International Journal of Innovative Computing, Information and Control
Volume **9**, Number **10**, October **2013**

ICIC International ©2013 ISSN 1349-4198
pp. 3941–3953

NETWORKED OUTPUT FEEDBACK ROBUST PREDICTIVE CONTROLLER DESIGN

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Received September 2012; revised January 2013

Abstract. *This paper studies the problem of robust networked static output feedback model predictive controller design that stabilizes uncertain system with guaranteed cost and Parameter-Dependent Quadratic Stability (PDQS). The upper bound on the time-delay is assumed to be bigger than sampling time. Control design is based on sufficient robust stability condition formulated as a solution of bilinear matrix inequality BMI. The example illustrates the viability of the proposed output feedback design method.*
Keywords: Robust control, Network systems, Predictive controller, Polytopic systems, Lyapunov-Krasovskii functional, Guaranteed cost

1. **Introduction.** Both model predictive control (MPC) and network control systems (NCSs) have recently attracted notable attention in control of dynamic systems. In this paper we study robust MPC design for system with uncertain time-delays induced in NCSs.

The idea of MPC can be summarized as follows [3]:

- Predict the future behavior of the process state/output over the finite time horizon.
- Compute the future input signals on-line at each step by minimizing a cost function under inequality constraints on the control and/or controlled variables.
- Apply on the controlled plant only the first element of vector control variable and repeat the previous step with new measured input/state/output variables.

The presence of the plant model is a necessary condition for the development of the predictive control. The success of MPC depends on the plant model precision. Real plants inherently include uncertainties, which should be considered in the model and consequently in the robust controller design. Two typical descriptions of uncertainty: polytopic uncertainty and bounded unstructured uncertainty, are extensively considered in the field of robust model predictive control. Most of the existing techniques for robust MPC assume measurable state, and apply plant state feedback or when the state estimator is utilized output feedback is applied. The survey of optimal and robust MPC design can be consulted in [11].

With developments in digital networks and computing devices, communication networks and control tend to be integrated into a new class of system known as networked control systems (NCSs). Some main problems emerging with NCSs are network induced time-delay and loss of data packet. Various approaches have been proposed to deal with these problems [2, 5, 9, 18]. In [13], a predictive control algorithm is constructed to generate a sequence of control predictions, some of which can suitably be selected to compensate the induced time-delay. In [4], an infinite horizon min-max model predictive control with a polytopic uncertainty and time delay is considered. The authors have used the classical solution: state feedback. To stabilize a plant with time delay, [6] proposed predictive control which is composed of an observer, a Smith predictor and a controller. In [19] a state-based networked predictive control approach is proposed to actively compensate the network communication delay. Based on switched system approach, stability analysis result is also established via the average dwell time technique. In [7], the stability of discrete-time systems with uncertain time-delays is presented. It is shown that results obtained from quadratic separation approach are equivalent to conditions obtained from standard Lyapunov-Krasovskii functional (LKF). This fact opens the possibility to determine different LKF to get less conservative results.

In this paper, a new MPC scheme for an uncertain time delay polytopic system with constrained control is developed. Based on the three terms parameter-dependent LKF, a novel less conservative robust stability condition with performance is derived for time delay uncertain NCS system with MPC. The received robust stability condition enables to design a static output feedback controller which robustly stabilizes the uncertain time delay system with guaranteed cost, and feedback gain matrices are received via a solution of bilinear matrix inequality (BMI). The other contribution of the present paper is that in the proposed MPC scheme, all the time demanding computations of output feedback gain matrices are realized off-line (for constrained control and unconstrained control cases). The actual value of control variable is obtained through simple on-line computation of scalar parameter and respective convex combination of already computed matrix gains. The developed control design scheme provides less conservative robust stability condition for parameter dependent quadratic stability (PDQS) and guarantees the robustness and performance (guaranteed cost) over the whole uncertainty domain.

The paper is organized as follows. A problem formulation and preliminaries on a predictive output/state model as a polytopic time-delay system are given in Section 2. In Section 3, the robust output feedback predictive controller design for networked predictive control systems using bilinear matrix inequality is presented. In Section 4, the input constraints (input rate constraints) are applied to linear matrix inequality (LMI) and (BMI) feasible solution. The example in Section 5 illustrates the effectiveness of the proposed method. Section 6 concludes the paper.

Hereafter, the following notational conventions will be adopted: given a symmetric matrix $P = P^T \in R^{n\times n}$, the inequality $P > 0$ ($P \geq 0$) denotes matrix positive definiteness (semi-definiteness). The notation $x(t+k)$ will be used to define at time t k-steps ahead prediction of a system variable x from time t onwards under specified initial state and input scenario. I denotes the identity matrix of corresponding dimensions.

2. **Problem Statement and Preliminaries.** Let us start with formulating uncertain system model respective to MPC. Consider the following linear discrete-time uncertain system

$$x(t+1) = A_v(\alpha)x(t) + B_v(\alpha)u(t) \qquad (1)$$
$$y(t) = C_v x(t)$$

where $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^l$ are state, control and output variables of system, respectively; $A_v(\alpha), B_v(\alpha)$ belong to the convex polytopic set S

$$S = \{(A_v(\alpha), B_v(\alpha)) : A_v(\alpha) \in R^{n\times n}, B_v(\alpha) \in R^{n\times m}\} \qquad (2)$$

$$(A_v(\alpha), B_v(\alpha)) = \sum_{j=1}^{K}(A_j, B_j)\alpha_j, \quad j = 1, 2, \ldots, K, \quad \alpha_j \geq 0, \quad \sum_{j=1}^{K}\alpha_j = 1$$

Simultaneously with (1) we consider the nominal model of system (1) in the form

$$x(t+1) = A_o x(t) + B_o u(t) \qquad (3)$$

where matrices A_o, B_o are any constant matrices from the convex bounded domain S (2). The nominal model serves for a model prediction of output variable $y(t+1), y(t+2), y(t+3), \dots, y(t+N)$, while (1) is considered as a real plant model description providing plant output $y(t)$. Therefore, in the robust controller design, we assume that for output feedback at time t, output $y(t)$ is obtained from uncertain real plant and predicted outputs for times $t+1, t+2, \dots, t+N$ will be obtained from model prediction, where the nominal model (3) is used. Thus, from robust stability point of view the predicted states and outputs of the system (1) for the instants $t+k$, $k = 1, 2, \dots, N$ are given as follows:

$$x(t+k+1) = A_o^k A_v(\alpha)x(t) + A_o^k B_v(\alpha)u(t) + \sum_{i=0}^{k-1} A_o^{k-i-1} B_o u(t+1+i) \tag{4}$$

and corresponding output is

$$y(t+k) = C_v x(t+k) \tag{5}$$

Prediction models (4) and (5) for $k = 0, 1, 2, \dots, N$ can be written in a compact form as

$$\begin{aligned} z(t+1) &= A_f(\alpha)z(t) + B_f(\alpha)v(t) \\ y_f(t) &= C_f z(t) \end{aligned} \tag{6}$$

where

$$\begin{aligned} z(t)^T &= [x(t)^T ... x(t+N)^T] \\ v(t)^T &= [u(t)^T ... u(t+N_u)^T] \\ y_f(t)^T &= [y(t)^T ... y(t+N)^T] \end{aligned} \tag{7}$$

and

$$B_f(\alpha) = \begin{bmatrix} B_v(\alpha) & 0 & ... & 0 \\ A_o B_v(\alpha) & B_o & ... & 0 \\ ... & ... & ... & 0 \\ A_o^N B_v(\alpha) & A_o^{N-1} B_o & ... & A_o^{N-N_u} B_o \end{bmatrix} \tag{8}$$

$$A_f(\alpha) = \begin{bmatrix} A_v(\alpha) & 0 & ... & 0 \\ A_o A_v(\alpha) & 0 & ... & 0 \\ ... & ... & ... & ... \\ A_o^N A_v(\alpha) & 0 & ... & 0 \end{bmatrix} \tag{9}$$

$$C_f = blockdiag\{C_v\}$$

where N, N_u are output and control prediction horizons of model predictive control, respectively. Matrices dimensions are $A_f(\alpha) \in R^{n(N+1)\times n(N+1)}$, $B_f(\alpha) \in R^{n(N+1)\times m(N_u+1)}$ and $C_f \in R^{l(N+1)\times n(N+1)}$. Other approach to construct the model predictions can be consulted in [14].

Consider the cost function associated with the system (6) over the optimization horizon N_{op}

$$J = F(z(N_{op}+1)) + \sum_{t=0}^{N_{op}} J(t) \tag{10}$$

where $F(z(N_{op}+1))$ is a given terminal penalty at time $N_{op}+1$ and

$$J(t) = z(t)^T Q z(t) + v(t)^T R v(t) \tag{11}$$

$Q = blockdiag\{Q_i\}$, $R = blockdiag\{R_j\}$, $i = 1, 2, \dots, N$; $j = 1, 2, \dots, N_u$ are corresponding weighting matrices. For more details see [11].

In the following we consider output feedback network control with uncertain network-induced time delay with control algorithm

$$v(t) = F y_f(t-\tau) = F C_f z(t-\tau) \tag{12}$$

where τ is unknown induced delay with maximum delay value τ_M $0 \le \tau \le \tau_M \in N^+$. Real value of maximum delay in seconds is calculated by formula $\tau_{Ms} = \tau_M T_s[sec]$, T_s-sampling time; $F^T = [F_0^T ... F_{N_u}^T]$, $F_i = [F_{i0} ... F_{iN}]$, $i = 0, 1, 2, \ldots, N_u$ are the output feedback gain matrices. Control $v(t)$ or rate of $v(t)$ is constrained to evolve in one or both of the following two sets

$$\Gamma_1 = \{v(t) \in R^{mN_u} : |v_i(t)| \le U_i > 0\}$$
$$\Gamma_2 = \{v(t) \in R^{mN_u} : |v_i(t+1) - v_i(t)| \le U_{di} > 0\}$$
$$i = 1, 2, \ldots, mN_u$$

Note that the proposed control algorithm (12) is more general than those proposed in [8, 12] where only state feedback is considered in the simple form

$$u(t+k) = Kx(t+k)$$

To obtain the set of stabilizing controllers which guarantee all constraints, the approach of stabilizing tubes, [10] and references therein, can be used. For stabilizing tubes approach, an approximating control law is given by the state feedback polynomial of pre-specified degree. In this paper, for guaranteeing the closed-loop system stability, we follow the idea of Lyapunov function, guaranteed cost and robust controller design.

Closed-loop system obtained from (6) and (12) is

$$z(t+1) = A_f(\alpha)z(t) + B_f(\alpha)FC_f z(t-\tau) \tag{13}$$

The guaranteed cost concept is given in the next definition.

Definition 2.1. *Consider the system (6) with control algorithm (12). If there exists a control law (12) with $v(t)^*$ and a positive scalar J^* such that the closed-loop system (13) is stable and the value of closed-loop cost function (10) J satisfies $J \le J^*$ then J^* is said to be the guaranteed cost and $v(t)^*$ is said to be the guaranteed cost control law for the system (6).*

The problem studied in this paper can be summarized as follows. Design the robust model predictive controller with static output feedback and input constraints which for given prediction N, control N_u and optimization N_{op} horizons guarantees the closed-loop system (13) stability, robustness and guaranteed cost when the communication real-time network is integrated into feedback control loop. We will consider below that $N_{op} \to \infty$, $F(z(N_{op}+1)) \to 0$.

In Section 3, where robust stability condition for MPC is developed, the following approach to time delay will be adopted. Consider that unknown time delay τ can be subdivided into N_d parts, that is

$$z(t) - z(t-\tau) = \sum_{i=1}^{N_d} \Delta z\left(t - \tau + \frac{\tau}{N_d} i\right) = z_s(t) \tag{14}$$

with

$$\Delta z\left(t - \tau + \frac{\tau}{N_d} i\right) = z\left(t - \tau + \frac{\tau}{N_d} i\right) - z\left(t - \tau + \frac{\tau}{N_d}(i-1)\right)$$

Relation (14) is a discrete-time counterpart of the Leibnitz-Newton formula used for continuous systems where the integral is replaced by the sum of the r.h.s. of (14). Using (14) the closed-loop system (13) can be rewritten as

$$z(t+1) = A_c(\alpha)z(t) - B_f(\alpha)FC_f z_s(t) \tag{15}$$

where $A_c(\alpha) = A_f(\alpha) + B_f(\alpha)FC_f$.

Let us turn our attention to robust stability for a system with time-delay. Considering Lyapunov function approach, the extended Lyapunov-Krasovskii functional (LKF) is used to include variable time-delay. To guarantee closed-loop stability of uncertain time delay system in the whole domain S (2), the concept of quadratic stability is frequently used. That is one Lyapunov-Krasovskii functional works for whole uncertainty domain. Experience has shown that in many cases quadratic stability is rather conservative. Therefore, to decrease the conservativeness for uncertain systems, robust stability with parameter-dependent quadratic stability (PDQS) has been introduced. Using the concept of Lyapunov -Krasovskii functional (LKF) it is possible to formulate the following definition and lemma.

Definition 2.2. *The closed-loop system (15) is robustly stable in the convex uncertainty domain S with parameter dependent quadratic stability (PDQS) if and only if there exists a positive definite parameter-dependent Ljapunov-Krasovskii functional (LKF) $V(\alpha,t)$ such that*

$$\Delta V(\alpha,t) = V(\alpha,t+1) - V(\alpha,t) < 0 \tag{16}$$

Lemma 2.1. [16] *Consider the closed-loop system (15). The respective control algorithm (12) is the guaranteed cost control law for the cost function (10) if and only if there exists a positive definite LKF $V(\alpha,t)$ and gain matrix F (12) such that the following condition holds*

$$B_e = z(t)^T(Q + C_f^T F^T R F C_f)z(t) + \Delta V(\alpha,t) \le 0 \tag{17}$$

Moreover, summarizing (17) from initial time t_o to $t \to \infty$ the following inequality is obtained

$$-V(\alpha,t_o) + J \le 0 \tag{18}$$

Definition 2.1 and (18) imply the guaranteed cost bound

$$J^* \le V(\alpha,t_o) \tag{19}$$

Based on the above preliminaries, the problem of robust MPC design for variable time-delay systems can be formulated as: Find an output feedback gain matrix F solving

$$\min_F \{J(t) = z(t)^T Q z(t) + v(t)^T R v(t)\} \tag{20}$$

subject to

- closed-loop system

$$z(t+1) = A_c(\alpha)z(t) - B_f(\alpha)FC_f z_s(t)$$

- robust stability condition (16)
- guaranteed cost condition (17)
- input constraints condition

$$\Gamma_1 = \{v(t) \in R^{mN_u} : |v_i(t)| \le U_i > 0\}$$

- rate of input constraints condition

$$\Gamma_2 = \{v(t) \in R^{mN_u} : |v_i(t+1) - v_i(t)| \le U_{di} > 0\}$$

$$i = 1, 2, \ldots, mN_u$$

- or other constraints conditions which are not handled in this paper.

3. **Robust Output Predictive Controller Design.** In this section a networked robust output feedback predictive controller design procedure is given. Main results are summarized in the following theorem.

Theorem 3.1. *Consider the uncertain system (6), the control algorithm (12) and cost function (10). The closed-loop system (13) is PDQS with guaranteed cost for time delay* τ, $0 \le \tau \le \tau_M$ *if there exist matrices* N_1, N_2, N_3, *symmetric positive definite matrices* Q_{oi}, Q_{1i}, P_i, $i = 1, 2, \dots, K$ *and a gain matrix* F *such that for each* $i = 1, 2, \dots, K$ *the following bilinear matrix inequality (BMI) holds*

$$W_i = \{w^i_{kj}\}_{3\times 3} \le 0, \quad i = 1, 2, \dots, K \tag{21}$$

where

$$w^i_{11} = N_1^T + N_1 + P_i + \tau_M^2 Q_{1i}$$
$$w^i_{12} = -N_1^T A_{ci} + N_2 - \tau_M^2 Q_{1i}$$
$$w^i_{13} = N_1^T B_{fi} F C_f + N_3$$
$$w^i_{22} = -N_2^T A_{ci} - A_{ci}^T N_2 + Q + C_f^T F^T R F C_f - P_i + \tau_M^2 Q_{1i}$$
$$w^i_{23} = N_2^T B_{fi} F C_f - A_{ci}^T N_3 - C_f^T F^T R F C_f + Q_{oi}$$
$$w^i_{33} = N_3^T B_{fi} F C_f + C_f^T F^T B_f^T N_3 + C_f^T F^T R F C_f - Q_{oi} - \tau_M Q_{1i}$$

and

$$(A_c(\alpha), B_f(\alpha)) = \sum_{i=1}^{K} (A_{ci}, B_{fi})\alpha_i$$

$$(P(\alpha), Q_o(\alpha), Q_1(\alpha)) = \sum_{i=1}^{K} (P_i, Q_{oi}, Q_{1i})\alpha_i$$

Note that for the case of $P_i = P_j$, $Q_{oi} = Q_{oj}$, $Q_{1i} = Q_{1j}$, $i \ne j$, $i, j = 1, 2, \dots, K$ *the quadratic stability is obtained.*

Proof: To prove (21), it is sufficient to construct such LKF $V(\alpha, t)$ for closed-loop system (15) that the inequality (17) holds. Main idea of the proof is presented; technical details are omitted. Consider the following parameter-dependent LKF in the form

$$V(\alpha, t) = V_1(\alpha, t) + V_2(\alpha, t) + V_3(\alpha, t) \tag{22}$$

where

$$V_1(\alpha, t) = z(t)^T P(\alpha) z(t),$$

its first difference

$$\Delta V_1(\alpha, t) = z(t+1)^T P(\alpha) z(t+1) - z(t)^T P(\alpha) z(t) \tag{23}$$

$$V_2(\alpha, t) = \sum_{j=t-\tau_M}^{t-1} z(j)^T Q_o(\alpha) z(j)$$

with first difference

$$\Delta V_2(\alpha, t) = -z(t-\tau_M)^T Q_o(\alpha) z(t-\tau_M) + z(t)^T Q_o(\alpha) z(t) \tag{24}$$

and

$$V_3(\alpha, t) = \tau_M \sum_{m=-\tau_M}^{-1} \sum_{j=t+m}^{t-1} y(j)^T Q_1(\alpha) y(j)$$

with first difference

$$\Delta V_3(\alpha, t) = \tau_M^2 y(t)^T Q_1(\alpha) y(t) - \tau_M \sum_{j=t-\tau_M}^{t-1} y(j)^T Q_1(\alpha) y(j) \tag{25}$$

where $y(j) = z(j+1) - z(j)$.

Applying Jensens inequality to (25), one obtains

$$\Delta V_3(\alpha, t) \le \tau_M^2 y(t)^T Q_1(\alpha) y(t) - \tau_M z_s(t)^T Q_1(\alpha) z_s(t) \tag{26}$$

(z_s was defined in (14)). The following equality is used to derive (21)

$$2[z(t+1)^T N_1^T + z(t)^T N_2^T + z_s(t)^T N_3^T][z(t+1) - A_c(\alpha) z(t) + B_f(\alpha) F C_f z_s(t)] = 0 \tag{27}$$

Substituting (23)-(27) to (17) and due to the linearity of obtained result with respect to α_i, the inequality (21) is obtained, which proves Theorem 3.1.

If the solution of (21) in Theorem 3.1 is feasible with respect to positive definite matrices P_i, Q_{oi}, Q_{1i}, $i = 1, 2, \ldots, K$, matrices N_1, N_2, N_3 and gain matrix F then for the uncertain time-delay $0 \le \tau \le \tau_M$ closed-loop system is PDQS in the convex set S (2) with guaranteed cost.

4. **MPC Design for Input and Input Rate Constraints.** In standard MPC approach, input or input rate constraints (as well as other constraints: state, output variables) are treated by solving open-loop optimal control problem in each sampling period over the defined finite horizon. The first element of the optimal control sequence is applied to the plant and the next time step the computation is repeated with new measured variables. Thus, the implementation of the standard MPC strategy requires a QP solver for the on-line optimization which requires significant on-line computational effort. The novel approach to guarantee the stability and feasibility of MPC and reduce the on line computation approximate state feedback polynomial controllers employing the concept of stability tubes has been introduced in [10] and references therein.

In this section, we propose the control algorithm, (based on the idea from [17]) where the actual output feedback control gain matrix is obtained as a convex combination of two gain matrices computed off line. One of this matrices is computed for constrained and one for unconstrained cases, such that both gains guarantee PDQS, guaranteed cost and robustness properties of the closed-loop system. Thus, the computationally demanding tasks- solution of matrix inequalities providing control gain matrices are realized off line which significantly reduces computational burden. The convex combination of these two gain matrices is determined by a scalar parameter which is updated on-line in each step. The respective procedure is described below.

Consider the system (6) where the control $v(t)$ or rate of $v(t)$ is constrained to evolve in the following two sets

$$\Gamma_1 = \{v(t) \in R^{mN_u} : |v_i(t)| \le U_i\} \tag{28}$$

$$\Gamma_2 = \{v(t) \in R^{mN_u} : |v_i(t+1) - v_i(t)| \le U_{di}\}$$

The aim of this part of paper is to design the robust stabilizing static output feedback control law for system (6) with guaranteed performance, in the form

$$v(t) = F C_f z(t - \tau) \tag{29}$$

considering the constraints (28). To derive sufficient stability conditions for input (input rate) constraints for networked control system we consider that the positive invariant region, [15], with respect to closed-loop system motion can be defined by the ellipsoidal Lyapunov function set given by $V_1(\alpha, t)$ as follows:

$$\Omega(P(\alpha)) = \{z(t) \in R^{nN} : z(t)^T P(\alpha) z(t) \le \theta\} \tag{30}$$

where θ is a positive real parameter which determines the size of $\Omega(P(\alpha))$. Note that $V_1(\alpha, t)$ is only a part of LKF (22); therefore, for the networked system the obtained results are only sufficient and may be conservative. Using the full LKF to obtain a positive invariant region or in other words design the robust output feedback gain matrix with input (rate of input) constraints for networked control systems is under research. Consider that vector F_i denotes the i-th row of matrix F and define the first input constraint

$$L(F) = \{z(t) \in R^{nN} : |F_i C_f z(t)| \leq U_i, i = 1, 2, \ldots, mN_u\}$$

or

$$L(F) = \{z(t) \in R^{nN} : |D_i F C_f z(t)| \leq U_i\} \quad (31)$$
$$i = 1, 2, \ldots, mN_u$$

where $D_i \in R^{1\times mN_u} = \{d_{ij}\}$; $d_{ij} = 1$, $i = j$, $d_{ij} = 0$, $i \neq j$. The input constraint reduces to LMI given by the following theorem [17].

Theorem 4.1. *The inclusion* $\Omega(P(\alpha)) \subseteq L(F)$ *is for output feedback control equivalent to*

$$\begin{bmatrix} P(\alpha) & C_f^T F^T D_i^T \\ D_i F C & \lambda_i \end{bmatrix} \geq 0, \quad i = 1, 2, \ldots, mN_u \quad (32)$$

where $\lambda_i \in < 0, \frac{U_i^2}{\theta} >$.

Note that Theorem 4.1 for a time delay-free system gives necessary and sufficient conditions, for networked system only sufficient ones.

Let us now formulate conditions for input variable rate constraints. Define the respective set

$$L_r(F) = \{z(t) \in R^{nN} : |D_i F C_f (z(t+1) - z(t))| \leq U_{di}, i = 1, 2, \ldots, mN_u\} \quad (33)$$

Condition defining the set $L_r(F)$ in (33) can be rewritten as

$$g_i(z) = z(t)^T (A_c(\alpha) - I)^T C_f^T F^T D_i^T D_i F C_f (A_c(\alpha) - I) z(t) - U_{di}^2 \leq 0 \quad (34)$$

Analogically to the previous case for input variable constraint, we search such controller that guarantees $\Omega(P(\alpha)) \subseteq L_r(F)$. Define $p(z) = z(t)^T P(\alpha) z(t) - \theta \leq 0$. According to the S-procedure the above inclusion is equivalent to the existence of a positive scalar γ_i such that

$$g_i(z) - \gamma_i p(z) \leq 0 \quad (35)$$

After some manipulation and using Schur complement formula the following BMI condition is obtained.

Theorem 4.2. *The inclusion* $\Omega(P(\alpha)) \subseteq L_r(F)$ *is for output feedback control equivalent to*

$$\begin{bmatrix} P(\alpha) & (A_c(\alpha) - I)^T C_f^T F^T D_i^T \\ * & \gamma_i \end{bmatrix} \geq 0 \quad (36)$$
$$i = 1, 2, \ldots, mN_u$$

where $\gamma_i \in < 0, \frac{U_{di}^2}{\theta} >$.

If the solutions of (21), together with (32) or (36) are feasible, the closed-loop system is PDQS with guaranteed cost and simultaneously constraints for input (input rate) hold.

Assume that we calculate two output feedback gain matrices: F_1 for unconstrained case and F_2 for the constrained one. Consider the real output feedback gain matrix F in the form [17]:

$$F = \delta F_1 + (1 - \delta) F_2, \quad \delta \in < 0, 1 > \quad (37)$$

Actual output feedback control gain matrix is determined as a convex combination of two gain matrices computed off-line: one for constrained and one for unconstrained cases, such that both gains guarantee PDQS, guaranteed cost and robustness properties of the closed-loop system. The convex combination of these two gain matrices is determined by a scalar parameter δ which is updated on-line in each step by (39). For gain matrices F_k, $k = 1, 2$ we obtain two matrices $A_{ck} = A_f(\alpha) + B_f(\alpha)F_kC_f$. Closed-loop system (15) is then

$$z(t+1) = A_{ck}z(t) - B_fF_kC_fz_s(t) \qquad (38)$$
$$k = 1, 2$$

For a closed-loop model respective to each k one obtains K vertices, that is finally we have $2K$ vertices. The following lemma gives the sufficient stability conditions to (38).

Lemma 4.1. *Assume that for $k = 1, 2$ inequality (21) holds. If there exists positive definite matrices P, Q_o, Q_1 such that for $2K$ vertices condition (21) holds then (38) is quadratically stable with gain matrix F given by (37). Scalar δ in (37) may be changed with any rate without violating the closed-loop stability of (38).*

Variable δ in (37) for input constraints could be calculated as

$$\delta = \min_i \frac{U_i - |u_i(t)|}{U_i} \qquad (39)$$

The resulting control design procedure is given by next steps:

- Off-line computation stage: compute output feedback gain matrices F_1 for unconstrained and F_2 for constrained case as a solution of (21) for F_1 and (21) with (32) or (36) for F_2.
- Check the robust stability conditions given by Lemma 4.1.
- On-line computation – in each step: measure the real value of $u_i(t)$ (rate of $u_i(t)$) and compute the actual value of scalar parameter δ, e.g., from (39); it is recommended to use the first order filter for δ; compute the actual feedback gain matrix F from (37).
- Time constant of the mentioned first order filter can be used for tuning the closed-loop system dynamic behavior.

All on-line computations follows the general MPC scheme, i.e., the first part of control vector $v(t)$ is applied on the real plant and the other part of control vector is used for model prediction.

4.1. **Hard input constraints.** For the obtained value of $u(t)$ (12) in the previous section, control algorithm with hard input constraints is constructed as follows [14]:

$$u_c(t) = k_u u(t) \qquad (40)$$

where k_u is defined as follows:

$$k_u = \left\{ \begin{array}{ccc} 1 & \text{if } |u_i(t)| \le U_M & i = 1, 2, \dots, m \\ \min_j \frac{U_M}{|u_j(t)|} & \text{if } |u_j(t)| > U_M & j = 1, 2, \dots, m \end{array} \right\} \qquad (41)$$

where U_M is the constraint on the input element of vector $u(t)$. For a given positive number k_{umin}, suppose k_u satisfies

$$k_{umin} \le k_u \le 1 \qquad (42)$$

Substituting $u_c(t)$ for $u(t)$ into (13) one obtains a new closed-loop system

$$z(t+1) = A_c(\alpha, F, k_u)z(t) - B_f(\alpha)F(k_u)C_fz_s(t) \qquad (43)$$

where F is the gain matrix calculated for the unconstrained case and k_u plays a role of new bounded uncertainty defined by (41). For this case, the number of vertices increases to $2K$ putting $k_u = k_{umin}$, $k_u = 1$ and a problem is to find such value of k_{umin} that guarantees the closed-loop robust stability with performance. With small modification Lemma 4.1 can guarantee the quadratic closed-loop stability when k_u varies within (41).

5. **Numerical Example.** The developed control design approach is illustrated on the model of double integrator controlled through NCS. Three simulation experiments are compared for closed-loop system with static output feedback control and input constraints:

- Case 1. Unconstrained case for output feedback gain matrix F_1.
- Case 2. Constrained case for output feedback gain matrix F_2.
- Case 3. In the paper proposed control algorithm (37) for output feedback gain matrix F.

The model of double integrator turns to (3) where

$$A_o = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}; \quad B_o = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad C_v = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and uncertain matrices for affine model are

$$A_{1u} = \begin{bmatrix} 0.001 & 0 \\ 0.005 & 0.0003 \end{bmatrix}; \quad B_{1u} = \begin{bmatrix} 0.0005 \\ 0 \end{bmatrix}$$

For the case when the number of uncertainties is $p = 1$, for polytopic model the number of vertices is $K = 2^p = 2$. Vertex matrices (2) are calculated as

$$A_1 = A_o - A_{1u}, \quad A_2 = A_o + A_{1u}, \quad B_1 = B_o - B_{1u}, \quad B_2 = B_o + B_{1u}$$

Maximal value of time delay (12) is $\tau_M = 2$ (two sampling periods). For the parameters $N = N_u = 1$, $Q = qI$, $q = 0.5$, $R = rI$, $r = 1$ and sampling time $T_s = 0.1s$ the following results are obtained for unconstrained and constrained cases:

- Unconstrained case. Maximal eigenvalue of time delay free closed-loop system $maxeig = 0.8394$, gain matrix F_1

$$F_1 = \begin{bmatrix} 0.2965 & -0.3586 \\ -0.2203 & 0.2922 \end{bmatrix}$$

  Maximal value of control variable $u(t)$ is about $u_{\max} = 0.06$.
- Case 2. Constrained case with input variable bound $U_1 = 0.03$, $\theta = 50$. Closed-loop $maxeig = 0.9725$, gain matrix F_2

$$F_2 = \begin{bmatrix} 0.0555 & -0.0633 \\ 0.614 & -0.6357 \end{bmatrix}$$

  and maximal value of control variable is about $u_{\max} = 0.008$.

Closed-loop step responses for control algorithm (37) proposed in the paper are given in Figures 1-6. Maximal value of control variable is about $u_{\max} = 0.03 = U_1$. Input constraints conditions were applied only for control variable $u(t)$. Comparison of responses respective to input constraints (Figure 3 and Figure 5) shows that the proposed control gain (Case 3) provides the response much closer to the unconstrained one than using only the feedback F_2 (Case 2). Thus we can conclude that the proposed approach provides "reasonable results", while keeping the very simple on-line computation.

6. **Conclusions.** The guaranteed cost robust control problem is studied in this paper for a class of linear time-delay uncertain polytopic predictive control systems. Based on Lyapunov-Krasovskii functional, sufficient parameter-dependent quadratic stability conditions with input (input rate) constraints are given. In the paper proposed design procedures are given in terms of bilinear matrix inequalities, which are solved off-line. The on-line stage for each input variable requires only computation of scalar parameter and addition of two pre-computed matrices. This quality makes the proposed robust MPC design method promising for applications with faster dynamics. Finally, numerical example is presented to show the effectiveness of the proposed methods.

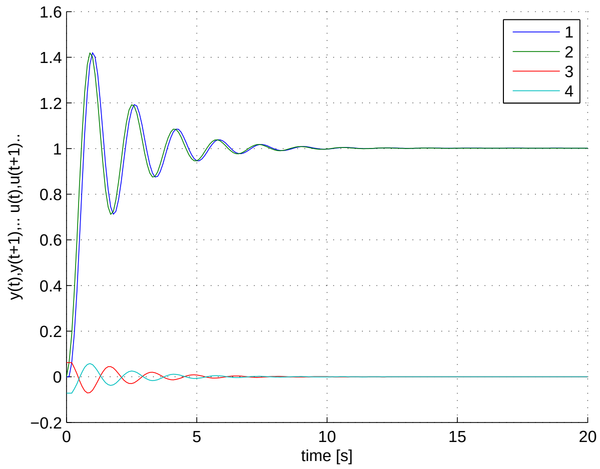

FIGURE 1. Time responses of the outputs (1, 2) and inputs (3, 4) without constraints

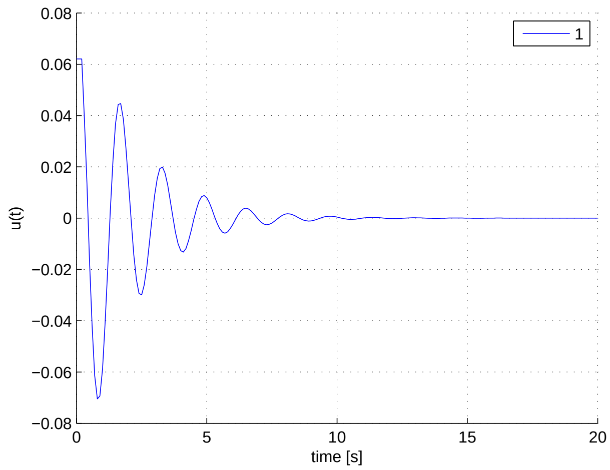

FIGURE 2. Time responses of the input $u(t)$ without constraints

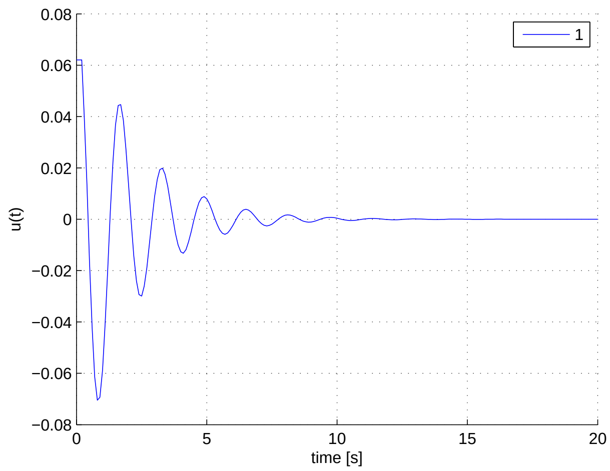

FIGURE 3. Time responses of the outputs (1, 2) and inputs (3, 4) with constraints

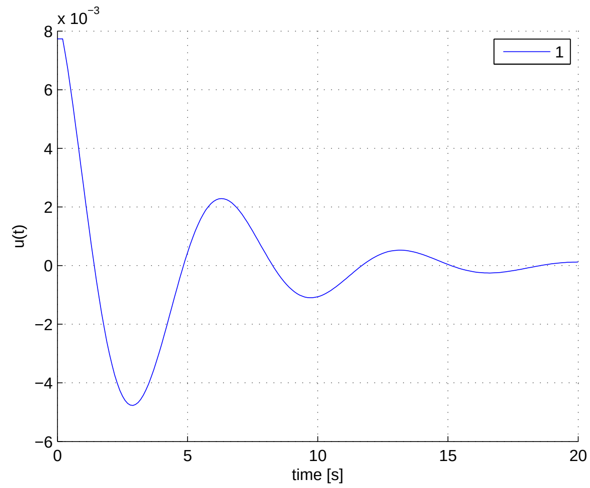

FIGURE 4. Time responses of the input $u(t)$ with constraints

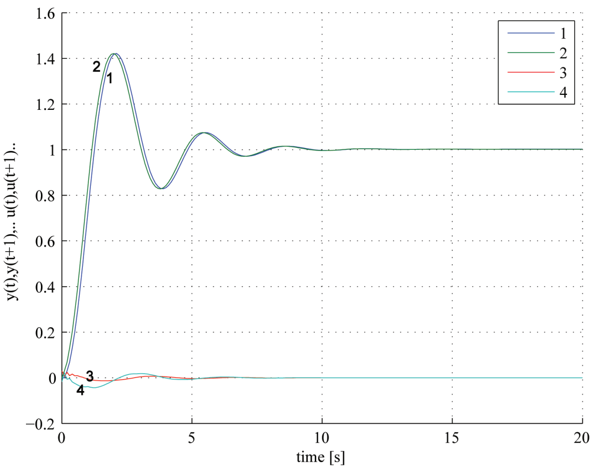

FIGURE 5. Time responses of the outputs (1, 2) and inputs (3, 4) for the proposed algorithm

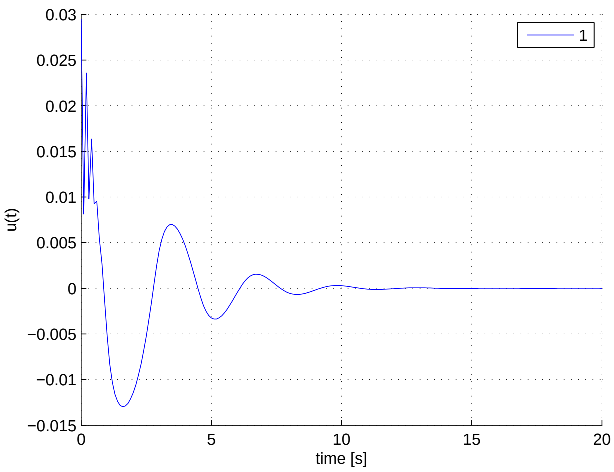

FIGURE 6. Time responses of the input $u(t)$ for the proposed algorithm

Acknowledgment. This work is partially supported by Grant N $APVV-0211-10$ and 1/2256/12 of the Slovak Scientific Grant Agency. The authors also gratefully acknowledge the helpful comments and suggestions of the reviewers and editor, which have improved the presentation.

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