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DESIGN OF PD-TYPE ANTI-WINDUP COMPENSATORS FOR CONTROL SYSTEMS WITH INPUT AND OUTPUT SATURATION NONLINEARITIES VIA LINEAR MATRIX INEQUALITY

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Abstract. *This paper presents a methodology for the design of proportional-derivative-type anti-windup compensators (PDAWCs) for linear time-invariant systems with input and output saturation nonlinearities. A linear matrix inequality (LMI) condition for a closed-loop control system representation is given to take into account the nonlinearities and disturbance. A PDAWC is designed by solving the LMI problem. A numerical example is shown to confirm the effectiveness of the proposed design method.*
Keywords: Anti-windup compensator, Input saturation, Output saturation, PD control, Linear matrix inequality

1. **Introduction.** Over the past few decades, several studies have been made on the design of anti-windup control systems, and various kinds of design methods have been proposed as in [1, 2, 3, 4, 5] and references therein.

In [1], one of the authors proposed a method of designing a proportional-derivative-type anti-windup compensators (PDAWCs) for linear time-invariant plant with input saturation as a strategy of the avoidance of a difficulty in the generation of control input. Although, in [2, 3], the problems of designing anti-windup control systems for input saturation were considered, constructive strategies for the problem about the difficulty in the generation of control input were not dealt with. Furthermore, in [1, 2, 3], the input saturation was only investigated as the nonlinearity. On the other hand, the analysis and design of control systems with output saturation have also been studied [4, 5] as another important issue. From the practical point of view, design methods for systems with input and output saturation nonlinearities can be applied to the consideration of actuator and sensor saturations in actual control systems, respectively.

In this paper, the authors provide a method of designing PDAWCs for linear time-invariant systems with not only input saturation but also output saturation. A linear matrix inequality (LMI) condition to formulate the design problem is derived. It is well-known that LMI problems can be solved efficiently by the interior-point method [6]. A numerical example is given to check the effectiveness of the proposed design method via the comparison with the conventional one.

The notation used in this paper is as follows: $\mathbb{R}^{\ell}$ is the set of all ℓ-dimensional real vectors, $\mathbb{R}^{q\times r}$ is the set of all $q \times r$ real matrices and I_q indicates the q-dimensional identity matrix, where the subscript q is sometimes omitted for notational simplicity. For a matrix A (or a vector x), A^{T} (or x^{T}) means the transpose. For a symmetric matrix A,

$A > 0$ ($A \geq 0$) means that A is positive (positive-semi) definite. For a symmetric matrix A, $A < 0$ ($A \leq 0$) indicates $-A > 0$ ($-A \geq 0$). $\|w\|_{L_2}$ is the L_2-norm of a signal $w(t)$.

2. Problem Statement.

Consider a linear time-invariant plant

$$\begin{bmatrix} \dot{x}_{\rm p}(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A_{\rm p} & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{\rm p}(t) \\ w(t) \\ u_{\rm s}(t) \end{bmatrix}, \tag{1}$$

where $x_{\rm p}(t) \in \mathbb{R}^n$ is the plant state, $w(t) \in \mathbb{R}^k$ is the exogenous disturbance, $u_{\rm s}(t) \in \mathbb{R}^\ell$ is the control input, $z(t) \in \mathbb{R}^m$ is the controlled output and $y(t) \in \mathbb{R}^r$ is the measured output.

The control input $u_{\rm s}(t)$ is subjected to the following saturation nonlinearity:

$$u_{\rm s} = \Phi_{\rm c}(u), \tag{2}$$

where

$$\Phi_{\rm c}(u) = \begin{bmatrix} \Phi_{{\rm c},1}(u_1) \\ \vdots \\ \Phi_{{\rm c},\ell}(u_\ell) \end{bmatrix}, \quad \Phi_{{\rm c},i}(u_i) = \begin{cases} \sigma_{{\rm c},i} & (u_i \geq \sigma_{{\rm c},i}) \\ u_i & (|u_i| \leq \sigma_{{\rm c},i}) \\ -\sigma_{{\rm c},i}, & (u_i \leq -\sigma_{{\rm c},i}) \end{cases} \quad (i = 1, 2, \ldots, \ell)$$

and $\sigma_{{\rm c},i} (> 0)$ $(i = 1, 2, \ldots, \ell)$ are given constants which represent the limitation on the amplitude of control input $u_{\rm s}(t)$.

The signal $u(t)$ in (2) is generated by a dynamic controller

$$\begin{bmatrix} \dot{x}_{\rm c}(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_{\rm c} & B_{\rm c} \\ C_{\rm c} & D_{\rm c} \end{bmatrix} \begin{bmatrix} x_{\rm c}(t) \\ y_{\rm s}(t) \end{bmatrix}, \tag{3}$$

where $x_{\rm c}(t) \in \mathbb{R}^q$ is the controller state.

The input $y_{\rm s}(t)$ in (3) is subjected to the following saturation nonlinearity:

$$y_{\rm s} = \Phi_{\rm m}(y), \tag{4}$$

where

$$\Phi_{\rm m}(y) = \begin{bmatrix} \Phi_{{\rm m},1}(y_1) \\ \vdots \\ \Phi_{{\rm m},r}(y_r) \end{bmatrix}, \quad \Phi_{{\rm m},i}(y_i) = \begin{cases} \sigma_{{\rm m},i} & (y_i \geq \sigma_{{\rm m},i}) \\ y_i & (|y_i| \leq \sigma_{{\rm m},i}) \\ -\sigma_{{\rm m},i}, & (y_i \leq -\sigma_{{\rm m},i}) \end{cases} \quad (i = 1, 2, \ldots, r)$$

and $\sigma_{{\rm m},i} (> 0)$ $(i = 1, 2, \ldots, r)$ are given constants which represent the limitation on the amplitude of output $y_{\rm s}(t)$.

The coefficient matrices $(A_{\rm c}, B_{\rm c}, C_{\rm c}, D_{\rm c})$ in (3) are assumed to be determined in advance using one of appropriate linear control system theories without considering the saturation nonlinearities (2) and (4) (i.e., $u_{\rm s}(t) \equiv u(t)$ and $y_{\rm s}(t) \equiv y(t)$).

Therefore, the nonlinearities (2) and (4) may deteriorate the performance of overall control system.

In order to cope with the adverse effect, in this paper, we propose the compensation by a proportional-derivative-type anti-windup compensator (PDAWC):

$$v(t) = \Lambda_{\rm P} d(t) + \Lambda_{\rm D} \dot{d}(t), \tag{5}$$

where

$$\Lambda_{\rm P} = \begin{bmatrix} \Lambda_{\rm Pc} & \Lambda_{\rm Pm} \end{bmatrix} \in \mathbb{R}^{q \times (\ell + r)}, \quad \Lambda_{\rm D} = \begin{bmatrix} \Lambda_{\rm Dc} & \Lambda_{\rm Dm} \end{bmatrix} \in \mathbb{R}^{q \times (\ell + r)},$$

and the signal $d(t)$ in (5) is defined as the output of a nonlinearity defined by

$$d = \Psi(\zeta) \tag{6}$$

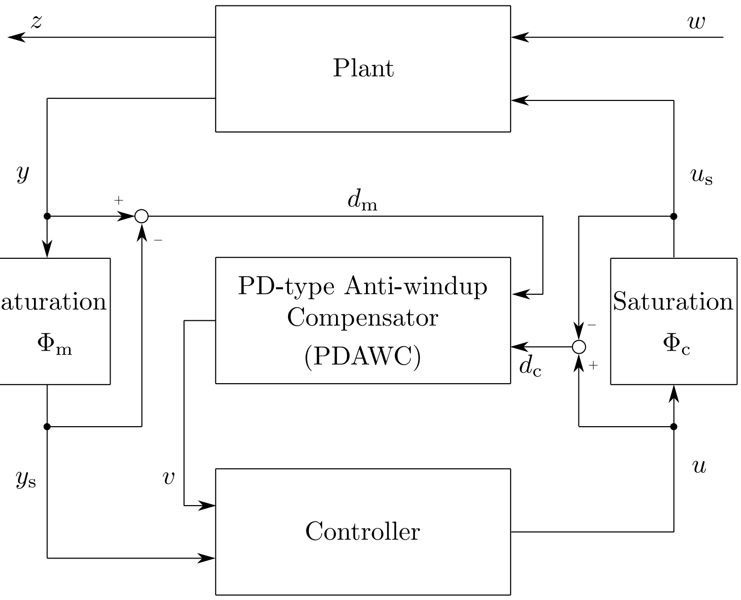

FIGURE 1. Anti-windup control system

with

$$d = \begin{bmatrix} d_c \\ d_m \end{bmatrix} \in \mathbb{R}^{\ell+r}, \quad \zeta = \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbb{R}^{\ell+r}$$

$$\Psi(\zeta) = \begin{bmatrix} \Psi_c(u) \\ \Psi_m(y) \end{bmatrix} = \begin{bmatrix} u - \Phi_c(u) \\ y - \Phi_m(y) \end{bmatrix}$$

$$\Psi_c(u) = \begin{bmatrix} \Psi_{c,1}(u_1) \\ \vdots \\ \Psi_{c,\ell}(u_\ell) \end{bmatrix}, \quad \Psi_m(y) = \begin{bmatrix} \Psi_{m,1}(y_1) \\ \vdots \\ \Psi_{m,r}(y_r) \end{bmatrix}.$$

Then, by the PDAWC (5), the controller (3) is compensated as follows:

$$\begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c(t) \\ y_s(t) \end{bmatrix} + \begin{bmatrix} v(t) \\ 0 \end{bmatrix}. \tag{7}$$

The block diagram of overall anti-windup control system is illustrated in Figure 1.

Our problem is to design a PDAWC (5) (or equivalently, to determine coefficient matrices (Λ_P, Λ_D)) by taking into account the nonlinearities and the disturbance.

3. **Design of PDAWC.**

In this section, we derive an LMI condition to determine coefficient matrices (Λ_P, Λ_D) of PDAWC.

3.1. **Analysis condition.**

The following lemma gives an analysis condition of the performance for systems with a class of nonlinearities.

Lemma 3.1. *Assume that a scalar $\gamma_a(> 0)$, a diagonal matrix R_a $(0 \le R_a < I)$ and coefficient matrices of a system*

$$\begin{bmatrix} \dot{x}_a(t) \\ \zeta_a(t) \\ \dot{\zeta}_a(t) \\ z_a(t) \end{bmatrix} = \begin{bmatrix} A_a & B_{a1} & B_{a2} & B_{a3} \\ C_{a1} & D_{a11} & D_{a12} & D_{a13} \\ C_{a2} & D_{a21} & D_{a22} & D_{a23} \\ C_{a3} & D_{a31} & D_{a32} & D_{a33} \end{bmatrix} \begin{bmatrix} x_a(t) \\ d_a(t) \\ \dot{d}_a(t) \\ w_a(t) \end{bmatrix} \tag{8}$$

are given. If there exists a positive-definite matrix $P_{\mathrm{a}}(>0)$*, a positive-definite diagonal matrix* $W_{\mathrm{a}}(>0)$ *and a nonsingular diagonal matrix* Z_{a} *satisfying a matrix inequality*

$$\begin{bmatrix} \Theta_{\mathrm{a}11} & \Theta_{\mathrm{a}12} & \Theta_{\mathrm{a}13} & \Theta_{\mathrm{a}14} & \Theta_{\mathrm{a}15} \\ {\Theta_{\mathrm{a}12}}^{\mathrm{T}} & \Theta_{\mathrm{a}22} & \Theta_{\mathrm{a}23} & \Theta_{\mathrm{a}24} & \Theta_{\mathrm{a}25} \\ {\Theta_{\mathrm{a}13}}^{\mathrm{T}} & {\Theta_{\mathrm{a}23}}^{\mathrm{T}} & \Theta_{\mathrm{a}33} & \Theta_{\mathrm{a}34} & \Theta_{\mathrm{a}35} \\ {\Theta_{\mathrm{a}14}}^{\mathrm{T}} & {\Theta_{\mathrm{a}24}}^{\mathrm{T}} & {\Theta_{\mathrm{a}34}}^{\mathrm{T}} & \Theta_{\mathrm{a}44} & \Theta_{\mathrm{a}45} \\ {\Theta_{\mathrm{a}15}}^{\mathrm{T}} & {\Theta_{\mathrm{a}25}}^{\mathrm{T}} & {\Theta_{\mathrm{a}35}}^{\mathrm{T}} & {\Theta_{\mathrm{a}45}}^{\mathrm{T}} & \Theta_{\mathrm{a}55} \end{bmatrix} < 0, \tag{9}$$

then for all $\tilde{\Psi}_{\mathrm{a}}$ *in*

$$\Delta_{\mathrm{a}} = \{\tilde{\Psi}_{\mathrm{a}} :\ d_{\mathrm{a}} = \tilde{\Psi}_{\mathrm{a}}(\zeta_{\mathrm{a}}),\ {d_{\mathrm{a}}}^{\mathrm{T}} W_{\mathrm{a}}(d_{\mathrm{a}} - R_{\mathrm{a}}\zeta_{\mathrm{a}}) \le 0,\ {\dot{d}_{\mathrm{a}}}^{\mathrm{T}} Z_{\mathrm{a}}(\dot{d}_{\mathrm{a}} - \dot{\zeta}_{\mathrm{a}}) = 0\}, \tag{10}$$

the system consisting of (8) *and* $d_{\mathrm{a}} = \tilde{\Psi}_{\mathrm{a}}(\zeta_{\mathrm{a}})$ *(for all* $\tilde{\Psi}_{\mathrm{a}} \in \Delta_{\mathrm{a}}$*) satisfies the condition* $\|z_{\mathrm{a}}\|_{L_2} \le \gamma_{\mathrm{a}}\|w_{\mathrm{a}}\|_{L_2}$ *for* $x_{\mathrm{a}}(0) = 0$*. The matrices in the matrix inequality* (9) *are given by*

$$\begin{aligned}
&\Theta_{\mathrm{a}11} = P_{\mathrm{a}}A_{\mathrm{a}} + {A_{\mathrm{a}}}^{\mathrm{T}}P_{\mathrm{a}}, \quad \Theta_{\mathrm{a}12} = P_{\mathrm{a}}B_{\mathrm{a}1} + {C_{\mathrm{a}1}}^{\mathrm{T}}R_{\mathrm{a}}W_{\mathrm{a}} \\
&\Theta_{\mathrm{a}13} = P_{\mathrm{a}}B_{\mathrm{a}2} + {C_{\mathrm{a}2}}^{\mathrm{T}}Z_{\mathrm{a}}, \quad \Theta_{\mathrm{a}14} = P_{\mathrm{a}}B_{\mathrm{a}3}, \quad \Theta_{\mathrm{a}15} = {C_{\mathrm{a}3}}^{\mathrm{T}} \\
&\Theta_{\mathrm{a}22} = W_{\mathrm{a}}R_{\mathrm{a}}D_{\mathrm{a}11} + {D_{\mathrm{a}11}}^{\mathrm{T}}R_{\mathrm{a}}W_{\mathrm{a}} - 2W_{\mathrm{a}} \\
&\Theta_{\mathrm{a}23} = W_{\mathrm{a}}R_{\mathrm{a}}D_{\mathrm{a}12} + {D_{\mathrm{a}21}}^{\mathrm{T}}Z_{\mathrm{a}}, \quad \Theta_{\mathrm{a}24} = W_{\mathrm{a}}R_{\mathrm{a}}D_{\mathrm{a}13}, \quad \Theta_{\mathrm{a}25} = {D_{\mathrm{a}31}}^{\mathrm{T}} \\
&\Theta_{\mathrm{a}33} = Z_{\mathrm{a}}D_{\mathrm{a}22} + {D_{\mathrm{a}22}}^{\mathrm{T}}Z_{\mathrm{a}} - 2Z_{\mathrm{a}}, \quad \Theta_{\mathrm{a}34} = Z_{\mathrm{a}}D_{\mathrm{a}23}, \quad \Theta_{\mathrm{a}35} = {D_{\mathrm{a}32}}^{\mathrm{T}} \\
&\Theta_{\mathrm{a}44} = -\gamma_{\mathrm{a}}I, \quad \Theta_{\mathrm{a}45} = {D_{\mathrm{a}33}}^{\mathrm{T}}, \quad \Theta_{\mathrm{a}55} = -\gamma_{\mathrm{a}}I.
\end{aligned}$$

Proof: It follows from the Schur complement formula [7] that the matrix inequality (9) is equivalent to

$$\begin{bmatrix} \Theta_{\mathrm{a}11} & \Theta_{\mathrm{a}12} & \Theta_{\mathrm{a}13} & \Theta_{\mathrm{a}14} \\ {\Theta_{\mathrm{a}12}}^{\mathrm{T}} & \Theta_{\mathrm{a}22} & \Theta_{\mathrm{a}23} & \Theta_{\mathrm{a}24} \\ {\Theta_{\mathrm{a}13}}^{\mathrm{T}} & {\Theta_{\mathrm{a}23}}^{\mathrm{T}} & \Theta_{\mathrm{a}33} & \Theta_{\mathrm{a}34} \\ {\Theta_{\mathrm{a}14}}^{\mathrm{T}} & {\Theta_{\mathrm{a}24}}^{\mathrm{T}} & {\Theta_{\mathrm{a}34}}^{\mathrm{T}} & \Theta_{\mathrm{a}44} \end{bmatrix} + \frac{1}{\gamma_{\mathrm{a}}} \begin{bmatrix} \Theta_{\mathrm{a}15} \\ \Theta_{\mathrm{a}25} \\ \Theta_{\mathrm{a}35} \\ \Theta_{\mathrm{a}45} \end{bmatrix} \begin{bmatrix} {\Theta_{\mathrm{a}15}}^{\mathrm{T}} & {\Theta_{\mathrm{a}25}}^{\mathrm{T}} & {\Theta_{\mathrm{a}35}}^{\mathrm{T}} & {\Theta_{\mathrm{a}45}}^{\mathrm{T}} \end{bmatrix} < 0. \tag{11}$$

Multiplying (11) by a vector $[{x_{\mathrm{a}}}^{\mathrm{T}}\ {d_{\mathrm{a}}}^{\mathrm{T}}\ {\dot{d}_{\mathrm{a}}}^{\mathrm{T}}\ {w_{\mathrm{a}}}^{\mathrm{T}}]$ from the left and its transpose from the right, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left({x_{\mathrm{a}}}^{\mathrm{T}}P_{\mathrm{a}}x_{\mathrm{a}}\right) - \gamma_{\mathrm{a}}{w_{\mathrm{a}}}^{\mathrm{T}}w_{\mathrm{a}} + \frac{1}{\gamma_{\mathrm{a}}}{z_{\mathrm{a}}}^{\mathrm{T}}z_{\mathrm{a}} \le 2{d_{\mathrm{a}}}^{\mathrm{T}}W_{\mathrm{a}}(d_{\mathrm{a}} - R_{\mathrm{a}}\zeta_{\mathrm{a}}) + 2{\dot{d}_{\mathrm{a}}}^{\mathrm{T}}Z_{\mathrm{a}}(\dot{d}_{\mathrm{a}} - \dot{\zeta}_{\mathrm{a}}). \tag{12}$$

Therefore, for all $\tilde{\Psi} \in \Delta$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left({x_{\mathrm{a}}}^{\mathrm{T}}P_{\mathrm{a}}x_{\mathrm{a}}\right) - \gamma_{\mathrm{a}}{w_{\mathrm{a}}}^{\mathrm{T}}w_{\mathrm{a}} + \frac{1}{\gamma_{\mathrm{a}}}{z_{\mathrm{a}}}^{\mathrm{T}}z_{\mathrm{a}} \le 0. \tag{13}$$

By integrating both sides of (13) from 0 to t, we obtain

$$-\gamma_{\mathrm{a}}\int_0^t {w_{\mathrm{a}}}^{\mathrm{T}}(\tau)w_{\mathrm{a}}(\tau)\,\mathrm{d}\tau - \frac{1}{\gamma_{\mathrm{a}}}\int_0^t {z_{\mathrm{a}}}^{\mathrm{T}}(\tau)z_{\mathrm{a}}(\tau)\,\mathrm{d}\tau \le -{x_{\mathrm{a}}}^{\mathrm{T}}(t)P_{\mathrm{a}}x_{\mathrm{a}}(t) \le 0$$

or $\|z\|_{L_2} \le \gamma_{\mathrm{a}}\|w\|_{L_2}$, which completes the proof.

3.2. **Derivation of LMI condition for PDAWC design.** The closed-loop system which consists of (1), (2), (4), (5), (6) and (7) is represented as a system

$$\begin{bmatrix} \dot{x}(t) \\ \zeta(t) \\ \dot{\zeta}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 & B_3 \\ C_1 & D_{11} & D_{12} & D_{13} \\ C_2 & D_{21} & D_{22} & D_{23} \\ C_3 & D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \\ \dot{d}(t) \\ w(t) \end{bmatrix} \tag{14}$$

with (6), where

$$x(t) = \begin{bmatrix} x_{\mathrm{p}}(t) \\ x_{\mathrm{c}}(t) \end{bmatrix} \in \mathbb{R}^{n+q}, \quad B_1 = \hat{B}_1 + \hat{B}_2\Lambda_{\mathrm{P}}, \quad B_2 = \hat{B}_2\Lambda_{\mathrm{D}}$$

$$D_{21} = \hat{D}_1 + \hat{D}_3\Lambda_{\mathrm{P}}, \quad D_{22} = \hat{D}_2 + \hat{D}_3\Lambda_{\mathrm{D}}$$

$$\hat{B}_1 = \begin{bmatrix} -B_u & -B_u D_{\mathrm{c}} \\ 0 & -B_{\mathrm{c}} \end{bmatrix}, \quad \hat{B}_2 = \begin{bmatrix} 0 \\ I_q \end{bmatrix}$$

$$\hat{D}_1 = \begin{bmatrix} -D_{\mathrm{c}}C_y B_u & -C_{\mathrm{c}}B_{\mathrm{c}} - D_{\mathrm{c}}C_y B_u D_{\mathrm{c}} \\ -C_y B_u & -C_y B_u D_{\mathrm{c}} \end{bmatrix}$$

$$\hat{D}_2 = \begin{bmatrix} 0 & -D_{\mathrm{c}} \\ 0 & 0 \end{bmatrix}, \quad \hat{D}_3 = \begin{bmatrix} C_{\mathrm{c}} \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{\mathrm{p}} + B_u D_{\mathrm{c}} C_y & B_u C_{\mathrm{c}} \\ B_{\mathrm{c}} C_y & A_{\mathrm{c}} \end{bmatrix}$$

$$B_3 = \begin{bmatrix} B_w \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} D_{\mathrm{c}} C_y & C_{\mathrm{c}} \\ C_y & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} C_{\mathrm{c}}B_{\mathrm{c}}C_y + D_{\mathrm{c}}C_y A_{\mathrm{p}} + D_{\mathrm{c}}C_y B_u D_{\mathrm{c}} C_y & C_{\mathrm{c}}A_{\mathrm{c}} + D_{\mathrm{c}}C_y B_u C_{\mathrm{c}} \\ C_y A_{\mathrm{p}} + C_y B_u D_{\mathrm{c}} C_y & C_y B_u C_{\mathrm{c}} \end{bmatrix}$$

$$C_3 = \begin{bmatrix} C_z + D_{zu} D_{\mathrm{c}} C_y & D_{zu} D_{\mathrm{c}} \end{bmatrix}$$

$$D_{11} = \begin{bmatrix} 0 & -D_{\mathrm{c}} \\ 0 & 0 \end{bmatrix}, \quad D_{12} = 0, \quad D_{13} = 0, \quad D_{23} = \begin{bmatrix} D_{\mathrm{c}} C_y B_w \\ C_y B_w \end{bmatrix}$$

$$D_{31} = \begin{bmatrix} -D_{zu} & -D_{zu} D_{\mathrm{c}} \end{bmatrix}, \quad D_{32} = 0, \quad D_{33} = D_{zw}.$$

The following theorem includes an LMI condition to design a PDAWC satisfying $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$ under some assumptions.

Theorem 3.1. *Assume that a diagonal matrix*

$$R = \text{block diag}\{R_{\mathrm{c}}, R_{\mathrm{m}}\} = \text{diag}\{R_{\mathrm{c},1}, \ldots, R_{\mathrm{c},\ell}, R_{\mathrm{m},1}, \ldots, R_{\mathrm{m},r}\}$$

$(0 \le R < I_{\ell+r})$, *the system matrices in* (1) *and* (3) *are given, and*

$$|u_i| \le \frac{\sigma_{\mathrm{c},i}}{1 - R_{\mathrm{c},i}} \quad (i = 1, 2, \ldots, \ell) \tag{15}$$

$$|y_i| \le \frac{\sigma_{\mathrm{m},i}}{1 - R_{\mathrm{m},i}} \quad (i = 1, 2, \ldots, r) \tag{16}$$

$$\frac{\mathrm{d}\Psi_{\mathrm{c},i}(u_i)}{\mathrm{d}t} = 0 \ \ (\text{or } \dot{u}_i) \ \ \text{at} \ \ u_i = \pm\sigma_{\mathrm{c},i} \quad (i = 1, 2, \ldots, \ell) \tag{17}$$

$$\frac{\mathrm{d}\Psi_{\mathrm{m},i}(y_i)}{\mathrm{d}t} = 0 \ \ (\text{or } \dot{y}_i) \ \ \text{at} \ \ y_i = \pm\sigma_{\mathrm{m},i} \quad (i = 1, 2, \ldots, r) \tag{18}$$

hold. If there exists a positive-definite matrix $Q(>0)$, a positive-definite diagonal matrix V, a nonsingular diagonal matrix U, a scalar γ, matrices $\tilde{\Lambda}_{\mathrm{P}}$, $\tilde{\Lambda}_{\mathrm{D}}$ satisfying

$$\tilde{\Theta}(Q, V, U, \tilde{\Lambda}_{\mathrm{P}}, \tilde{\Lambda}_{\mathrm{D}}, \gamma) < 0, \tag{19}$$

where

$$\tilde{\Theta}(Q, V, U, \tilde{\Lambda}_{\mathrm{P}}, \tilde{\Lambda}_{\mathrm{D}}, \gamma) = \begin{bmatrix} \tilde{\Theta}_{11} & \tilde{\Theta}_{12} & \tilde{\Theta}_{13} & \tilde{\Theta}_{14} & \tilde{\Theta}_{15} \\ \tilde{\Theta}_{12}{}^{\mathrm{T}} & \tilde{\Theta}_{22} & \tilde{\Theta}_{23} & \tilde{\Theta}_{24} & \tilde{\Theta}_{25} \\ \tilde{\Theta}_{13}{}^{\mathrm{T}} & \tilde{\Theta}_{23}{}^{\mathrm{T}} & \tilde{\Theta}_{33} & \tilde{\Theta}_{34} & \tilde{\Theta}_{35} \\ \tilde{\Theta}_{14}{}^{\mathrm{T}} & \tilde{\Theta}_{24}{}^{\mathrm{T}} & \tilde{\Theta}_{34}{}^{\mathrm{T}} & \tilde{\Theta}_{44} & \tilde{\Theta}_{45} \\ \tilde{\Theta}_{15}{}^{\mathrm{T}} & \tilde{\Theta}_{25}{}^{\mathrm{T}} & \tilde{\Theta}_{35}{}^{\mathrm{T}} & \tilde{\Theta}_{45}{}^{\mathrm{T}} & \tilde{\Theta}_{55} \end{bmatrix}$$

$$\tilde{\Theta}_{11} = AQ + QA^{\mathrm{T}}, \quad \tilde{\Theta}_{12} = QC_1^{\mathrm{T}}R + \hat{B}_1 V + \hat{B}_2 \tilde{\Lambda}_{\mathrm{P}}$$

$$\tilde{\Theta}_{13} = QC_2^{\mathrm{T}} + \hat{B}_2 \tilde{\Lambda}_{\mathrm{D}}, \quad \tilde{\Theta}_{14} = B_3, \quad \tilde{\Theta}_{15} = QC_3^{\mathrm{T}}$$

$$\tilde{\Theta}_{22} = RD_{11}V + VD_{11}{}^{\mathrm{T}}R - 2V, \quad \tilde{\Theta}_{23} = V\hat{D}_1^{\mathrm{T}} + RD_{12}U + \tilde{\Lambda}_{\mathrm{P}}{}^{\mathrm{T}}\hat{D}_3^{\mathrm{T}}$$

$$\tilde{\Theta}_{24} = RD_{13}, \quad \tilde{\Theta}_{25} = VD_{31}{}^{\mathrm{T}}, \quad \tilde{\Theta}_{33} = \hat{D}_2 U + U\hat{D}_2^{\mathrm{T}} + \hat{D}_3\tilde{\Lambda}_{\mathrm{D}} + \tilde{\Lambda}_{\mathrm{D}}{}^{\mathrm{T}}\hat{D}_3^{\mathrm{T}} - 2U$$

$$\tilde{\Theta}_{34} = D_{23}, \quad \tilde{\Theta}_{35} = UD_{32}{}^{\mathrm{T}}, \quad \tilde{\Theta}_{44} = -\gamma I_k, \quad \tilde{\Theta}_{45} = D_{33}{}^{\mathrm{T}}, \quad \tilde{\Theta}_{55} = -\gamma I_m,$$

then there exists a PDAWC (5) *which achieves the specification that the system* (14) *with* (6) *satisfies $\|z\|_{L_2} \le \gamma\|w\|_{L_2}$ for $x(0) = 0$. The coefficient matrices Λ_{P} and Λ_{D} in* (5) *can be determined by*

$$\Lambda_{\mathrm{P}} = \tilde{\Lambda}_{\mathrm{P}} V^{-1}, \quad \Lambda_{\mathrm{D}} = \tilde{\Lambda}_{\mathrm{D}} U^{-1}, \tag{20}$$

respectively.

Proof: The congruence transformation [8] with

$$\text{block diag}\{Q^{-1}, V^{-1}, U^{-1}, I_k, I_m\}$$

for (19) yields the following equivalent inequality:

$$\Theta(P, W, Z, \gamma; \Lambda_{\mathrm{P}}, \Lambda_{\mathrm{D}}) < 0, \tag{21}$$

where

$$P = Q^{-1}, \quad W = V^{-1}, \quad Z = U^{-1}, \quad \Lambda_{\mathrm{P}} = \tilde{\Lambda}_{\mathrm{P}} V^{-1}, \quad \Lambda_{\mathrm{D}} = \tilde{\Lambda}_{\mathrm{D}} U^{-1}$$

$$\Theta(P, W, Z, \gamma; \Lambda_{\mathrm{P}}, \Lambda_{\mathrm{D}}) = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & \Theta_{15} \\ \Theta_{12}{}^{\mathrm{T}} & \Theta_{22} & \Theta_{23} & \Theta_{24} & \Theta_{25} \\ \Theta_{13}{}^{\mathrm{T}} & \Theta_{23}{}^{\mathrm{T}} & \Theta_{33} & \Theta_{34} & \Theta_{35} \\ \Theta_{14}{}^{\mathrm{T}} & \Theta_{24}{}^{\mathrm{T}} & \Theta_{34}{}^{\mathrm{T}} & \Theta_{44} & \Theta_{45} \\ \Theta_{15}{}^{\mathrm{T}} & \Theta_{25}{}^{\mathrm{T}} & \Theta_{35}{}^{\mathrm{T}} & \Theta_{45}{}^{\mathrm{T}} & \Theta_{55} \end{bmatrix}$$

$$\Theta_{11} = PA + A^{\mathrm{T}}P, \quad \Theta_{12} = PB_1 + C_1^{\mathrm{T}}RW, \quad \Theta_{13} = PB_2 + C_2^{\mathrm{T}}Z$$

$$\Theta_{14} = PB_3, \quad \Theta_{15} = C_3^{\mathrm{T}}, \quad \Theta_{22} = WRD_{11} + D_{11}{}^{\mathrm{T}}RW - 2W$$

$$\Theta_{23} = WRD_{12} + D_{21}{}^{\mathrm{T}}Z, \quad \Theta_{24} = WRD_{13}, \quad \Theta_{25} = D_{31}{}^{\mathrm{T}}$$

$$\Theta_{33} = ZD_{22} + D_{22}{}^{\mathrm{T}}Z - 2Z, \quad \Theta_{34} = ZD_{23}, \quad \Theta_{35} = D_{32}{}^{\mathrm{T}}$$

$$\Theta_{44} = -\gamma I_k, \quad \Theta_{45} = D_{33}{}^{\mathrm{T}}, \quad \Theta_{55} = -\gamma I_m.$$

If (15) and (16) hold, then, for any $W_{\mathrm{c},i}(>0)$ and $W_{\mathrm{m},i}(>0)$, we have

$$d_{\mathrm{c},i} W_{\mathrm{c},i}(d_{\mathrm{c},i} - R_{\mathrm{c},i} u_i) \le 0 \quad (i = 1, 2, \ldots, \ell) \tag{22}$$

and

$$d_{\mathrm{m},i}W_{\mathrm{m},i}(d_{\mathrm{m},i} - R_{\mathrm{m},i}y_i) \leq 0 \quad (i = 1, 2, \ldots, r) \tag{23}$$

from the definition of (6). The inequalities (22) and (23) imply

$$\sum_{i=1}^{\ell} d_{\mathrm{c},i}W_{\mathrm{c},i}(d_{\mathrm{c},i} - R_{\mathrm{c},i}u_i) + \sum_{i=1}^{r} d_{\mathrm{m},i}W_{\mathrm{m},i}(d_{\mathrm{m},i} - R_{\mathrm{m},i}y_i) = d^{\mathrm{T}}W(d - R\zeta) \leq 0, \tag{24}$$

where $W = \mathrm{diag}\{W_{\mathrm{c},1}, \ldots, W_{\mathrm{c},\ell}, W_{\mathrm{m},1}, \ldots, W_{\mathrm{m},r}\}$. On the other hand, if (17) and (18) hold, then, for any $Z_{\mathrm{c},i}$ and $Z_{\mathrm{m},i}$, we have

$$\dot{d}_{\mathrm{c},i}Z_{\mathrm{c},i}(\dot{d}_{\mathrm{c},i} - \dot{u}_i) = 0 \quad (i = 1, 2, \ldots, \ell) \tag{25}$$

and

$$\dot{d}_{\mathrm{m},i}Z_{\mathrm{m},i}(\dot{d}_{\mathrm{m},i} - \dot{y}_i) = 0 \quad (i = 1, 2, \ldots, r) \tag{26}$$

from the definition of (6). The equalities (25) and (26) imply

$$\sum_{i=1}^{\ell} \dot{d}_{\mathrm{c},i}Z_{\mathrm{c},i}(\dot{d}_{\mathrm{c},i} - \dot{u}_i) + \sum_{i=1}^{r} \dot{d}_{\mathrm{m},i}Z_{\mathrm{m},i}(\dot{d}_{\mathrm{m},i} - \dot{y}_i) = \dot{d}^{\mathrm{T}}Z(\dot{d} - \dot{u}) = 0, \tag{27}$$

where $Z = \mathrm{diag}\{Z_{\mathrm{c},1}, \ldots, Z_{\mathrm{c},\ell}, Z_{\mathrm{m},1}, \ldots, Z_{\mathrm{m},r}\}$. Therefore, if (15), (16), (17) and (18) are met, then the nonlinearity Ψ in (6) belongs to

$$\Delta = \{\tilde{\Psi} :\ d = \tilde{\Psi}(\zeta),\ d^{\mathrm{T}}W(d - R\zeta) \leq 0,\ \dot{d}^{\mathrm{T}}Z(\dot{d} - \dot{\zeta}) = 0\}. \tag{28}$$

Consequently, Lemma 3.1 shows that if the inequality (21) holds for some P, W, Z, Λ_{P}, Λ_{D} and γ under the assumptions (15), (16), (17) and (18), then the system (14) with (6) satisfies $\|z\|_{L_2} \leq \gamma\|w\|_{L_2}$ for $x(0) = 0$.

3.3. **Problem formulation.** It is noted that the matrix inequality (19) is an LMI in Q, V, U, $\tilde{\Lambda}_{\mathrm{P}}$, $\tilde{\Lambda}_{\mathrm{D}}$ and γ.

Thus, by virtue of Theorem 3.1, the problem of designing a PDAWC (5) can be formulated as follows:

find Q, V, U, $\tilde{\Lambda}_{\mathrm{P}}$, $\tilde{\Lambda}_{\mathrm{D}}$ and γ so as to

minimize γ

subject to LMI (19).

After solving this LMI problem, we can determine coefficient matrices Λ_{P} and Λ_{D} in (5) by (20).

4. **Numerical Example.** Consider a two-mass-spring-damper system illustrated in Figure 2 as the plant (1), where x_1 and x_2 are the displacements of masses m_1 and m_2, respectively, w is the displacement disturbance, u_{s} is the control force, k_1 and k_2 are the spring constants, and c_2 is the damping coefficient. Such a model can be applied to the vibration control of suspensions. The values of the physical parameters are given by $m_1 = 15.8\,[\mathrm{kg}]$, $m_2 = 12.6\,[\mathrm{kg}]$, $k_1 = 5.2\,[\mathrm{N/m}]$, $k_2 = 5.8\,[\mathrm{N/m}]$, $c_2 = 0.05\,[\mathrm{Ns/m}]$, $\sigma_{\mathrm{c}} = 0.25\,[\mathrm{N}]$ and $\sigma_{\mathrm{m}} = 0.026\,[\mathrm{m}]$. The plant state $x_{\mathrm{p}}(t)$, the controlled output $z(t)$ and the measured output $y(t)$ are selected as

$$x_{\mathrm{p}}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}, \quad z(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ 0.1u_{\mathrm{s}}(t) \end{bmatrix}, \quad y(t) = x_1(t) - x_2(t),$$

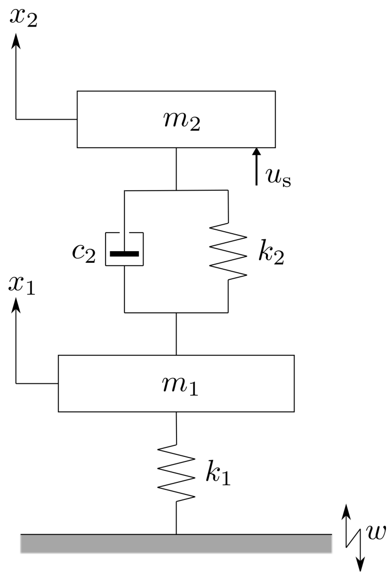

Figure 2. A plant model for numerical example

respectively.

The dynamic controller (3) was constructed by not considering the nonlinearities (2) and (4). The coefficient matrices (A_c, B_c, C_c, D_c) of the controller (3) were obtained by solving a problem of minimizing the upper bound of an H_∞-norm $\|C_3(sI_{n+q} - A)^{-1}B_3 + D_{33}\|_\infty$. As is well-known, the H_∞ control problem for $q = n$ can be reduced to LMI one by using some appropriate techniques such as the elimination of variables [9]. By solving the LMI problem [6], the coefficient matrices were obtained as follows:

$$A_c = \begin{bmatrix} 3.2454 & 0.4251 & 3.8343 & 1.9577 \\ 17.6131 & -1.9059 & 13.1615 & 0.5400 \\ -8.8843 & 0.9485 & -5.8152 & 0.5995 \\ 1.8223 & -0.6524 & 1.1524 & -276.8294 \end{bmatrix}, \quad B_c = \begin{bmatrix} 13.8014 \\ 76.0085 \\ -34.9302 \\ 6.9049 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 25.8701 & -2.6556 & 19.0254 & 1.6743 \end{bmatrix}, \quad D_c = 104.5199.$$

By setting $R_c = 0.75$, $R_m = 0.5$, and solving the LMI problem formulated in the previous section, we have

$$W = \mathrm{diag}\{0.2741,\ 488.2579\}, \quad Z = \mathrm{diag}\{0.0007,\ 4.1861\}, \quad \gamma = 567.3658$$

$$\Lambda_P = \begin{bmatrix} -0.0004 & -0.0086 \\ -0.0006 & 0.0074 \\ 0.0001 & -0.0081 \\ 0.0377 & 1.3981 \end{bmatrix} \times 10^3, \quad \Lambda_D = \begin{bmatrix} 0.0050 & -0.4546 \\ -0.0025 & 0.1127 \\ 0.0036 & -0.2648 \\ -0.7624 & 71.6831 \end{bmatrix}.$$

To demonstrate the effectiveness and efficiency of design method proposed in this paper, a simulation was conducted. Figures 4-7 are the response for a disturbance w depicted in Figure 3, where, in these figures, without sat. (dash-dotted line) is the response without any saturation nonlinearities (i.e., $u_s(t) \equiv u(t)$ and $y_s(t) \equiv y(t)$), with sat. without comp. (broken line) is the response with input and output saturations but without any compensation, with sat. with comp. (proposed) (solid line) is the response with input and

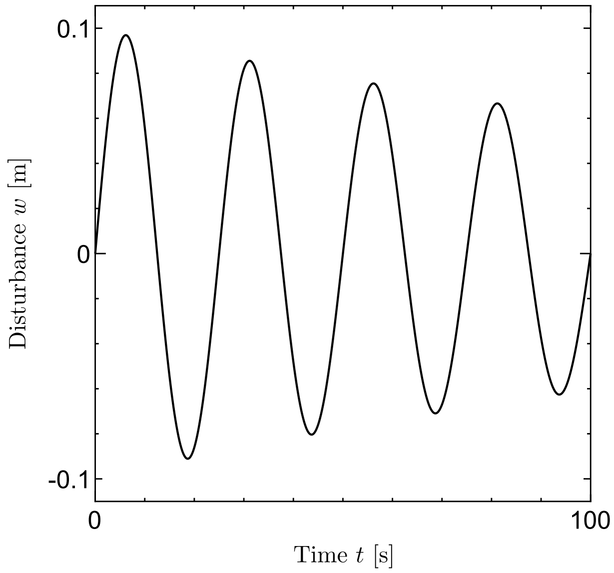

FIGURE 3. Disturbance w

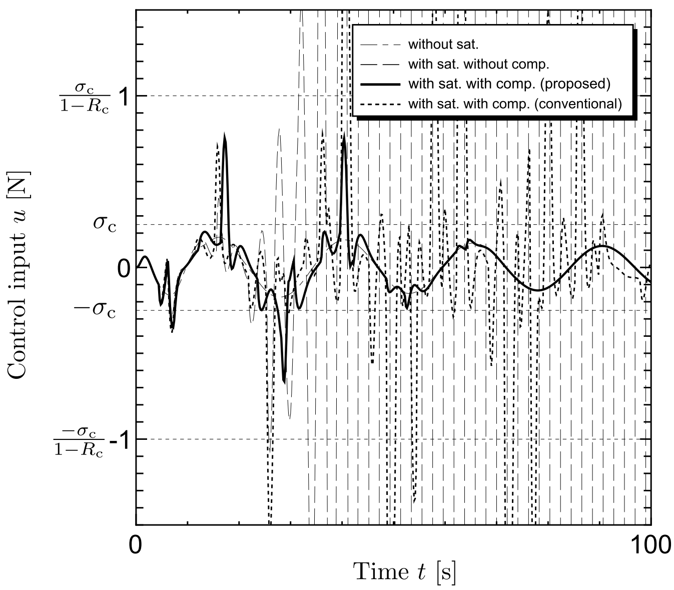

FIGURE 4. Output u of controller

output saturations and PDAWC designed in this paper, and with sat. with comp. (conventional) is the response with input and output saturations and PDAWC proposed in [1].

The performances evaluated by $\|z\|_{L_2}/\|w\|_{L_2}$ were calculated for saturated cases:

$$\frac{\|z\|_{L_2}}{\|w\|_{L_2}} = 2.8812 \quad \text{(with sat. without comp.)}$$

$$\frac{\|z\|_{L_2}}{\|w\|_{L_2}} = 1.7177 \quad \text{(with sat. with comp. (proposed))}$$

$$\frac{\|z\|_{L_2}}{\|w\|_{L_2}} = 2.4025. \quad \text{(with sat. with comp. (conventional))}$$

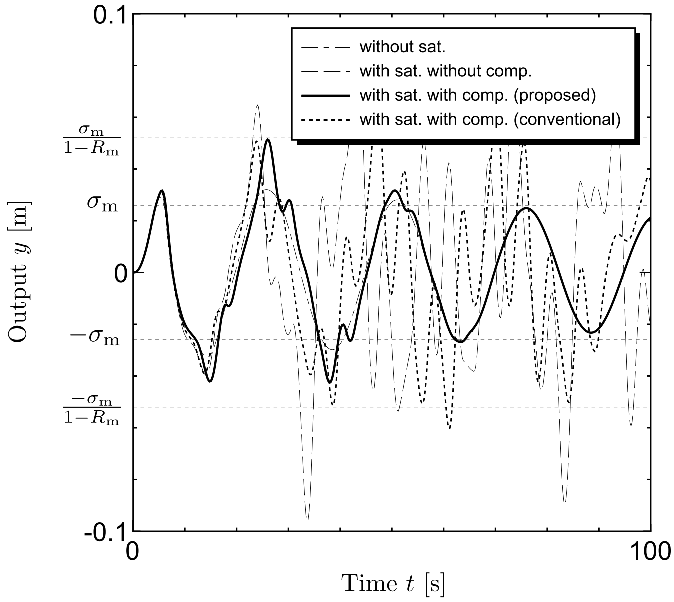

FIGURE 5. Output y of plant

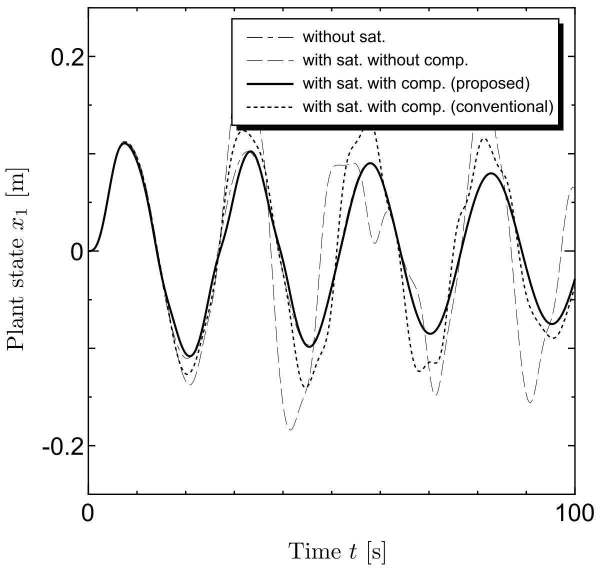

FIGURE 6. Plant state x_1

5. **Discussion.** The anti-windup compensation by (5) proposed in this paper includes the strategy in [1] as a special case. Indeed, by setting $\Lambda_{\mathrm{P_m}} \equiv 0$ and $\Lambda_{\mathrm{D_m}} \equiv 0$, we have a PDAWC for the plant (1) with input saturation but without output saturation.

Furthermore, the theoretical property in the conventional PDAWC strategy of [1] for the avoidance of the difficulty in the generation of control input u_{s} is still maintained. The difficulty arises in the anti-windup control system with the control mechanism

$$\begin{bmatrix} \dot{x}_{\mathrm{c}}(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_{\mathrm{c}} & B_{\mathrm{c}} \\ C_{\mathrm{c}} & D_{\mathrm{c}} \end{bmatrix} \begin{bmatrix} x_{\mathrm{c}}(t) \\ y_{\mathrm{s}}(t) \end{bmatrix} + \begin{bmatrix} v_{\mathrm{c1}}(t) \\ v_{\mathrm{c2}}(t) \end{bmatrix} \tag{29}$$

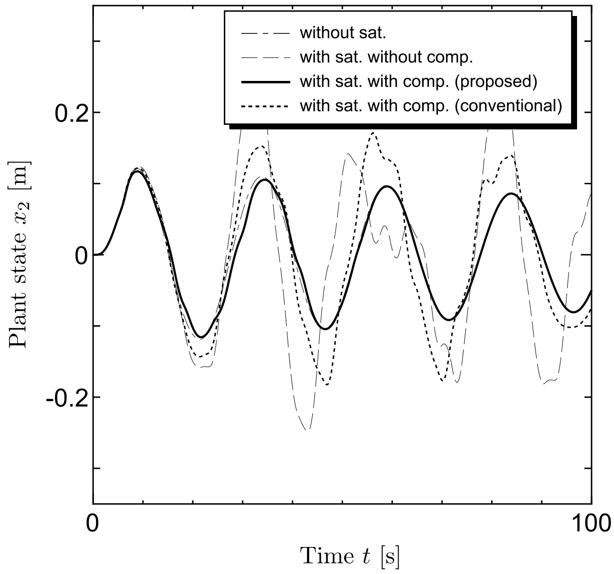

FIGURE 7. Plant state x_2

compensated by a static anti-windup compensator

$$\begin{bmatrix} v_{\mathrm{c1}}(t) \\ v_{\mathrm{c2}}(t) \end{bmatrix} = \begin{bmatrix} \Lambda_{\mathrm{c1}} \\ \Lambda_{\mathrm{c2}} \end{bmatrix} d(t). \tag{30}$$

For (29) and (30), since we can represent the output u of controller as

$$u = \xi + G u_{\mathrm{s}}, \tag{31}$$

where

$$\begin{aligned} \xi &= (I - \Lambda_{\mathrm{c21}})^{-1}(C_{\mathrm{c}} x_{\mathrm{c}} + D_{\mathrm{c}} y_{\mathrm{s}} + \Lambda_{\mathrm{c22}} d_{\mathrm{m}}) \\ G &= -(I - \Lambda_{\mathrm{c21}})^{-1}\Lambda_{\mathrm{c21}}, \quad \Lambda_{\mathrm{c2}} = \begin{bmatrix} \Lambda_{\mathrm{c21}} & \Lambda_{\mathrm{c22}} \end{bmatrix} \in \mathbb{R}^{q \times (\ell + r)} \end{aligned}$$

under the assumption of the invertibility of the matrix $I - \Lambda_{\mathrm{c21}}$, the determination of the output u of controller must be governed by

$$u_{\mathrm{s}} = \Phi(\xi + G u_{\mathrm{s}}) \tag{32}$$

if we adopt the control mechanism (29) with (30). The determination of the control input u_{s} under the relationship (32) is not, in general, necessarily easy. On the other hand, for (7) and (5), since the output u of controller does not depend on u_{s} explicitly, we can avoid such a difficulty. However, the structure of control mechanism is not different. In fact, for (29) and (30), we have a relationship

$$u(t) = C_{\mathrm{c}} \int_0^t \mathrm{e}^{A_{\mathrm{c}}(t-\tau)} \{B_{\mathrm{c}} y_{\mathrm{s}}(\tau) + \Lambda_{\mathrm{c1}} d(\tau)\} \mathrm{d}\tau + D_{\mathrm{c}} y_{\mathrm{s}}(t) + \Lambda_{\mathrm{c2}} d(t). \tag{33}$$

On the other hand, for (7) and (5), we have a relationship

$$u(t) = C_{\mathrm{c}} \int_0^t \mathrm{e}^{A_{\mathrm{c}}(t-\tau)} \{B_{\mathrm{c}} y_{\mathrm{s}}(\tau) + (\Lambda_{\mathrm{P}} + A_{\mathrm{c}}\Lambda_{\mathrm{D}}) d(\tau)\} \mathrm{d}\tau + D_{\mathrm{c}} y_{\mathrm{s}}(t) + C_{\mathrm{c}}\Lambda_{\mathrm{D}} d(t). \tag{34}$$

The simulation result in the previous section shows that the condition $\|z\|_{L_2} \leq \gamma \|w\|_{L_2}$ was satisfied, and the compensation by PDAWC designed in this paper was successfully done. Furthermore, although the conventional PDAWC improved the behavior of the control system, the proposed one brought better performance in the sense of the ratio $\|z\|_{L_2}/\|w\|_{L_2}$. This is the important advantage of this paper.

6. **Conclusion.** In this paper, a methodology of designing PDAWCs for control systems with not only input saturation but also output saturation by means of LMI was developed. The LMI condition to determine the coefficient matrices of PDAWC was derived under some assumptions for a closed-loop representation. By using the LMI condition, the design problem was formulated as the minimization problem of the performance index concerning the disturbance attenuation. A numerical example was shown to check the effectiveness of the proposed design method. The structural relationship of control system with PDAWC was discussed. The discussion revealed that the design method proposed in this paper was a successful extension of the result in [1]. The comparison of the proposed PDAWC result with the conventional one in the numerical simulation indicated that the proposed PDAWC brought better performance.

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