_z}(4 + \varnothing) \tag{47}$$

Shear deformation and flexure are added up in the joint "B" and the simplification is realized:

$$\theta_{B1} = \frac{M'_{AB}L}{12EI_z}(2 - \varnothing) \tag{48}$$

The member of Figure 2(e) is analyzed to find "θ_{A2}" and "θ_{B2}" in function of "M'_{BA}" of the same way as was done in Figure 2(d), it is obtained:

$$\theta_{A2} = \frac{M'_{BA}L}{12EI_z}(2 - \varnothing) \tag{49}$$

$$\theta_{B2} = \frac{M'_{BA}L}{12EI_z}(4 + \varnothing) \tag{50}$$

Now, substituting Equations (47) and (48) into Equation (7a) and Equations (49) and (50) into Equation (7b) are presented:

$$\theta_A = -\frac{M'_{AB}L}{12EI_z}(4 + \varnothing) + \frac{M'_{BA}L}{12EI_z}(2 - \varnothing) \tag{51}$$

$$\theta_B = \frac{M'_{AB}L}{12EI_z}(2 - \varnothing) - \frac{M'_{BA}L}{12EI_z}(4 + \varnothing) \tag{52}$$

We develop Equations (51) and (52) to find "M'_{AB}" and "M'_{BA}" in function of "θ_A" and "θ_B" is shown:

$$M'_{AB} = \frac{EI_z}{L}\left[-\left(\frac{4+\varnothing}{1+\varnothing}\right)\theta_A - \left(\frac{2-\varnothing}{1+\varnothing}\right)\theta_B\right] \tag{53a}$$

$$M'_{BA} = \frac{EI_z}{L}\left[-\left(\frac{4+\varnothing}{1+\varnothing}\right)\theta_B - \left(\frac{2-\varnothing}{1+\varnothing}\right)\theta_A\right] \tag{53b}$$

Finally, substituting Equations (26a), (26b) and (53a), (53b) into Equations (8a), (8b), respectively, we obtain the Equations for statically indeterminate rigid frames:

$$M_{AB} = M_{FAB} + \frac{EI_z}{L}\left[-\left(\frac{4+\varnothing}{1+\varnothing}\right)\theta_A - \left(\frac{2-\varnothing}{1+\varnothing}\right)\theta_B\right] + \frac{6EI_z}{L^2(\varnothing+1)}(\Delta_{VB}-\Delta_{VA}) \quad (54a)$$

$$M_{BA} = M_{FBA} + \frac{EI_z}{L}\left[-\left(\frac{4+\varnothing}{1+\varnothing}\right)\theta_B - \left(\frac{2-\varnothing}{1+\varnothing}\right)\theta_A\right] + \frac{6EI_z}{L^2(\varnothing+1)}(\Delta_{VB}-\Delta_{VA}) \quad (54b)$$

2.4. **Equations for shear forces.** The equations for shear forces acting in the ends of the members are expressed in terms of the rotations, displacements perpendicular to the axis of member and loads on the members. Then, the member "AB" that was showed in Figure 2(a) now is presented in Figure 3, with shear forces and moments acting on the ends of the member.

The sum of moments in "B" is realized to obtain "V_{AB}", gives:

$$V_{AB} = \frac{M_{AB}+M_{BA}}{L} + \frac{P_1(L-a)+P_2(L-a-b)}{L} \quad (55)$$

Later, the sum of vertical forces is developed to find "V_{BA}":

$$V_{BA} = V_{AB} - P_1 - P_2 \quad (56)$$

2.5. **Equations for axial forces.** The equations for axial forces acting in the ends of the members are expressed in terms of the displacements parallel to the axis of bar. Then, the member "AB", which is showed in Figure 2(a), now the axial forces acting on the ends of the member is presented in Figure 4.

By sum of horizontal forces is presented:

$$H_{AB} = H_{BA} \quad (57)$$

using the principles of materials strength is shown as follows:

$$H_{AB} = H_{BA} = \frac{EA}{L}(\Delta_{HA}-\Delta_{HB}) \quad (58)$$

2.6. **General procedure of method.** Procedure to the analysis of statically indeterminate rigid frame is presented [9,10]:

1. Determine the fixed-end moments at the ends of each span, using the expressions as shown in Figure 5.

2. Express all end moments in terms of the fixed-end moments, rotations and displacements perpendicular to the axis of the bar of each one the members using Equations (54a) and (54b).

3. Find all shear forces in terms of the moments, the sum of moments in a bar end is realized and simplifying to obtain the shear forces in the opposite end. Then the shear forces are presented in terms of the rotations and displacements perpendicular to the axis

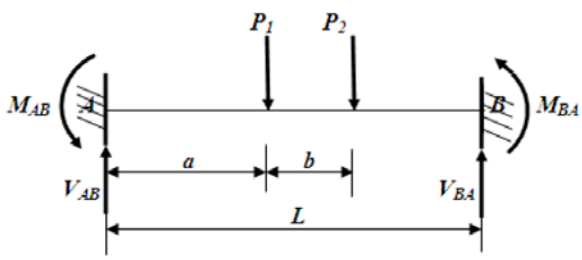

FIGURE 3. Shear forces and moments acting in ends of the member

FIGURE 4. Axial forces acting in ends of member

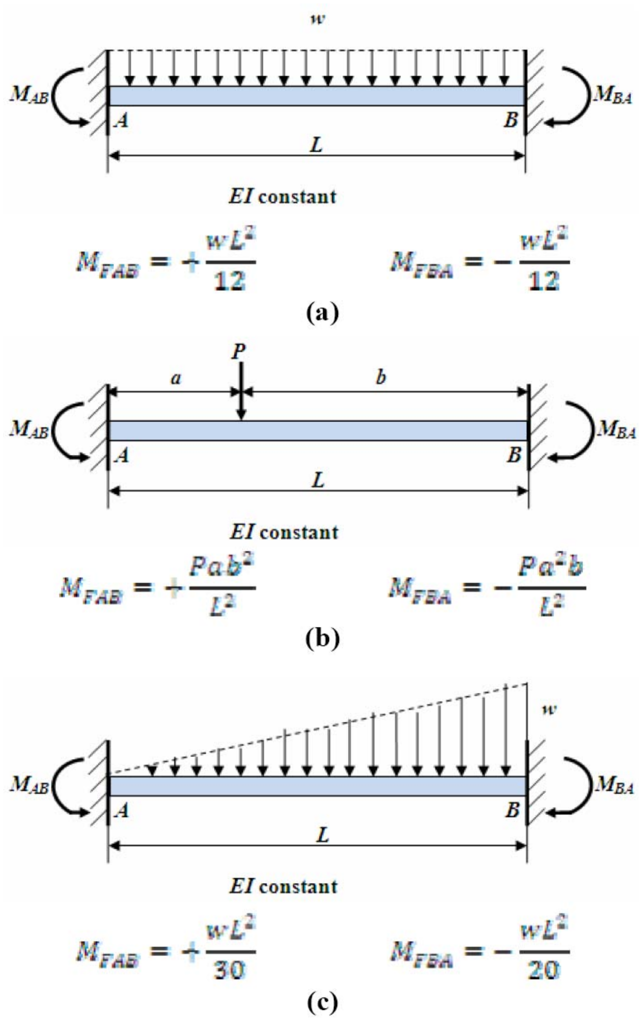

FIGURE 5. Moments fixed-end: (a) uniformly distributed load, (b) concentrated load, (c) triangular distributed load

of the bar of each one the members, since the moments are function of these unknowns, using Equations (55) and (56).

4. Present all axial forces in terms of the displacements parallel to the axis of the bar of each one the members, using Equation (58).

5. Apply the conditions of equilibrium acting on the joints: sum of moments, sum of vertical forces and sum of horizontal forces, should be equal zero.

6. Establish simultaneous equations with the rotations, perpendicular displacements and displacements parallel to the axis of the bar of each one the members at the joints as unknowns.

7. Solve the equations for the rotations, perpendicular displacements and parallel to the axis of the bar at all joints.

8. Substitute the rotations, perpendicular displacements and parallel to the axis of the bar into the equations corresponding, and obtain the moments, shear forces and axial forces final.
9. Draw the diagrams axial forces, shear forces and moments.

3. **Application.** The structural analysis of a steel rigid frame in three different problems is presented in Figure 6, it is developed by: the classic method that considers the deformations by flexure (Method 1); the Luévanos-Rojas method that takes into account the deformations by flexure and shear (Method 2) and the proposed method taking into account the deformations by flexure, shear and axial (Method 3) in basis to the following data that appear next:

w = 3500kg/m
L = 10.00m; 5.00m; 3.00m
P = 5000kg
h = 5.00m
E = 2040734kg/cm^2
Properties of the beam W24X94
A = 178.71cm^2
A_s = 80.77cm^2
I_z = 112382cm^4
Properties of the column W24X62
A = 117.42cm^2
A_s= 65.86cm^2
I_z = 64516cm^4
ν = 0.32

Known conditions: $\theta_E = \theta_F = \theta_G = \theta_H = \Delta_{HE} = \Delta_{VE} = \Delta_{HF} = \Delta_{VF} = \Delta_{HG} = \Delta_{VG} = \Delta_{HH} = \Delta_{VH} = 0$.

Unknown conditions: $\theta_A, \theta_B, \theta_C, \theta_D$, Δ_{HA}, Δ_{VA}, Δ_{HB}, Δ_{VB}, Δ_{HC}, Δ_{VC}, $\Delta_{HD} y \Delta_{VD}$.

Using Equation (23), the shear modulus is obtained:

$$G = \frac{2040734}{2(1 + 0.32)} = 773005.303\text{kg/cm}^2$$

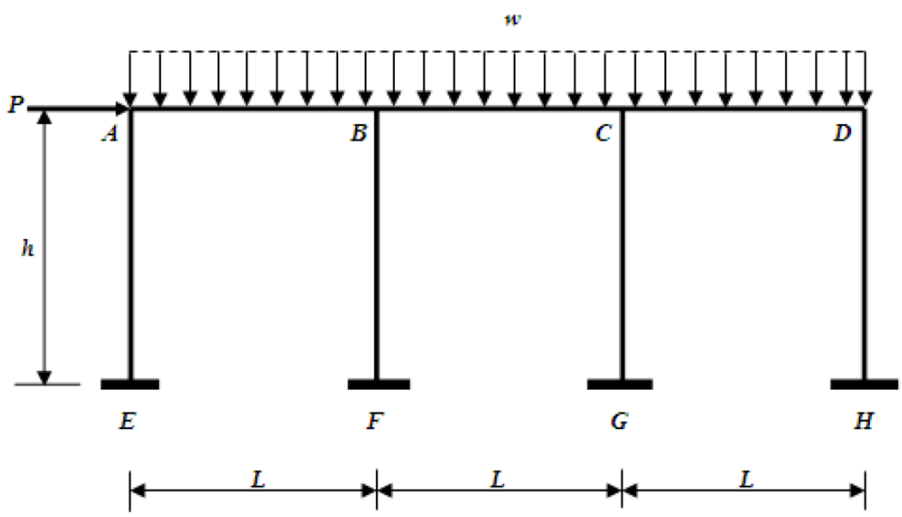

FIGURE 6. The rigid frame of steel of three lengths equal for beams, with uniformly distributed load and a concentrated load in joint "A"

Once that the shear modulus is obtained, the form factor through Equation (22) is found:

For beams of 10.00m is:

$$\varnothing_{AB} = \varnothing_{BC} = \varnothing_{CD} = \frac{12(2040734)(112382)}{(773005.303)(80.77)(1000)^2} = 0.04407901152$$

For beam of 5.00m is:

$$\varnothing_{AB} = \varnothing_{BC} = \varnothing_{CD} = \frac{12(2040734)(112382)}{(773005.303)(80.77)(500)^2} = 0.1763160461$$

For beam of 3.00m is:

$$\varnothing_{AB} = \varnothing_{BC} = \varnothing_{CD} = \frac{12(2040734)(112382)}{(773005.303)(80.77)(300)^2} = 0.4897667946$$

For columns of 5.00m is:

$$\varnothing_{AE} = \varnothing_{BF} = \varnothing_{CG} = \varnothing_{DH} = \frac{12(2040734)(64516)}{(773005.303)(65.86)(500)^2} = 0.1241340346$$

The fixed-end moments for beams with uniformly distributed load are shown:

For beams of 10.00m is:

$$M_{FAB} = M_{FBC} = M_{FCD} = +\frac{(3500)(10.00)^2}{12} = +29166.67\text{kg-m}$$

$$M_{FBA} = M_{FCB} = M_{FDC} = -\frac{(3500)(10.00)^2}{12} = -29166.67\text{kg-m}$$

For beams of 5.00m is:

$$M_{FAB} = M_{FBC} = M_{FCD} = +\frac{(3500)(5.00)^2}{12} = +7291.67\text{kg-m}$$

$$M_{FBA} = M_{FCB} = M_{FDC} = -\frac{(3500)(5.00)^2}{12} = -7291.67\text{kg-m}$$

For beams of 3.00m is:

$$M_{FAB} = M_{FBC} = M_{FCD} = +\frac{(3500)(3.00)^2}{12} = +2625\text{kg-m}$$

$$M_{FBA} = M_{FCB} = M_{FDC} = -\frac{(3500)(3.00)^2}{12} = -2625\text{kg-m}$$

For all the columns is:

$$M_{FAE} = M_{FEA} = M_{FBF} = M_{FFB} = M_{FCG} = M_{FGC} = M_{FDH} = M_{FHD} = 0$$

Calculation of "EI" is:

For all beams is:

$$EI_z = (2040734)(112382) = 229341768400\text{kg-cm}^2 = 22934176.84\text{kg-m}^2$$

For all columns is:

$$EI_z = (2040734)(64516) = 131659994700\text{kg-cm}^2 = 13165999.47\text{kg-m}^2$$

Evaluation of "EA" is:

For all beams is:

$$EA = (2040734)(178.71) = 364699573.1\text{kg}$$

For all columns is:

$$EA = (2040734)(117.42) = 239622986.3\text{kg}$$

3.1. **Method 1.** The classic method considers the deformations by flexure (method of slope-deflection) [17-22].

The values of "EI" and "M_{FAB}" are substituted to obtain the moments "M_{AB}" for each member in function of rotations and perpendicular displacements in the equations following are:

$$M_{AB} = M_{FAB} + \frac{EI_z}{L}(-4\theta_A - 2\theta_B) + \frac{6EI_z}{L^2}\Delta$$
$$M_{BA} = M_{FBA} + \frac{EI_z}{L}(-4\theta_B - 2\theta_A) + \frac{6EI_z}{L^2}\Delta$$

Once the moments are obtained, the condition equilibrium of moments is applied at joints, are:

Joint A:

$$M_{AB} + M_{AE} = 0$$

Joint B:

$$M_{BA} + M_{BC} + M_{BF} = 0$$

Joint C:

$$M_{CB} + M_{CD} + M_{CG} = 0$$

Joint D:

$$M_{DC} + M_{DH} = 0$$

Then, the equilibrium condition of shear forces are generated at the base of frame, as shown in Figure 7, is:

$$P - H_{EA} - H_{FB} - H_{GC} - H_{HD} = 0$$

Shear forces on the base of frame are expressed in terms of the final moments, as shown in Figure 8:

$$H_{EA} = \frac{M_{AE} + M_{EA}}{h}; \quad H_{FB} = \frac{M_{BF} + M_{FB}}{h};$$
$$H_{GC} = \frac{M_{CG} + M_{GC}}{h}; \quad H_{HD} = \frac{M_{DH} + M_{HD}}{h}$$

These equations are presented in terms of rotations and displacement, in this case there are 5 equations with 5 unknowns (θ_A, θ_B, θ_C, θ_D and Δ), and these are developed to find their values. Once the rotations and displacement are found, subsequently, these are substituted into the equations corresponding to localize the final moments at the ends of

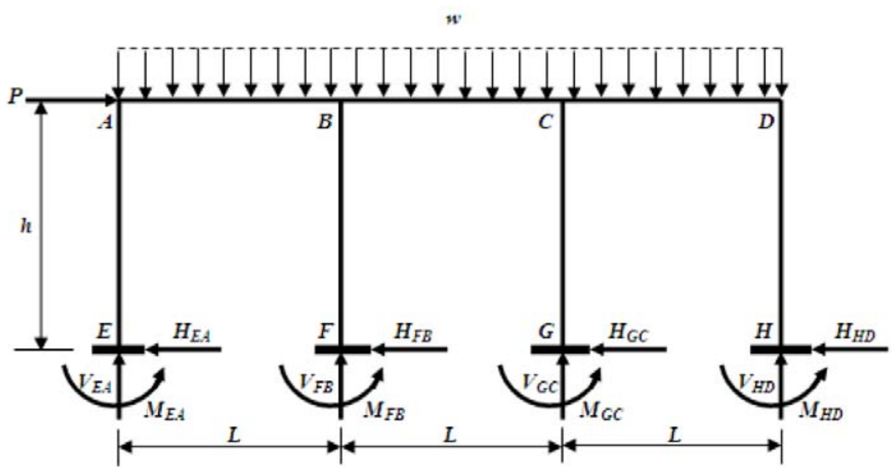

FIGURE 7. Free body diagram of whole frame

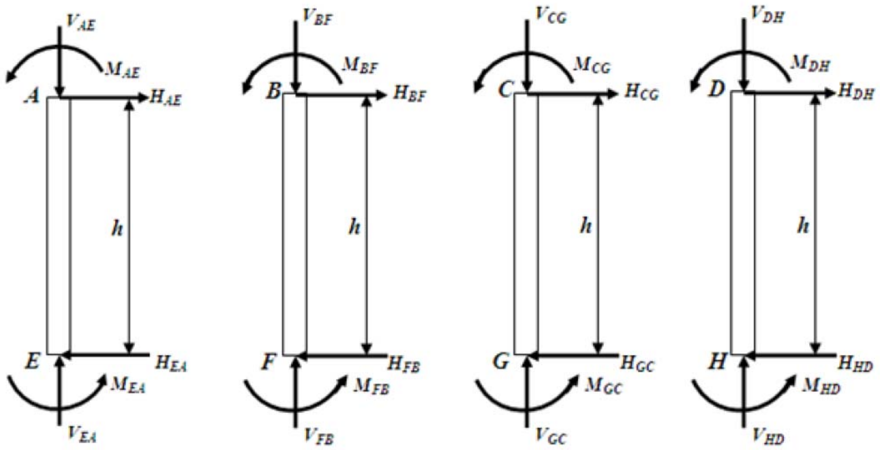

Figure 8. Free body diagram of each column

the members. Now by static equilibrium, axial forces and shear forces are obtained for each member. Then, the diagrams of axial forces, shear forces and moments are obtained.

3.2. **Method 2.** The Luévanos-Rojas method takes into account the deformations by flexure and shear [9,10].

The values of "EI", "$\varnothing$" and "M_{FAB}" are substituted to obtain the moments "M_{AB}" for each member in function of rotations and displacements in the equations following:

$$M_{AB} = M_{FAB} + \frac{EI_z}{L}\left[-\left(\frac{4+\varnothing}{1+\varnothing}\right)\theta_A - \left(\frac{2-\varnothing}{1+\varnothing}\right)\theta_B\right] + \frac{6EI_z}{L^2(\varnothing+1)}\Delta$$

$$M_{BA} = M_{FBA} + \frac{EI_z}{L}\left[-\left(\frac{4+\varnothing}{1+\varnothing}\right)\theta_B - \left(\frac{2-\varnothing}{1+\varnothing}\right)\theta_A\right] + \frac{6EI_z}{L^2(\varnothing+1)}\Delta$$

Once the moments are obtained, the condition equilibrium of moments is applied at joints, and then it follows the same procedure that was done for Method 1.

3.3. **Method 3.** The proposed method takes into account the deformations by flexure, shear and axial.

The values of "EI", "$\varnothing$" and "M_{FAB}" are substituted to obtain the moments "M_{AB}" for each member into Equations (54a) and (54b) are presented in function of rotations and displacements.

Now, using the free body diagrams of each member, the shear forces are obtained in function of moments, but the moments are in function of rotations and displacements, then the shear forces can be expressed in function of rotations and displacements.

Equation (58) is used to found the axial forces in each one of members in function of axial displacements.

The free body diagrams of all free joints are presented in Figure 9.

The equilibrium conditions for each of joints are used, these are: sum of moments, sum of vertical forces and sum of horizontal forces should be equal zero, the equations system is found in function of rotations and displacements.

Then, solve the equations for the rotations, perpendicular displacements and parallel to the axis of the bar at all joints.

Now, the rotations and displacements are substituted in equations corresponding, the moments, shear forces and axial forces are obtained at the ends of members.

The results are presented in Tables 1-4, for the three different examples were performed by the three methods. Those are shown in Appendix.

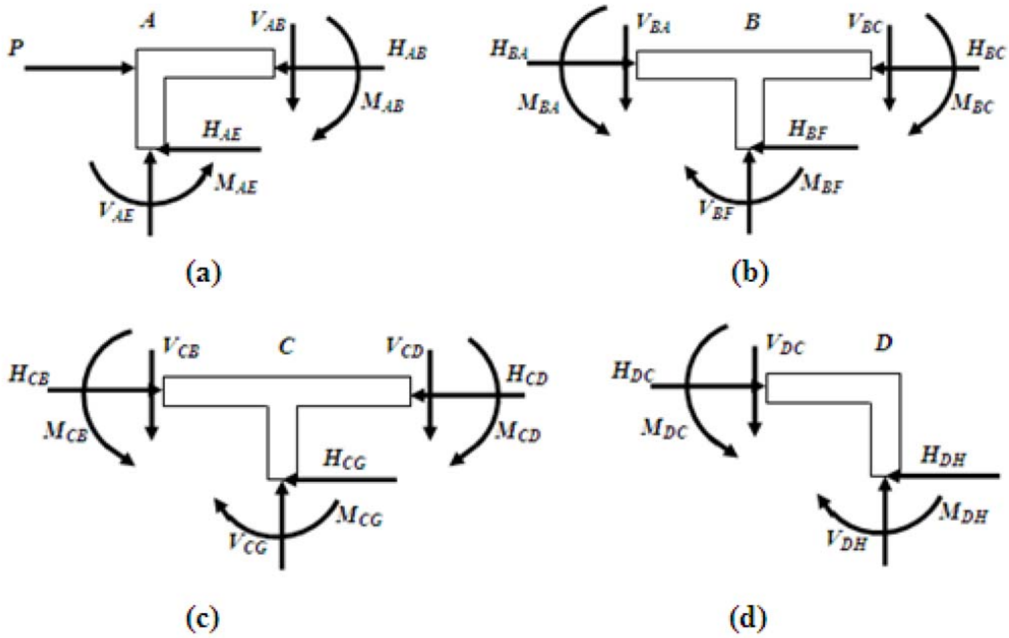

FIGURE 9. The free body diagram of joints: (a) Joint "A", (b) Joint "B", (c) Joint "C", (d) Joint "D"

Nomenclature that is used in Appendix:

θ_i = the angle that forms the tangent due to the deformation in the joint "i", in radians, (+) the rotations are shown as clockwise; (−) the rotations are shown as counterclockwise.

Δ_{Hi} = the horizontal displacement in the joint "i", in meters, (+) the displacement to the right of the axis of reference; (−) the displacement to the left of the axis of reference.

Δ_{Vi} = the vertical displacement in the joint "i", in meters, (+) the displacement is above of the axis of reference; (−) the displacement is below of the axis of reference.

$M1$ = Method 1 considers the deformations by flexure.

$M2$ = Method 2 considers the deformations by flexure and shear.

$M3$ = Method 3 considers the deformations by flexure, shear and axial.

V_{ij} = shear force the member "ij", in the joint i: for beams, (+) shear force is above of the axis of reference; (−) shear force is below of the axis of reference; for columns, (+) shear force to the right of the axis of reference; (−) shear force to the left of the axis of reference.

Being: $V_{AE} = -V_{EA}$; $V_{BF} = -V_{FB}$; $V_{CG} = -V_{GC}$ and $V_{DH} = -V_{HD}$.

N_{ij} = axial force the member "ij", (+) compression force, (−) tension force.

M_{ij} = negative moment the member "ij", in the joint "i".

$M\text{¢}_{ij}$ = positive moment in center of length the member "ij".

Nomenclature for moment: for horizontal members, (+) moment that is above of the axis of reference (compression in superior fiber and tension in inferior fiber), (−) moment that is below the axis of reference (tension in superior fiber and compression in inferior fiber); for vertical members, (+) moment that is to the right of the axis of reference (compression in fiber of right side and tension in fiber of left side), (−) moment that is to the left of the axis of reference (tension in fiber of right side, and compression in fiber of left side).

4. **Results.** According to Table 1, the rotations and displacements in each of the joints are presented; the differences are smaller in all cases between the method 3 and method

2. However, in short members are quite considerable. By example, for $L = 3.00$m, the largest difference exists in the joint "B", due to the rotation of a 75% and the smallest difference for horizontal displacement which occurs in the joint "C" of 0%.

The shear forces in the ends of the members for the three methods are shown in Table 2; also the differences are greater in all cases between the method 3 and method 1. Then, the differences are presented between the method 3 and method 2. However, in short beams are greatest in all case. By example, for $L = 3.00$m is lower a 5% in the member "BA" in absolute value for the method 3 with respect to the method 2, and for $L = 3.00$m is greater in a 12% in the member "AB" for the method 3 with respect to the method 2. For the columns are presented the greatest difference, for $L = 5.00$m in the member "AE" of a 12%, the method 3 is lower with respect to the method 2 in absolute value.

Table 3 shows the axial forces in the members between the three methods; the differences are smaller in all cases between the method 3 and method 2. However, the greatest difference in short members. By example, for $L = 3.00$m is smaller a 4% for the member "BF" in absolute value for the method 3 with respect to the method 2, and for $L = 3.00$m is greater of a 12% in the member "AE" for the method 3 with respect to the method 2.

The negative moments and positive for the three methods are illustrated in Table 4; also the differences are smaller in all cases between the methods 2 and 3. According to the results, for $L = 5.00$m is greater in a 63% for the member "AB" and "AE" in the joint "A" in absolute value in the method 3 with respect to the method 2, and for $L = 3.00$m is smaller in a 39% in the member "BC" of the joint "B" in absolute value for the method 3 with respect to the method 2.

5. **Conclusions.** In this paper is proposed a new method based in principles of mechanic, i.e., using equilibrium systems static, which considers the three conditions basic of equilibrium, these are: sum of the moments, sum of the vertical forces and sum of the horizontal forces, these are equal zero. Also, it presents mathematical development to find the equations of moments, shear forces and axial forces, that are shown in function of rotations, vertical displacements and horizontal displacements, which takes into account the deformations by flexure, shear and axial. Finally, a numerical example for structural analysis of rigid frames was presented.

The traditional method considers the deformations by flexure, also is called the method slope-deflection; this is how usually develops the structural analysis of statically indeterminate rigid frames, the differences with respect the proposed method are quite considerable.

The method developed by Luévanos-Rojas considers the deformations by flexure and shear for statically indeterminate rigid frames; the differences with respect the proposed method are smaller that between the traditional method and the proposed method, such a situation is logic because the method 2 is more close to the real conditions that Method 1.

Structural members are designed due to the actions to which are subject, these are: the moments, the shear forces and the axial forces. This paper considers the conditions mentioned above because are implicit in the deformations: 1) flexure deformations are presented due to the moments; 2) shear deformations appear due to the shear forces acting on member; 3) axial deformations are shown due axial forces acting on member. Hence, the proposed method is more precise, since it considers total actions acting on structural members and therefore adheres more to the real conditions.

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Appendix.

TABLE 1. The rotations and displacements in each one of the joints

Deformations	Case 1 $L = 10.00$m					Case 2 $L = 5.00$m					Case 3 $L = 3.00$m				
	$M1$	$M2$	$M3$	$\frac{M3}{M1}$	$\frac{M3}{M2}$	$M1$	$M2$	$M3$	$\frac{M3}{M1}$	$\frac{M3}{M2}$	$M1$	$M2$	$M3$	$\frac{M3}{M1}$	$\frac{M3}{M2}$
$\theta_A \times 10^4$(rad)	+17.42	+18.46	+19.18	1.10	1.04	+3.89	+4.28	+4.75	1.22	1.11	+1.47	+1.84	+2.35	1.60	1.28
$\Delta_{HA} \times 10^4$(m)	+13.59	+14.96	+17.94	1.32	1.20	+11.93	+13.47	+14.32	1.20	1.06	+11.18	+12.94	+13.53	1.21	1.05
$\Delta_{VA} \times 10^4$(m)	0	0	+3.20	–	–	0	0	+1.44	–	–	0	0	+0.71	–	–
$\theta_B \times 10^4$(rad)	−1.91	−2.00	−1.79	0.94	0.90	−0.18	−0.02	+0.21	1.17	10.5	+0.08	+0.32	+0.56	7.00	1.75
$\Delta_{HB} \times 10^4$(m)	+13.59	+14.96	+15.65	1.15	1.05	+11.93	+13.47	+13.68	1.15	1.02	+11.18	+12.94	+13.19	1.18	1.02
$\Delta_{VB} \times 10^4$(m)	0	0	+7.70	–	–	0	0	+3.92	–	–	0	0	+2.36	–	–
$\theta_C \times 10^4$(rad)	+3.94	+4.10	+3.86	0.98	0.94	+1.14	+1.18	+0.95	0.83	0.81	+0.47	+0.58	+0.39	0.83	0.67
$\Delta_{HC} \times 10^4$(m)	+13.59	+14.96	+13.97	1.03	0.93	+11.93	+13.47	+13.23	1.03	0.91	+11.18	+12.94	+12.96	1.16	1.00
$\Delta_{VC} \times 10^4$(m)	0	0	+7.64	–	–	0	0	+3.80	–	–	0	0	+2.22	–	–
$\theta_D \times 10^4$(rad)	−13.54	−14.41	−15.05	1.11	1.04	−1.59	−1.68	−2.03	1.28	1.21	+0.05	+0.17	+0.08	1.60	0.47
$\Delta_{HD} \times 10^4$(m)	+13.59	+14.96	+12.43	0.91	0.83	+11.93	+13.47	+12.95	1.09	0.96	+11.18	+12.94	+12.84	1.15	0.99
$\Delta_{VD} \times 10^4$(m)	0	0	+3.37	–	–	0	0	+1.80	–	–	0	0	+1.29	–	–

TABLE 2. The shear forces of each member in kg

Shear Force	Case 1 $L = 10.00$m					Case 2 $L = 5.00$m					Case 3 $L = 3.00$m				
	$M1$	$M2$	$M3$	$\frac{M3}{M1}$	$\frac{M3}{M2}$	$M1$	$M2$	$M3$	$\frac{M3}{M1}$	$\frac{M3}{M2}$	$M1$	$M2$	$M3$	$\frac{M3}{M1}$	$\frac{M3}{M2}$
V_{AB}	+15365	+15331	+15327	1.00	1.00	+6709	+6760	+6892	1.03	1.02	+2883	+3038	+3389	1.18	1.12
V_{BA}	−19635	−19669	−19673	1.00	1.00	−10791	−10740	−10608	0.98	0.99	−7617	−7462	−7111	0.93	0.95
V_{BC}	+17220	+17223	+17226	1.00	1.00	+8221	+8207	+8183	1.00	1.00	+4416	+4326	+4177	0.95	0.97
V_{CB}	−17780	−17777	−17774	1.00	1.00	−9279	−9293	−9317	1.00	1.00	−6084	−6174	−6323	1.04	1.02
V_{CD}	+18821	+18858	+18861	1.00	1.00	+8993	+8981	+8880	0.99	0.99	+4466	+4475	+4300	0.96	0.96
V_{DC}	−16179	−16142	−16139	1.00	1.00	−8507	−8519	−8620	1.01	1.01	−6034	−6025	−6200	1.03	1.03
V_{AE}	+3788	+3507	+3375	0.89	0.96	−280	−312	−275	0.98	0.88	−949	−937	−861	0.91	0.92
V_{BF}	−2320	−2245	−2263	0.98	1.01	−1564	−1521	−1478	0.95	0.97	−1387	−1366	−1325	0.96	0.97
V_{CG}	−473	−529	−486	1.03	0.92	−1148	−1182	−1220	1.06	1.03	−1266	−1290	−1347	1.06	1.04
V_{DH}	−5996	−5733	−5626	0.94	0.98	−2008	−1985	−2027	1.01	1.02	−1398	−1407	−1467	1.05	1.04

TABLE 3. The axial forces of each member in kg

Axial Force	Case 1 $L = 10.00$m					Case 2 $L = 5.00$m					Case 3 $L = 3.00$m				
	$M1$	$M2$	$M3$	$\frac{M3}{M1}$	$\frac{M3}{M2}$	$M1$	$M2$	$M3$	$\frac{M3}{M1}$	$\frac{M3}{M2}$	$M1$	$M2$	$M3$	$\frac{M3}{M1}$	$\frac{M3}{M2}$
N_{AB}	+8788	+8507	+8375	0.95	0.98	+4720	+4688	+4725	1.00	1.01	+4051	+4063	+4139	1.02	1.02
N_{BC}	+6468	+6262	+6112	0.94	0.98	+3156	+3167	+3247	1.03	1.03	+2664	+2697	+2814	1.06	1.04
N_{CD}	+5996	+5733	+5626	0.94	0.98	+2008	+1985	+2027	1.01	1.02	+1398	+1407	+1467	1.05	1.04
N_{AE}	+15365	+15331	+15327	1.00	1.00	+6709	+6760	+6892	1.03	1.02	+2883	+3038	+3389	1.18	1.12
N_{BF}	+36855	+36892	+36899	1.00	1.00	+19012	+18947	+18791	0.99	0.99	+12033	+11788	+11288	0.94	0.96
N_{CG}	+36601	+36636	+36635	1.00	1.00	+18272	+18273	+18197	1.00	1.00	+10550	+10648	+10624	1.01	1.00
N_{DH}	+16179	+16142	+16139	1.00	1.00	+8507	+8519	+8620	1.01	1.01	+6034	+6025	+6200	1.03	1.03

TABLE 4. The moments of each member in *kg-m*

Moment	Case 1 $L = 10.00$m					Case 2 $L = 5.00$m					Case 3 $L = 3.00$m				
	$M1$	$M2$	$M3$	$\frac{M3}{M1}$	$\frac{M3}{M2}$	$M1$	$M2$	$M3$	$\frac{M3}{M1}$	$\frac{M3}{M2}$	$M1$	$M2$	$M3$	$\frac{M3}{M1}$	$\frac{M3}{M2}$
M_{AB}	−14058	−13628	−13489	0.96	0.99	−324	−345	−564	1.74	1.63	+1986	+1859	+1534	0.77	0.83
M℄$_{AB}$	+19668	+19948	+20069	1.02	1.01	+6106	+6183	+6222	1.02	1.01	+3173	+3177	+3174	1.00	1.00
M_{BA}	−35410	−35320	−35223	0.99	1.00	−10531	−10295	−9853	0.94	0.96	−5115	−4777	−4050	0.79	0.85
M_{BC}	−29108	−29181	−29093	1.00	1.00	−6573	−6488	−6213	0.95	0.96	−1666	−1445	−885	0.53	0.61
M℄$_{BC}$	+13254	+13193	+13297	1.00	1.01	+3081	+3135	+3352	1.09	1.07	+1118	+1228	+1608	1.44	1.31
M_{CB}	−31907	−31953	−31834	1.00	1.00	−9220	−9201	−9049	0.98	0.98	−4171	−4217	−4104	0.98	0.97
M_{CD}	−31762	−31712	−31636	1.00	1.00	−6651	−6558	−6249	0.94	0.95	−1129	−1145	−839	0.74	0.73
M℄$_{CD}$	+18841	+19093	+19183	1.02	1.00	+4903	+4964	+5015	1.02	1.01	+1720	+1715	+1803	1.05	1.05
M_{DC}	−18554	−18128	−18026	0.97	0.99	−5436	−5404	−5600	1.03	1.04	−3482	−3471	−3688	1.06	1.06
M_{AE}	+14058	+13628	+13489	0.96	0.99	+324	+345	+564	1.74	1.63	−1986	−1859	−1534	0.77	0.83
M_{EA}	−4882	−3906	−3386	0.69	0.87	+1723	+1907	+1938	1,12	1.02	+2759	+2828	+2771	1.00	0.98
M_{BF}	−6301	−6139	−6130	0.97	1.00	−3958	−3807	−3640	0.92	0.96	−3447	−3332	−3165	0.92	0.95
M_{FB}	+5298	+5085	+5186	0.98	1.02	+3864	+3800	+3752	0.97	0.99	+3490	+3498	+3461	0.99	0.99
M_{CG}	−144	−241	−197	1.37	0.82	−2569	−2643	−2800	1.09	1.06	−3042	−3071	−3265	1.07	1.06
M_{GC}	+2219	+2402	+2231	1.01	0.93	+3170	+3266	+3301	1.04	1.01	+3287	+3379	+3471	1.06	1.03
M_{DH}	−18554	−18128	−18026	0.97	0.99	−5436	−5404	−5600	1.03	1.04	−3482	−3471	−3688	1.06	1.06
M_{HD}	+11424	+10538	+10104	0.88	0.96	+4603	+4522	+4532	0.98	1.00	+3507	+3561	+3645	1.04	1.02