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ESTIMATING DELAY BOUNDS OF TIME-DELAY SYSTEMS WITH FUZZY MODEL-BASED APPROACH

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Abstract. *This paper aims at estimating the delay bounds of linear systems with unknown state delay via a novel fuzzy model-based approach. Both robust stability analysis and stabilization problems are investigated with the estimation. Fuzzy techniques are utilized novelly by setting the time-varying delay to be distributed on the fuzzy set, and a novel fuzzy model is proposed to describe the linear time-delay system, where the state with time-varying delay is represented by an interpolation of states with some local constant delays. Based on the model, a new approach incorporating with the developed fuzzy discretized Lyapunov functional and slack matrix techniques is presented, and new stability criteria of time-delay systems are obtained. Further, the controller design method is developed to guarantee the asymptotic stability of the closed-loop system. The obtained stabilization conditions developed by the proposed fuzzy model-based approach are much less conservative than those obtained by the traditional methods in some literature. The results are presented in the form of linear matrix inequality, which can easily be solved by standard software packages. Simulation examples are provided to show the less conservativeness and effectiveness of the obtained results.*
Keywords: Time-delay systems, Fuzzy model, Robust stabilization, Fuzzy discretized method

1. Introduction.

After-effect phenomena often appear in various engineering, communication and chemical processes. In a control system, time delays often degrade the system's performance and even cause instability; therefore they have been regarded as an important issue in control community [31, 32, 33]. Over the past years, many researchers have paid great attention to time-delay systems and various problems have been investigated. For instance, stability analysis is carried out in [10, 37]; stabilizing and H_∞ control are addressed in [36]; model reduction is presented in [38] and filtering problems are investigated in [9]. Various approaches have been proposed for the analysis and synthesis of time-delay systems based on the theories of Lyapunov-Krasovskii method [21, 27], Lyapunov-Razumikhin method, etc. Both delay-dependent [24] and delay-independent approaches are proposed while the former is more preferred by the researchers. Much attention has been academically devoted to reducing the conservatism by developing new Lyapunov functional methods [29].

Recently in the literature, delay partitioning ideas are involved in the construction of Lyapunov functionals to obtain less conservative stability criteria for time-delay systems. The topology separation approach is proposed in [18], and is extended to stability analysis of general linear systems in [11]. Motivated by that, the delay term of the Lyapunov functional is artificially partitioned into several parts in [6] to obtain less conservative stability conditions. Later the approach is further extended to stability analysis and stabilization of time-delay fuzzy systems [39]. Besides, an alternative approach is the so-called "discretized" method [7] with the kernel of the Lyapunov functional being piecewise linear. This method can lead to much less conservative results. More recently, this discretized method has been extended straightforwardly to the stabilization of fuzzy systems with time-varying delay in [22]. It is noted that the delays concerned are mostly constant in the above mentioned literature.

In the last decades, growing interests have been seen towards fuzzy control of complex nonlinear systems. In particular, Takagi-Sugeno (T-S) fuzzy model based control has drawn great attention [26, 34]. It has been proved that T-S fuzzy models can approximate any smooth nonlinear systems to any accuracy on a compact set, which is realized through smoothly connecting a family of local linear models by fuzzy membership functions. This "blending" makes T-S fuzzy models in appearances of linear systems, and the stability analysis and synthesis can be derived by making full use of the fruitful results on linear systems. So far, a great number of results have been reported for T-S fuzzy systems. To mention a few, the problem of stability analysis is investigated in [12]; stabilizing and H_∞ control are reported in [5, 15, 40]; reliable control strategies are presented in [28] and sampled-data control is considered in [4]. On the other hand, complex delay phenomena often appear in complex systems such as biology and synthetic biology systems. Some information of the delay cannot be obtained such as the derivative of the time-varying delay. In this case, an alternative approach which does not need the detailed information of the time delay may be more interesting.

Inspired by the above observations, in this paper, we investigate the problem of robust stability and stabilization of linear systems with unknown state delay. Fuzzy techniques are utilized novelly by setting the time-varying delay to be distributed on the fuzzy set, and a novel fuzzy model is proposed to describe the linear time-delay system, where the state with time-varying delay is represented by an interpolation of states with some local constant delays. Based on the model, a new approach incorporating with the developed fuzzy discretized Lyapunov functional and slack matrix techniques is proposed, and new stability criteria of time-delay systems are obtained. Further, the controller design method is developed to guarantee the asymptotic stability of the closed-loop system. The obtained stabilization condition developed by the proposed fuzzy model-based approach is much less conservative than those by the traditional methods in some literature.

The contribution of this research is that the discretized idea is extended to the system with unknown state delays via the fuzzy techniques, which has not been mentioned in the existing literature to the best of the author's knowledge, and the obtained stabilization results are less conservative than the existing ones especially for the small delay case. Moreover, because the method proposed in this paper does not need the detailed information of the delay, such as the derivative of the time-varying delay, the results are more suitable for those complex systems with time delay. For example, some constant delays can be measured in the kiln system, but what form of the delay is cannot be determined. In this case, the model can be established as the fuzzy model proposed in this paper and using the method proposed therein.

The remainder of the paper is organized as follows. Section 2 formulates the problem under consideration. The robust stability analysis and stabilization results are presented in Section 3. Illustrative examples are given in Section 4 and some concluding remarks are given in Section 5.

Notation: The superscript "T" stands for matrix transposition; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; the notation $P > 0$ (≥ 0) means that P is positive definite (semi-definite). In symmetric block matrices or complex matrix expressions, we use an asterisk ($*$) to represent a term that is induced by symmetry and diag$\{\ldots\}$ stands for a block-diagonal matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. **Problem Formulation.**

In this section, the problem will be presented mathematically, and a model transformation will be given.

2.1. **Plant model.**

Consider a linear system with a time-varying state delay which can be described by the following linear model:

$$\begin{cases} \dot{x}(t) = (A + \Delta A(t))\, x(t) + (A_d + \Delta A_d(t))\, x(t - \tau(t)) + Bu(t), \quad t > 0, \\ x(t) = \varphi(t), \quad t \in [-\bar{\tau}, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $\varphi(t)$ is the initial condition; $\tau(t)$ is the time-varying delay satisfying $0 \leq \tau_0 \leq \tau(t) \leq \bar{\tau}$ and $\bar{\tau}$ is a real constant number; A and A_d are known constant matrices with appropriate dimensions and $\varphi(t)$ denotes a continuous vector-valued initial function of $t \in [-\bar{\tau}, 0]$. $\Delta A(t)$ and $\Delta A_d(t)$ denote the uncertainties in the system and they are of the form [20]

$$\Delta A(t) = D_a F(t) E_a, \quad \Delta A_d(t) = D_d F(t) E_d, \tag{2}$$

where D_a, D_d, E_a and E_d are known constant matrices and $F(t)$ is an unknown real time-varying matrix with Lebesgue measurable elements bounded by

$$F^T(t) F(t) \leq I. \tag{3}$$

2.2. **Model transformation.**

The fuzzy model rules for the system in (1) is represented as follows:

Fuzzy Rule i: IF $\tau(t)$ is about τ_i, THEN

$$\begin{cases} \dot{x}(t) = (A + \Delta A(t)) x(t) + (A_d + \Delta A_d(t)) x(t - \tau_i) + Bu(t), \quad i = 0, 1, \ldots, r \\ x(t) = \varphi(t), \quad t \in [-\bar{\tau}, 0], \end{cases} \tag{4}$$

The compact form of the system model in (4) can be further obtained by the fuzzy inference engine as follows:

$$\begin{aligned} &\dot{x}(t) = (A + \Delta A(t))\, x(t) + (A_d + \Delta A_d(t)) \sum_{i=0}^{r} h_i(\tau(t))\, x(t - \tau_i) + Bu(t), \\ &x(t) = \varphi(t), \quad t \in [-\bar{\tau}, 0]. \end{aligned} \tag{5}$$

Notice that the delay term $x(t - \tau(t))$ is represented by the fuzzy interpolation of some constant delay terms $x(t - \tau_i)$ with the constant delays τ_i, for $i = 1, \ldots, m - 1$ where m is an integer, distributed equally in the range $[\tau_0, \bar{\tau}]$. The approximation error is

$$\left| x(t - \tau(t)) - \sum_{i=0}^{r} h_i(\tau(t))\, x(t - \tau_i) \right| < \sum_{i=0}^{r} h_i(\tau(t)) \left| x(t - \tau(t)) - x(t - \tau_i) \right|,$$

and since the system in (1) is continuous, we know that the approximation error is as small as possible only if the interval between $\tau(t)$ and τ_i is as small as possible. Note that we have slightly abused the notation by using $h_i(\tau(t))$ with $i = 0$ to denote the membership function distributing around the point τ_0 for clarity.

2.3. **Controller.** The state-feedback control strategy is utilized as follows:

$$u(t) = Kx(t), \tag{6}$$

where $x(t) \in \mathbb{R}^n$ is the input of the controller; $u(t) \in \mathbb{R}^m$ is the output of the controller; K is the gain matrix of the state-feedback controller.

Therefore, we can obtain the closed-loop system as follows:

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A(t) + BK)\, x(t) + (A_d + \Delta A_d(t)) \sum_{i=0}^{r} h_i(\tau(t))\, x(t - \tau_i), \\ x(t) &= \varphi(t), \quad t \in [-\bar{\tau}, 0]. \end{aligned} \tag{7}$$

3. **Main Results.** In this section, we will give our main results. Asymptotical stability conditions of system (5) will be given first, based on which a state-feedback controller will be designed such that the system in (1) can be stabilized asymptotically. The proposed discretized fuzzy Lyapunov functional approach will be utilized. Before proceeding further, we first give the following lemmas needed for the subsequent derivations.

Lemma 3.1. [8] *For any constant matrix $M \in \mathbb{R}^{m\times m}$, $M = M^T > 0$, scalar $\gamma > 0$, vector function $\omega : [0,\ \gamma] \to \mathbb{R}^m$ such that the integrations concerned are well defined, then*

$$\gamma \int_0^{\gamma} \omega^T(\beta)\, M\omega(\beta)\, d\beta \geq \left(\int_0^{\gamma} \omega(\beta)\, d\beta \right)^T M \left(\int_0^{\gamma} \omega(\beta)\, d\beta \right).$$

Lemma 3.2. *For any constant matrix $M \in \mathbb{R}^{m\times m}$, $M = M^T > 0$, scalar $\tau > 0$, vector function $\omega : [-\tau_{i+1},\ -\tau_{i-1}] \to \mathbb{R}^m$ such that the integrations concerned are well defined, then*

$$\begin{aligned} &\tau \int_{-\tau_{i+1}}^{-\tau_i} \omega^T(\beta)\, M\omega(\beta)\, d\beta + \tau \int_{-\tau_i}^{-\tau_{i-1}} \omega^T(\beta)\, M\omega(\beta)\, d\beta \\ &\geq \left(\int_{-\tau_i}^{-\tau_{i-1}} \omega(\beta)\, d\beta + \int_{-\tau_{i+1}}^{-\tau_i} \omega(\beta)\, d\beta \right)^T M \left(\int_{-\tau_i}^{-\tau_{i-1}} \omega(\beta)\, d\beta + \int_{-\tau_{i+1}}^{-\tau_i} \omega(\beta)\, d\beta \right) \end{aligned} \tag{8}$$

Proof: From Lemma 3.1, (8) can be obtained straightforwardly. □

Lemma 3.3. [35] *Given matrices $\Phi = \Phi^T$, D, E and $R = R^T > 0$ of appropriate dimensions,*

$$\Phi + DFE + E^T F^T D^T < 0,$$

for all F satisfying $F^T F \leq R$, if and only if there exists a scalar $\varepsilon > 0$ such that

$$\Phi + \varepsilon DD^T + \varepsilon^{-1} E^T R E < 0.$$

3.1. **Stability analysis.** In this subsection, stability of the unforced system of (5) will be analyzed with the help of a fuzzy discretized Lyapunov functional approach and slack matrix techniques. The following theorem shows the criteria guaranteeing the asymptotical stability of the closed-loop system.

Theorem 3.1. *The system with interval time-varying delay in (5) is asymptotically stable if there exist matrices $R_{ij} = R_{ji}^T$, Q_i, M, $S_i > 0$, scalars $\varepsilon_{1,2} > 0$, for i, $j = 0, \ldots, r$,*

$P > 0$, $Z_i > 0$, $G_i > 0$, for $i = 1, \ldots, r$, where $S_i > \frac{1}{\tau_{i-1}} G_i$, $Z_i < G_i$ and $S_{i-1} < S_i$, such that the following inequalities are satisfied:

$$\begin{bmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \tilde{R} + \tilde{S} \end{bmatrix} > 0, \tag{9}$$

$$\begin{bmatrix} \Omega & N_1 & \cdots & N_r & W_x^T E_a^T & \bar{W}_{H_1}^T E_d^T \\ * & -\tau_0^{-1} Z_1 & \cdots & 0 & 0 & 0 \\ * & * & \ddots & \vdots & \vdots & \vdots \\ * & * & * & -\tau_{r-1}^{-1} Z_r & 0 & 0 \\ * & * & * & * & -\varepsilon_1 I & 0 \\ * & * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \quad i = 0, 1, \ldots, r, \tag{10}$$

where

$$\tilde{R} = \begin{bmatrix} R_{00} & R_{01} & \ldots & R_{0r} \\ R_{10} & R_{11} & \ldots & R_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ R_{r0} & R_{r1} & \cdots & R_{rr} \end{bmatrix}, \quad \tilde{S} = \begin{bmatrix} \frac{S_0}{\tau} & 0 & 0 & 0 \\ 0 & \frac{S_1}{\tau} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \frac{S_r}{\tau} \end{bmatrix}, \quad \tilde{Q} = [Q_0, Q_1, \ldots Q_r],$$

$$\begin{aligned} \Omega = {} & \hat{W}_x^T P W_x + W_x^T P \hat{W}_x + 2\hat{W}_x^T \bar{Q}_1 W_{f_1} + 2\hat{W}_x^T \bar{Q}_2 W_{f_2} + W_x^T (S_1 + S_2 + \ldots + S_r) W_x \\ & + 2W_{H_1}^T \bar{S}_1 W_{H_1} - 2W_x^T \bar{S}_2 W_{H_1} + 2(N_1 + N_2 + \ldots + N_r) W_x - W_{H_1}^T \bar{S}_3 W_{H_1} \\ & - 2\begin{bmatrix} N_1 & N_2 & N_3 & \ldots & N_r & 0 \end{bmatrix} W_{H_1} + W_{H_2}^T \bar{S}_4 W_{H_2} + W_{f_1}^T \bar{S}_5 W_{f_1} - 2M\hat{W}_x \\ & + 2M(A + BK) W_x + 2W_{f_2}^T \bar{R}_1 W_{H_1} - 2W_{f_2}^T \bar{R}_1 W_{H_2} - 2W_{f_2}^T \bar{R}_2 W_{H_2} + 2W_{f_1}^T \bar{R}_3 W_{H_1} \\ & - 2W_{f_1}^T \bar{R}_4 W_{H_2} - 2W_{f_2}^T \bar{R}_5 W_{H_1} + 2W_{f_2}^T \bar{R}_6 W_{H_2} + 2W_{f_2}^T \bar{R}_7 W_{H_2} + 2W_{f_2}^T \bar{R}_5 W_{H_2} \\ & - 2W_{f_1}^T \bar{R}_9 W_{H_1} + 2W_{f_1}^T \bar{R}_9 W_{H_2} - 2W_{f_2}^T \bar{R}_8 W_{H_2} + 2W_{f_2}^T \bar{R}_9 W_{H_1} - 2W_{f_2}^T \bar{R}_9 W_{H_2} \\ & + 2\frac{1}{\tau} W_{f_1}^T \mathcal{R}_4 W_{f_1} - 2\frac{1}{\tau} W_{f_2}^T \mathcal{R}_4 W_{f_1} + \frac{1}{\tau} W_{f_2}^T \mathcal{R}_3 W_{f_1} + \frac{1}{\tau} W_{f_1}^T \mathcal{R}_3 W_{f_2} - \frac{1}{\tau} W_{f_1}^T \mathcal{R}_3 W_{f_1} \\ & + \frac{1}{\tau} W_{f_1}^T \mathcal{R}_2 W_{f_2} + \frac{1}{\tau} W_{f_2}^T \mathcal{R}_2 W_{f_1} - \frac{1}{\tau} W_{f_1}^T \mathcal{R}_2 W_{f_1} + 2MA_d \begin{bmatrix} \ldots & I_{n(i)} & \ldots \end{bmatrix} W_{H_1} \\ & + \varepsilon_1 (MD_a)(MD_a)^T + \varepsilon_2 (MD_d)(MD_d)^T - 2W_x^T \frac{1}{\tau} \bar{Q}_2 W_{f_1} + 2W_x^T \bar{Q}_3 W_{H_1} \\ & + 2W_{f_1}^T \bar{R}_8 W_{H_2} - 2W_{f_2}^T \bar{R}_3 W_{H_1} - 2\frac{1}{\tau} W_{f_2}^T \mathcal{R}_1 W_{f_1} + 2W_x^T \bar{Q}_4 W_{H_2}, \end{aligned}$$

$$\bar{Q}_1 = \begin{bmatrix} Q_0 & Q_1 & \ldots & Q_{r-1} \end{bmatrix}, \quad \bar{Q}_2 = \begin{bmatrix} Q_1 - Q_0 & Q_2 - Q_1 & \ldots & Q_r - Q_{r-1} \end{bmatrix},$$

$$\bar{Q}_3 = \begin{bmatrix} Q_1 & Q_2 - Q_0 & Q_3 - Q_1 & \ldots & Q_r - Q_{r-2} & -Q_{r-1} \end{bmatrix},$$

$$\bar{Q}_4 = \begin{bmatrix} Q_0 - Q_1 & Q_0 - Q_2 & Q_1 - Q_3 & Q_{r-2} - Q_r & Q_{r-1} - Q_r \end{bmatrix},$$

$$\bar{S}_1 = \mathrm{diag}\{S_1, S_2, \ldots, S_r, 0\}, \quad \bar{S}_2 = \begin{bmatrix} S_1 & S_2 & S_3 & \ldots & S_r & 0 \end{bmatrix},$$

$$\bar{S}_4 = \mathrm{diag}\{S_0 - S_1, S_0 - S_2, S_1 - S_3, \ldots, S_{r-2} - S_r, S_{r-1} - S_r\},$$

$$\bar{S}_5 = \mathrm{diag}\left\{\frac{1}{\tau}(S_0 - S_1), \frac{1}{\tau}(S_1 - S_2), \ldots, \frac{1}{\tau}(S_{r-1} - S_r)\right\},$$

$$\bar{R}_1 = \begin{bmatrix} \mathcal{R}_1 & 0 \end{bmatrix}, \quad \bar{R}_2 = \begin{bmatrix} 0 & \mathcal{R}_1 \end{bmatrix}, \quad \bar{R}_3 = \begin{bmatrix} \mathcal{R}_2 & 0 \end{bmatrix}, \quad \bar{S}_3 = \mathrm{diag}\{0, S_0, S_1, \ldots, S_{r-1}\},$$

$$\bar{R}_4 = \begin{bmatrix} R_{01} & \cdots & 2R_{0r} & R_{0r} \\ \vdots & \ddots & \vdots & \vdots \\ R_{(r-1)1} & \cdots & 2R_{(r-1)r} & R_{(r-1)r} \end{bmatrix}, \quad \bar{R}_7 = \begin{bmatrix} \mathcal{R}_3 & 0 \end{bmatrix}, \quad \bar{R}_8 = \begin{bmatrix} \mathcal{R}_4 & 0 \end{bmatrix},$$

$$\bar{R}_5 = \begin{bmatrix} 0 & \mathcal{R}_3 \end{bmatrix},\ \bar{R}_6 = \begin{bmatrix} R_{01} & \cdots & 2R_{0r} & R_{0r} \\ \vdots & \ddots & \vdots & \vdots \\ R_{(r-1)1} & \cdots & 2R_{(r-1)r} & R_{(r-1)r} \end{bmatrix},$$

$$\bar{R}_9 = \begin{bmatrix} 0 & \mathcal{R}_4 \end{bmatrix},\ \mathcal{R}_1 = \begin{bmatrix} R_{11} & \cdots & R_{1r} \\ \vdots & \ddots & \vdots \\ R_{r1} & \cdots & R_{rr} \end{bmatrix},\ \mathcal{R}_2 = \begin{bmatrix} R_{01} & \cdots & R_{0r} \\ \vdots & \ddots & \vdots \\ R_{(r-1)1} & \cdots & R_{(r-1)r} \end{bmatrix},$$

$$\mathcal{R}_3 = \begin{bmatrix} R_{10} & \cdots & R_{1(r-1)} \\ \vdots & \ddots & \vdots \\ R_{r0} & \cdots & R_{r(r-1)} \end{bmatrix},\ \mathcal{R}_4 = \begin{bmatrix} R_{00} & \cdots & R_{0(r-1)} \\ \vdots & \ddots & \vdots \\ R_{(r-1)0} & \cdots & R_{(r-1)(r-1)} \end{bmatrix},$$

$$W_{H_1} = \begin{bmatrix} 0_{rn,n} & I_{(r+1)n} & 0_{rn,(3r+2)n} \end{bmatrix},\ W_{f_1} = \begin{bmatrix} 0_{rn,(2r+2)n} & I_{rn} & 0_{rn,(r+1)n} \end{bmatrix},$$

$$W_{H_2} = \begin{bmatrix} 0_{(r+1)n} & I_{(r+1)n} & 0_{(r+1)n,(2r+1)n} \end{bmatrix},\ W_{f_2} = \begin{bmatrix} 0_{rn,(3r+2)n} & I_{rn} & 0_{rn,r} \end{bmatrix},$$

$$\bar{W}_{H_1}^T = W_{H_1}^T \begin{bmatrix} \vdots \\ I_{n(i)} \\ \vdots \end{bmatrix},\ \hat{W}_x = \begin{bmatrix} 0_{n,(4r+2)n} & I_n \end{bmatrix},\ W_x = \begin{bmatrix} I_n & 0_{n,(4r+3)n} \end{bmatrix}.$$

Proof: Choose a quadratic functional as follows:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t), \tag{11}$$

where

$$\begin{aligned} V_1(t) &= x^T(t) P x(t), \\ V_2(t) &= \sum_{i=1}^{r} 2x^T(t) \int_{-\tau_i}^{-\tau_{i-1}} (h_{i-1}Q_{i-1} + h_i Q_i) x(t+\xi)\, d\xi, \\ V_3(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r} \int_{-\tau_i}^{-\tau_{i-1}} \int_{-\tau_j}^{-\tau_{j-1}} x^T(t+\xi) \\ &\quad \times (h_i h_j R_{ij} + h_{i-1} h_j R_{i-1,j} + h_i h_{j-1} R_{i,j-1} + h_{i-1} h_{j-1} R_{i-1,j-1}) x(t+\eta)\, d\xi d\eta, \\ V_4(t) &= \sum_{i=1}^{r} \int_{-\tau_i}^{-\tau_{i-1}} x^T(t+\xi)(h_{i-1}S_{i-1} + h_i S_i) x(t+\xi)\, d\xi, \end{aligned}$$

h_i, $i = 1, \ldots, r$ is the brevity of $h_i(\xi + t)$, $i = 1, \ldots, r$, which are the membership functions satisfying the similar triangular distribution. First, we prove that the above functional is a Lyapunov functional. Introducing the following vector function

$$\Phi = \begin{bmatrix} \int_{-\tau_1}^{-\tau_0} h_0 x(t+\xi)\, d\xi \\ \int_{-\tau_1}^{-\tau_0} h_1 x(t+\xi)\, d\xi + \int_{-\tau_2}^{-\tau_1} h_1 x(t+\xi)\, d\xi \\ \int_{-\tau_2}^{-\tau_1} h_2 x(t+\xi)\, d\xi + \int_{-\tau_3}^{-\tau_2} h_2 x(t+\xi)\, d\xi \\ \vdots \\ \int_{-\tau_r}^{-\tau_{r-1}} h_r x(t+\xi)\, d\xi \end{bmatrix},$$

and after some transformation, we can obtain

$$V_2(t) = 2x^T(t)\tilde{Q}\Phi,\quad V_3(t) = \Phi^T \tilde{R} \Phi. \tag{12}$$

According to Lemma 3.2, we have

$$\Phi^T \tilde{S} \Phi = \int_{-\tau_1}^{-\tau_0} h_0 x(t+\xi)\, d\xi^T \frac{S_0}{\tau} \int_{-\tau_1}^{-\tau_0} h_0 x(t+\xi)\, d\xi$$

$$
\begin{aligned}
&+\left(\int_{-\tau_1}^{-\tau_0} h_1 x\left(t+\xi\right) d\xi+\int_{-\tau_2}^{-\tau_1} h_1 x\left(t+\xi\right) d\xi\right)^T \frac{S_1}{\tau} \\
&\times\left(\int_{-\tau_1}^{-\tau_0} h_1 x\left(t+\xi\right) d\xi+\int_{-\tau_2}^{-\tau_1} h_1 x\left(t+\xi\right) d\xi\right)+\ldots \\
&+\int_{-\tau_r}^{-\tau_{r-1}} h_r x\left(t+\xi\right) d\xi^T \frac{S_r}{\tau} \int_{-\tau_r}^{-\tau_{r-1}} h_r x\left(t+\xi\right) d\xi \\
\leq &\int_{-\tau_1}^{-\tau_0} h_0^2 x\left(t+\xi\right)^T S_0 x\left(t+\xi\right) d\xi+\int_{-\tau_1}^{-\tau_0} h_1^2 x\left(t+\xi\right)^T S_1 x\left(t+\xi\right) d\xi \\
&+\int_{-\tau_2}^{-\tau_1} h_1^2 x\left(t+\xi\right)^T S_1 h_1 x\left(t+\xi\right) d\xi+\ldots \\
&+\int_{-\tau_r}^{-\tau_{r-1}} h_r^2 x\left(t+\xi\right)^T S_r h_r x\left(t+\xi\right) d\xi \\
\leq &\int_{-\tau_1}^{-\tau_0} x\left(t+\xi\right)^T\left(h_0^2 S_0+h_1^2 S_1\right) x\left(t+\xi\right) d\xi \\
&+\int_{-\tau_2}^{-\tau_1} x\left(t+\xi\right)^T\left(h_1^2 S_1+h_2^2 S_2\right) x\left(t+\xi\right) d\xi \\
&+\ldots+\int_{-\tau_r}^{-\tau_{r-1}} x\left(t+\xi\right)^T\left(h_{r-1}^2 S_{r-1}+h_r^2 S_r\right) x\left(t+\xi\right) d\xi,
\end{aligned}
$$

and thus we can obtain

$$
\Phi^T \tilde{S} \Phi \leq V_4\left(t\right), \tag{13}
$$

which together with (11) and (12) implies

$$
V_1\left(t\right)+V_2\left(t\right)+V_3\left(t\right)+V_4\left(t\right) \geq \left[\begin{array}{ll} x^T\left(t\right) & \Phi^T \end{array}\right]\left[\begin{array}{cc} P & \tilde{Q} \\ \tilde{Q}^T & \tilde{R}+\tilde{S} \end{array}\right]\left[\begin{array}{c} x\left(t\right) \\ \Phi \end{array}\right]. \tag{14}
$$

From inequality (9), we can conclude $V\left(t\right) \geq 0$.

Next, we prove that the derivative of the quadratic functional in (11) is negative. The derivative of the quadratic functional can be obtained as follows:

$$
\begin{aligned}
\dot{V}_1\left(t\right)= &\ \dot{x}^T\left(t\right) P x\left(t\right)+x^T\left(t\right) P \dot{x}\left(t\right), \\
\dot{V}_2\left(t\right)= &\sum_{i=1}^r 2 \dot{x}^T\left(t\right) \int_{-\tau_i}^{-\tau_{i-1}}\left(h_{i-1} Q_{i-1}+h_i Q_i\right) x\left(t+\xi\right) d\xi \\
&+\sum_{i=1}^r 2 x^T\left(t\right) \int_{-\tau_i}^{-\tau_{i-1}}\left(\frac{1}{\tau} Q_{i-1}-\frac{1}{\tau} Q_i\right) x\left(t+\xi\right) d\xi \\
&+\sum_{i=1}^r 2 x^T\left(t\right)\left(h_{i-1} Q_{i-1}+h_i Q_i\right)\left[x\left(t-\tau_{i-1}\right)-x\left(t-\tau_i\right)\right], \\
\dot{V}_3\left(t\right)= &\sum_{i=1}^r \sum_{j=1}^r \int_{-\tau_i}^{-\tau_{i-1}} x^T\left(t+\xi\right) d\xi \\
&\times\left(h_i h_j R_{ij}+h_{i-1} h_j R_{i-1,j}+h_i h_{j-1} R_{i,j-1}+h_{i-1} h_{j-1} R_{i-1,j-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \left(x\left(t-\tau_{j-1}\right)-x\left(t-\tau_{j}\right)\right)+\sum_{i=1}^{r} \sum_{j=1}^{r}\left(x\left(t-\tau_{i-1}\right)-x\left(t-\tau_{i}\right)\right) \\
&\times\left(h_{i} h_{j} R_{i j}+h_{i-1} h_{j} R_{i-1, j}+h_{i} h_{j-1} R_{i, j-1}+h_{i-1} h_{j-1} R_{i-1, j-1}\right) \\
&\int_{-\tau_{j}}^{-\tau_{j-1}} x(t+\eta) d \eta++\left(h_{i}-h_{j-1}\right) R_{i, j-1}+\left(h_{j}-h_{i-1}\right) R_{i-1, j}\right) x(t+\eta) d \xi d \eta, \\
&+\sum_{i=1}^{r} \sum_{j=1}^{r} \int_{-\tau_{i}}^{-\tau_{i-1}} \int_{-\tau_{j}}^{-\tau_{j-1}} x^{T}(t+\xi) \frac{1}{\tau}\left(-\left(h_{i}+h_{j}\right) R_{i j}+\left(h_{i-1}+h_{j-1}\right) R_{i-1, j-1}\right.
\end{aligned}
$$

$$
\begin{aligned}
\dot{V}_{4}(t)= & \sum_{i=1}^{r}\left(x^{T}\left(t-\tau_{i-1}\right)\left(h_{i-1} S_{i-1}+h_{i} S_{i}\right) x\left(t-\tau_{i-1}\right)\right) \\
& -\sum_{i=1}^{r}\left(x^{T}\left(t-\tau_{i}\right)\left(h_{i-1} S_{i-1}+h_{i} S_{i}\right) x\left(t-\tau_{i}\right)\right) \\
& +\sum_{i=1}^{r} \int_{-\tau_{i}}^{-\tau_{i-1}} x^{T}(t+\xi) \frac{1}{\tau}\left(S_{i-1}-S_{i}\right) x(t+\xi) d \xi .
\end{aligned} \tag{15}
$$

Since $0 \leq h_i \leq 1$, the following inequality is apparently true:

$$
\begin{aligned}
\dot{V}_{4}(t) \leq & \sum_{i=1}^{r}\left\{x^{T}\left(t-\tau_{i-1}\right) S_{i} x\left(t-\tau_{i-1}\right)-x^{T}\left(t-\tau_{i}\right) S_{i-1} x\left(t-\tau_{i}\right)\right\} \\
& +\sum_{i=1}^{r} h_{i-1}^{2}\left\{x^{T}\left(t-\tau_{i-1}\right)\left(S_{i-1}-S_{i}\right) x\left(t-\tau_{i-1}\right)\right\} \\
& +\sum_{i=1}^{r} h_{i}^{2}\left\{x^{T}\left(t-\tau_{i}\right)\left(S_{i-1}-S_{i}\right) x\left(t-\tau_{i}\right)\right\} \\
& +\sum_{i=1}^{r} \int_{-\tau_{i}}^{-\tau_{i-1}} x^{T}(t+\xi) \frac{1}{\tau}\left(S_{i-1}-S_{i}\right) x(t+\xi) d \xi,
\end{aligned} \tag{16}
$$

and $x^T(t-\tau_{i-1}) S_i x(t-\tau_{i-1})$ can be further relaxed by considering $Z_i < G_i$ with the help of slack matrices N_i and M_i:

$$
\begin{aligned}
& x^{T}\left(t-\tau_{i-1}\right) S_{i} x\left(t-\tau_{i-1}\right) \\
\leq\ & 2 x^{T}\left(t-\tau_{i-1}\right) S_{i} x\left(t-\tau_{i-1}\right)+x^{T}(t) S_{i} x(t)-2 x^{T}(t) S_{i} x\left(t-\tau_{i-1}\right) \\
& -\left(\int_{t-\tau_{i-1}}^{t} \dot{x}(w) d w\right)^{T} S_{i} \int_{t-\tau_{i-1}}^{t} \dot{x}(w) d w-\int_{t-\tau_{i-1}}^{t} \phi^{T} N_{i} G_{i}^{-1} N_{i}^{T} \phi d w \\
& +2 \phi^{T} N_{i} x(t)-2 \phi^{T} N_{i} x\left(t-\tau_{i-1}\right)-2 \phi^{T} N_{i} \int_{t-\tau_{i-1}}^{t} \dot{x}(w) d w+\tau_{i-1} \phi^{T} N_{i} Z_{i}^{-1} N_{i}^{T} \phi \\
& +2 \phi^{T} M\left(\dot{x}(t)-(A+\Delta A(t)) x(t)+\left(A_{d}+\Delta A_{d}(t)\right) x\left(t-\tau_{i}\right)\right) \\
=\ & 2 x^{T}\left(t-\tau_{i-1}\right) S_{i} x\left(t-\tau_{i-1}\right)+x^{T}(t) S_{i} x(t)-2 x^{T}(t) S_{i} x\left(t-\tau_{i-1}\right)+2 \phi^{T} N_{i} x(t) \\
& -2 \phi^{T} N_{i} x\left(t-\tau_{i-1}\right)+\tau_{i-1} \phi^{T} N_{i} Z_{i}^{-1} N_{i}^{T} \phi+2 \phi^{T} M\left(\dot{x}(t)-A x(t)+A_{d} x\left(t-\tau_{i}\right)\right) \\
& -\left[\begin{array}{ll}\left(\int_{t-\tau_{i-1}}^{t} \dot{x}(w) d w\right)^{T} & \phi^{T}\end{array}\right]\left[\begin{array}{cc}S_{i} & N_{i}^{T} \\ N_{i} & \tau_{i-1} N_{i} G_{i}^{-1} N_{i}^{T}\end{array}\right]\left[\begin{array}{c}\left(\int_{t-\tau_{i-1}}^{t} \dot{x}(w) d w\right) \\ \phi\end{array}\right] d w .
\end{aligned}
$$

$S_i > \frac{1}{\tau_{i-1}} G_i$ implies the last term in the above inequality is negative. The derivative of the Lyapunov functional in (15) can be rewritten as the following compact form:

$$\dot{V}(t) \le \phi^T \Omega \phi + \phi^T \left(\tau_1 N_2 Z_2^{-1} N_2^T + \ldots + \tau_{r-1} N_r Z_r^{-1} N_r^T \right) \phi - 2\phi^T M \hat{W}_x \phi$$
$$+2\phi^T M \left(A + \Delta A(t) \right) W_x \phi + \sum_{i=0}^{r} h_i(\tau(t)) 2\phi^T M \left(A_d + \Delta A_d(t) \right) x(t-\tau_i),$$

where

$$\phi = \left[\begin{array}{cccccc} x^T(t) & H_1^T & H_2^T & f_1^T & f_2^T & \dot{x}^T(t) \end{array} \right]^T, \quad H_1 = \begin{bmatrix} x(t-\tau_0) \\ x(t-\tau_1) \\ \vdots \\ x(t-\tau_r) \end{bmatrix},$$

$$H_2 = \begin{bmatrix} h_0 x(t-\tau_0) \\ h_1 x(t-\tau_1) \\ \vdots \\ h_r x(t-\tau_r) \end{bmatrix}, \quad f_1 = \begin{bmatrix} \int_{-\tau_1}^{-\tau_0} x(t+\xi)\, d\xi \\ \int_{-\tau_2}^{-\tau_1} x(t+\xi)\, d\xi \\ \vdots \\ \int_{-\tau_r}^{-\tau_{r-1}} x(t+\xi)\, d\xi \end{bmatrix}, \quad f_2 = \begin{bmatrix} \int_{-\tau_1}^{-\tau_0} h_1 x(t+\xi)\, d\xi \\ \int_{-\tau_2}^{-\tau_1} h_2 x(t+\xi)\, d\xi \\ \vdots \\ \int_{-\tau_r}^{-\tau_{r-1}} h_r x(t+\xi)\, d\xi \end{bmatrix}.$$

According to Lemma 3.3 and Schur complement, the inequality in (10) guarantees $\dot{V}(t) < 0$. Thus, one can always find a small scalar $\varepsilon > 0$ such that $V(t) \ge \varepsilon \|x(t)\|^2$ and $\dot{V}(t) \le -\varepsilon \|x(t)\|^2$. The proof is completed. □

3.2. **Controller design.** In this subsection, based on the stability criteria, the state-feedback controller will be designed so that the closed-loop system is asymptotically stable.

Theorem 3.2. *There exists a state-feedback controller such that the system with time-varying delay in (1) is asymptotically stable if there exist matrices $\breve{R}_{ij} = \breve{R}_{ji}^T$, $\breve{Q}_i$, X, Y, $\breve{S}_i > 0$, scalar $\varepsilon_{1,2,3} > 0$, for $i, j = 0, \ldots, r$, $\breve{P} > 0$, $\breve{Z}_i > 0$, $\breve{G}_i > 0$, for $i = 1, \ldots, r$, where $\breve{S}_i > \frac{1}{\tau_{i-1}} \breve{G}_i$, $\breve{Z}_i < \breve{G}_i$ and $\breve{S}_{i-1} < \breve{S}_i$, such that for $i = 0, 1, \ldots, r$, the following inequalities are satisfied:*

$$\begin{bmatrix} \breve{P} & \mathcal{Q} \\ * & \tilde{\mathcal{R}} + \mathcal{S} \end{bmatrix} > 0, \tag{17}$$

$$\begin{bmatrix} \breve{\Omega} & \breve{N}_1 & \cdots & \breve{N}_r & \Gamma_1 & \Gamma_2 \\ * & -\tau_0^{-1} \breve{Z}_1 & \ldots & 0 & 0 & 0 \\ * & * & \ddots & \vdots & \vdots & \vdots \\ * & * & * & -\tau_{r-1}^{-1} \breve{Z}_r & 0 & 0 \\ * & * & * & * & -\varepsilon_1 I & 0 \\ * & * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{18}$$

where

$$\tilde{\mathcal{R}} = \begin{bmatrix} \breve{R}_{00} & \breve{R}_{01} & \ldots & \breve{R}_{0r} \\ * & \breve{R}_{11} & \ldots & \breve{R}_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \breve{R}_{rr} \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} \frac{\breve{S}_0}{\tau} & 0 & 0 & 0 \\ * & \frac{\breve{S}_1}{\tau} & 0 & 0 \\ * & * & \ddots & 0 \\ * & * & * & \frac{\breve{S}_r}{\tau} \end{bmatrix}, \quad \mathcal{Q} = \left[\breve{Q}_0, \breve{Q}_1, \ldots \breve{Q}_r \right],$$

$$\Gamma_1 = W_x^T X^T E_a^T, \quad \Gamma_2 = W_{H_1}^T \bar{X}^T, \quad \bar{X} = X \left[\begin{array}{ccc} \ldots & I_{n(i)} & \ldots \end{array} \right] E_d^T,$$

$$\breve{\Omega} = \hat{W}_x^T \breve{P} W_x + W_x^T \breve{P} \hat{W}_x + 2\hat{W}_x^T \breve{Q}_1 W_{f_1} + 2\hat{W}_x^T \breve{Q}_2 W_{f_2} + 2W_x^T \breve{Q}_3 W_{H_1} + 2W_x^T \breve{Q}_4 W_{H_2}$$

$$
\begin{aligned}
&+W_x^T\left(\breve{S}_1+\ldots+\breve{S}_r\right)W_x+2W_{H_1}^T\breve{S}_1W_{H_1}-2W_x^T\breve{S}_2W_{H_1}-2W_x^T\frac{1}{\tau}\breve{Q}_2W_{f_1}\\
&+2\left(\breve{N}_1+\breve{N}_2+\ldots+\breve{N}_r\right)W_x-W_{H_1}^T\breve{S}_3W_{H_1}-2\begin{bmatrix}\breve{N}_1 & \ldots & \breve{N}_r & 0\end{bmatrix}W_{H_1}\\
&-2W_{f_2}^T\breve{R}_2W_{H_2}+2W_{f_2}^T\breve{R}_6W_{H_2}+2W_{f_1}^T\breve{R}_3W_{H_1}-2W_{f_2}^T\breve{R}_3W_{H_1}-2W_{f_1}^T\breve{R}_4W_{H_2}\\
&+2W_{f_2}^T\breve{R}_5W_{H_2}+2W_{f_1}^T\breve{R}_8W_{H_2}-2W_{f_1}^T\breve{R}_9W_{H_1}+2W_{f_1}^T\breve{R}_9W_{H_2}-2W_{f_2}^T\breve{R}_8W_{H_2}\\
&-2\frac{1}{\tau}W_{f_2}^T\underline{R}_1W_{f_1}+2\frac{1}{\tau}W_{f_1}^T\underline{R}_4W_{f_1}-2\frac{1}{\tau}W_{f_2}^T\underline{R}_4W_{f_1}+\frac{1}{\tau}W_{f_2}^T\underline{R}_3W_{f_1}+\frac{1}{\tau}W_{f_1}^T\underline{R}_3W_{f_2}\\
&+\frac{1}{\tau}W_{f_1}^T\underline{R}_2W_{f_2}+\frac{1}{\tau}W_{f_2}^T\underline{R}_2W_{f_1}-\frac{1}{\tau}W_{f_1}^T\underline{R}_2W_{f_1}+2UA_dX\begin{bmatrix}\ldots & I_{n(i)} & \ldots\end{bmatrix}W_{H_1}\\
&+W_{H_2}^T\breve{S}_4W_{H_2}+W_{f_1}^T\breve{S}_5W_{f_1}-2UX\hat{W}_x+2UAXW_x+2UBYW_x+2W_{f_2}^T\breve{R}_1W_{H_1}\\
&-2W_{f_2}^T\breve{R}_1W_{H_2}-2W_{f_2}^T\breve{R}_5W_{H_1}+2W_{f_2}^T\breve{R}_7W_{H_2}+2W_{f_2}^T\breve{R}_9W_{H_1}-2W_{f_2}^T\breve{R}_9W_{H_2}\\
&-\frac{1}{\tau}W_{f_1}^T\underline{R}_3W_{f_1}+\varepsilon_1(UD_aD_a^TU^T)+\varepsilon_2(UD_dD_d^TU^T)+\varepsilon_3(UD_bD_b^TU^T)
\end{aligned}
$$

$$
\begin{aligned}
\breve{Q}_1&=\begin{bmatrix}\breve{Q}_0 & \breve{Q}_1 & \ldots & \breve{Q}_{r-1}\end{bmatrix},\ \breve{Q}_2=\begin{bmatrix}\breve{Q}_1-\breve{Q}_0 & \breve{Q}_2-\breve{Q}_1 & \ldots & \breve{Q}_r-\breve{Q}_{r-1}\end{bmatrix},\\
\breve{Q}_3&=\begin{bmatrix}\breve{Q}_1 & \breve{Q}_2-\breve{Q}_0 & \breve{Q}_3-\breve{Q}_1 & \ldots & \breve{Q}_r-\breve{Q}_{r-2} & -\breve{Q}_{r-1}\end{bmatrix},\\
\breve{Q}_4&=\begin{bmatrix}\breve{Q}_0-\breve{Q}_1 & \breve{Q}_0-\breve{Q}_2 & \breve{Q}_1-\breve{Q}_3 & \breve{Q}_{r-2}-\breve{Q}_r & \breve{Q}_{r-1}-\breve{Q}_r\end{bmatrix},\\
\breve{S}_1&=\mathrm{diag}\{\breve{S}_1,\ldots,\breve{S}_r,0\},\ \breve{S}_2=\begin{bmatrix}\breve{S}_1 & \ldots & \breve{S}_r & 0\end{bmatrix},\ \breve{S}_3=\mathrm{diag}\{0,\breve{S}_0,\ldots,\breve{S}_{r-1}\},\\
\breve{S}_4&=\mathrm{diag}\{\breve{S}_0-\breve{S}_1,\breve{S}_0-\breve{S}_2,\breve{S}_1-\breve{S}_3,\ldots,\breve{S}_{r-2}-\breve{S}_r,\breve{S}_{r-1}-\breve{S}_r\},\\
\breve{S}_5&=\mathrm{diag}\left\{\frac{1}{\tau}\left(\breve{S}_0-\breve{S}_1\right),\ldots,\frac{1}{\tau}\left(\breve{S}_{r-1}-\breve{S}_r\right)\right\},\ \breve{R}_1=\begin{bmatrix}\underline{R}_1 & 0\end{bmatrix},\ \breve{R}_2=\begin{bmatrix}0 & \underline{R}_1\end{bmatrix},\\
\breve{R}_3&=\begin{bmatrix}\underline{R}_2 & 0\end{bmatrix},\ \breve{R}_4=\begin{bmatrix}\breve{R}_{01} & \cdots & 2\breve{R}_{0r} & \breve{R}_{0r}\\ \vdots & \ddots & \vdots & \vdots\\ \breve{R}_{(r-1)1} & \cdots & 2\breve{R}_{(r-1)r} & \breve{R}_{(r-1)r}\end{bmatrix},\ \breve{R}_5=\begin{bmatrix}0 & \underline{R}_3\end{bmatrix},\\
\breve{R}_6&=\begin{bmatrix}\breve{R}_{01} & \cdots & 2\breve{R}_{0r} & \breve{R}_{0r}\\ \vdots & \ddots & \vdots & \vdots\\ \breve{R}_{(r-1)1} & \cdots & 2\breve{R}_{(r-1)r} & \breve{R}_{(r-1)r}\end{bmatrix},\ \breve{R}_7=\begin{bmatrix}\underline{R}_3 & 0\end{bmatrix},\ \breve{R}_8=\begin{bmatrix}\underline{R}_4 & 0\end{bmatrix},\\
\breve{R}_9&=\begin{bmatrix}0 & \underline{R}_4\end{bmatrix},\ \underline{R}_1=\begin{bmatrix}\breve{R}_{11} & \cdots & \breve{R}_{1r}\\ \vdots & \ddots & \vdots\\ \breve{R}_{r1} & \cdots & \breve{R}_{rr}\end{bmatrix},\ \underline{R}_2=\begin{bmatrix}\breve{R}_{01} & \cdots & \breve{R}_{0r}\\ \breve{R}_{11} & \cdots & \breve{R}_{1r}\\ \vdots & \ddots & \vdots\\ \breve{R}_{(r-1)1} & \cdots & \breve{R}_{(r-1)r}\end{bmatrix},\\
\underline{R}_3&=\begin{bmatrix}\breve{R}_{10} & \cdots & \breve{R}_{1(r-1)}\\ \vdots & \ddots & \vdots\\ \breve{R}_{r0} & \cdots & \breve{R}_{r(r-1)}\end{bmatrix},\ \underline{R}_4=\begin{bmatrix}\breve{R}_{00} & \cdots & \breve{R}_{0(r-1)}\\ \vdots & \ddots & \vdots\\ \breve{R}_{(r-1)0} & \cdots & \breve{R}_{(r-1)(r-1)}\end{bmatrix}.
\end{aligned}
$$

Furthermore, if the above inequalities are satisfied, the controller gain in (6) is given by

$$K=YX^{-1} \tag{19}$$

Proof: Suppose there exist matrices $\breve{P}>0$, $\breve{Z}_i>0$, $\breve{G}_i>0$, $\breve{S}_i>0$, $\breve{R}_{ij}=\breve{R}_{ji}^T$, $\breve{Q}_i$, X and Y satisfying the inequalities in (17) and (18). Without loss of generality, we assume

X is invertible. Define $\Pi = X^{-T}$ and the following matrices:

$$\begin{aligned} \Xi &= \operatorname{diag}\{\Pi, \Pi, \ldots, \Pi\} \in \mathbb{R}^{(7r+4)n\times(5r+4)n}, \\ \Xi_1 &= \operatorname{diag}\{\Pi, \Pi, \ldots, \Pi\} \in \mathbb{R}^{(r+2)n\times(r+2)n}, \\ \Xi_2 &= \operatorname{diag}\{\Pi, \Pi, \ldots, \Pi\} \in \mathbb{R}^{(r+1)n\times(r+1)n}, \\ \Xi_3 &= \operatorname{diag}\{\Pi, \Pi, \ldots, \Pi\} \in \mathbb{R}^{(4r+4)n\times(4r+4)n}, \end{aligned}$$

and pre- and post-multiplying (17) with Ξ_1 and Ξ_1^T, and (18) with $\operatorname{diag}\{\Xi, I, I\}$ and its transposition, then we have

$$\begin{bmatrix} \Pi\breve{P}\Pi^T & \Pi\mathcal{Q}\Xi_2^T \\ * & \Xi_2\left(\tilde{\mathcal{R}} + \mathcal{S}\right)\Xi_2^T \end{bmatrix} > 0, \tag{20}$$

$$\begin{bmatrix} \Xi_3\breve{\Omega}\Xi_3^T & \Xi_3\breve{N}_1\Pi^T & \cdots & \Xi_3\breve{N}_r\Pi^T & \Xi_3\Gamma_1 & \Xi_3\Gamma_2 \\ * & -\tau_0^{-1}\Pi\breve{Z}_1\Pi^T & \ldots & 0 & 0 & 0 \\ * & * & \ddots & \vdots & \vdots & \vdots \\ * & * & * & -\tau_{r-1}^{-1}\Pi\breve{Z}_r\Pi^T & 0 & 0 \\ * & * & * & * & -\varepsilon_1 I & 0 \\ * & * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \quad i = 0, 1, \ldots, r. \tag{21}$$

By defining the following matrix variables

$$\begin{gathered} P = \Pi\breve{P}\Pi^T, \ \ \tilde{Q} = \Pi\mathcal{Q}\Xi_2^T, \ \ \tilde{R} = \Xi_2\tilde{\mathcal{R}}\Xi_2^T, \ \ \tilde{S} = \Xi_2\mathcal{S}\Xi_2^T, \ \ \Pi\breve{R}_{ij}\Pi^T = R_{ij}, \\ Q_i = \Pi\breve{Q}_i\Pi^T, \ \ S_i = \Pi\breve{S}_i\Pi^T, \ \ Z_i = \Pi\breve{Z}_i\Pi^T, \ \ Z_i = \Pi\breve{Z}_i\Pi^T, \ \ N_i = \Xi_3\breve{N}_i\Pi^T, \end{gathered}$$

and calculating the items in inequalities (20) and (21), the following equality can be obtained by considering $Y = KX$:

$$\Xi_3\breve{\Omega}\Xi_3^T = \Omega,$$

where a transfermation is used that $U\Pi = M$. Besides, we have

$$\Xi_3\Gamma_1 = W_x^T\Pi X^T E_a^T = W_x^T E_a^T, \ \ \Xi_3\Gamma_2 = W_{H_1}^T\Pi X \begin{bmatrix} \ldots & I_{n(i)} & \ldots \end{bmatrix} E_d^T = W_{H_1}^T E_d^T.$$

Thus we can obtain (9) and the following inequality

$$\begin{bmatrix} \Omega & N_1 & \cdots & N_r & W_x^T E_a^T & \bar{W}_{H_1}^T E_d^T \\ * & -\tau_0^{-1}Z_1 & \cdots & 0 & 0 & 0 \\ * & * & \ddots & \vdots & \vdots & \vdots \\ * & * & * & -\tau_{r-1}^{-1}Z_r & 0 & 0 \\ * & * & * & * & -\varepsilon_1 I & 0 \\ * & * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \quad i = 0, 1, \ldots, r,$$

which guarantee Theorem 3.1. The proof is completed. □

4. **Numerical Examples.** In this section, the examples will be provided to show the less conservatism and the effectiveness of the proposed method.

Example 4.1.

Consider the following system:

$$\dot{x}(t) = Ax(t) + A_d x(t - \tau(t)),$$

with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \ \ A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

TABLE 1. Maximum delay values $\bar{\tau}$

Methods	τ_0	2	3	4	5	6
[19]	$\bar{\tau}$	2.505	3.259	4.074	–	–
[23]	$\bar{\tau}$	2.566	3.341	4.169	5.028	6.014
Theorem 3.1	$\bar{\tau}$	6.803	7.803	8.803	9.809	10.809

We suppose that the complex delay system has the above form, and the purpose is to show the effectiveness of the proposed fuzzy modelling method.

First, suppose that the time-varying delay satisfies $\tau_0 \le \tau(t) \le \bar{\tau}$ and $\tau_0 > 0$. With the given delay lower bound, we obtain the upper bound containing the asymptotical stability of the system from Theorem 3.1 in this paper is 6.803, where the division number of the delay range is 3 and the division interval is 1.601 for $\tau_0 = 2, 3, 4$ and 1.603 for $\tau_0 = 5, 6$. Comparably, with the same lower delay bound, we also give the results obtained by the methods in most recent available literature, shown in Table 1. Table 1 shows that the delay upper bounds obtained by our method are larger than those obtained in [19, 23]; for example, the delay upper bounds obtained by the methods in [19, 23] are 2.5048 and 2.5663 respectively. Next, we assume that the time-varying delay satisfies $\tau_0 \le \tau(t) \le \bar{\tau}$ and $\tau_0 = 0$. In this case, the maximum value obtained by Theorem 3.1 is 4.982, where the division number of the delay range is 3 and the equal interval is 1.6608. Comparably, the maximum delay upper bounds obtained from Corollary 3 in [17] and Corollary 2 in [23] are 4.472 and 3.918 respectively.

The delay upper bounds obtained based on the fuzzy model are larger, which indicates that under some special situations that the current delay information is unknown, the proposed fuzzy model method can be effective. The simulation also confirms that the stability of the local systems is a sufficient condition not a necessary one for a T-S fuzzy system; readers who are interested in that are referred to the example in [13] for details.

Example 4.2.

Consider the following system [3, 14]:

$$\dot{x}(t) = (A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t-\tau) + Bu(t),$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -2 & -0.5 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$D_a = 0.2I, \quad E_a = I, \quad E_d = 0.$$

The purpose is to compare the conservatism of the result obtained in this paper and those in the recent available results. More specifically, we compare the maximum delay bounds obtained by Theorem 3.2 in this paper and the results in other literature. The maximum delay upper bounds obtained from Theorem 3.2 in [17] is 0.90, while the upper bound obtained by our results is 2.163 with the delay lower bound 0.3. The detailed comparison data is presented in Table 2.

Example 4.3.

Consider a continuous stirred tank reactor (CSTR) system, whose model is:

$$\dot{x}(t) = Ax(t) + A_d x(t - h(t)) + Bu(t),$$

TABLE 2. Maximum delay values $\bar{\tau}$

Case (τ_0)	Cases ($\bar{\tau}$)		K
0	$\bar{\tau}$ by [16]	0.45	–
	$\bar{\tau}$ by [3]	0.55	$[-0.020\ 52.86]$
	$\bar{\tau}$ by [2]	0.58	$[-0.31\ -4.44]$
	$\bar{\tau}$ by [14]	0.84	$[-34.72\ -18.41]$
	$\bar{\tau}$ by Theorem 3.2	1.86	$[-754.59\ -272.27]$
0.3	$\bar{\tau}$ by [14]	0.90	$[-70.22\ -33.14]$
	$\bar{\tau}$ by Theorem 3.2	2.16	$[-756.61\ -272.83]$

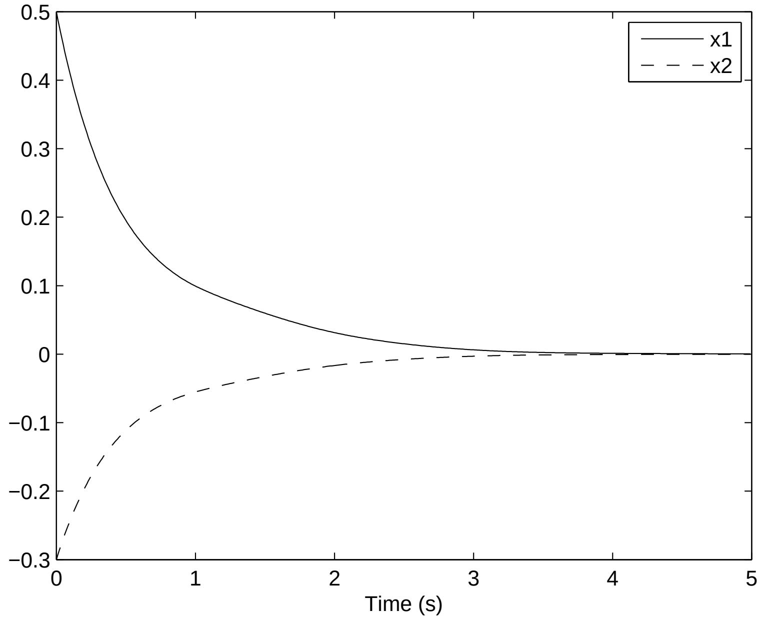

FIGURE 1. The trajectories of the closed-loop system

where $x_1(t)$ corresponds to the conversion rate of the reaction, and $x_2(t)$ is the dimensionless temperature. The parameters are the same as those in [1], and

$$A = \begin{bmatrix} -2.0508 & 0.3958 \\ -6.4066 & 1.6168 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix}.$$

It is assumed that $h(t) = 1.2\sin(t)$. The goal is to design an controller such that the closed-loop system is asymptotically stable with the time-varying delay. When choosing the upper bound of the time delay $\bar{h} = 1.2$ and solving the LMIs in Theorem 3.2, we obtain the controller gain $K = \begin{bmatrix} 21.28 & -13.5 \end{bmatrix}$. Figure 1 shows the state variables of the closed-loop system, where the initial condition is assumed to be in $[0.5, -0.3]$. From Figure 1, we can see that the state variables of the closed-loop system converge to zero. The effectiveness of the proposed controller design approach is apparent.

Except the practical use for systems or processes with time-varying delay, the proposed method can be used for other complex systems such as the biology system, kiln system, where the detailed information of the time delay cannot be obtained, and only some local constant delays can be measured.

5. **Conclusions.** This paper has investigated the problem of robust stability and stabilization of linear systems with unknown state delay. The fuzzy techniques have been utilized novelly to construct a model by setting the time-varying delay to be distributed on the fuzzy set. Concerned with the stability analysis, a new approach incorporating with the fuzzy discretized Lyapunov functional is proposed, and with the help of slack matrix techniques, the stability conditions are obtained. Further, the controller has been designed based on the fuzzy model which can stabilize the approximated system effectively with much less conservatism, which is the main contribution of this paper. Future work should include dealing with the time delay in nonlinear systems or filtering problems [30].

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