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STABILIZATION WITH POSITIVITY OF nD SYSTEMS

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Abstract. *The problem of synthesizing stabilizing controllers is solved for a class of nD systems when the closed-loop system is required to remain positive. More precisely, the stabilizing local state-feedback controllers are characterized in terms of Linear Programs, for a generic class of nD systems.*

Keywords: Control systems, Multidimensional systems, Stability, Positive systems, Linear programming

1. **Introduction.** During the last few decades, n-dimensional (nD) systems theory (also refereed to as multidimensional systems theory) has received considerable attention from many researchers. The particular case when $n = 2$ was introduced in the seventies [15, 29] and then it was generalized to the general nD case ($n \geq 2$). These models have found many applications in, for instance, digital data filtering, image processing [25], and in systems described by partial differential equations [23]. Some important problems, such as realization, controllability or minimum energy control, have been extensively studied for specific classes of nD systems [15, 18, 25]. Even though most of the classes considered are recursive in the upper right quadrant of the 2D plane, a great variety of models have been used: Roesser [25], Fornasini and Marchesini [10], Kurek [22], 2D general models [17, 20], and many other variations. It is remarkable that even if these models are closely related the existing results apply only to the particular model in consideration.

The main goal of this paper is to present a numerically reliable framework to deal with these classes of nD systems for nonnegative states. One of the main advantages of using such unified framework is that one is free to use the more convenient model for a particular task and still apply the same methodology. Hence, it is obvious that many of the existing results turn out to be just particular cases of this more general approach, as we shall show later on.

As mentioned above, this paper concentrates on *positive* nD systems, that is, nD systems that keep invariant the positive orthant, or in other words, that their trajectories evolve in the positive orthant "when they start in it". Recently, there has been outstanding growing interest in both the theory and application of positive 1D systems (see

[2, 3, 4, 30, 31]) and nD systems (see [1, 8, 13, 19, 20, 26, 27] and references therein). In particular, we concentrate on the derivation of conditions for the stabilization of positive nD systems described by a generic model. Albeit the stabilization problem has previously been investigated, it is not completely solved for several classes of nD systems. The previous works concentrate on specific classes of 2D/nD systems, so they are not generally applicable. Starting from the results in [2], and using ideas borrowed from [14], easily checkable conditions for stabilization are derived in terms of Linear Programs (LP) for a generic class of nD systems. This represents a significant innovation with respect to the previous contributions since dealing with LP instead of the classical LMI conditions allows us to tackle the important problem of designing *stabilizing state-feedback controllers with structured or bounded gains*.

It is well known that many real systems involve bounded controls, with these bounds arising from physical constraints. These constraints must be considered during controller design, as they have a significant destabilizing effect. Although the design of controllers for 1D systems with constraints has been extensively studied (see for example [6, 11] and references therein), this is not the case for n-D systems, where they have not been fully considered. These constraints are allowed in this paper to be nonsymmetric, as it is frequent in practical control problems [7, 24, 28]. As far as the authors know, these types of constraints have not been considered for n-D systems: see [1, 5, 8, 12, 13], for related problems in the context of 2-D systems.

This paper is organized as follows. Section 2 presents the problem formulation and some preliminary results, whereas Section 3 presents the stabilization problem of positive nD systems. Section 4 derives the main results of positive state-feedback stabilization. In Section 5, we extend these results for a control law forced to be bounded. Finally, after providing an illustrative example in Section 6, some conclusions are given in Section 7.

Notation: $\mathbb{R}^n_+$ denotes the non-negative orthant of the n-dimensional real space $\mathbb{R}^n$ and $\mathbb{N}$ the set of natural numbers. M^T denotes the transpose of the real matrix M. For a real matrix M, $M > 0$ denotes a positive matrix, that is, a matrix with all its components positive (i.e., $m_{ij} > 0$), whereas $M \geq 0$ denotes a nonnegative matrix, with none of its components negative (i.e., $m_{ij} \geq 0$); $\rho(M)$ denotes its spectral radius. $\mathbb{I}$ denotes the identity matrix of appropriate order.

2. **Problem Formulation and Preliminaries.**

Consider a linear homogeneous nD system described by the following general model

$$\begin{aligned} x(i_1+N_1,\dots,i_n+N_n) = \sum_{p_1=0}^{N_1}\cdots\sum_{p_n=0}^{N_n} & (A_{p_1\dots p_n}x(i_1+p_1,\dots,i_n+p_n) \\ & + (B_{p_1\dots p_n}u(i_1+p_1,\dots,i_n+p_n)) \end{aligned} \tag{1}$$

where $i_1,\dots,i_n,N_1,\dots,N_n \in \mathbb{N}$, $A_{p_1\dots p_n} \in \mathbb{R}^{q\times q}$ and $B_{p_1\dots p_n} \in \mathbb{R}^{q\times m}$ are given matrices, $x(i_1,\dots,i_n) \in \mathbb{R}^q$ is the nD state vector and $u(i_1,\dots,i_n) \in \mathbb{R}^m$ is the nD input vector. In order to enhance readability we shall assume that $N = N_1 = \dots = N_n$. The results can be easily extended to different values of N.

The boundary conditions for (1) are assigned in $\mathcal{X} = \mathcal{X}_{-N}\bigcup\mathcal{X}_{-N+1}\bigcup\cdots\bigcup\mathcal{X}_0$ where

$$\mathcal{X}_t := \{(k_1,k_2,\dots,k_n) \in \mathbb{N}^n : \exists\, i \in \{1,\dots,n\} \text{ such that } k_i = t \text{ and } k_j \geq t\ \forall j \neq i\}.$$

Remark 2.1. *The proposed model of the nD system considered here is quite generic, in the sense that it comprises standard 2D and nD systems, such as the general 2D state-space model, 2D general models, Fornasini-Marchesini, Roesser and Atasi representations* [8, 16, 17, 19, 22, 25, 29]. *Thus, the results provided in the rest of the paper comprise* *previous results in the literature. This class of system is selected as it simplifies the development of necessary and sufficient stabilization conditions and provides an efficient solution based on linear programming (LP).*

Remark 2.2. *For simplicity,* $A_{00\dots0}$ *is assumed to be zero.*

For further development, we need to introduce the following useful notation which can be considered as a generalization of the Hurwitz product of two matrices.

Definition 2.1. *Let* $\{A_{p_1,p_2,\dots,p_n}\}_{0\le p_1,p_2,\dots,p_n\le N} = \{A_{100\dots0}, A_{010\dots0}, \dots, A_{NN\dots N}\}$ *be a set of matrices. Then we define*

$$S_{t_1,t_2,\dots,t_n}(A_{100\dots0}, A_{010\dots0}, \dots, A_{NN\dots N})$$

to be the sum of all matrix products

$$\prod_{0\le j_1,\dots,j_n\le N} A_{j_1,\dots,j_n}^{k_{j_1,\dots,j_n}} \tag{2}$$

such that

$$\sum_{0\le j_1,\dots,j_n\le N} k_{j_1,\dots,j_n}\cdot(j_1,\dots,j_n) = (t_1,t_2,\dots,t_n).$$

The matrix products in (2) represent all possible paths from the set of boundary conditions to the point $(t_1,t_2,\dots,t_n)$.

Now, making use of the above definition, the explicit formula of the trajectories of System (1), with $u = 0$, with respect to assigned boundary conditions in $\mathcal{X}$, is given by

$$\begin{aligned} x(i_1,\dots,i_n) = \sum_{(\ell_1,\ell_2,\dots,\ell_n)\in\mathcal{X}} & S_{(i_1-\ell_1,i_2-\ell_2,\dots,i_n-\ell_n)} \\ & (A_{100\dots0}, A_{010\dots0}, \dots, A_{NN\dots N})x(\ell_1,\ell_2,\dots,\ell_n). \end{aligned} \tag{3}$$

Next, we introduce the notion of positivity that we will use throughout the paper.

Definition 2.2. *System (1) with zero input* $(u = 0)$ *is said to be positive, if for any given nonnegative boundary conditions, the resulting state is always nonnegative, that is,* $x(i_1,\dots,i_n)\ge 0$ *for all* $i_1,\dots,i_n\in\mathbb{N}$.

The following result shows how one can check the positiveness of System (1). The conditions can be easily deduced from previous results in the literature, for example see [19].

Proposition 2.1. *System (1) with zero input* $(u = 0)$ *is positive if and only if* $A_{p_1\dots p_n}\ge 0$, $\forall 0\le p_1,\dots,p_n\le N$.

3. Stabilization of Positive nD Systems.

This section studies the stabilization problem of the class of linear homogeneous nD systems described by the generic model (1).

We aim to study the (asymptotic) stability of System (1) in the sense that, once the boundary conditions are assigned, then for every $\epsilon > 0$ there exists an integer $T\in\mathbb{N}$ such that

$$\|x(i_1,i_2,\dots,i_n)\|\le\epsilon \quad \forall\ \ (i_1,i_2,\dots,i_n)\in\mathcal{X}_h \text{ with } h\ge T.$$

As a consequence, in order for this definition to make sense, one necessarily needs to assume that the given boundary conditions are bounded, i.e., there exists K such that for $i = 1,2,\dots,n$ it must hold that

$$\|x_i(i_1,i_2,\dots,i_n)\|\le K$$

for all $(i_1, i_2, \ldots, i_n) \in \mathcal{X}_\ell$, $\ell = -N, -N+1, \ldots, 0$ where

$$x(i_1, i_2, \ldots, i_n) = [x_1(i_1, i_2, \ldots, i_n), x_2(i_1, i_2, \ldots, i_n), \ldots, x_n(i_1, i_2, \ldots, i_n)]^T.$$

This will be assumed throughout the rest of the manuscript.

Before we present a result on the stability of the nD systems studied here, we need to introduce an auxiliary simple result that will be used later.

Lemma 3.1. *Consider System (1) with $u = 0$; Assume that this system is positive (i.e., $A_{p_1\ldots p_n} \geq 0$, $\forall 0 \leq p_1, \ldots, p_n \leq N$) and $\sum_{p_1=0}^{N} \cdots \sum_{p_n=0}^{N} A_{p_1\ldots p_n} = (A_{100\ldots0} + A_{010\ldots0} + \cdots + A_{NN\ldots N})$ is Schur. Then,*

1) $\rho(A_{100\ldots0} + A_{010\ldots0} + \cdots + A_{NN\ldots N}) \geq \rho(A_{i_1,\ldots,i_n})$ for any $0 \leq i_1, \ldots, i_n \leq N$.
2) $\rho(A^k_{i_1,\ldots,i_n}) \geq \rho(A_{i_1,\ldots,i_n})$ for any $0 \leq i_1, \ldots, i_n \leq N$ and $k \in \mathbb{N}$.
3) $\rho((A_{100\ldots0} + A_{010\ldots0} + \cdots + A_{NN\ldots N})^k) \geq \rho(A_{100\ldots0} + A_{010\ldots0} + \cdots + A_{NN\ldots N})$ for any $0 \leq i_1, \ldots, i_n \leq N$ and $k \in \mathbb{N}$.

Proof: It is well known that every nonnegative matrix has a positive real eigenvalue whose modulus is greater than or equal to the modulus of any other eigenvalue. Suppose now that v_0 is the maximal eigenvector of $A_{100\ldots0}$ associated with λ_0. Then,

$$(A_{100\ldots0} + A_{010\ldots0} + \cdots + A_{NN\ldots N})v_0 = \lambda_0 v_0 + A_{010} v_0 + \cdots + A_{NN\ldots N} v_0 \geq \lambda_0 v_0$$

which implies that the maximal eigenvalue of $(A_{100\ldots0} + A_{010\ldots0} + \cdots + A_{NN\ldots N})$ is larger than the eigenvalue of $A_{100\ldots0}$. Obviously, this also holds for $A_{i_1,\ldots,i_n}$, which proves 1). Note that, in particular, this implies that each one of $A_{100\ldots0}, A_{010\ldots0}, \ldots, A_{NN\ldots N}$ is Schur. One can easily prove, using the same type of arguments, statements 2) and 3).

The following theorem characterizes the stability of the unforced System (1).

Theorem 3.1. *Consider System (1) with $u = 0$ and assume that the system is positive. If $\sum_{p_1=0}^{N} \cdots \sum_{p_n=0}^{N} A_{p_1\ldots p_n}$ is a Schur matrix, then System (1) is asymptotically stable.*

Proof: Let $\epsilon > 0$ be given. We need to show that there exists $T \in \mathbb{N}$ such that

$$\begin{aligned}\|x(i_1, \ldots, i_n)\| &= \| \sum_{(\ell_1,\ell_2,\ldots,\ell_n)\in\mathcal{X}} S_{(i_1-\ell_1, i_2-\ell_2, \ldots, i_n-\ell_n)}(A_{100\ldots0}, \ldots, A_{NN\ldots N}) x(\ell_1, \ell_2, \ldots, \ell_n)\| \\ &\leq \epsilon,\end{aligned}$$

for $(i_1, \ldots, i_n) \in \mathcal{X}_h$ with $h \geq T$. One can verify, by using Lemma 3.1, that $(A_{100\ldots0} + A_{010} \ldots 0 + \cdots + A_{NN\ldots N})$ is Schur, or equivalently, $(A_{100\ldots0} + A_{010} \ldots 0 + \cdots + A_{NN\ldots N})^s \to 0$ as $s \to \infty$, implies that if $(i_1, i_2, \ldots, i_n) \in \mathcal{X}_h$ then $S_{(i_1-\ell_1, i_2-\ell_2, \ldots, i_n-\ell_n)}(A_{100\ldots0}, \ldots, A_{NN\ldots N}) \to 0$ as $h \to \infty$ and therefore $\|x(i_1, \ldots, i_n)\|$, and, consequently, $x(i_1, \ldots, i_n)$, also goes to zero. This concludes the proof.

Remark 3.1. *It is easy to check that previous results on stability of $2n/n$D positive systems are particular cases of the previous theorem, see for instance* [29, Proposition 2], [21, Theorem 3]*,* [13, Lemma 3.2] *or* [27, Lemma 5]*. Hence, all the examples in these previous works perfectly fit in the proposed framework. We remark that existing conditions in the literature are shown to be efficiently checkable via LMI conditions. In contrast, we propose a different approach based on LP conditions which allow us to address the nontrivial problem of state-feedback stabilization with constraints on the controls and states.*

4. **State-Feedback Stabilization.** In this section, we investigate the existence of local state-feedback control laws of the form

$$u(i_1,\dots,i_n) = \mathbf{K}x\ \ (i_1,\dots,i_n), \tag{4}$$

such that the resulting nD closed-loop system is positive and asymptotically stable, where $\mathbf{K}$ is the controller gain to be determined. First, if one uses directly the results of Theorem 3.1 and Proposition 2.1, the following sufficient conditions for the nD closed-loop system to be positive and asymptotically stable are obtained:

$$\begin{cases} i)\ \text{For all } 0 \le p_1,\dots,p_n \le N \text{ the matrices } A_{p_1\dots p_n} + B_{p_1\dots p_n}\mathbf{K} \text{ are nonnegative.} \\ ii)\ \sum_{p_1=0}^{N}\cdots\sum_{p_n=0}^{N}(A_{p_1\dots p_n} + B_{p_1\dots p_n}\mathbf{K}) \text{ is a Schur matrix.} \end{cases} \tag{5}$$

In order to derive a constructive solution to this problem we shall make use of LP techniques. For this purpose, we now recall a classical result for 1-D systems that will be used in the sequel.

Proposition 4.1. [2] *Let M be a nonnegative matrix. Then, the following conditions are equivalent:*

1. *The 1-D system $x(k+1) = Mx(k)$ is asymptotically stable (or equivalently, $\rho(M) < 1$).*
2. *There exists a positive vector $\lambda > 0$ such that $(M - \mathbb{I})\lambda < 0$.*

The matrices $\mathbf{K}$ that provide positivity and asymptotic stability for the resulting closed-loop system are now characterized.

Theorem 4.1. *System (1), under the feedback law (4), is positive and asymptotically stable for any nonnegative boundary conditions if there exist vectors $d \in \mathbb{R}^q$ and $y_1,\dots,y_q \in R^m$ such that*

$$\begin{cases} d > 0 \\ \left(\sum_{p_1=0}^{N}\cdots\sum_{p_n=0}^{N} A_{p_1\dots p_n} - \mathbb{I}\right) d + \left(\sum_{p_1=0}^{N}\cdots\sum_{p_n=0}^{N} B_{p_1\dots p_n}\right)\sum_{i=0}^{q} y_i < 0 \\ a_{i,j}^{i_1,\dots,i_n} d_j + b_i^{i_1,\dots,i_n} y_j \ge 0,\ \ 0 \le i_1,\dots,i_n \le N,\ \ 1 \le i,j \le q, \end{cases} \tag{6}$$

with $d = [d_1\ \dots\ d_q]^T$, $A_{i_1,\dots,i_n} = [a_{i,j}^{i_1,\dots,i_n}]$ and $B_{i_1,\dots,i_n}^T = [(b_1^{i_1,\dots,i_n})^T \dots\ (b_q^{i_1,\dots,i_n})^T]$. Moreover, a stabilizing gain matrix $\boldsymbol{K}$ is given by

$$\boldsymbol{K} = [d_1^{-1}y_1\ \dots\ d_q^{-1}y_q]. \tag{7}$$

Proof: Define the appropriate matrix $\mathbf{K} = [k_1\ \dots\ k_q]$ with columns $k_i = d_i^{-1}y_i$, for $i = 1,\dots,q$. It holds that all the $A_{p_1\dots p_n} + B_{p_1\dots p_n}\mathbf{K}$ matrices are nonnegative since, from the last set of inequalities in condition (6), we have that, for $i,j = 1,\dots,q$, it is fulfilled that

$$0 \le (a_{i,j}^{i_1,\dots,i_n} d_j + b_i^{i_1,\dots,i_n} y_j)d_j^{-1} = a_{i,j}^{i_1,\dots,i_n} + b_i^{i_1,\dots,i_n} k_j = [A_{i_1,\dots,i_n} + B_{i_1,\dots,i_n}\mathbf{K}]_{i,j},$$

i.e., the closed-loop system is positive. Next, we show the stability of the closed-loop system. Using the defined gain $\mathbf{K}$, we obtain by simple calculations that

$$\sum_{p_1=0}^{N}\cdots\sum_{p_n=0}^{N} B_{p_1\dots p_n}\mathbf{K}d = \sum_{p_1=0}^{N}\cdots\sum_{p_n=0}^{N} B_{p_1\dots p_n}\left(\sum_{i=1}^{q} y_i\right),$$

which, substituted in the second inequality of condition (6), leads to

$$\left(\sum_{p_1=0}^{N}\cdots\sum_{p_n=0}^{N}[A_{p_1\dots p_n} + B_{p_1\dots p_n}\mathbf{K}] - \mathbb{I}\right) d < 0.$$

As $d > 0$ (first inequality in (6)), one can apply Proposition 4.1 to concluding that the matrix $\sum_{p_1=0}^{N} \cdots \sum_{p_n=0}^{N}[A_{p_1 \cdots p_n} + B_{p_1 \dots p_n}\mathbf{K}]$ is Schur, which, by Theorem 3.1, amounts to saying that the closed-loop system is asymptotically stable.

Remark 4.1. *We emphasize that the LP formulation proposed in Theorem 4.1 does not impose any restriction on the dynamics of the governed system. In fact, any matrix $A_{p_1 \dots p_n}$ may have negative components, or equivalently, the free system may not be positive. In this case, the proposed synthesis methodology can be viewed as enforcing a nonnegative system to be positive (This is the controlled positivity problem studied for other systems in* [4, 9]*).*

Remark 4.2. *We must point out that it is possible to obtain an alternative formulation in terms of LMIs by using diagonal Lyapunov matrices. However, we have selected a Linear Programming formulation, as it simplifies adding bounds on the controls or states, as is now shown.*

5. **Controlled Positivity under Bounded Controls and States.** Results are now provided to solve the stabilization problem under bounded controls and nonnegative states. More precisely, the stabilizing controller is specifically designed to respect non-symmetric bounds on the controls. That is, Theorem 4.1 is now extended to include the presence of nonsymmetric bounds on controls.

Thus, consider the following general set of nD systems with control constraints:

$$\begin{cases} x(i_1 + N_1, \dots, i_n + N_n) = \\ \sum_{p_1=0}^{N_1} \cdots \sum_{p_n=0}^{N_n} (A_{p_1 \dots p_n} x(i_1 + p_1, \dots, i_n + p_n) + B_{p_1 \dots p_n} u(i_1 + p_1, \dots, i_n + p_n)), \\ -\underline{u} \le u(i_1, \dots, i_n) \le \bar{u} \end{cases} \tag{8}$$

The problem that is solved now is the following: determine a bounded state-feedback control law $u(i_1, \dots, i_n) = \mathbf{K}x\ (i_1, \dots, i_n)$, such that the resulting closed-loop system is positive and asymptotically stable, together with a bound $\bar{x}$ on the the set of boundary conditions, to ensure that $-\underline{u} \le u(i_1, \dots, i_n) \le \bar{u}$, with $\underline{u} > 0$ and $\bar{u} > 0$ given, as long as the boundary conditions fulfill $0 \le x(\ell_1, \ell_2, \dots, \ell_n) \le \bar{x}$, for $(\ell_1, \ell_2, \dots, \ell_n) \in \mathcal{X}$.

First, we need to introduce an auxiliary result.

Lemma 5.1. *Consider System (1) with $u = 0$ and assume that the system is positive and there exists $\overline{x}$ such that $((A_{100\dots0} + A_{010\dots0} + \cdots + A_{NN\dots N}) - \mathbb{I})\overline{x} \le 0$. Then,*

$$[0 \le x(\ell_1, \ell_2, \dots, \ell_n) \le \overline{x}, (\ell_1, \dots, \ell_n) \in \mathcal{X}] \Rightarrow [0 \le x(i_1, i_2, \dots, i_n) \le \overline{x}, \\ \forall (i_1, i_2, \dots, i_n) \in \mathbb{N}^n]$$

Proof: Obviously, if $(A_{100\dots0} + A_{010\dots0} + \cdots + A_{NN\dots N})\overline{x} \le \overline{x}$, then $(A_{100\dots0} + A_{010\dots0} + \cdots + A_{NN\dots N})^k \overline{x} \le \overline{x}$, for all $k \in \mathbb{N}$. Hence, it can be checked by contradiction that $\sum_{(\ell_1, \ell_2, \dots, \ell_n) \in \mathcal{X}} (S_{(i_1-\ell_1, i_2-\ell_2, \dots, i_n-\ell_n)}(A_{100\dots0}, A_{010\dots0}, \dots, A_{NN\dots N}))\overline{x} \le \overline{x}$ for all $(i_1, \dots, i_n) \in \mathbb{N}$. Therefore,

$$\begin{aligned} x(i_1, \dots, i_n) &= \sum_{(\ell_1, \ell_2, \dots, \ell_n) \in \mathcal{X}} S_{(i_1-\ell_1, i_2-\ell_2, \dots, i_n-\ell_n)}(A_{100\dots0}, A_{010\dots0}, \dots, A_{NN\dots N}) x(\ell_1, \ell_2, \dots, \ell_n). \\ &\le \sum_{(\ell_1, \ell_2, \dots, \ell_n) \in \mathcal{X}} S_{(i_1-\ell_1, i_2-\ell_2, \dots, i_n-\ell_n)}(A_{100\dots0}, A_{010\dots0}, \dots, A_{NN\dots N})\overline{x} \le \overline{x} \end{aligned}$$

for all $(i_1, \dots, i_n) \in \mathbb{N}$.

Theorem 5.1. *Consider the following LP problem in the variables $\bar{x} = [\bar{x_1}\ \dots\ \bar{x_q}]^T \in \mathbb{R}^q$, $y_1, \dots, y_q \in \mathbb{R}^m$ and $z_1, \dots, z_q \in \mathbb{R}^m$:*

$$\begin{cases} \bar{x} > 0, \\ y_i \geq 0 \ \mathit{for}\ i = 1, \ldots, q, \\ z_i \geq 0 \ \mathit{for}\ i = 1, \ldots, q, \\ \left(\sum_{p_1=0}^{N} \cdots \sum_{p_n=0}^{N} A_{p_1 \ldots p_n} - \mathbb{I}_n\right) \bar{x} + \left(\sum_{p_1=0}^{N} \cdots \sum_{p_n=0}^{N} B_{p_1 \ldots p_n}\right) \sum_{i=0}^{q} (y_i - z_i) < 0, \\ \sum_{i=0}^{q} y_i \leq \bar{u}, \\ \sum_{i=0}^{q} z_i \leq \underline{u}, \\ a_{i,j}^{i_1, \ldots, i_n} \bar{x}_j + b_i^{i_1, \ldots, i_n} (y_j - z_j) \geq 0 \ \mathit{for}\ 0 \leq i_1, \ldots, i_n \leq N, \ 1 \leq i, j \leq q, \end{cases} \tag{9}$$

with $A_{i_1, \ldots, i_n} = [a_{i,j}^{i_1, \ldots, i_n}]$ *and* $B_{i_1, \ldots, i_n}^T = [(b_1^{i_1, \ldots, i_n})^T \ldots \ (b_q^{i_1, \ldots, i_n})^T]$*. Then, system (1) under the state-feedback control* $u(i_1, \ldots, i_n) = \boldsymbol{K} x(i_1, \ldots, i_n)$ *is positive and asymptotically stable, when*

$$\boldsymbol{K} = [\bar{x}_1^{-1}(y_1 - z_1) \ \cdots \ \bar{x}_n^{-1}(y_n - z_n)].$$

Moreover, it holds that $-\underline{u} \leq u(i_1, \ldots, i_n) \leq \bar{u}$ *for any boundary condition satisfying* $0 \leq x(\ell_1, \ell_2, \ldots, \ell_n) \leq \bar{x}$, $(\ell_1, \ell_2, \ldots, \ell_n) \in \mathcal{X}$.

Proof: Select any $\bar{x} = [\bar{x}_1 \ \cdots \ \bar{x}_n]^T$, $y_1, \ldots, y_n$ and $z_1, \ldots, z_n$ that solve (9) and define $\mathbf{K}_1 = [\bar{x}_1^{-1} y_1 \cdots \ \bar{x}_n^{-1} y_n]$ and $\mathbf{K}_2 = [\bar{x}_1^{-1} z_1 \cdots \ \bar{x}_n^{-1} z_n]$. As $\bar{x} > 0$, $a_{i,j}^{i_1, \ldots, i_n} \bar{x}_j + b_i^{i_1, \ldots, i_n}(y_i - z_i) \geq 0$, if and only if $a_{i,j}^{i_1, \ldots, i_n} + b_i^{i_1, \ldots, i_n} \bar{x}_j^{-1}(y_i - z_i) \geq 0$, we obtain that the matrix $A_{i_1, \ldots, i_n} + B_{i_1, \ldots, i_n} \mathbf{K}$ is nonnegative, with $\mathbf{K} = \mathbf{K}_1 - \mathbf{K}_2$. The inequality

$$\left(\sum_{p_1=0}^{N} \cdots \sum_{p_n=0}^{N} A_{p_1 \ldots p_n} - \mathbb{I}_n\right) \bar{x} + \left(\sum_{p_1=0}^{N} \cdots \sum_{p_n=0}^{N} B_{p_1 \ldots p_n}\right) \sum_{i=0}^{q} (y_i - z_i) < 0$$

is equivalent to

$$\left(\sum_{p_1=0}^{N} \cdots \sum_{p_n=0}^{N} (A_{p_1 \ldots p_n} + B_{p_1 \ldots p_n} \mathbf{K})\right) \bar{x} < 0.$$

Since $\bar{x} > 0$ and $\left(\sum_{p_1=0}^{N} \cdots \sum_{p_n=0}^{N} (A_{p_1 \ldots p_n} + B_{p_1 \ldots p_n} \mathbf{K})\right) \geq 0$ are positive, then by using Proposition 4.1, we can conclude that the matrix $\left(\sum_{p_1=0}^{N} \cdots \sum_{p_n=0}^{N} (A_{p_1 \ldots p_n} + B_{p_1 \ldots p_n} \mathbf{K})\right)$ is also Schur.

Furthermore, by Lemma 5.1, the trajectory of the closed-loop system is such that $0 \leq x(i_1, i_2, \ldots, i_n) \leq \bar{x}$ for any boundary condition satisfying $0 \leq x(\ell_1, \ell_2, \ldots, \ell_n) \leq \bar{x}$, $(\ell_1, \ell_2, \ldots, \ell_n) \in \mathcal{X}$. Using this fact and recalling that $y_i \geq 0$ and $z_i \geq 0$, or equivalently, $\mathbf{K}_1 \geq 0, \mathbf{K}_2 \geq 0$ and $\sum_{i=0}^{q} y_i \leq \bar{u}$, $\sum_{i=0}^{q} z_i \leq \underline{u}$, it is easy to see that the state-feedback control $u(i_1, \ldots, i_n) = \mathbf{K} x(i_1, \ldots, i_n)$ fulfills $-\underline{u} \leq u(i_1, \ldots, i_n) \leq \bar{u}$, for any boundary condition satisfying $0 \leq x(\ell_1, \ell_2, \ldots, \ell_n) \leq \bar{x}$.

6. **Numerical Example.** In order to illustrate the proposed design methodology for controlled positivity, we deal with an nD system described by (1) with $N = 2$, $n = 3$ and the following matrices:

$$A_{021} = \begin{bmatrix} -1.5 & 0.1 & 0.2 \\ 0.2 & 0.5 & 0.3 \\ 0 & 0 & 0.1 \end{bmatrix}, \ A_{102} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0.2 \end{bmatrix}, \ A_{110} = \begin{bmatrix} 0.1 & 0.02 & 0 \\ 0 & 0.1 & 0.08 \\ 0 & 0.05 & 0.1 \end{bmatrix},$$

$$B_{021} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ B_{102} = \begin{bmatrix} 0 \\ 0 \\ 0.5 \end{bmatrix}, \ B_{110} = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix}.$$

Of course, as the matrix A_{021} has a negative component, the free system (i.e., when $u = 0$) is not positive. So, its stabilization with positivity via state-feedback is a challenging problem, specially in the presence of bounded controls, e.g., $-1 \leq u(i,j) \leq 1000$.

The objective is, therefore, to design a state-feedback controller that stabilizes the system and enforces it to be positive in the presence of bounded controls ($\underline{u} = -1$ and $\overline{u} = 1000$). For this purpose, it suffices to use the results of Theorem 5.1. Thus, it is only necessary to find a feasible solution of the inequalities (9), such as the following:

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ y_1 \\ y_2 \\ y_3 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 666.666 \\ 1 \\ 1 \\ 1000 \\ 0 \\ 0 \\ 0 \\ 0.1 \\ 0.15 \end{bmatrix}.$$

For this numerical solution, the following stabilizing controller provides the desired positivity to the closed-loop system under bounded control:

$$\mathbf{K} = \begin{bmatrix} 1.5 & -0.1 & -0.15 \end{bmatrix}.$$

The corresponding system matrices in closed-loop are

$$A_{021} + B_{021}\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 \\ 1.7 & 0.4 & 0.15 \\ 0 & 0 & 0.1 \end{bmatrix},$$

$$A_{102} + B_{102}\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1.75 & 0.95 & 0.125 \end{bmatrix},$$

$$A_{110} + B_{110}\mathbf{K} = \begin{bmatrix} 0.1 & 0.02 & 0 \\ 0.75 & 0.05 & 0.005 \\ 0.75 & 0 & 0.025 \end{bmatrix}.$$

Hence, it suffices to look at the entries of the matrices $A_{021} + B_{021}\mathbf{K}$ and $A_{102} + B_{102}\mathbf{K}$, $A_{110} + B_{110}\mathbf{K}$ to confirm that the closed-loop system is positive (according to Proposition 2.1). In addition, according to Theorem 3.1, the closed-loop system is asymptotically stable (it can be checked that the matrix $A_{021} + B_{021}\mathbf{K} + A_{102} + B_{102}\mathbf{K} + A_{110} + B_{110}\mathbf{K}$ has all the eigenvalues inside the unit circle). Moreover, all the states of the feedback system fulfill

$$0 \leq x(i,j) \leq \begin{bmatrix} 666.666 \\ 1 \\ 1 \end{bmatrix},$$

as long as the boundary states also fulfill these bounds.

7. **Conclusions.** This paper has provided a sound and practical approach for the synthesis of stabilizing state-feedback controllers for nD systems described by a general model under the requirement of positivity of the closed-loop system. For this, conditions for the solvability of the controlled positivity problem have been proposed, including the presence of bounds (maybe nonsymmetric) on the controls. These conditions are expressed in terms of Linear Programs. An illustrative numerical example has been looked at.

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