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PROPERTIES OF THE INTRINSIC VALUE FUNCTION AND THE DURATION FUNCTION DETERMINED BY CONTINUOUS CASH FLOW

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Abstract. *This paper uses the point of view of function to examine some changing rules of the intrinsic value of assets with time. From the continuous yield equation, this paper will derive the formula of the intrinsic value function for assets. On this basis, the definition formula of the duration function determined by continuous cash flow is given and the differential equation that connects the duration function and the intrinsic value function of the assets is got. Meanwhile, the situations of finite and infinite time period are respectively discussed and the relationship between the duration of a single asset and the duration of an assets portfolio is also discussed. Then a number of conclusions that the changes of cash flow cause the changes in duration are achieved. And for some common cash earnings function, the analytical formulas of the corresponding intrinsic value functions and the duration functions are given.*

Keywords: Discounted cash flow model, Cash earnings function, Continuous cash flow, Intrinsic value function, Duration function

1. **Introduction.** The discounted cash flow model is the most important mathematical model to assess the value of assets in investment analysis. And there is dividend discount model for the evaluation of stock and free cash flow discount model for the evaluation of bonds and other financial assets. The cash earnings of assets in the future, the price of the assets at the final moment and the level of the discount rate in the future are the three important factors that determine the intrinsic value of the assets at the present moment. One can expediently study the quantitative relationship between changes in interest rate and changes in the intrinsic value of assets through the discounted cash flow model. And also, one can regard the market price of the assets as its intrinsic value and get the corresponding internal rate of return through the discounted cash flow model. Comparing the obtained internal rate of return and the market interest rate, considering the risk premium factors required by investors who invest in risky assets, one could estimate whether the market price of assets is overvalued or undervalued. Thereby, this can guide the investor's investment actions.

Originally, the concept of duration is used as an indicator to measure the average time period of the cash flow for bonds. The greater the value of the duration is, the longer the average period of the cash flow is, and so the greater the risk of interest rate that the corresponding financial assets face is. For any financial assets, as long as the future cash flows can be known or expected, one can use the duration formula to estimate the duration of the cash flow. For the common bonds, Macaulay has given the definition of duration [8]. That is, duration is determined by weighted average of each coupon interest of bonds and the payment time of the principal. CF_t is used as the cash flow at time

t and $t = 1, 2, \ldots, T$. The Macaulay duration is denoted by D and the discount rate is denoted by r. $P(r)$ means the bond price (the intrinsic value) when $t = 0$. The definition formula of the Macaulay duration is as follows:

$$D = \sum_{t=1}^{T} t \cdot \frac{CF_t}{(1+r)^t P(r)} \tag{1}$$

The following discounted cash flow model gives the intrinsic value of the bond:

$$P(r) = \sum_{t=1}^{T} \frac{CF_t}{(1+r)^t} \tag{2}$$

It is the infinite period discounted cash flow model without principal return when $T = +\infty$. In order to separate the bond coupon interest and the par value in CF_t, there is the following discounted cash flow formula:

$$P(r) = \sum_{t=1}^{T} \frac{\text{coupon interest of time } t}{(1+r)^t} + \frac{\text{par value}}{(1+r)^T} \tag{3}$$

For the duration D and the intrinsic value $P(r)$ and the discount rate r, there is the following differential equation:

$$\frac{d \ln P(r)}{dr} = -\frac{D}{(1+r)} \tag{4}$$

Equation (4) shows that there is a strict negative correlation between the changing rate of the intrinsic value and duration D. The greater the duration is, the more sensitive the impact that the interest rate fluctuations have on the intrinsic value is. Because the duration matching for assets and liabilities can immunize interest rate, the estimates of the duration for assets and liabilities have great significance for the risk management.

After Macaulay introduces the duration concept of cash flow, some scholars give different forms of the duration formula to more subtly research the relationship between discount rate and bond price [6]. The duration formula is widely applied to bond portfolio management [2-4,9]. Although the concept of duration has been proposed for a long time and many theoretical and applied research achievements have been obtained on this basis, no research has been done based on the duration of continuous cash flow. It can be expected that we can make the best use of calculus to conduct the research on the duration formula for continuous cash flow and get the formula of continuous form between the intrinsic value of the assets and the duration.

In general, commercial banks or institutional investors possess many kinds of assets and every asset has a unique cash flow. Some of the cash flows arrive earlier while some arrive later; some use a year, half a year or a quarter as the unit of time while some use a month, a week or a day as the unit of time. These cash flows can be fully distributed in time when being synthesized together. Then the total cash flows can be approximately described in a continuous function with time-parameter. Liabilities of commercial banks or institutional investors can also carry on the same analysis. For every enterprise, due to the continuous productive and sales process and the activities of continuous financing and debt payment, the monetary income flow and the monetary expenditure flow can be approximately described by a continuous function with time-parameter. The research of continuous cash flow is not only just an interest in theoretical studies, but also has a wide and practical application background.

More in-depth discussions can be made in some aspects of the discounted cash flow model. The discounted cash flow model in the discrete form is familiar to financial experts while the discounted cash flow model in continuous form has not been systematically studied. The cash earnings function in continuous form needs defining and the discounted cash flow formula in continuous form needs deducing. Another point has been ignored by financial economists, that is, for cash earnings of each given period in the future, it is feasible to use the discounted cash flow model to estimate the intrinsic value of the assets in different time of the future and use the viewpoint of function to study the properties of the intrinsic value function which is composed by the intrinsic value of assets at all the different time.

For the cash flow in continuous form, the formula of the intrinsic value function is in integral form, which is got by the author of this paper [10]. Various properties of the intrinsic value function are systematically studied and the application mode of the intrinsic value function in the risky assets valuation is given. On the basis of the concept of intrinsic value function, this paper introduces the concept of duration function corresponding to the cash flow in continuous form and studies the relationships between the intrinsic value function and the duration function and the discount rate. The deducing of formulas and the proof of theorems depend on the use of calculus; such a research has a clear background of investment principles.

Although the discounted cash flow model is established on the basis of the time value principle of cash, the model is still a definition formula. It is a significant attempt if we can deduce the discounted cash flow formula through some more fundamental financial principles. In addition, such attempts may help the more in-depth study on the discounted cash flow model. The yield equation in continuous form can be derived from the definition formula of the rate of return on assets. This is a differential equation, from whose solution the intrinsic value function of the assets can be determined. On the basis of the concept of the intrinsic value function, we proposed the concepts of duration function by imitating the Macaulay definition of duration in the form of discrete cash flow. This paper is made up of the intrinsic value function, the deducing of the duration function formula and the research on the relationship between the intrinsic value function and the duration function.

In the second section, the yield equation in continuous form is given according to the definition of the rate of returns on assets, and the mathematical formulas of the intrinsic value function of the assets are gained through solving the yield equations in the cases of finite time period and infinite time period. In the third section, the definitions of duration and the duration function in continuous form are given based on the concept of the intrinsic value function, a number of differential equations that the intrinsic value function and the duration function satisfies are deduced. In the fourth section, we introduced the concept of high-order duration of cash flow and obtained the Taylor's expansion for intrinsic value function. In the fifth section, the mathematical formula of the duration function for portfolio is deduced. In the sixth section, the relationship between the duration and the cash flow is discussed, and several properties that the changes of cash flow lead to the regular changes of the value of duration are given. In the seventh section, under the conditions of finite time period and infinite time period, a number of corresponding intrinsic value functions and duration functions of some common cash earnings functions are respectively given. The last section is the conclusion.

2. The Intrinsic Value Function of the Assets Determined by Yield Equation in Continuous Form.

Suppose there is an asset A whose cash earnings are realized continuously. Using $C(t)$ to represent the cash earnings of each share of assets in unit time at t, we call function $C(t)$ the cash earnings function and also call function $C(t)$ the cash flow. $P(t)$ represents the price of each share of assets in time t. The cash earnings are approximately equal to $C(t)\Delta t$ during the time interval $[t, t+\Delta t]$. We use $r(t)$ to represent the instantaneous investment rate of return in time t, the investment rate of return is $r(t)\Delta t$ during the time interval $[t, t+\Delta t]$. Estimating the time interval $[t, t+\Delta t]$, we can get the following formula of the rate of return for assets A:

$$r(t)\Delta t = \frac{C(t)\Delta t + [P(t+\Delta t) - P(t)]}{P(t)}$$

$P(t+\Delta t) - P(t)$ is the capital gain during the time interval $[t, t+\Delta t]$. Taking a fixed discount rate (or the expected rate of return) r to substitute $r(t)$ in the above expression, we have:

$$r\Delta t = \frac{C(t)\Delta t + [P(t+\Delta t) - P(t)]}{P(t)}$$

Simplifying the above-mentioned formula and letting $\Delta t \to 0$, we get the following differential equation:

$$\frac{dP(t)}{dt} = rP(t) - C(t) \tag{5}$$

We call Equation (5) the continuous-form yield equation, which is a differential equation about the assets price $P(t)$. In regard to r, the fixed discount rate, if the cash earnings function $C(t)$ is given, the price function $P(t)$ can be solved with Equation (5). Under certain conditions, the price function $P(t)$ satisfying Equation (5) is fair and reasonable, so it should be the intrinsic value function of the asset A. Similarly, for fixed discount rate r, if the intrinsic value function $P(t)$ is known, the cash earnings function $C(t)$ that the asset A can accomplish can be solved through Equation (5).

The mathematical expression of the intrinsic value function $P(t)$ will be derived from Equation (5). First of all, the infinite time period is taken into consideration, that is, the cash earnings function is $C(t)$ and $t \in [0, +\infty)$. Based on the actual background of investments, we assume that $C(t) \geq 0$, $P(t) \geq 0$ and $r > 0$, when $t \in [0, +\infty)$. Equation (5) is a first order ordinary differential equation. For the common first order ordinary differential equation $y' + p(x)y = q(x)$, its general solution is as follows:

$$y = e^{-\int p(x)dx}\left[\int q(x)\, e^{\int p(x)dx}dx + c\right]$$

The time parameter t is used to represent the independent variable of Equation (5) and accordingly there is $p(t) = -r$ and $q(t) = -C(t)$. And then we substitute them into the general solution's expression. Then we have:

$$P(t) = e^{rt}\left[\int_0^t -C(\xi)\, e^{-r\xi}d\xi + C_0\right] \tag{6}$$

In this expression, C_0 is a constant. Let $t = 0$, we can obtain $C_0 = P(0)$ from the above-mentioned expression, and then we have:

$$P(t) = e^{rt}\left[\int_0^t -C(\xi)\, e^{-r\xi}d\xi + P(0)\right]$$

The reasonable assumption is $0 < r < 1$ and $C(t) \geq 0$, so $P(t) \geq 0$ is also reasonable. And once ascertaining the value of $P(0)$, we can obtain the solution $P(t)$. At the same time, the integration $\int_0^\infty C(\xi)\, e^{-r\xi}d\xi$ is convergent as long as the growth rate of the cash flow $C(t)$ is less than e^{rt}. We take $P(0) = \int_0^\infty C(\xi)\, e^{-r\xi}d\xi + C$, in which C is an undetermined constant. We substitute expression of $P(0)$ into expression of $P(t)$, and simplify it. Then we have:

$$\begin{aligned} P(t) &= e^{rt}\left[\int_0^{+\infty} C(\xi)\, e^{-r\xi} d\xi - \int_0^{t} C(\xi)\, e^{-r\xi} d\xi + C\right] \\ &= \int_t^{+\infty} e^{-r(\xi-t)} C(\xi)\, d\xi + e^{rt}C \end{aligned}$$

When $C(t) \equiv 0$, $P(t)$ should be equal to 0. However, at this time, $P(t) = e^{rt} \cdot C$; therefore, $C = 0$. In this way, we acquire the investments-accorded solution $P(t)$ of the differential Equation (5). Distinctly, $P(t)$ contains the parameter r (the discount rate), and then we use $P(t, r)$ to represent the solution of Equation (5) and the derived intrinsic value function $P(t, r)$ of the assets A is given by the following formula:

$$P(t, r) = \int_t^{+\infty} e^{-r(\xi-t)} C(\xi)\, d\xi \tag{7}$$

The Expression (7) is the discounted cash flow formula of the intrinsic value function of the assets in continuous form. Especially, when $t = 0$, we have:

$$P(0, r) = \int_0^{+\infty} C(\xi)\, e^{-r\xi} d\xi$$

It is the discounted cash flow formula of the intrinsic value function of the assets in continuous form in time 0.

Secondly, the finite time period is taken into account. That is, $t \in [0, T]$. We take C_0 in the Expression (6) of general solution as:

$$C_0 = \int_0^{T} C(\xi)\, e^{-r\xi} d\xi + C$$

C is an undetermined constant. We substitute it into the Expression (6) of general solution, and simplify it. The result is as follows:

$$\begin{aligned} P(t) &= e^{rt}\left[\int_0^{T} C(\xi)\, e^{-r\xi} d\xi - \int_0^{t} C(\xi)\, e^{-r\xi} d\xi + C\right] \\ &= \int_t^{T} e^{-r(\xi-t)} C(\xi)\, d\xi + e^{rt}C \end{aligned}$$

Suppose $C(t)$ is a bounded continuous function in the $[0, T]$, then there must be:

$$\lim_{t \to T} \int_t^{T} e^{-r(\xi-t)} C(\xi)\, d\xi = 0$$

Hence, $P(T) = Ce^{rT}$, that is, $C = e^{rT}P(T)$. $P(T)$ is the price of the asset A in time T, and the value C is the price discounts of the asset A from time T to 0. Therefore, we have:

$$P(t, r) = \int_t^{T} e^{-r(\xi-t)} C(\xi)\, d\xi + e^{-r(T-t)} P(t) \tag{8}$$

The Expression (8) is the formula of the intrinsic value function of assets in continuous form under the condition of finite time period. Particularly, when $t = 0$, the discount cash flow formula of the intrinsic value in continuous form in time 0 is as follows:

$$P(0, r) = \int_0^{T} C(\xi)\, e^{-r\xi} d\xi + e^{-rT} P(t)$$

TABLE 1. Several cash earnings functions in the case of infinite time period and their corresponding intrinsic value functions

The cash earnings function $C(t)$	The intrinsic value function $P(t,r)$	The intrinsic value $P(0,r)$
$C(t)=D_0$ $D_0>0$	$\frac{D_0}{r}$	$\frac{D_0}{r}$
$C(t)=a+bt$ $a>0,\ b>0$	$\frac{a+bt}{r}+\frac{b}{r^2}$	$\frac{1}{r}\left(a+\frac{b}{r}\right)$
$C(t)=\begin{cases} a & 0\le t\le t_0 \\ a+b(t-t_0) & t>t_0 \end{cases}$ $a>0,\ b>0$	$\begin{cases} \frac{1}{r}\left[a+\frac{b}{r}e^{r(t-t_0)}\right] & 0\le t\le t_0 \\ \frac{1}{r}\left[a+b(t-t_0)+\frac{b}{r}\right] & t>t_0 \end{cases}$	$\frac{1}{r}\left(a+\frac{b}{r}e^{-rt_0}\right)$
$C(t)=a+bt+ct^2$ $c>0$	$\frac{a+bt+ct^2}{r}+\frac{b+2ct}{r^2}+\frac{2c}{r^3}$	$\frac{a}{r}+\frac{b}{r^2}+\frac{2c}{r^3}$
$C(t)=Ae^{\lambda(t-t_0)}$ $\lambda<r$	$\frac{Ae^{\lambda(t-t_0)}}{r-\lambda}$	$\frac{Ae^{-\lambda t_0}}{r-\lambda}$
$C(t)=\begin{cases} 0 & 0\le t\le t_0 \\ A & t>t_0 \end{cases}$	$\begin{cases} \frac{A}{r}e^{r(t-t_0)} & 0\le t\le t_0 \\ \frac{A}{r} & t>t_0 \end{cases}$	$\frac{A}{r}e^{-rt_0}$
$C(t)=A_0+A\sin\omega t$ $A_0>A>0$	$\frac{A_0}{r}+\frac{A}{r^2+\omega^2}(r\sin\omega t+\omega\cos\omega t)$	$\frac{A_0}{r}+\frac{A\omega}{r^2+\omega^2}$
$C(t)=D_0+A\delta(t-t_0)$ $\delta(t-t_0)=\begin{cases} 0 & t\ne t_0 \\ +\infty & t=t_0 \end{cases}$ and $\int_{-\infty}^{+\infty}\delta(t-t_0)dt=1$	$\begin{cases} \frac{D_0}{r}+Ae^{r(t-t_0)} & 0\le t\le t_0 \\ \frac{D_0}{r} & t>t_0 \end{cases}$	$\frac{D_0}{r}+Ae^{-rt_0}$

In accordance with the Expression (7), we have:

$$\frac{\partial P(t,r)}{\partial r}=-\int_t^{+\infty}(\xi-t)\,C(\xi)\,e^{r(t-\xi)}d\xi$$

Because $\xi\ge t$, $C(\xi)\ge 0$, $e^{-r(t-\xi)}>0$, the integrand in the integral sign of the above-mentioned expression is non-negative, it turns out that $\frac{\partial P(t,r)}{\partial r}\le 0$, and there must be $\frac{\partial P(t,r)}{\partial r}<0$ when $C(\xi)\ge 0$ does not vanish identically. This conclusion shows that the intrinsic value function is a decreasing function about the discount rate r. Based on the Expression (8), the conclusion $\frac{\partial P(t,r)}{\partial r}\le 0$ can be also easily drawn.

In terms of the functional transformation, Formula (7) defines the transformation from the cash earnings function $C(t)$ to the intrinsic value function $P(t)$. Some commonly used cash earnings functions $C(t)$ are selected to deduce the corresponding intrinsic value functions $P(t)$. The results of several intrinsic value functions determined by several cash earnings functions are listed in Table 1.

3. **The Duration Function Determined by the Intrinsic Value Function and Its Properties.** For the discrete cash flow, the definition formula of Macaulay duration is (1), and it can be written in the following form:

$$D\cdot P(r)=\sum_{t=1}^{T}t\cdot\frac{CF_t}{(1+r)^t}$$

Following the definition Formula (1) and combining the obtained intrinsic value function (7) and (8), we give several definitions of duration for continuous cash flow according to the meaning of the duration in investments. Transforming the summation in (1) into the integration and matching CF_t with $C(t)$, $\frac{1}{(1+r)^t}$ with e^{-rt}, the corresponding duration definitions for continuous cash flow are introduced respectively in the case of infinite time period and finite time period.

Definition 3.1. *For $C(t)$, $t \in [0, +\infty)$, the continuous cash flow of the infinite time period, the duration $D_\infty(r)$ can be confirmed by the following expression when $t = 0$:*

$$D_\infty(r) = \frac{\int_0^{+\infty} \xi e^{-r\xi} C(\xi) d\xi}{\int_0^{+\infty} e^{-r\xi} C(\xi) d\xi} \tag{9}$$

The denominator $\int_0^{+\infty} e^{-r\xi} C(\xi) d\xi$ is $P(0, r)$ which is the value of the intrinsic value function determined by the cash flow $C(t)$, and the numerator $\int_0^{+\infty} \xi e^{-r\xi} C(\xi) d\xi$ is the weighted average of the time parameter ξ in $[0, +\infty)$.

Definition 3.2. *For $C(t)$, $t \in [0, +\infty)$, the continuous cash flow of the infinite time period, the duration $D_\infty(t, r)$ at any time t can be confirmed by the following expression:*

$$D_\infty(t, r) = \frac{\int_t^{+\infty} (\xi - t) e^{-r(\xi - t)} C(\xi) d\xi}{\int_t^{+\infty} e^{-r(\xi - t)} C(\xi) d\xi} \tag{10}$$

In the Expression (10), the denominator $\int_t^{+\infty} e^{-r(\xi-t)} C(\xi) d\xi$ is $P(t, r)$ which is the intrinsic value function provided in the Expression (7), and the numerator $\int_t^{+\infty} (\xi - t) e^{-r(\xi-t)} C(\xi) d\xi$ is the weighted average of the time parameter $(\xi - t)$ in $[t, +\infty)$. $D_\infty(t, r)$ is the function of the time parameter t and the discount rate parameter r. Therefore, it is called the duration function determined by the cash flow $C(t)$, $t \in [0, +\infty)$.

Definition 3.3. *For $C(t)$, $t \in [0, T]$, the continuous cash flow of the finite time period, P_{face} represents the value of the assets A at time T (if the assets A is a kind of bond, then P_{face} represents the par value of the bonds). And the duration $D_T(r)$ can be confirmed by the following expression when $t = 0$:*

$$D_T(r) = \frac{\int_0^T \xi e^{-r\xi} C(\xi) d\xi + T e^{-rT} P_{face}}{\int_0^T e^{-r\xi} C(\xi) d\xi + e^{-rT} P_{face}} \tag{11}$$

In the Expression (11), the denominator $\int_0^T e^{-r\xi} C(\xi) d\xi + e^{-rT} P_{face}$ is the value $P(0, r)$ of the intrinsic value function ($t = 0$) provided in the Expression (8). And $D_T(r)$ is a function of the discount rate parameter r.

Definition 3.4. *For $C(t)$, $t \in [0, T]$, the continuous cash flow of the finite time period, P_{face} represents the value of the assets A at time T. And the duration $D_T(t, r)$ at any time t can be confirmed by the following expression:*

$$D_T(t, r) = \frac{\int_t^T (\xi - t) e^{-r(\xi-t)} C(\xi) d\xi + (T - t) e^{-r(T-t)} P_{face}}{\int_t^T e^{-r(\xi-t)} C(\xi) d\xi + e^{-r(T-t)} P_{face}} \tag{12}$$

In the Expression (12), the denominator $\int_t^T e^{-r(\xi-t)} C(\xi) d\xi + e^{-r(T-t)} P_{face}$ is the value $P(t, r)$ of the intrinsic value function provided in the Expression (8). $D_T(t, r)$ is a function of the time parameter t and the discount rate parameter r. Therefore, it is called the duration function determined by the cash flow $C(t)$, $t \in [0, T]$.

Next, the relationship between the intrinsic value functions and the duration functions are deduced from Definition 3.2 and Definition 3.4, and we have the following two theorems.

Theorem 3.1. *For $C(t)$, $t \in [0, +\infty)$, the continuous cash flow of the infinite time period, the intrinsic value function $P(t, r)$ determined by the Expression (7) satisfies the following differential equation:*

$$\frac{\partial \ln P(t,r)}{\partial r} = -D_\infty(t,r) \tag{13}$$

In Equation (13), $D_\infty(t,r)$ is the duration function identified by the Expression (10) in Definition 3.2.

Proof: According to the Expression (7), we have:

$$P(t,r) = \int_t^{+\infty} e^{-r(\xi-t)} C(\xi) d\xi$$

Solving the partial derivative respect to the discount rate r, we have:

$$\frac{\partial P(t,r)}{\partial r} = \frac{\partial}{\partial r} \int_t^{+\infty} e^{-r(\xi-t)} C(\xi) d\xi = \int_t^{+\infty} -(\xi - t) e^{-r(\xi-t)} C(\xi) d\xi$$

On the basis of the definition Formula (10), we have:

$$\int_t^{+\infty} (\xi - t) e^{-r(\xi-t)} C(\xi) d\xi = D_\infty(t,r) \int_t^{+\infty} e^{-r(\xi-t)} C(\xi) d\xi = D_\infty(t,r) P(t,r)$$

Therefore, we have:

$$\frac{\partial P(t,r)}{\partial r} = -D_\infty(t,r) P(t,r)$$

Simplifying the above-mentioned expression, we have:

$$\frac{\partial \ln P(t,r)}{\partial r} = -D_\infty(t,r)$$

The theorem is proved.

Theorem 3.2. *For $C(t)$, $t \in [0, T]$, the continuous cash flow of the finite time period, the intrinsic value function $P(t, r)$ determined by the Expression (8) satisfies the following differential equation:*

$$\frac{\partial \ln P(t,r)}{\partial r} = -D_T(t,r) \tag{14}$$

In Equation (14), $D_T(t,r)$ is the duration function identified by the Expression (12) in Definition 3.4.

Proof: The proving processes are the same with Theorem 3.1.

Theorem 3.1 and Theorem 3.2 have accounted for the relationship among the intrinsic value, duration and discount rate in the form of continuous cash flow, and the derived formula corresponds to Formula (4) of discrete cash flow. As the conclusions of Theorem 3.1 and Theorem 3.2 indicate, for the continuous cash flow of infinite time period and finite time period, the relationship between the intrinsic value function of the assets and the corresponding duration function is embodied in the same differential equation. And the relative changing rate of the intrinsic value function of the assets respect to the discount rate is right equal to the value of the duration times -1.

From Theorem 3.1 and Theorem 3.2, the partial derivative of the duration function respect to the discount rate can be derived. The result is as follows:

$$\frac{\partial D_\infty(t,r)}{\partial r} = -\frac{\partial^2 \ln P(t,r)}{\partial r^2} \tag{15}$$

In the Expression (15), the intrinsic value function $P(t,r)$ is determined by the Expression (7).

$$\frac{\partial D_T(t,r)}{\partial r} = -\frac{\partial^2 \ln P(t,r)}{\partial r^2} \tag{16}$$

In the Expression (16), the intrinsic value function $P(t,r)$ is determined by the Expression (8).

As the Expressions (15) and (16) show, if $\ln P(t,r)$ is a second order differentiable concave function of r, then the duration function is an increasing function respect to the discount rate r. Correspondingly, if $\ln P(t,r)$ is a second order differentiable convex function of r, then the duration function is a decreasing function respect to the discount rate r.

Next, the changing rate of the duration function respect to the time parameter is deduced and we have the following two theorems.

Theorem 3.3. *For $C(t)$, $t \in [0,+\infty)$, the continuous cash flow of the infinite time period, $P(t,r)$ is the intrinsic value function determined by the Expression (7). The duration function $D_\infty(t,r)$ determined by the Expression (10) satisfies the following differential equation:*

$$\frac{\partial D_\infty(t,r)}{\partial t} = \frac{C(t)}{P(t,r)} D_\infty(t,r) - 1 \tag{17}$$

Proof: According to the definition Formula (10), we solve the partial derivative of $D_\infty(t,r)$ respect to the time parameter t. For the common parameter integral, the following derivation rules are tenable:

$$\frac{d}{dt}\int_{\varphi(t)}^{+\infty} f(\xi,t)d\xi = \int_{\varphi(t)}^{+\infty} \frac{d}{dt} f(\xi,t)d\xi - f(\varphi(t),t)\varphi'(t)$$

The derivative of the denominator of $D_\infty(t,r)$ respect to the parameter t is as follows:

$$\frac{\partial}{\partial t}\int_t^{+\infty} e^{-r(\xi-t)}C(\xi)d\xi = r\int_t^{+\infty} e^{-r(\xi-t)}C(\xi)d\xi - C(t)$$

The derivative of the numerator of $D_\infty(t,r)$ respect to the parameter t is as follows:

$$\begin{aligned}\frac{\partial}{\partial t}\int_t^{+\infty} (\xi-t)e^{-r(\xi-t)}C(\xi)d\xi &= \int_t^{+\infty} ((\xi-t)r-1)e^{-r(\xi-t)}C(\xi)d\xi \\ &= r\int_t^{+\infty} (\xi-t)e^{-r(\xi-t)}C(\xi)d\xi - \int_t^{+\infty} e^{-r(\xi-t)}C(\xi)d\xi\end{aligned}$$

The denominator of $D_\infty(t,r)$ is represented by M and the numerator of $D_\infty(t,r)$ is represented by Z, and then we have:

$$\begin{aligned}\frac{\partial D_\infty(t,r)}{\partial t} &= \frac{M\int_t^{+\infty}((\xi-t)r-1)e^{-r(\xi-t)}C(\xi)d\xi - Z\left(r\int_t^{+\infty} e^{-r(\xi-t)}C(\xi)d\xi - C(t)\right)}{M^2} \\ &= \frac{M(rZ-M) - Z(rM - C(t))}{M^2}\end{aligned}$$

We simplify the above-mentioned expression. Notice that:

$$M = P(t,r), \quad Z = D_\infty(t,r)P(t,r)$$

We have:

$$\frac{\partial D_\infty(t,r)}{\partial t} = \frac{C(t)}{P(t,r)} D_\infty(t,r) - 1$$

The theorem is proved.

Theorem 3.4. *For $C(t)$, $t \in [0,T]$, the continuous cash flow of the finite time period, $P(t,r)$ is the intrinsic value function determined by the Expression (8), and the duration function $D_T(t,r)$ determined by the Expression (12) satisfies the following differential equation:*

$$\frac{\partial D_T(t,r)}{\partial t} = \frac{C(t)}{P(t,r)} D_T(t,r) - 1 \tag{18}$$

Proof: The proving processes are the same with Theorem 3.3.

As the conclusions of Theorem 3.3 and Theorem 3.4 indicate, for the continuous cash flow of infinite time period and finite time period, the partial derivative of the duration function respect to time satisfies the same differential equation.

4. The High-Order Duration of Cash Flow and Taylor's Expansion for Intrinsic Value Function.

Next, the mathematical relation that the intrinsic value function transforms with the change of discount rate will be discussed. According to the definition Formula (7) of the intrinsic value function, we manage to solve the Taylor's expansion of the intrinsic value function $P(t,r)$ for any r and use the values of $P(t,r)$'s every-derivative at r to calculate the values of the intrinsic value function at $r+\Delta r$ with high accuracy. Imitating the high-order moments' concept of the random variable, we introduce the concept of the high-order duration of cash flow.

Definition 4.1. *For $C(t), t \in [0,+\infty)$, the continuous cash flow of the infinite duration, the n-order duration $D_\infty^n(t,r)$ for any t can be confirmed by the following expression:*

$$D_\infty^n(t,r) = \frac{\int_t^{+\infty} (\xi - t)^n e^{-r(\xi-t)} C(\xi) d\xi}{\int_t^{+\infty} e^{-r(\xi-t)} C(\xi) d\xi} \tag{19}$$

In the Expression (19), the denominator $\int_t^{+\infty} e^{-r(\xi-t)} C(\xi) d\xi$ is $P(t,r)$, the intrinsic value function provided in the Expression (7), and the numerator $\int_t^{+\infty} (\xi - t)^n e^{-r(\xi-t)} C(\xi) d\xi$ is the weighted average of the time parameter $(\xi - t)^n$ in $[t,+\infty)$. $D_\infty^n(t,r)$ is the function of the time parameter t and the discount rate parameter r. Therefore, it is called the n-order duration function determined by the cash flow $C(t)$, $t \in [0,+\infty)$.

It is obvious that $D_\infty(t,r) = D_\infty^1(t,r)$. And according to the Expression (7), $P(t,r) = \int_t^{+\infty} e^{-r(\xi-t)} C(\xi) d\xi$, we compute the derivatives for r and we have:

$$\frac{\partial P}{\partial r} = \int_t^{+\infty} -(\xi - t) e^{-r(\xi-t)} C(\xi) d\xi$$

Generally, for $k \geq 1$, we have:

$$\frac{\partial^k P}{\partial r^k} = \int_t^{+\infty} (-1)^k (\xi - t)^k e^{-r(\xi-t)} C(\xi) d\xi$$

In accordance with the definition of the n-order duration function $D_\infty^n(t,r)$, we have:

$$\frac{\partial^k P}{\partial r^k} = (-1)^k D_\infty^k(t,r) P(t,r)$$

We have the following Taylor's expansion:

$$P(t,r+\Delta r) = P(t,r) + \sum_{k=1}^{+\infty} \frac{(-1)^k}{k!} D_\infty^k(t,r) P(t,r) (\Delta r)^k$$

Simplifying the above-mentioned expression, we have:

$$\frac{\Delta P}{P} = \sum_{k=1}^{+\infty} \frac{(-1)^k}{k!} D_\infty^k(t,r)(\Delta r)^k \tag{20}$$

The Expression (20) is the formula of the intrinsic value function's relative changing quantity that is represented by the values of each-order duration of cash flow $C(t)$ and the changing quantity of discount rate Δr. Taking the first item of the Expression (20), we have:

$$\frac{\Delta P}{P} = -D_\infty(t,r)\Delta r$$

This expression is the same as the Expression (13). Taking the first two items of the Expression (20), we have:

$$\frac{\Delta P}{P} = -D_\infty(t,r)\Delta r + \frac{1}{2}D_\infty^2(t,r)(\Delta r)^2$$

In the case of the discrete cash flow, the values of duration are stood for by D. And when $k=1$, the given formula is as follows:

$$\frac{\Delta P}{P} = -D\frac{\Delta r}{1+r}$$

When $k=2$, the given formula is as follows:

$$\frac{\Delta P}{P} = -D\frac{\Delta r}{1+r} + \frac{1}{2}\frac{\sum\limits_{t=1}^{T}\frac{t(t+1)CF_t}{(1+r)^t}}{\sum\limits_{t=1}^{T}\frac{CF_t}{(1+r)^t}}\left(\frac{\Delta r}{1+r}\right)^2$$

Under the circumstances of continuous cash flow, through the Expression (20), we can select any $k \geq 1$, and use the first k items of the Expression (20) to accurately compute the value of $\frac{\Delta P}{P}$. For the cash flow $C(t)$, $t \in [0,T]$ of the finite time period, we can also introduce the corresponding concept of the n-order duration and acquire the same formula as Expression (20). And this paper will not give more details.

5. **The Duration of the Assets Portfolio.** In this section, the relationship between duration and cash flow is discussed, the relationship between the duration of the assets portfolio and the duration of each asset is given.

Theorem 5.1. *For $t \in [0,+\infty)$, the duration function corresponding to the continuous cash flow $C(t) \geq 0$ of the assets A is $D_\infty(t,r)$. Then for any positive number ρ, the duration function corresponding to the continuous cash flow $\rho \cdot C(t)$ is still $D_\infty(t,r)$.*

Proof: According to the definition formula of the duration function $D_\infty(t,r)$, the duration function corresponding to the continuous cash flow $C(t)$ is:

$$D_\infty(t,r) = \frac{\int_t^{+\infty}(\xi-t)e^{-r(\xi-t)}C(\xi)d\xi}{\int_t^{+\infty}e^{-r(\xi-t)}C(\xi)d\xi}$$

The corresponding duration function of the continuous cash flow $\rho \cdot C(t)$ is denoted by $D_\infty^\rho(t,r)$, and we have:

$$\begin{aligned} D_\infty^\rho(t,r) &= \frac{\int_t^{+\infty}(\xi-t)e^{-r(\xi-t)}\rho\cdot C(\xi)d\xi}{\int_t^{+\infty}e^{-r(\xi-t)}\rho\cdot C(\xi)d\xi} = \frac{\rho\cdot\int_t^{+\infty}(\xi-t)e^{-r(\xi-t)}C(\xi)d\xi}{\rho\cdot\int_t^{+\infty}e^{-r(\xi-t)}C(\xi)d\xi} \\ &= D_\infty(t,r) \end{aligned}$$

The theorem is proved.

Theorem 5.2. *For $t \in [0, T]$, the duration function corresponding to the continuous cash flow $C(t) \geq 0$ of the asset B is $D_T(t, r)$. Then for any positive number ρ, the corresponding duration function of the assets is still $D_T(t, r)$ when the continuous cash flow is $\rho \cdot C(t)$ and the price in time T is $\rho \cdot P_{face}$.*

Proof: The proving processes are the same with Theorem 5.1.

The conclusions of Theorem 5.1 and Theorem 5.2 indicate that the duration function value of the assets is determined by the cash flow mode. The cash flow that enlarges or shrinks the same multiple does not change the value of duration function.

Theorem 5.3. *For $t \in [0, +\infty)$, there are n kinds of assets, $A_1, A_2, \ldots, A_n$. The intrinsic value function corresponding to the continuous cash flow $C_i(t) \geq 0$ of the asset A_i is $P_i(t, r)$, and the duration function is $D_{\infty i}(t, r)$, $i = 1, 2, \ldots, n$. Then for any assets portfolio mode $(\rho_1, \rho_2, \ldots, \rho_n) \geq 0$, the duration function corresponding to the continuous cash flow $\sum_{i=1}^{n} \rho_i \cdot C_i(t)$ of the assets portfolio is denoted by $D_\infty^P(t, r)$, and we have:*

$$D_\infty^P(t, r) = \frac{\sum_{i=1}^{n} \rho_i \cdot D_{\infty i}(t, r) \cdot P_i(t, r)}{\sum_{i=1}^{n} \rho_i \cdot P_i(t, r)} \tag{21}$$

Proof: According to the definition formula of the duration function $D_\infty(t, r)$, the duration function $D_\infty^P(t, r)$ corresponding to the continuous cash flow $\sum_{i=1}^{n} \rho_i \cdot C_i(t)$ is:

$$D_\infty^P(t, r) = \frac{\int_t^{+\infty} (\xi - t) e^{-r(\xi - t)} \left(\sum_{i=1}^{n} \rho_i \cdot C_i(\xi) \right) d\xi}{\int_t^{+\infty} e^{-r(\xi - t)} \left(\sum_{i=1}^{n} \rho_i \cdot C_i(\xi) \right) d\xi}$$

It is known that:

$$D_{\infty i}(t, r) = \frac{\int_t^{+\infty} (\xi - t) e^{-r(\xi - t)} C_i(\xi) d\xi}{\int_t^{+\infty} e^{-r(\xi - t)} C_i(\xi) d\xi} = \frac{\int_t^{+\infty} (\xi - t) e^{-r(\xi - t)} C_i(\xi) d\xi}{P_i(t, r)}$$

Therefore, we have:

$$\int_t^{+\infty} (\xi - t) e^{-r(\xi - t)} C_i(\xi) d\xi = D_{\infty i}(t, r) \cdot P_i(t, r)$$

Substituting the above-mentioned expression of $D_\infty^P(t, r)$, we have:

$$D_\infty^P(t, r) = \frac{\sum_{i=1}^{n} \rho_i \cdot D_{\infty i}(t, r) \cdot P_i(t, r)}{\sum_{i=1}^{n} \rho_i \cdot P_i(t, r)}$$

The theorem is proved.

Theorem 5.4. *For $t \in [0, +\infty)$, there are n kinds of assets, $A_1, A_2, \ldots, A_n$. The intrinsic value function corresponding to the continuous cash flow $C_i(t) \geq 0$ of the asset A_i is $P_i(t, r)$, and the duration function is $D_{\infty i}(t, r)$, $i = 1, 2, \ldots, n$. Then for any assets portfolio mode $(\rho_1, \rho_2, \ldots, \rho_n) \geq 0$, the duration function corresponding to the continuous cash flow $\sum_{i=1}^{n} \rho_i \cdot C_i(t)$ of assets portfolio is denoted by $D_\infty^P(t, r)$, and we have:*

$$\min(D_{\infty i}(t, r), 1 \leq i \leq n) \leq D_\infty^P(t, r) \leq \max(D_{\infty i}(t, r), 1 \leq i \leq n)$$

Proof: In the Expression (21) derived from Theorem 5.3, it is denoted:

$$W_i = \frac{\rho_i \cdot P_i(t,r)}{\sum_{j=1}^{n} \rho_j \cdot P_j(t,r)}$$

Obviously, there are $W_i \geq 0$ and $\sum_{i=1}^{n} W_i = 1$. It illustrates that $(W_1, W_2, \ldots, W_n)$ is a group of weight. The Expression (21) can be denoted in the following form:

$$D_\infty^P(t,r) = \sum_{i=1}^{n} W_i D_{\infty i}(t,r)$$

Therefore, $D_\infty^P(t,r)$ is the convex combination of all the duration function $D_{\infty i}(t,r)$ for $i = 1, 2, \ldots, n$. According to the properties of convex combination, we have:

$$D_\infty^P(t,r) \geq \min(D_{\infty i}(t,r), 1 \leq i \leq n)$$

$$D_\infty^P(t,r) \leq \max(D_{\infty i}(t,r), 1 \leq i \leq n)$$

The theorem is proved.

The conclusion of Theorem 5.4 indicates that the duration function value of assets portfolio is the combination of the duration function values of each asset, and the duration function value of assets portfolio is between the maximum and minimum of the duration function values of each asset.

6. **The Relationship between the Duration and the Cash Flow.** In this section, the relationship between duration and cash flow is discussed, and some change rules of duration when cash flow changes in accordance with some special requirements are probed.

From the meaning of the duration of cash flow, we know that the delayed cash flow makes the duration value increase. For a given cash flow $C(t)$, supposing that the value of $C(t)$ is changed to $\tilde{C}(t)$, $\tilde{C}(t) = C(t)q(t)$, $q(t) > 0$, and that the value of $q(t)$ is required to increase with the increase of t, which means that the cash flow $\tilde{C}(t)$ delays compared with $C(t)$, then the duration of the cash flow $\tilde{C}(t)$ should be greater than the duration of $C(t)$. Similarly, from the meaning of the duration of cash flow, we know that early-realized cash flow makes the duration value decrease. For a given cash flow $C(t)$, supposing that the value of $C(t)$ is changed to $\tilde{C}(t)$, $\tilde{C}(t) = C(t)q(t)$, $q(t) > 0$, and that the value of $q(t)$ is required to decrease with the increase of t, which means that the cash flow $\tilde{C}(t)$ advances compared with $C(t)$, then the duration of the cash flow $\tilde{C}(t)$ should be less than the duration of $C(t)$. Taking it into account, we have the following Theorem 6.1 and Theorem 6.2.

Theorem 6.1. *For $C(t) \geq 0$, $t \in [0, +\infty)$, the continuous cash earnings function, $q(t) > 0$ is a monotonic increasing continuous function. Then the duration function value of the cash earnings function composed by $C(t)q(t)$ must be greater than the duration function value of the cash earnings function $C(t)$.*

Proof: $q(t) > 0$ is a monotonic increasing continuous function and it can be expressed in the form of the sum of simple functions. $\eta(t_0, t)$ is used to represent the unit step function. That is:

$$\eta(t_0, t) = \begin{cases} 0, & t \leq t_0 \\ 1, & t > t_0 \end{cases}$$

For any partition $t_1 < t_2 < t_3 < \ldots < t_n < \ldots$, $t_1 = 0$, $\lim_{n\to\infty} t_n = \infty$ when $t \in [0, +\infty)$, $q(t)$ can be approximately represented in the following form:

$$q(t) \approx \sum_{i=1}^{\infty} a_i \eta(t_i, t)$$

In the expression, there is $a_i > 0$. When the partition $\{t_1, t_2, t_3, \ldots, t_n, \ldots\}$ is fully populated, $t_{i+1} - t_i$ is sufficient small for every i. At this time, $\sum_{i=1}^{\infty} a_i \eta(t_i, t)$ can approach to $q(t)$ infinitely. Then we have $C(t)q(t) \approx \sum_{i=1}^{\infty} a_i \cdot C(t) \cdot \eta(t_i, t)$.

Next, we estimate the duration function of the cash flow $a_i \cdot C(t) \cdot \eta(t_i, t)$. The duration function of the cash flow $C(t)$ is denoted by $D_\infty(t, r)$ while the duration function of the cash flow $a_i \cdot C(t) \cdot \eta(t_i, t)$ is denoted by $D_{i\infty}(t, r)$. According to the definition Formula (10), we have:

$$D_{i\infty}(t, r) = \frac{\int_t^{+\infty} (\xi - t) e^{-r(\xi - t)} C(\xi) \cdot a_i \cdot \eta(t_i, \xi) d\xi}{\int_t^{+\infty} e^{-r(\xi - t)} C(\xi) \cdot a_i \cdot \eta(t_i, \xi) d\xi}$$

When $t \geq t_i$, $\eta(t_i, t) = 1$. And we have:

$$D_{i\infty}(t, r) = \frac{\int_t^{+\infty} (\xi - t) e^{-r(\xi - t)} C(\xi) \cdot a_i d\xi}{\int_t^{+\infty} e^{-r(\xi - t)} C(\xi) \cdot a_i d\xi} = D_\infty(t, r)$$

When $t < t_i$, we have:

$$\begin{aligned} D_{i\infty}(t, r) &= \frac{\int_t^{+\infty} (\xi - t) e^{-r(\xi - t)} C(\xi) \cdot a_i \cdot \eta(t_i, \xi) d\xi}{\int_t^{+\infty} e^{-r(\xi - t)} C(\xi) \cdot a_i \cdot \eta(t_i, \xi) d\xi} \\ &= \frac{\int_{t_i}^{+\infty} (\xi - t) e^{-r(\xi - t)} C(\xi) \cdot a_i d\xi}{\int_{t_i}^{+\infty} e^{-r(\xi - t)} C(\xi) \cdot a_i d\xi} = \frac{\int_{t_i}^{+\infty} (\xi - t) e^{-r(\xi - t)} C(\xi) d\xi}{\int_{t_i}^{+\infty} e^{-r(\xi - t)} C(\xi) d\xi} \\ &= \frac{\int_t^{+\infty} (\xi - t) e^{-r(\xi - t)} C(\xi) d\xi - \int_t^{t_i} (\xi - t) e^{-r(\xi - t)} C(\xi) d\xi}{\int_t^{+\infty} e^{-r(\xi - t)} C(\xi) d\xi - \int_t^{t_i} e^{-r(\xi - t)} C(\xi) d\xi} \end{aligned}$$

We estimate the following expression:

$$F(t, t_i) = \frac{\int_t^{t_i} (\xi - t) e^{-r(\xi - t)} C(\xi) d\xi}{\int_t^{t_i} e^{-r(\xi - t)} C(\xi) d\xi}$$

It is denoted by:

$$\begin{aligned} M_1 &= \left(\int_t^{t_i} e^{-r(\xi - t)} C(\xi) d\xi \right) \frac{d}{dt_i} \int_t^{t_i} (\xi - t) e^{-r(\xi - t)} C(\xi) d\xi \\ &= \left(\int_t^{t_i} e^{-r(\xi - t)} C(\xi) d\xi \right) (t_i - t) e^{-r(t_i - t)} C(t_i) \end{aligned}$$

$$\begin{aligned} M_2 &= \left(\int_t^{t_i} (\xi - t) e^{-r(\xi - t)} C(\xi) d\xi \right) \frac{d}{dt_i} \int_t^{t_i} e^{-r(\xi - t)} C(\xi) d\xi \\ &= \left(\int_t^{t_i} (\xi - t) e^{-r(\xi - t)} C(\xi) d\xi \right) e^{-r(t_i - t)} C(t_i) \end{aligned}$$

We have:

$$\frac{\partial}{\partial t_i}F(t,t_i) = \frac{M_1 - M_2}{\left(\int_t^{t_i} e^{-r(\xi-t)}C(\xi)d\xi\right)^2} = \frac{e^{-r(t_i-t)}C(t_i)\int_t^{t_i}(t_i-\xi)e^{-r(\xi-t)}C(\xi)d\xi}{\left(\int_t^{t_i} e^{-r(\xi-t)}C(\xi)d\xi\right)^2} > 0$$

Notice that $\lim_{t_i\to\infty} F(t,t_i) = D_\infty(t,r)$, and because $\frac{\partial}{\partial t_i}F(t,t_i) > 0$, we have: $F(t,t_i) < D_\infty(t,r)$. That is:

$$\int_t^{t_i}(\xi-t)e^{-r(\xi-t)}C(\xi)d\xi < D_\infty(t,r)\int_t^{t_i} e^{-r(\xi-t)}C(\xi)d\xi$$

According to the definition Formula (10), we have:

$$\int_t^{\infty}(\xi-t)e^{-r(\xi-t)}C(\xi)d\xi = D_\infty(t,r)\int_t^{\infty} e^{-r(\xi-t)}C(\xi)d\xi$$

From the above-mentioned two expressions, we have:

$$\begin{aligned}&\int_t^{\infty}(\xi-t)e^{-r(\xi-t)}C(\xi)d\xi - \int_t^{t_i}(\xi-t)e^{-r(\xi-t)}C(\xi)d\xi\\ &> D_\infty(t,r)\int_t^{\infty} e^{-r(\xi-t)}C(\xi)d\xi - D_\infty(t,r)\int_t^{t_i} e^{-r(\xi-t)}C(\xi)d\xi\end{aligned}$$

Simplifying the above-mentioned expression, we obtained the following inequality:

$$D_{i\infty}(t,r) = \frac{\int_t^{+\infty}(\xi-t)e^{-r(\xi-t)}C(\xi)d\xi - \int_t^{t_i}(\xi-t)e^{-r(\xi-t)}C(\xi)d\xi}{\int_t^{+\infty} e^{-r(\xi-t)}C(\xi)d\xi - \int_t^{t_i} e^{-r(\xi-t)}C(\xi)d\xi} > D_\infty(t,r)$$

According to Theorem 4.4, the duration value $D_\infty^q(t,r)$ of the cash earnings function composed by $C(t)q(t)$ is the convex combination of all $D_{i\infty}(t,r)$. So there must be $D_\infty^q(t,r) > D_\infty(t,r)$. The theorem is proved.

Theorem 6.2. *For $C(t) \geq 0$, $t \in [0,+\infty)$, the continuous cash earnings function, $q(t) > 0$ is a monotonic decreasing continuous function. Then the duration function value of the cash earnings function composed by $C(t)q(t)$ must be less than the duration function value of the cash earnings function $C(t)$.*

Proof: $q(t) > 0$ is a monotonic decreasing continuous function and it can be expressed in the form of the sum of simple functions. $\eta(t_0,t)$ is used to represent the unit step function. That is:

$$\eta(t_0,t) = \begin{cases} 0, & t \leq t_0 \\ 1, & t > t_0 \end{cases}$$

For any partition $t_1 < t_2 < t_3 < \ldots < t_n < \ldots$, $t_1 = 0$, $\lim_{n\to\infty} t_n = \infty$ when $t \in [0,+\infty)$, $q(t)$ can be approximately represented in the following form:

$$q(t) \approx \sum_{i=1}^{\infty}(b_i + a_i(1-\eta(t_i,t)))$$

In this expression, $b_i > 0$ and $a_i > 0$. When the partition $\{t_1,t_2,t_3,\ldots,t_n,\ldots\}$ is fully populated, $t_{i+1} - t_i$ is sufficient small for every i. At this time, $\sum_{i=1}^{\infty}(b_i + a_i(1-\eta(t_i,t)))$ can approach to $q(t)$ infinitely. Then we have:

$$C(t)q(t) \approx \sum_{i=1}^{\infty} C(t)(b_i + a_i(1-\eta(t_i,t))).$$

Next, we estimate the duration function of the cash flow $C(t)(b_i + a_i(1 - \eta(t_i, t)))$. The duration function of the cash flow $C(t)$ is denoted by $D_\infty(t, r)$ while the duration function of the cash flow $C(t)(b_i + a_i(1 - \eta(t_i, t)))$ is denoted by $D_{i\infty}(t, r)$. According to the definition Formula (10), we have:

$$D_{i\infty}(t, r) = \frac{\int_t^{+\infty} (\xi - t)e^{-r(\xi-t)}C(\xi)(b_i + a_i(1 - \eta(t_i, \xi)))d\xi}{\int_t^{+\infty} e^{-r(\xi-t)}C(\xi)(b_i + a_i(1 - \eta(t_i, \xi)))d\xi}$$

When $t \geq t_i$, $\eta(t_i, t) = 1$, $1 - \eta(t_i, t) = 0$, we have:

$$D_{i\infty}(t, r) = \frac{\int_t^{+\infty} (\xi - t)e^{-r(\xi-t)}C(\xi) \cdot b_i d\xi}{\int_t^{+\infty} e^{-r(\xi-t)}C(\xi) \cdot b_i d\xi} = D_\infty(t, r)$$

When $t < t_i$, we have:

$$\begin{aligned} D_{i\infty}(t, r) &= \frac{\int_t^{+\infty} (\xi - t)e^{-r(\xi-t)}C(\xi)(b_i + a_i(1 - \eta(t_i, \xi)))d\xi}{\int_t^{+\infty} e^{-r(\xi-t)}C(\xi)(b_i + a_i(1 - \eta(t_i, \xi)))d\xi} \\ &= \frac{\int_t^{+\infty} b_i(\xi - t)e^{-r(\xi-t)}C(\xi)d\xi + \int_t^{+\infty} (\xi - t)e^{-r(\xi-t)}C(\xi)a_i(1 - \eta(t_i, \xi))d\xi}{\int_t^{+\infty} b_i e^{-r(\xi-t)}C(\xi)d\xi + \int_t^{+\infty} e^{-r(\xi-t)}C(\xi)a_i(1 - \eta(t_i, \xi))d\xi} \\ &= \frac{\int_t^{+\infty} b_i(\xi - t)e^{-r(\xi-t)}C(\xi)d\xi + \int_t^{t_i} a_i(\xi - t)e^{-r(\xi-t)}C(\xi)d\xi}{\int_t^{+\infty} b_i e^{-r(\xi-t)}C(\xi)d\xi + \int_t^{i_i} a_i e^{-r(\xi-t)}C(\xi)d\xi} \end{aligned}$$

We estimate the following expression:

$$F(t, t_i) = \frac{\int_t^{t_i} (\xi - t)e^{-r(\xi-t)}C(\xi)d\xi}{\int_t^{t_i} e^{-r(\xi-t)}C(\xi)d\xi}$$

According to the conclusion that is drawn from the process of proving Theorem 6.1, we have:

$$\frac{\partial}{\partial t_i}F(t, t_i) = \frac{e^{-r(t_i-t)}C(t_i)\int_t^{t_i} (t_i - \xi)e^{-r(\xi-t)}C(\xi)d\xi}{(\int_t^{t_i} e^{-r(\xi-t)}C(\xi)d\xi)^2} > 0$$

Notice that $\lim_{t_i\to\infty} F(t, t_i) = D_\infty(t, r)$, and because $\frac{\partial}{\partial t_i}F(t, t_i) > 0$, we have $F(t, t_i) < D_\infty(t, r)$. That is:

$$\int_t^{t_i} (\xi - t)e^{-r(\xi-t)}C(\xi)d\xi < D_\infty(t, r)\int_t^{t_i} e^{-r(\xi-t)}C(\xi)d\xi$$

According to definition Formula (10), we have:

$$\int_t^{\infty} (\xi - t)e^{-r(\xi-t)}C(\xi)d\xi = D_\infty(t, r)\int_t^{\infty} e^{-r(\xi-t)}C(\xi)d\xi$$

From the above-mentioned two expressions, we have:

$$\begin{aligned} &\int_t^{\infty} b_i(\xi - t)e^{-r(\xi-t)}C(\xi)d\xi + \int_t^{t_i} a_i(\xi - t)e^{-r(\xi-t)}C(\xi)d\xi \\ &< D_\infty(t, r)\int_t^{\infty} b_i e^{-r(\xi-t)}C(\xi)d\xi + D_\infty(t, r)\int_t^{t_i} a_i e^{-r(\xi-t)}C(\xi)d\xi \end{aligned}$$

Simplifying the above-mentioned expression, we obtained the following inequality:

$$D_{i\infty}(t, r) = \frac{\int_t^{+\infty} b_i(\xi - t)e^{-r(\xi-t)}C(\xi)d\xi + \int_t^{t_i} a_i(\xi - t)e^{-r(\xi-t)}C(\xi)d\xi}{\int_t^{+\infty} b_i e^{-r(\xi-t)}C(\xi)d\xi + \int_t^{i_i} a_i e^{-r(\xi-t)}C(\xi)d\xi} < D_\infty(t, r)$$

According to Theorem 5.4, the duration value $D^q_\infty(t,r)$ of the cash earnings function composed by $C(t)q(t)$ is the convex combination of all $D_{i\infty}(t,r)$. So there must be $D^q_\infty(t,r) < D_\infty(t,r)$. The theorem is proved.

Theorem 6.3. *The discount rate is denoted by r. If the duration value of the cash earnings function $C(t) \geq 0$, $t \in [0,+\infty)$ is less than $\frac{1}{r}$, then for any positive constant H, the duration value of the cash earnings function $C(t)+H$ is less than $\frac{1}{r}$. If the duration value of the cash earnings function $C(t) \geq 0$, $t \in [0,+\infty)$ is greater than $\frac{1}{r}$, then for any positive constant H, the duration value of the cash earnings function $C(t)+H$ is greater than $\frac{1}{r}$.*

Proof: If $C(t) = H$, the intrinsic value function corresponding to the cash earnings function $C(t) = H$ is:

$$P(t,r) = \int_t^{+\infty} e^{-r(\xi-t)} H d\xi = \frac{H}{r}$$

The duration function corresponding to the cash earnings function $C(t) = H$ is denoted by $D_\infty(t,r)$. According to Theorem 3.1, we have:

$$D_\infty(t,r) = -\frac{\partial \ln P(t,r)}{\partial r} = \frac{1}{r}$$

According to Theorem 5.4, the duration function of the cash earnings function $C(t)+H$ is the convex combination of the duration function of $C(t)$ and $\frac{1}{r}$ (the duration value of H). Therefore, if the duration value of the cash earnings function $C(t) \geq 0$, $t \in [0,+\infty)$ is less than $\frac{1}{r}$, then for any positive constant H, the duration value of the cash earnings function $C(t)+H$ is less than $\frac{1}{r}$. If the duration value of the cash earnings function $C(t) \geq 0$, $t \in [0,+\infty)$ is greater than $\frac{1}{r}$; then for any positive constant H, the duration value of the cash earnings function $C(t)+H$ is greater than $\frac{1}{r}$. The theorem is proved.

7. Several Cash Earnings Function and the Corresponding Intrinsic Value Function and Duration Function.

There are two ways to solve the duration of an asset. One is to solve it directly according to the definition formula. To solve the duration function $D_\infty(t,r)$ with Formula (10) needs to solve two integrals. Similarly, to solve the duration function $D_T(t,r)$ with Formula (12) also needs to solve two integrals. The other way is to solve the duration function according to the conclusions of Theorem 3.1 and Theorem 3.2. First of all, we solve the intrinsic value function by using Formula (7) or Formula (8). Then we solve the derivative of the intrinsic value function with respect to the discount rate parameter. Finally, after the derivative is divided by the intrinsic value function and then multiplied by -1, we can obtain the duration function.

Take $C(t) = C_0$, $t \in [0,+\infty)$ for example. The intrinsic value function $P(t,r)$ is as follows:

$$P(t,r) = \int_t^{+\infty} e^{-r(\xi-t)} C_0 d\xi = C_0 \cdot \left[-\frac{1}{r} e^{-r(\xi-t)}\right]\Big|_t^{+\infty} = \frac{C_0}{r}$$

According to the definition Formula (10) of the duration function, the integral of the numerator is:

$$\begin{aligned}\int_t^{+\infty} (\xi-t) e^{-r(\xi-t)} \cdot C_0 \cdot d\xi &= C_0 \int_0^{+\infty} u e^{-ru} du \\ &= \frac{C_0}{r^2} e^{-ru}(-ru-1)\,\Big|_0^{+\infty} = \frac{C_0}{r^2}\end{aligned}$$

Therefore, the duration function is as follows:

$$D_\infty(t,r) = \frac{\int_t^{+\infty} (\xi - t)e^{-r(\xi-t)} \cdot C_0 \cdot d\xi}{\int_t^{+\infty} e^{-r(\xi-t)} C_0 d\xi} = \left(\frac{C_0}{r^2}\right) \Big/ \left(\frac{C_0}{r}\right) = \frac{1}{r}$$

According to Theorem 3.1, $D_\infty(t,r) = -\frac{\partial \ln P(t,r)}{\partial r}$. Substituting $P(t,r) = \frac{C_0}{r}$ into it, we can also get $D_\infty(t,r) = \frac{1}{r}$. Therefore, it can be seen that when the cash flow is a constant, the duration of the infinite cash flow is the reciprocal of the discount rate, and irrelevant with the value of cash flow.

For the finite cash flow $C(t) = 0$, $t \in [0,T]$, P_{face} is the par value of bonds, which is a zero coupon bond. In the case of discrete cash flow, the duration of zero coupon bonds equals T. According to the definition Formula (12), we have:

$$D_T(t,r) = \frac{\int_t^T (\xi - t)e^{-r(\xi-t)} C(\xi) d\xi + (T-t)e^{-r(T-t)} P_{face}}{\int_t^T e^{-r(\xi-t)} C(\xi) d\xi + e^{-r(T-t)} P_{face}}$$

Substituting $C(t) = 0$ into the above-mentioned expression, we have:

$$D_T(t,r) = \frac{(T-t)e^{-r(T-t)} P_{face}}{e^{-r(T-t)} P_{face}} = T - t$$

That is to say, the duration function of zero coupon bonds equals the remaining length of period of validity. Especially when $t = 0$, we have $D_T(t,r) = T$. This is consistent with the conclusion of discrete cash flow.

For the finite cash flow $C(t) = D_0 P_{face}$, $t \in [0,T]$, P_{face} is the par value of bonds. The proportion that each bond interest occupies in the par value is D_0, and it is actually the coupon rate. According to the definition Formula (12), we have:

$$\begin{aligned} D_T(t,r) &= \frac{\int_t^T (\xi - t)e^{-r(\xi-t)} D_0 P_{face} d\xi + (T-t)e^{-r(T-t)} P_{face}}{\int_t^T e^{-r(\xi-t)} D_0 P_{face} d\xi + e^{-r(T-t)} P_{face}} \\ &= \frac{D_0 \int_t^T (\xi - t)e^{-r(\xi-t)} d\xi + (T-t)e^{-r(T-t)}}{D_0 \int_t^T e^{-r(\xi-t)} d\xi + e^{-r(T-t)}} \\ &= \frac{\frac{D_0}{r^2}[1 - (1 + r(T-t))e^{-r(T-t)}] + (T-t)e^{-r(T-t)}}{\frac{D_0}{r}(1 - e^{-r(T-t)}) + e^{-r(T-t)}} \end{aligned}$$

The duration function $D_T(t,r)$ has nothing to do with the par value P_{face}. When $t = T$, we have $D_T(T,r) = 0$.

Some commonly used cash earnings functions $C(t)$ are selected to provide the intrinsic value function and the duration function in analytical form. The results of the intrinsic value function and the duration function, both of which are determined by the cash earnings function, can be found in Table 2.

In Table 2, the formulas of the intrinsic value function and the duration function corresponding to the cash earnings function in the sixth, seventh and eighth line are relatively complicated. And the corresponding formulas are as follows:

$$H_{62} = P_{face} \left[\frac{D_0}{r}(1 - e^{-r(T-t)}) + e^{-r(T-t)} \right]$$

$$H_{63} = \frac{\frac{D_0}{r^2}\left[1 - (1 + r(T-t))e^{-r(T-t)}\right] + (T-t)e^{-r(T-t)}}{\frac{D_0}{r}(1 - e^{-r(T-t)}) + e^{-r(T-t)}}$$

$$H_{72} = P_{face} \left[\frac{D_0 e^{rt - \lambda t_0}}{r - \lambda} (e^{(\lambda - r)t} - e^{(\lambda - r)T}) + e^{-r(T-t)} \right]$$

$$H_{73} = \frac{\frac{D_0}{(r-\lambda)^2}(e^{\lambda(t-t_0)} - (1+(r-\lambda)(T-t))e^{-r(T-t)}e^{\lambda(T-t_0)}) + (T-t)e^{-r(T-t)}}{\frac{D_0 e^{rt-\lambda t_0}}{r-\lambda}(e^{(\lambda-r)t} - e^{(\lambda-r)T}) + e^{-r(T-t)}}$$

$$H_{82} = \begin{cases} P_{face}\left[\frac{D_0}{r}(e^{-r(t_0-t)} - e^{-r(T-t)}) + e^{-r(T-t)}\right] & 0 \le t \le t_0 \\ P_{face}\left[\frac{D_0}{r}(1 - e^{-r(T-t)}) + e^{-r(T-t)}\right] & t > t_0 \end{cases}$$

$$H_{83} = \begin{cases} = \frac{\frac{D_0}{r^2}[(1+r(t_0-t))e^{-r(t_0-t)} - (1+r(T-t))e^{-r(T-t)}] + (T-t)e^{-r(T-t)}}{\frac{D_0}{r}(e^{-r(t_0-t)} - e^{-r(T-t)}) + e^{-r(T-t)}} & 0 \le t \le t_0 \\ = \frac{\frac{D_0}{r^2}[1-(1+r(T-t))e^{-r(T-t)}] + (T-t)e^{-r(T-t)}}{\frac{D_0}{r}(1-e^{-r(T-t)}) + e^{-r(T-t)}} & t > t_0 \end{cases}$$

It can be seen that for the finite cash flow, the formulas of the intrinsic value function and duration function of the assets are relatively complex, while the formulas of the intrinsic value function and duration function of the assets are relatively simple for the infinite cash flow.

Table 2. Some cash earnings functions and the corresponding intrinsic value functions and the duration functions

The cash earnings function $C(t)$	The intrinsic value function $P(t,r)$	The duration function $D_\infty(t,r)$ or $D_T(t,r)$
$C(t) = D_0$ $D_0 > 0,\ t \in [0,+\infty)$	$\frac{D_0}{r}$	$\frac{1}{r}$
$C(t) = a + bt$ $a > 0,\ b > 0,\ t \in [0,+\infty)$	$\frac{a+bt}{r} + \frac{b}{r^2}$	$\frac{1}{r}\left(1 + \frac{b}{(a+bt)r+b}\right)$
$C(t) = Ae^{\lambda(t-t_0)}$ $\lambda < r,\ t \in [0,+\infty)$	$\frac{Ae^{\lambda(t-t_0)}}{r-\lambda}$	$\frac{1}{r-\lambda}$
$C(t) = \begin{cases} 0 & 0 \le t \le t_0 \\ A & t > t_0 \end{cases}$ $t \in [0,+\infty)$	$\begin{cases} \frac{A}{r}e^{r(t-t_0)} & 0 \le t \le t_0 \\ \frac{A}{r} & t > t_0 \end{cases}$	$\begin{cases} \frac{1}{r} + (t_0 - t) & 0 \le t \le t_0 \\ \frac{1}{r} & t > t_0 \end{cases}$
$C(t) = 0$ $t \in [0,T]$	$e^{-r(T-t)}P_{face}$	$T - t$
$C(t) = D_0 P_{face}$ $t \in [0,T]$	H_{62}	H_{63}
$C(t) = D_0 P_{face} e^{\lambda(t-t_0)}$ $\lambda < r,\ t \in [0,T]$	H_{72}	H_{73}
$C(t) = \begin{cases} 0 & 0 \le t \le t_0 \\ D_0 P_{face} & t > t_0 \end{cases}$ $t \in [0,T]$	H_{82}	H_{83}

8. **Conclusion.** This paper focuses on the studying of the intrinsic value and duration that correspond to continuous cash flow. In the case of the finite time period and infinite time period, we respectively deduce the corresponding intrinsic value function formulas from the yield equation in the continuous form; the formulas are given in the form of parameter integral. On the basis of the intrinsic value function of the assets, we proposed the definitions of the duration and the duration function corresponding to the form of continuous cash flow with imitating the definition of Macaulay duration in discrete cash flow form. Meanwhile, the differential equations, which the intrinsic value function and the duration function that correspond to continuous cash flow satisfy, are deduced. And the differential equations include the partial derivative of duration function respect to the discount rate and the time parameter. In addition, the formula between the change of

the intrinsic value of the assets and the change of the discount rate is also obtained; the formula is signified in the high-order duration. The relationship between duration and cash flow are discussed and the relationship between the duration of the assets portfolio and the duration of each individual assets is derived. For a series of commonly used continuous cash flow functions and even more, the corresponding intrinsic value function and duration function are derived. And these results are tabulated to the application, which is undoubtedly significant for ensuring that the content is rich and the reference is convenient. The tools of the calculus have been fully applied in this study. The derived differential equations and mathematical formulas are all in simple forms and the use of them makes it more convenient to study the rules of the duration of the cash flow for general functions.

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