International Journal of Innovative
Computing, Information and Control
Volume **9**, Number **11**, November **2013**

ICIC International ©2013 ISSN 1349-4198
pp. 4389–4400

FURTHER RESULTS ON GLOBAL STATE FEEDBACK STABILIZATION OF NONLINEAR SYSTEMS WITH LOW-ORDER AND HIGH-ORDER NONLINEARITIES

Xing-Hui Zhang[1] and Na Duan[2]

[1]Institute of Automation
Qufu Normal University
No. 57, West Jingxuan Rd., Qufu 273165, P. R. China
lyzhangxinghui@163.com

[2]School of Electrical Engineering and Automation
Jiangsu Normal University
No. 101, Shanghai Rd., Tongshanxin Dist., Xuzhou 221116, P. R. China
duanna08@163.com

Received December 2012; revised April 2013

Abstract. *This paper discusses the state feedback control problem for a class of nonlinear systems with low-order and high-order nonlinearities, and multiple time-varying delays. The introduction of sign function together with the method of adding a power integrator and Lyapunov-Krasovskii theorem makes the closed-loop system globally asymptotically stable.*

Keywords: Nonlinear systems, Low-order and high-order nonlinearities, Multiple time-varying delays, State feedback, Sign function

1. Introduction.

Consider a class of nonlinear systems:

$$\begin{aligned} \dot{x}_i(t) &= x_{i+1}^{p_i}(t) + f_i(t, x(t), x_1(t-\tau_1(t)), \cdots, x_n(t-\tau_n(t))), \quad i = 1, \cdots, n-1, \\ \dot{x}_n(t) &= u^{p_n}(t) + f_n(t, x(t), x_1(t-\tau_1(t)), \cdots, x_n(t-\tau_n(t))), \end{aligned} \tag{1}$$

where $x(t) = [x_1(t), \cdots, x_n(t)]^\top \in R^n$ and $u(t) \in R$ are the system state and control input, respectively. For $i = 1, \cdots, n$, $\tau_i(t) : R^+ \to R^+$ is time-varying delay with $0 \le \tau_i(t) \le \varepsilon_i$, where ε_i is a positive constant, $p_i \in R_{odd}^{\ge 1} \triangleq \{\frac{p}{q} \in R^+$: p and q are odd integers, $p \ge q\}$, $f_i : R^+ \times R^n \times R^n \to R$ is an unknown continuous function. The initial value $x_0(\theta)$ is a continuous function on $\theta \in [-\tau, 0]$ with $\tau = \max\{\varepsilon_1, \cdots, \varepsilon_n\}$.

In recent years, the global stabilization for system (1) with both lower-order and higher-order nonlinearities has become a hot topic being studied. In the latest papers [1, 2], the authors discussed that these two papers encompass and substantially generalize the existing results in [3-9].

In [1, 2], the following condition on the uncertain term f_i is assumed:

$$|f_i(\cdot)| \le c \sum_{j=1}^{i} \left(|x_j|^{\frac{1}{p_j \cdots p_{i-1}}} + |x_j|^{\frac{r_i+\omega}{r_j}} \right), \tag{2}$$

where $r_1 = 1$, $r_i p_{i-1} = r_{i-1} + \omega$, $i = 1, \cdots, n$. However, (2) needs the condition of $\omega = \frac{s}{o}$ with s being an even integer and o being an odd integer, which results in $\frac{r_i+\omega}{r_j}$ in (2) being always a ratio of odd integers. Naturally, an interesting problem may be proposed:

Is it possible to relax the assumption on ω in (2)? Under the weaker assumption, can one design a stabilizing controller?

In this paper, by introducing the sign function approach, and overcoming several troublesome obstacles in the design and analysis procedure, we focus on solving the above problem under the assumption of the restriction on ω being relaxed to any real number.

2. **Mathematical Preliminaries.**

The following notations and lemmas are to be used throughout the paper.

Notations: R^+ stands for the set of all the nonnegative real numbers. For any vector $x = [x_1, \cdots, x_n]^\top \in R^n$, denote $\bar{x}_i \triangleq [x_1, \ldots, x_i]^\top \in R^i$, $i = 1, \ldots, n-1$, $\|x\| \triangleq \sqrt{\sum_{i=1}^n x_i^2}$, $x_t \triangleq x(t+\theta)$, $\theta \in [-\tau, 0]$, and $\|x_t\|_{\mathcal{C}} \triangleq \sup_{-\tau\le\theta\le 0} \|x(t+\theta)\|$, $\forall t \ge 0$. Use c or $c(\bar{h}_i)$ to represent any positive constant or constant dependent on $[h_1, \cdots, h_i]^\top$, which may be implicitly changed from place to place. A sign function $\mathrm{sgn}(x)$ is defined as: $\mathrm{sgn}(x) = 1$ if $x > 0$, $\mathrm{sgn}(x) = 0$ if $x = 0$, and $\mathrm{sgn}(x) = -1$ if $x < 0$. The arguments of functions (or functionals) are sometimes omitted or simplified; for instance, we sometimes denote a function $f(x(t))$ by $f(x)$, $f(\cdot)$, or f.

Lemma 2.1. [10] *For $x, y \in R$, $p \ge 1$ is a constant, then $|x+y|^p \le 2^{p-1}|x^p + y^p|$, $(|x|+|y|)^{\frac{1}{p}} \le |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}}$. If $p \in R^{\ge 1}_{odd}$, then $|x-y|^p \le 2^{p-1}|x^p - y^p|$, $|x^{\frac{1}{p}} - y^{\frac{1}{p}}| \le 2^{1-\frac{1}{p}}|x-y|^{\frac{1}{p}}$.*

Lemma 2.2. [10] *Let m, n be positive constants. Given any positive number $\gamma > 0$, then $|x|^m|y|^n \le \frac{m}{m+n}\gamma|x|^{m+n} + \frac{n}{m+n}\gamma^{-\frac{m}{n}}|y|^{m+n}$.*

Lemma 2.3. [11] *For the continuous function $f : [a, b] \to R$ $(a \le b)$, if it is monotonically increasing and satisfies $f(a) = 0$, then $|\int_a^b f(x)dx| \le |f(b)||b-a|$.*

Lemma 2.4. *$f(x) = \mathrm{sgn}(x)|x|^a$ is continuously differentiable and satisfies $\dot{f}(x) = a|x|^{a-1}$, where $a \ge 1$, $x \in R$.*

Proof: See the Appendix. □

Lemma 2.5. [2] *Let $0 \le \mu_1 \le \cdots \le \mu_n$ be real numbers and $c_j > 0$, $j = 1, \cdots, n$. Then for any $x \in R$, one has $c_1|x|^{\mu_1} + c_n|x|^{\mu_n} \le \sum_{j=1}^n c_j|x|^{\mu_j} \le (\sum_{j=1}^n c_j)(|x|^{\mu_1} + |x|^{\mu_n})$.*

3. **Design of State Feedback Controller.**

3.1. **Problem formulation.**

In this paper, we need the following assumptions:

Assumption 3.1. *For each $i = 1, \cdots, n$ and any $\omega > 0$, there exists a known constant M such that*

$$
\begin{aligned}
&|f_i(t, x(t), x_1(t-\tau_1(t)), \cdots, x_n(t-\tau_n(t)))| \\
&\le M \sum_{j=1}^{i} \left(|x_j(t)|^{\frac{1}{p_j \cdots p_{i-1}}} + |x_j(t-\tau_j(t))|^{\frac{1}{p_j \cdots p_{i-1}}} + |x_j(t)|^{\frac{r_i+\omega}{r_j}} + |x_j(t-\tau_j(t))|^{\frac{r_i+\omega}{r_j}} \right),
\end{aligned}
$$

where

$$
r_1 = 1, \quad r_{i+1} = \frac{r_i + \omega}{p_i}, \quad i = 1, \cdots, n. \tag{3}
$$

Assumption 3.2. *For $\tau_i(t)$, $i = 1, \cdots, n$, there is a constant δ_i such that $\dot{\tau}_i(t) \le \delta_i < 1$.*

Remark 3.1. *Compared with Assumption 1 in [2], two ingredients make the assumption in this paper much weaker. One is that the restriction on ω is removed. In (2), $\omega = \frac{s}{o}$ with s being an even integer and o being an odd integer results in $\frac{r_i+\omega}{r_j}$ always being a ratio of odd integers. The other is the appearance of multiple time-varying delays in state variables.* □

3.2. **State feedback controller design.** We now design a state feedback controller to stabilize system (1) under Assumptions 3.1 and 3.2.

Step 1: Define $z_1(t) = x_1(t)$ and $\sigma = r_{n+1}p_1 \cdots p_n$, and choose

$$V_1(t, z_{1t}) = \frac{z_1^2}{2} + \frac{r_1}{2\sigma - \omega}|z_1|^{\frac{2\sigma-\omega}{r_1}} + \frac{b_1}{1-\delta_1}\int_{t-\tau_1(t)}^{t}\left(z_1^2(s) + z_1^{\frac{2\sigma}{r_1}}(s)\right)\mathrm{d}s$$
$$+\frac{c_1}{1-\delta_2}\int_{t-\tau_2(t)}^{t}\left(z_1^2(s) + z_1^{\frac{2\sigma}{r_1}}(s)\right)\mathrm{d}s, \tag{4}$$

where b_1, c_1 are positive constants. (4) and $0 \le \tau_i(t) \le \varepsilon_i$ imply that

$$\begin{aligned} V_1(t, z_{1t}) &\ge \frac{z_1^2}{2} + \frac{r_1}{2\sigma-\omega}|z_1|^{\frac{2\sigma-\omega}{r_1}} \triangleq \pi_{11}(\|z_1\|), \\ V_1(t, z_{1t}) &\le \frac{\|z_{1t}\|_{\mathcal{C}}^2}{2} + \frac{r_1}{2\sigma-\omega}\|z_{1t}\|_{\mathcal{C}}^{\frac{2\sigma-\omega}{r_1}} + \left(\frac{b_1\varepsilon_1}{1-\delta_1} + \frac{c_1\varepsilon_2}{1-\delta_2}\right) \le \left(\|z_{1t}\|_{\mathcal{C}}^2 + \|z_{1t}\|_{\mathcal{C}}^{\frac{2\sigma}{r_1}}\right) \\ &\triangleq \pi_{12}(\|z_{1t}\|_{\mathcal{C}}), \end{aligned} \tag{5}$$

where $\pi_{11}(\cdot)$, $\pi_{12}(\cdot)$ are class $\mathcal{K}_\infty$ functions.

From Lemma 2.2 and Assumption 3.1, it follows that

$$\begin{aligned} &\left(z_1 + \mathrm{sgn}(z_1)|z_1|^{\frac{2\sigma - r_2p_1}{r_1}}\right) f_1 \\ &\le c\left(z_1 + \mathrm{sgn}(z_1)|z_1|^{\frac{2\sigma - r_2p_1}{r_1}}\right)\left(|z_1| + |z_1|^{\frac{r_2p_1}{r_1}} + |z_1(t-\tau_1(t))| + |z_1(t-\tau_1(t))|^{\frac{r_2p_1}{r_1}}\right) \\ &\le \lambda_1(\bar{b}_{11})\left(z_1^2 + z_1^{\frac{2\sigma}{r_1}}\right) + \bar{b}_{11}\left(z_1^2(t-\tau_1(t)) + z_1^{\frac{2\sigma}{r_1}}(t-\tau_1(t))\right), \end{aligned} \tag{6}$$

where $0 < \bar{b}_{11} < b_1$, $\lambda_1(\bar{b}_{11}) > 0$. By (1), (4), (6) and Assumption 3.2, we have

$$\begin{aligned} \dot{V}_1 &\le -b_1\left(z_1^2(t-\tau_1(t)) + z_1^{\frac{2\sigma}{r_1}}(t-\tau_1(t))\right) - c_1\left(z_1^2(t-\tau_2(t)) + z_1^{\frac{2\sigma}{r_1}}(t-\tau_2(t))\right) \\ &\quad + \left(\frac{b_1}{1-\delta_1} + \frac{c_1}{1-\delta_2}\right)\left(z_1^2 + z_1^{\frac{2\sigma}{r_1}}\right) + \left(z_1 + \mathrm{sgn}(z_1)|z_1|^{\frac{2\sigma - r_2p_1}{r_1}}\right)\alpha_1^{p_1} \\ &\quad + \left(z_1 + \mathrm{sgn}(z_1)|z_1|^{\frac{2\sigma - r_2p_1}{r_1}}\right)(x_2^{p_1} - \alpha_1^{p_1}) + \left(z_1 + \mathrm{sgn}(z_1)|z_1|^{\frac{2\sigma - r_2p_1}{r_1}}\right) f_1 \\ &\le -b_{11}\left(z_1^2(t-\tau_1(t)) + z_1^{\frac{2\sigma}{r_1}}(t-\tau_1(t))\right) - c_{11}\left(z_1^2(t-\tau_2(t)) + z_1^{\frac{2\sigma}{r_1}}(t-\tau_2(t))\right) \\ &\quad + \left(\lambda_1(\bar{b}_{11}) + \frac{b_1}{1-\delta_1} + \frac{c_1}{1-\delta_2}\right)\left(z_1^2 + z_1^{\frac{2\sigma}{r_1}}\right) + \left(z_1 + \mathrm{sgn}(z_1)|z_1|^{\frac{2\sigma - r_2p_1}{r_1}}\right)\alpha_1^{p_1} \\ &\quad + \left(z_1 + \mathrm{sgn}(z_1)|z_1|^{\frac{2\sigma - r_2p_1}{r_1}}\right)(x_2^{p_1} - \alpha_1^{p_1}), \end{aligned} \tag{7}$$

where $b_{11} = b_1 - \bar{b}_{11}$, $c_{11} = c_1$. Then the first virtual controller α_1 defined by

$$\alpha_1(z_1) = -h_1^{\frac{1}{p_1}}\left(z_1 + \mathrm{sgn}(z_1)|z_1|^{\frac{r_2p_1}{r_1}}\right)^{\frac{1}{p_1}}, \tag{8}$$

results in

$$\begin{aligned} \dot{V}_1 &\le -a_{11}\left(z_1^2 + z_1^{\frac{2\sigma}{r_1}}\right) - b_{11}\left(z_1^2(t-\tau_1(t)) + z_1^{\frac{2\sigma}{r_1}}(t-\tau_1(t))\right) \\ &\quad -c_{11}\left(z_1^2(t-\tau_2(t)) + z_1^{\frac{2\sigma}{r_1}}(t-\tau_2(t))\right) + \left(z_1 + \mathrm{sgn}(z_1)|z_1|^{\frac{2\sigma - r_2p_1}{r_1}}\right)(x_2^{p_1} - \alpha_1^{p_1}) \end{aligned} \tag{9}$$

where $h_1 = a_{11} + \lambda_1(\bar{b}_{11}) + \frac{b_1}{1-\delta_1} + \frac{c_1}{1-\delta_2}$, $a_{11} > 0$.

Step k ($k = 2, \cdots, n$): We start with the following proposition:

Proposition 3.1. *Suppose that there is a continuously differentiable Lyapunov-Krasovskii functional* $V_{k-1}(t, \bar{z}_{k-1,t})$ *satisfying*

$$\pi_{k-1,1}(\|\bar{z}_{k-1}\|) \le V_{k-1}(t, \bar{z}_{k-1,t}) \le \pi_{k-1,2}(\|\bar{z}_{k-1,t}\|_{\mathcal{C}}), \tag{10}$$

and a series of continuous virtual controllers $\alpha_2, \cdots, \alpha_{k-1}$ *defined by*

$$\begin{aligned} &z_i(t) = x_i^{p_1\cdots p_{i-1}} - \alpha_{i-1}^{p_1\cdots p_{i-1}}(z_{i-1}(t)), \\ &\alpha_i(z_i(t)) = -h_i^{\frac{1}{p_1\cdots p_i}}\left(z_i(t) + \operatorname{sgn}(z_i(t))|z_i(t)|^{\frac{r_{i+1}p_i}{r_i}}\right)^{\frac{1}{p_1\cdots p_i}}, \quad i = 2, \cdots, k-1, \end{aligned} \tag{11}$$

such that

$$\begin{aligned} &\dot{V}_{k-1}(t, \bar{z}_{k-1,t}) \\ \le& -\sum_{j=1}^{k-1} a_{k-1,j}\left(z_j^2 + z_j^{\frac{2\sigma}{r_jp_1\cdots p_{j-1}}}\right) - \sum_{j=1}^{k-1} b_{k-1,j}\left(z_j^2(t-\tau_j(t)) + z_j^{\frac{2\sigma}{r_jp_1\cdots p_{j-1}}}(t-\tau_j(t))\right) \\ &- \sum_{j=1}^{k-1} c_{k-1,j}\left(z_j^2(t-\tau_{j+1}(t)) + z_j^{\frac{2\sigma}{r_jp_1\cdots p_{j-1}}}(t-\tau_{j+1}(t))\right) + \left(\operatorname{sgn}(z_{k-1})|z_{k-1}|^{\frac{2\sigma - r_kp_{k-1}}{r_{k-1}p_1\cdots p_{k-2}}}\right. \\ &\left. + z_{k-1}^{2-\frac{1}{p_1\cdots p_{k-2}}}\right)(x_k^{p_{k-1}} - \alpha_{k-1}^{p_{k-1}}), \end{aligned} \tag{12}$$

where $\pi_{k-1,1}(\cdot)$, $\pi_{k-1,2}(\cdot)$ *are class* $\mathcal{K}_\infty$ *functions,* h_i, $i = 2, \cdots, k-1$, $a_{k-1,j}$, $b_{k-1,j}$, $c_{k-1,j}$, $j = 1, \cdots, k-1$, *are positive constants. Then, by defining* $z_k = x_k^{p_1\ldots p_{k-1}} - \alpha_{k-1}^{p_1\ldots p_{k-1}}$, *the* k*th functional*

$$V_k(t, \bar{z}_{kt}) = V_{k-1}(t, \bar{z}_{k-1,t}) + W_{Lk}(\bar{z}_k) + W_{Hk}(\bar{z}_k) + W_{Dk}(t, \bar{z}_{kt}) \tag{13}$$

is continuously differentiable and satisfies

$$\pi_{k1}(\|\bar{z}_k\|) \le V_k(t, \bar{z}_{kt}) \le \pi_{k2}(\|\bar{z}_{kt}\|_{\mathcal{C}}), \tag{14}$$

and one can design a controller $\alpha_k(z_k) = -h_k^{\frac{1}{p_1\cdots p_k}}\left(z_k + \operatorname{sgn}(z_k)|z_k|^{\frac{r_{k+1}p_k}{r_k}}\right)^{\frac{1}{p_1\cdots p_k}}$ *such that*

$$\begin{aligned} \dot{V}_k(t, \bar{z}_{kt}) \le& -\sum_{j=1}^{k} a_{kj}\left(z_j^2 + z_j^{\frac{2\sigma}{r_jp_1\cdots p_{j-1}}}\right) - \sum_{j=1}^{k} b_{kj}\left(z_j^2(t-\tau_j(t)) + z_j^{\frac{2\sigma}{r_jp_1\cdots p_{j-1}}}(t-\tau_j(t))\right) \\ &- \sum_{j=1}^{k} c_{kj}\left(z_j^2(t-\tau_{j+1}(t)) + z_j^{\frac{2\sigma}{r_jp_1\cdots p_{j-1}}}(t-\tau_{j+1}(t))\right) + \left(\operatorname{sgn}(z_k)|z_k|^{\frac{2\sigma - r_{k+1}p_k}{r_kp_1\cdots p_{k-1}}}\right. \\ &\left. + z_k^{2-\frac{1}{p_1\cdots p_{k-1}}}\right)(x_{k+1}^{p_k} - \alpha_k^{p_k}), \end{aligned} \tag{15}$$

$\pi_{k1}(\cdot)$, $\pi_{k2}(\cdot)$ *are class* $\mathcal{K}_\infty$ *functions,* $a_{kj}, b_{kj}, c_{kj}, j = 1, \cdots, k$, *are positive constants,*

$$\begin{aligned} W_{Lk}(\cdot) &= \int_{\alpha_{k-1}}^{x_k} \left(s^{p_1\cdots p_{k-1}} - \alpha_{k-1}^{p_1\cdots p_{k-1}}\right)^{2-\frac{1}{p_1\cdots p_{k-1}}} \mathrm{d}s, \\ W_{Hk}(\cdot) &= \int_{\alpha_{k-1}}^{x_k} \operatorname{sgn}\left(s^{p_1\cdots p_{k-1}} - \alpha_{k-1}^{p_1\cdots p_{k-1}}\right)\left|s^{p_1\cdots p_{k-1}} - \alpha_{k-1}^{p_1\cdots p_{k-1}}\right|^{\frac{2\sigma - r_{k+1}p_k}{r_kp_1\cdots p_{k-1}}} \mathrm{d}s, \\ W_{Dk}(\cdot) &= \left(\frac{b_k\int_{t-\tau_k(t)}^t}{1-\delta_k} + \frac{c_k\int_{t-\tau_{k+1}(t)}^t}{1-\delta_{k+1}}\right)\left(z_k^2(s) + z_k^{\frac{2\sigma}{r_kp_1\cdots p_{k-1}}}(s)\right)\mathrm{d}s, \quad b_k, c_k > 0. \end{aligned} \tag{16}$$

Proof: See the Appendix. □

Hence, at step n, by choosing $V_n(\cdot) = V_{n-1}(\cdot) + W_{Ln}(\cdot) + W_{Hn}(\cdot) + W_{Dk}(\cdot)$, designing appropriate constants such that $a_{nj} > 0$, $b_{nj} = c_{nj} = 0$, $j = 1, \cdots, n$, and constructing the controller as

$$u(t) = \alpha_n(z_n(t)) = -h_n^{\frac{1}{p_1 \cdots p_n}} \left(z_n + \operatorname{sgn}(z_n)|z_n|^{\frac{r_{n+1}p_n}{r_n}} \right)^{\frac{1}{p_1 \cdots p_n}}, \tag{17}$$

it is easy to get

$$\dot{V}_n \leq -\sum_{j=1}^{n} a_{nj} \left(z_j^2 + z_j^{\frac{2\sigma}{r_j p_1 \cdots p_{j-1}}} \right). \tag{18}$$

4. **Stability Analysis.** We state the main result in this paper.

Theorem 4.1. *If Assumptions 3.1 and 3.2 hold for system (1), under the continuous state feedback controller (17), then the equilibrium at the origin of the closed-loop system is globally asymptotically stable.*

Proof: Firstly, by the existence and continuation of the solutions, the states $x(t)$ and $z(t)$ are defined on $[-\tau, t_M]$, where the number t_M may be infinite or not. The following analysis focuses on $[-\tau, t_M]$. Secondly, from Proposition 3.1, we know that

$$\pi_{n1}(\|z(t)\|) \leq V_n(t, z_t) \leq \pi_{n2}(\|z_t\|_{\mathcal{C}}), \tag{19}$$

where $\pi_{n1}(\cdot)$, $\pi_{n2}(\cdot)$ are class $\mathcal{K}_\infty$ functions. (18) and Lemma 4.3 in [12] imply that there exists a class $\mathcal{K}_\infty$ function $\pi_{n3}(\cdot)$ such that

$$\dot{V}(t, z_t) \leq -\pi_{n3}(\|z(t)\|). \tag{20}$$

Thirdly, for any $\varepsilon > 0$, since $\pi_{n1}(\cdot)$ is a class $\mathcal{K}_\infty$ function, one can always find a $\beta = \beta(\varepsilon)$ satisfying $\beta > \varepsilon > 0$ such that $\pi_{n2}(\varepsilon) \leq \pi_{n1}(\beta)$. If $\|z_0(\theta)\|_{\mathcal{C}} < \varepsilon$, (19) and (20) yield

$$\pi_{n1}(\|z(t)\|) \leq V_n(t, z_t(\theta)) \leq V_n(0, z_0(\theta)) \leq \pi_{n2}(\|z_0(\theta)\|_{\mathcal{C}}) \leq \pi_{n2}(\varepsilon) \leq \pi_{n1}(\beta), \quad \forall t \in [0, t_M],$$

which means that $\|z(t)\| \leq \beta$, $\forall t \in [-\tau, t_M]$. Suppose that t_M is finite, then $\lim_{t \to t_M} \|z(t)\| = +\infty$, which contradicts $\|z(t)\| \leq \beta$, $\forall t \in [-\tau, t_M]$. Hence, the state $z(t)$ is well defined on $[-\tau, +\infty)$, so is $x(t)$.

(19), (20) and Lyapunov-Krasovskii theorem in [13] result in the transformed closed-loop z-system being globally asymptotically stable at the equilibrium $z = 0$, which, together with (11), directly leads to the globally asymptotic stability of the closed-loop system at the origin $x = 0$. $\Box$

5. **Conclusions.** By combining the method of adding a power integrator together and the sign function design approach, this paper further discusses the global stabilization for a class of high-order nonlinear systems with multiple time-varying delays.

Recently, [14-41] discuss different control problems for stochastic nonlinear systems. However, all these references only consider the systems with linear or higher-order growth condition. An important problem is how to give the design and analysis of controller for stochastic nonlinear systems with both lower-order and higher-order nonlinearities by adopting this method in this paper.

Acknowledgment. This work is supported by the Program for the Scientific Research Innovation Team in Colleges and Universities of Shandong Province, the National Natural Science Foundation of China (No. 61273125, 61104222), the Specialized Research Fund for the Doctoral Program of Higher Education (No. 20103705110002), the Shandong Provincial Natural Science Foundation (No. ZR2012FM018), the Project of Taishan Scholar of Shandong Province.

REFERENCES

[1] H. Lei, *Universal Output Feedback Control of Nonlinear Systems*, Case Western Reserve University, Ph.D. Thesis, 2008.

[2] H. Lei and W. Lin, Reduced-order observer, homogeneous domination and nonsmooth output feedback stabilization of nonlinear systems, *The 7th World Congress on Intelligent Control and Automation*, pp.394-399, 2008.

[3] C. J. Qian and W. Lin, Output feedback control of a class of nonlinear systems: A non-separarion principle diagram, *IEEE Trans. on Automatic Control*, vol.47, no.10, pp.1710-1715, 2002.

[4] C. J. Qian, A homogeneous domination approach for global output feedback stabilization of a class of nonlinear systems, *Proc. of American Control Conference*, pp.4708-4715, 2005.

[5] V. Andrieu, A. Astolfi and L. Praly, Nonlinear output feedback design via domination and generalized weighted homogeneity, *Proc. of the 45th IEEE Conference on Decision and Control*, pp.6391-6396, 2006.

[6] H. Lei and W. Lin, Robust control of uncertain nonlinear systems by output feedback, *Proc. of the American Control Conference*, New York, USA, pp.5354-5359, 2007.

[7] J. Polendo and C. J. Qian, A generalized framework for global output feedback stabilization of genuinely nonlinear systems, *Proc. of the 44th IEEE Conference on Decision and Control*, pp.2646-2651, 2005.

[8] C. J. Qian and W. Lin, Recursive observer design, homogeneous approximation, and nonsmooth output feedback stabilization of nonlinear systems, *IEEE Trans. on Automatic Control*, vol.51, no.9, pp.1457-1471, 2006.

[9] B. Yang and W. Lin, Robust output feedback stabilization of nonlinear systems with uncontrollable/unobservable linearization, *IEEE Trans. on Automatic Control*, vol.50, no.5, pp.619-630, 2005.

[10] C. J. Qian and W. Lin, A continuous feedback approach to global strong stabilization of nonlinear systems, *IEEE Trans. on Automatic Control*, vol.46, no.7, pp.1061-1079, 2001.

[11] Z. Y. Sun and Y. G. Liu, Adaptive practical output tracking control for high-order nonlinear uncertain systems, *Acta Automatica Sinica*, vol.34, no.8, pp.984-988, 2008.

[12] H. K. Khalil, *Nonlinear Systems*, Publishing House of Electronics Industry, Beijing, 2007.

[13] K. Q. Gu, V. L. Kharitonov and J. Chen, *Stability of Time-Delay Systems*, Birkhauser, Berlin, 2003.

[14] Z. J. Wu, X. J. Xie and S. Y. Zhang, Stochastic adaptive backstepping controller design by introducing dynamic signal and changing supply function, *International Journal of Control*, vol.79, no.12, pp.1635-1646, 2006.

[15] X. J. Xie and J. Tian, State-feedback stabilization for high-order stochastic nonlinear systems with stochastic inverse dynamics, *International Journal of Robust and Nonlinear Control*, vol.17, no.14, pp.1343-1362, 2007.

[16] J. Tian and X. J. Xie, Adaptive state-feedback stabilization for high-order stochastic non-linear systems with uncertain control coefficients, *International Journal of Control*, vol.80, no.9, pp.1503-1516, 2007.

[17] Z. J. Wu, X. J. Xie and S. Y. Zhang, Adaptive backstepping controller design using stochastic small-gain theorem, *Automatica*, vol.43, no.4, pp.608-620, 2007.

[18] W. Q. Li and X. J. Xie, Inverse optimal stabilization for stochastic nonlinear systems whose linearizations are not stabilizable, *Automatica*, vol.45, no.2, pp.498-503, 2009.

[19] Z. J. Wu, X. J. Xie, P. Shi and Y. Q. Xia, Backstepping controller design for a class of stochastic nonlinear systems with Markovian switching, *Automatica*, vol.45, no.4, pp.997-1004, 2009.

[20] X. J. Xie and J. Tian, Adaptive state-feedback stabilization of high-order stochastic systems with nonlinear parameterization, *Automatica*, vol.45, no.1, pp.126-133, 2009.

[21] X. J. Xie and W. Q. Li, Output-feedback control of a class of high-order stochastic nonlinear systems, *International Journal of Control*, vol.82, no.9, pp.1692-1705, 2009.

[22] L. Liu and X. J. Xie, Decentralized adaptive stabilization for interconnected systems with dynamic input-output and nonlinear interactions, *Automatica*, vol.46, no.6, pp.1060-1067, 2010.

[23] L. Liu, N. Duan and X. J. Xie, Output-feedback stabilization for stochastic high-order nonlinear systems with a ratio of odd integers power, *Acta Automatic Sinica*, vol.36, no.6, pp.858-864, 2010.

[24] X. J. Xie and N. Duan, Output tracking of high-order stochastic nonlinear systems with application to benchmark mechanical system, *IEEE Trans. on Automatic Control*, vol.55, no.5, pp.1197-1202, 2010.

[25] X. Yu, X. J. Xie and Y. Q. Wu, Further results on output-feedback regulation of stochastic nonlinear systems with SiISS inverse dynamics, *International Journal of Control*, vol.83, no.10, pp.2140-2152, 2010.
[26] X. Yu and X. J. Xie, Output feedback regulation of stochastic nonlinear systems with stochastic iISS inverse dynamics, *IEEE Trans. on Automatic Control*, vol.55, no.2, pp.304-320, 2010.
[27] X. Yu, X. J. Xie and N. Duan, Small-gain control method for stochastic nonlinear systems with stochastic iISS inverse dynamics, *Automatica*, vol.46, no.11, pp.1790-1798, 2010.
[28] X. J. Xie, N. Duan and X. Yu, State-feedback control of high-order stochastic nonlinear systems with SiISS inverse dynamics, *IEEE Trans. on Automatic Control*, vol.56, no.8, pp.1921-1926, 2011.
[29] X. Yu, X. J. Xie and Y. Q. Wu, Decentralized adaptive output-feedback control for stochastic interconnected systems with stochastic unmodeled dynamic interactions, *International Journal of Adaptive Control and Signal Processing*, vol.25, no.8, pp.740-757, 2011.
[30] N. Duan and X. J. Xie, Further results on output-feedback stabilization for a class of stochastic nonlinear systems, *IEEE Trans. on Automatic Control*, vol.56, no.5, pp.1208-1213, 2011.
[31] N. Duan, X. J. Xie and X. Yu, State feedback stabilization of stochastic nonlinear systems with SiISS inverse dynamics, *International Journal of Robust and Nonlinear Control*, vol.21, no.16, pp.1903-1919, 2011.
[32] N. Duan, X. Yu and X. J. Xie, Output feedback control using small-gain conditions for stochastic nonlinear systems with SiISS inverse dynamics, *International Journal of Control*, vol.84, no.1, pp.47-56, 2011.
[33] L. Liu and X. J. Xie, Output-feedback stabilization for stochastic high-order nonlinear systems with time-varying delay, *Automatica*, vol.47, no.12, pp.2772-2779, 2011.
[34] Z. J. Wu, M. Y. Cui, X. J. Xie and P. Shi, Theory of stochastic dissipative systems, *IEEE Trans. on Automatic Control*, vol.56, no.7, pp.1650-1655, 2011.
[35] W. Q. Li, X. J. Xie and S. Y. Zhang, Output-feedback stabilization of stochastic high-order nonlinear systems under weaker conditions, *SIAM Journal on Control and Optimization*, vol.49, no.3, pp.1262-1282, 2011.
[36] L. Liu and X. J. Xie, State-feedback stabilization for stochastic high-order nonlinear systems with SISS inverse dynamics, *Asian Journal of Control*, vol.14, no.4, pp.1-11, 2012.
[37] X. J. Xie and L. Liu, Further result on output feedback stabilization for stochastic high-order nonlinear systems with time-varying delay, *Automatica*, vol.48, no.10, pp.2577-2586, 2012.
[38] M. Y. Cui, Z. J. Wu, X. J. Xie and P. Shi, Modeling and adaptive tracking for a class of stochastic lagrangian control systems, *Automatica*, vol.49, no.3, pp.770-779, 2013.
[39] C. R. Zhao and X. J. Xie, Output feedback stabilization using small-gain method and reduced-order observer for stochastic nonlinear systems, *IEEE Trans. on Automatic Control*, vol.58, no.2, pp.523-528, 2013.
[40] L. Liu and X. J. Xie, State feedback stabilization for stochastic feedforward nonlinear systems with time-varying delay, *Automatica*, vol.49, no.4, pp.936-942, 2013.
[41] X. J. Xie and L. Liu, A homogeneous domination approach to a class of stochastic time-varying delay nonlinear systems, *IEEE Trans. on Automatic Control*, vol.58, no.2, pp.494-499, 2013.

Appendix.

Proof of Lemma 2.4: It is easy to verify that $f(x)$ is continuously differentiable with $\dot{f}(x) = a|x|^{a-1}$ $(a \geq 1)$ for the case of $x \neq 0$. When $x = 0$, one can calculate $f'_+(0) = \lim_{x\to 0^+} x^{a-1} = 0 = \lim_{x\to 0^-}(-x)^{a-1} = f'_-(0)$, i.e., $\dot{f}(0) = 0$. □

Proof of Proposition 3.1: (i) Firstly, by $2 - \frac{1}{p_1\cdots p_{k-1}} \geq 1$, $\frac{2\sigma - r_{k+1}p_k}{r_k p_1\cdots p_{k-1}} \geq 1$ and Lemma 2.4, it is clear that W_{Lk}, W_{Lk}, W_{Dk} are continuously differentiable.

Secondly, when $x_k > \alpha_{k-1}$, from Lemma 2.1, it follows that

$$\begin{aligned} W_{Lk}(\cdot) &\geq c\int_{\alpha_{k-1}}^{x_k} (s-\alpha_{k-1})^{2p_1\cdots p_{k-1}-1}\mathrm{d}s \geq c(x_k-\alpha_{k-1})^{2p_1\cdots p_{k-1}}, \\ W_{Hk}(\cdot) &\geq c\int_{\alpha_{k-1}}^{x_k} (s-\alpha_{k-1})^{\frac{2\sigma - r_{k+1}p_k}{r_k}}\mathrm{d}s \geq c|x_k-\alpha_{k-1}|^{\frac{2\sigma-\omega}{r_k}}. \end{aligned} \tag{21}$$

Similarly, it can be shown that (21) still holds when $x_k \leq \alpha_{k-1}$. (16) together with (11), Lemmas 2.1, 2.3 and Assumption 3.2 implies that

$$\begin{aligned}&W_{Lk}(\cdot) \le |z_k|^{2-\frac{1}{p_1\cdots p_{k-1}}}|x_k-\alpha_{k-1}| \le cz_k^2 \le c\|\bar{z}_{kt}\|_{\mathcal{C}}^2,\\ &W_{Hk}(\cdot) \le |z_k|^{\frac{2\sigma-r_{k+1}p_k}{r_kp_1\cdots p_{k-1}}}|x_k-\alpha_{k-1}| \le c|z_k|^{\frac{2\sigma-\omega}{r_kp_1\cdots p_{k-1}}} \le c\|\bar{z}_{kt}\|_{\mathcal{C}}^{\frac{2\sigma-\omega}{r_kp_1\cdots p_{k-1}}},\end{aligned} \tag{22}$$

$$\begin{aligned}W_{Dk}(\cdot) &\le \left(\frac{b_k}{1-\delta_k}\int_{t-\varepsilon_k}^{t} + \frac{c_k}{1-\delta_{k+1}}\int_{t-\varepsilon_{k+1}}^{t}\right)\left(z_k^2(s)+z_k^{\frac{2\sigma}{r_kp_1\cdots p_{k-1}}}(s)\right)\mathrm{d}s\\ &\le \left(\frac{b_k\varepsilon_k}{1-\delta_k}+\frac{c_k\varepsilon_{k+1}}{1-\delta_{k+1}}\right)\left(\|\bar{z}_{kt}\|_{\mathcal{C}}^2+\|\bar{z}_{kt}\|_{\mathcal{C}}^{\frac{2\sigma}{r_kp_1\cdots p_{k-1}}}\right)\\ &\triangleq \bar{\pi}_{k2}(\|\bar{z}_{kt}\|_{\mathcal{C}}),\end{aligned} \tag{23}$$

where $\bar{\pi}_{k2}(\cdot)$ is a class $\mathcal{K}_\infty$ function obviously. It follows from (10), (13), (21)-(23) and $W_{Dk}(\cdot) \ge 0$ that

$$\begin{aligned}V_k(\cdot) &\le \pi_{k-1,2}(\|\bar{z}_{k-1,t}\|_{\mathcal{C}}) + c\|\bar{z}_{kt}\|_{\mathcal{C}}^2 + c\|\bar{z}_{kt}\|_{\mathcal{C}}^{\frac{2\sigma-\omega}{r_kp_1\cdots p_{k-1}}} + \bar{\pi}_{k2}(\|\bar{z}_{kt}\|_{\mathcal{C}})\\ &\le \pi_{k-1,2}(\|\bar{z}_{kt}\|_{\mathcal{C}}) + c\|\bar{z}_{kt}\|_{\mathcal{C}}^2 + c\|\bar{z}_{kt}\|_{\mathcal{C}}^{\frac{2\sigma-\omega}{r_kp_1\cdots p_{k-1}}} + \bar{\pi}_{k2}(\|\bar{z}_{kt}\|_{\mathcal{C}})\\ &\triangleq \pi_{k2}(\|\bar{z}_{kt}\|_{\mathcal{C}}),\end{aligned} \tag{24}$$

$$\begin{aligned}V_k(\cdot) &\ge \pi_{k-1,1}(\|\bar{z}_{k-1}\|) + c(x_k-\alpha_{k-1})^{2p_1\cdots p_{k-1}}\\ &= \pi_{k-1,1}(\|\bar{z}_{k-1}\|) + c\left(\left(z_k+\alpha_{k-1}^{p_1\cdots p_{k-1}}(z_k)\right)^{\frac{1}{p_1\cdots p_{k-1}}} - \alpha_{k-1}(z_k)\right)^{2p_1\cdots p_{k-1}}\\ &\triangleq W_k(\bar{z}_k).\end{aligned} \tag{25}$$

Obviously, $\pi_{k2}(\cdot)$ is a class $\mathcal{K}_\infty$ function. Next we show that $W_k(\bar{z}_k)$ is positive definite and radially unbounded. On one hand, it is easy to know that $W_k(\bar{z}_k) \ge 0$ and $W_k(\bar{z}_k) = 0$ if and only if $\bar{z}_k = 0$. On the other hand, $\|\bar{z}_k\| \to +\infty$ means $\|\bar{z}_{k-1}\| \to +\infty$ or $|z_k| \to +\infty$. When $\|\bar{z}_{k-1}\| \to +\infty$, we obtain from (25) that $W_k(\bar{z}_k) \ge \pi_{k-1,1}(\|\bar{z}_{k-1}\|) \to +\infty, \|z_k\| \to +\infty$. As for the case of $|z_k| \to +\infty$ and $\|\bar{z}_{k-1}\| \le \bar{M}$ for a finite positive constant $\bar{M}$, since $\alpha_{k-1}(z_{k-1})$ is continuous on z_{k-1} and $|z_{k-1}| \le \|\bar{z}_{k-1}\| \le \bar{M}$, we have

$$W_k(\bar{z}_k) \ge c\left(\left(z_k+\alpha_{k-1}^{p_1\cdots p_{k-1}}(z_{k-1})\right)^{\frac{1}{p_1\cdots p_{k-1}}} - \alpha_{k-1}(z_{k-1})\right)^{2p_1\cdots p_{k-1}} \to +\infty,\ \|\bar{z}_k\| \to +\infty \tag{26}$$

Above analysis implies that $W_k(\bar{z}_k)$ is radially unbounded. Hence, there is a class $\mathcal{K}_\infty$ function $\pi_{k1}(\cdot)$ such that $\pi_{k1}(\|\bar{z}_k(t)\|) \le W_k(\bar{z}_k) \le V_k$, which and (24) arrive at (14).

(ii) Using (1), (13), (14), (16) and Assumption 3.2, we have

$$\begin{aligned}&\dot{V}_k\\ &\le -\sum_{j=1}^{k-1}a_{k-1,j}\left(z_j^2+z_j^{\frac{2\sigma}{r_jp_1\cdots p_{j-1}}}\right) - \sum_{j=1}^{k-1}b_{k-1,j}\left(z_j^2(t-\tau_j(t))+z_j^{\frac{2\sigma}{r_jp_1\cdots p_{j-1}}}(t-\tau_j(t))\right)\\ &\quad -\sum_{j=1}^{k-1}c_{k-1,j}\left(z_j^2(t-\tau_{j+1}(t))+z_j^{\frac{2\sigma}{r_jp_1\cdots p_{j-1}}}(t-\tau_{j+1}(t))\right) - b_k\left(z_k^{\frac{2\sigma}{r_kp_1\cdots p_{k-1}}}(t-\tau_k(t))\right.\\ &\quad \left.+z_k^2(t-\tau_k(t))\right) - c_k\left(z_k^2(t-\tau_{k+1}(t))+z_k^{\frac{2\sigma}{r_kp_1\cdots p_{k-1}}}(t-\tau_{k+1}(t))\right)\\ &\quad +\left(\frac{b_k}{1-\delta_k}+\frac{c_k}{1-\delta_{k+1}}\right)\left(z_k^2+z_k^{\frac{2\sigma}{r_kp_1\cdots p_{k-1}}}\right) + \left(z_k^{2-\frac{1}{p_1\cdots p_{k-1}}}+\mathrm{sgn}(z_k)|z_k|^{\frac{2\sigma-r_{k+1}p_k}{r_kp_1\cdots p_{k-1}}}\right)\\ &\quad \cdot\alpha_k^{p_k} + \left(\mathrm{sgn}(z_k)|z_k|^{\frac{2\sigma-r_{k+1}p_k}{r_kp_1\cdots p_{k-1}}}+z_k^{2-\frac{1}{p_1\cdots p_{k-1}}}\right)(x_{k+1}^{p_k}-\alpha_k^{p_k}) + (x_k^{p_{k-1}}-\alpha_{k-1}^{p_{k-1}})\\ &\quad \cdot\left(z_{k-1}^{2-\frac{1}{p_1\cdots p_{k-2}}}+\mathrm{sgn}(z_{k-1})|z_{k-1}|^{\frac{2\sigma-r_kp_{k-1}}{r_{k-1}p_1\cdots p_{k-2}}}\right) + \left(\mathrm{sgn}(z_k)|z_k|^{\frac{2\sigma-r_{k+1}p_k}{r_kp_1\cdots p_{k-1}}}+z_k^{2-\frac{1}{p_1\cdots p_{k-1}}}\right)\end{aligned}$$

$$\cdot f_k + \sum_{j=1}^{k-1} \left(\frac{\partial W_{Lk}}{\partial x_j} + \frac{\partial W_{Hk}}{\partial x_j} \right) \cdot (x_{j+1}^{p_j} + f_j). \tag{27}$$

Next, we estimate the last three terms on the right-hand side of (27).

First of all, it follows from (11), Lemmas 2.1, 2.2, 2.5 that

$$\begin{aligned} &\left(z_{k-1}^{2-\frac{1}{p_1\cdots p_{k-2}}} + \operatorname{sgn}(z_{k-1})|z_{k-1}|^{\frac{2\sigma - r_k p_{k-1}}{r_{k-1}p_1\cdots p_{k-2}}} \right) (x_k^{p_{k-1}} - \alpha_{k-1}^{p_{k-1}}) \\ &\leq \bar{a}_{k,k-1,1} \left(z_{k-1}^2 + z_{k-1}^{\frac{2\sigma}{r_{k-1}p_1\cdots p_{k-2}}} \right) + \lambda_{k1}(\bar{a}_{k,k-1,1}) \left(z_k^2 + z_k^{\frac{2\sigma}{r_k p_1\cdots p_{k-1}}} \right), \end{aligned} \tag{28}$$

where $\bar{a}_{k,k-1,1} > 0$ and $\lambda_{k1}(\bar{a}_{k,k-1,1}) > 0$ are constants.

In view of (3), one can deduce that

$$\frac{1}{p_1\cdots p_{k-1}} \leq \frac{r_{j+1}p_j}{r_j p_1\cdots p_{k-1}}, \frac{r_{k+1}p_k}{r_j p_1\cdots p_{j-1}}, \frac{r_{k+1}p_k}{r_{j+1}p_1\cdots p_j} \leq \frac{r_{k+1}p_k}{r_j p_1\cdots p_{j-1}}, \quad 1\leq j\leq k-1 \tag{29}$$

which and (11), (29), Lemmas 2.1, 2.2, 2.5 yield

$$\begin{aligned} f_k(\cdot) \leq c(\bar{h}_{k-1})\Bigg(&\sum_{j=1}^{k-1} \left(|z_j(t-\tau_{j+1}(t))|^{\frac{1}{p_1\cdots p_{k-1}}} + |z_j(t-\tau_{j+1}(t))|^{\frac{r_{k+1}p_k}{r_j p_1\cdots p_{j-1}}} \right) \\ &+ \sum_{j=1}^{k} \Bigg(|z_j(t)|^{\frac{1}{p_1\cdots p_{k-1}}} + |z_j(t)|^{\frac{r_{k+1}p_k}{r_j p_1\cdots p_{j-1}}} + |z_j(t-\tau_j(t))|^{\frac{1}{p_1\cdots p_{k-1}}} \\ &+ |z_j(t-\tau_j(t))|^{\frac{r_{k+1}p_k}{r_j p_1\cdots p_{j-1}}} \Bigg)\Bigg). \end{aligned} \tag{30}$$

With the help of (30), Lemmas 2.2, 2.5, and $r_{k+1}p_1\cdots p_k \leq \sigma$, $k = 1, \cdots, n$, we get

$$\begin{aligned} &\left(z_k^{2-\frac{1}{p_1\cdots p_{k-1}}} + \operatorname{sgn}(z_k)|z_k|^{\frac{2\sigma - r_{k+1}p_k}{r_k p_1\cdots p_{k-1}}} \right) f_k \\ &\leq c(\bar{h}_{k-1}) \left(|z_k|^{2-\frac{1}{p_1\cdots p_{k-1}}} + |z_k|^{\frac{2\sigma - r_{k+1}p_k}{r_k p_1\cdots p_{k-1}}} \right) \Bigg(\sum_{j=1}^{k} \Bigg(|z_j(t)|^{\frac{1}{p_1\cdots p_{k-1}}} + |z_j(t)|^{\frac{r_{k+1}p_k}{r_j p_1\cdots p_{j-1}}} \\ &\quad + |z_j(t-\tau_j(t))|^{\frac{1}{p_1\cdots p_{k-1}}} + |z_j(t-\tau_j(t))|^{\frac{r_{k+1}p_k}{r_j p_1\cdots p_{j-1}}} \Bigg) + \sum_{j=1}^{k-1} \Bigg(|z_j(t-\tau_{j+1}(t))|^{\frac{1}{p_1\cdots p_{k-1}}} \\ &\quad + |z_j(t-\tau_{j+1}(t))|^{\frac{r_{k+1}p_k}{r_j p_1\cdots p_{j-1}}} \Bigg)\Bigg) \\ &\leq \sum_{j=1}^{k-2} \bar{a}_{kj1} \left(z_j^2 + z_j^{\frac{2\sigma}{r_j p_1\cdots p_{j-1}}} \right) + \sum_{j=1}^{k} \bar{b}_{kj1} \left(z_j^2(t-\tau_j(t)) + z_j^{\frac{2\sigma}{r_j p_1\cdots p_{j-1}}}(t-\tau_j(t)) \right) \\ &\quad + \bar{a}_{k,k-1,2} \left(z_{k-1}^2 + z_{k-1}^{\frac{2\sigma}{r_{k-1}p_1\cdots p_{k-2}}} \right) + \sum_{j=1}^{k-1} \bar{c}_{kj1} \left(z_j^2(t-\tau_{j+1}(t)) + z_j^{\frac{2\sigma}{r_j p_1\cdots p_{j-1}}}(t-\tau_{j+1}(t)) \right) \\ &\quad + \lambda_{k2}(\bar{a}_{k11}, \cdots, \bar{a}_{k,k-1,2}, b_{k11}, \cdots, b_{kk1}, c_{k11}, \cdots, c_{k,k-1,1}, \bar{h}_{k-1}) \left(z_k^2 + z_k^{\frac{2\sigma}{r_k p_1\cdots p_{k-1}}} \right) \end{aligned} \tag{31}$$

where $\bar{a}_{k11}, \cdots, \bar{a}_{k,k-2,1}, \bar{a}_{k,k-1,2}, \bar{b}_{k11}, \cdots, \bar{b}_{kk1}, \bar{c}_{k11}, \cdots, \bar{c}_{k,k-1,1}$, and $\lambda_{k2}(\bar{a}_{k11}, \cdots, \bar{a}_{k,k-2,1}, \bar{a}_{k,k-1,2}, \bar{b}_{k11}, \cdots, \bar{b}_{kk1}, \bar{c}_{k11}, \cdots, \bar{c}_{k,k-1,1}, \bar{h}_{k-1})$ are some positive constants to be designed.

By (11), (16) and Lemmas 2.1, 2.3, we have

$$\begin{aligned}
&\sum_{j=1}^{k-1}\left(\frac{\partial W_{Lk}}{\partial x_j}+\frac{\partial W_{Hk}}{\partial x_j}\right)(x_{j+1}^{p_j}+f_j)\\
=&-\int_{\alpha_{k-1}}^{x_k}\left(\frac{2p_1\cdots p_{k-1}-1}{p_1\cdots p_{k-1}}\cdot\left(s^{p_1\cdots p_{k-1}}-\alpha_{k-1}^{p_1\cdots p_{k-1}}\right)^{1-\frac{1}{p_1\cdots p_{k-1}}}+\frac{2\sigma-r_{k+1}p_k}{r_kp_1\cdots p_{k-1}}\right.\\
&\left.\cdot\left|s^{p_1\cdots p_{k-1}}-\alpha_{k-1}^{p_1\cdots p_{k-1}}\right|^{\frac{2\sigma-r_{k+1}p_k}{r_kp_1\cdots p_{k-1}}-1}\right)\mathrm{d}s\cdot\sum_{j=1}^{k-1}\frac{\partial\alpha_{k-1}^{p_1\cdots p_{k-1}}}{\partial x_j}(x_{j+1}^{p_j}+f_j)\\
\le& c\left(|z_k|+|z_k|^{\frac{2\sigma-\omega}{r_kp_1\cdots p_{k-1}}-1}\right)\cdot\sum_{j=1}^{k-1}\left|\frac{\partial\alpha_{k-1}^{p_1\cdots p_{k-1}}}{\partial x_j}\right|(|x_{j+1}^{p_j}|+|f_j|)\\
\le& c\left(|z_k|+|z_k|^{\frac{2\sigma-\omega}{r_kp_1\cdots p_{k-1}}-1}\right)\cdot\sum_{j=1}^{k-1}|x_j|^{p_1\cdots p_{j-1}-1}(|x_{j+1}^{p_j}|+|f_j|)\prod_{l=j}^{k-1}\left(1+|z_l|^{\frac{\omega}{r_l}}\right).
\end{aligned}\tag{32}$$

Next, we will prove that for $j=1,\cdots,k-1$, $k=2,\cdots,n$ or $j=1,\cdots,n-1$, $k=j+1,\cdots,n$,

$$\begin{aligned}
&|x_j|^{p_1\cdots p_{j-1}-1}(|x_{j+1}^{p_j}|+|f_j|)\prod_{l=j}^{k-1}\left(1+|z_l|^{\frac{\omega}{r_l}}\right)\\
\le& c(\bar{h}_{j-1})\left(\sum_{i=1}^{k}\left(|z_i|+|z_i|^{\frac{r_kp_1\cdots p_{k-1}+\omega}{r_ip_1\cdots p_{i-1}}}\right)+\sum_{i=1}^{j}\left(|z_i(t-\tau_i(t))|+|z_i(t-\tau_i(t))|^{\frac{r_kp_1\cdots p_{k-1}+\omega}{r_ip_1\cdots p_{i-1}}}\right)\right.\\
&\left.+\sum_{i=1}^{j-1}\left(|z_i(t-\tau_{i+1}(t))|+|z_i(t-\tau_{i+1}(t))|^{\frac{r_kp_1\cdots p_{k-1}+\omega}{r_ip_1\cdots p_{i-1}}}\right)\right).
\end{aligned}\tag{33}$$

The conclusion can be proved by an inductive argument on k.

First of all, by (11), (30) and Lemmas 2.2, 2.5, then for $j=1,\cdots,n-1$, $k=j+1,\cdots,n$,

$$\begin{aligned}
&|x_j|^{p_1\cdots p_{j-1}-1}(|x_{j+1}^{p_j}|+|f_j|)\prod_{l=j}^{k-1}\left(1+|z_l|^{\frac{\omega}{r_l}}\right)\\
\le& c(\bar{h}_{j-1})\left(|z_j|^{1-\frac{1}{p_1\cdots p_{j-1}}}+|z_{j-1}|^{1-\frac{1}{p_1\cdots p_{j-1}}}+|z_{j-1}|^{\frac{r_j(p_1\cdots p_{j-1}-1)}{r_{j-1}p_1\cdots p_{j-2}}}\right)\\
&\cdot\left(\sum_{l=1}^{j}\left(|z_l|^{\frac{1}{p_1\cdots p_{j-1}}}+|z_l|^{\frac{r_{j+1}p_j}{r_lp_1\cdots p_{l-1}}}+|z_l(t-\tau_l(t))|^{\frac{1}{p_1\cdots p_{j-1}}}+|z_l(t-\tau_l(t))|^{\frac{r_{j+1}p_j}{r_lp_1\cdots p_{l-1}}}\right)\right.\\
&\left.+|z_{j+1}|^{\frac{1}{p_1\cdots p_{j-1}}}+\sum_{l=1}^{j-1}\left(|z_l(t-\tau_{l+1}(t))|^{\frac{1}{p_1\cdots p_{j-1}}}+|z_l(t-\tau_{l+1}(t))|^{\frac{r_{j+1}p_j}{r_lp_1\cdots p_{l-1}}}\right)\right)\\
&\cdot\prod_{l=j}^{k-1}\left(1+|z_l|^{\frac{\omega}{r_l}}\right).
\end{aligned}\tag{34}$$

For any $j\in\{1,2,\cdots,n-1\}$, when $k=j+1$, (34) and Lemmas 2.2, 2.5 imply that

$$\begin{aligned}
&\left(1+|z_j|^{\frac{\omega}{r_j}}\right)|x_j|^{p_1\cdots p_{j-1}-1}(|x_{j+1}^{p_j}|+|f_j|)\\
\le& c(\bar{h}_{j-1})\left(1+|z_j|^{\frac{\omega}{r_j}}\right)\left(|z_j|^{1-\frac{1}{p_1\cdots p_{j-1}}}+|z_{j-1}|^{1-\frac{1}{p_1\cdots p_{j-1}}}+|z_{j-1}|^{\frac{r_j(p_1\cdots p_{j-1}-1)}{r_{j-1}p_1\cdots p_{j-2}}}\right)\\
&\cdot\left(\sum_{l=1}^{j}\left(|z_l|^{\frac{1}{p_1\cdots p_{j-1}}}+|z_l|^{\frac{r_{j+1}p_j}{r_lp_1\cdots p_{l-1}}}+|z_l(t-\tau_l(t))|^{\frac{1}{p_1\cdots p_{j-1}}}+|z_l(t-\tau_l(t))|^{\frac{r_{j+1}p_j}{r_lp_1\cdots p_{l-1}}}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=1}^{j-1}\left(|z_l(t-\tau_{l+1}(t))|^{\frac{1}{p_1\cdots p_{j-1}}}+|z_l(t-\tau_{l+1}(t))|^{\frac{r_{j+1}p_j}{r_lp_1\cdots p_{l-1}}}\right)+|z_{j+1}|^{\frac{1}{p_1\cdots p_{j-1}}}\Bigg)\\
&\le c(\bar{h}_{j-1})\left(\sum_{l=1}^{j+1}\left(|z_l|+|z_l|^{\frac{r_{j+1}p_1\cdots p_j+\omega}{r_lp_1\cdots p_{l-1}}}\right)+\sum_{l=1}^{j}\left(|z_l(t-\tau_l(t))|+|z_l(t-\tau_l(t))|^{\frac{r_{j+1}p_1\cdots p_j+\omega}{r_lp_1\cdots p_{l-1}}}\right)\right.\\
&\left.+\sum_{l=1}^{j-1}\left(|z_l(t-\tau_{l+1}(t))|+|z_l(t-\tau_{l+1}(t))|^{\frac{r_{j+1}p_1\cdots p_j+\omega}{r_lp_1\cdots p_{l-1}}}\right)\right),
\end{aligned}\tag{35}
$$

which means that (33) holds for $k=j+1$.

Suppose that (33) holds for $k=m$, $j+1\le m\le n$. Then when $k=m+1$, it can be shown from (34) and Lemmas 2.2, 2.5 that

$$
\begin{aligned}
&|x_j|^{p_1\cdots p_{j-1}-1}(|x_{j+1}^{p_j}|+|f_j|)\prod_{l=j}^{m}\left(1+|z_l|^{\frac{\omega}{r_l}}\right)\\
&\le c(\bar{h}_{j-1})(1+|z_m|^{\frac{\omega}{r_m}})\left(\sum_{i=1}^{j-1}\left(|z_i(t-\tau_{i+1}(t))|+|z_i(t-\tau_{i+1}(t))|^{\frac{r_mp_1\cdots p_{m-1}+\omega}{r_ip_1\cdots p_{i-1}}}\right)\right.\\
&\left.+\sum_{i=1}^{m}\left(|z_i|+|z_i|^{\frac{r_mp_1\cdots p_{m-1}+\omega}{r_ip_1\cdots p_{i-1}}}\right)+\sum_{i=1}^{j}\left(|z_i(t-\tau_i(t))|+|z_i(t-\tau_i(t))|^{\frac{r_mp_1\cdots p_{m-1}+\omega}{r_ip_1\cdots p_{i-1}}}\right)\right)\\
&\le c(\bar{h}_{j-1})\left(\sum_{i=1}^{m+1}\left(|z_i|+|z_i|^{\frac{r_{m+1}p_1\cdots p_m+\omega}{r_ip_1\cdots p_{i-1}}}\right)\right.\\
&+\sum_{i=1}^{j}\left(|z_i(t-\tau_i(t))|+|z_i(t-\tau_i(t))|^{\frac{r_{m+1}p_1\cdots p_m+\omega}{r_ip_1\cdots p_{i-1}}}\right)\\
&\left.+\sum_{i=1}^{j-1}\left(|z_i(t-\tau_{i+1}(t))|+|z_i(t-\tau_{i+1}(t))|^{\frac{r_{m+1}p_1\cdots p_m+\omega}{r_ip_1\cdots p_{i-1}}}\right)\right),
\end{aligned}\tag{36}
$$

which implies that (33) holds for $k=m+1$.

(32) and (33) together with Lemmas 2.2, 2.5 arrive at

$$
\begin{aligned}
&\sum_{j=1}^{k-1}\left(\frac{\partial W_{Lk}}{\partial x_j}+\frac{\partial W_{Hk}}{\partial x_j}\right)(x_{j+1}^{p_j}+f_j)\\
&\le c(\bar{h}_{k-2})\sum_{j=1}^{k-1}\left(|z_k|+|z_k|^{\frac{2\sigma-\omega}{r_kp_1\cdots p_{k-1}}-1}\right)\left(\sum_{i=1}^{j}\left(|z_i(t-\tau_i(t))|+|z_i(t-\tau_i(t))|^{\frac{r_kp_1\cdots p_{k-1}+\omega}{r_ip_1\cdots p_{i-1}}}\right)\right.\\
&\left.+\sum_{i=1}^{k}\left(|z_i|+|z_i|^{\frac{r_kp_1\cdots p_{k-1}+\omega}{r_ip_1\cdots p_{i-1}}}\right)+\sum_{i=1}^{j-1}\left(|z_i(t-\tau_{i+1}(t))|+|z_i(t-\tau_{i+1}(t))|^{\frac{r_kp_1\cdots p_{k-1}+\omega}{r_ip_1\cdots p_{i-1}}}\right)\right)\\
&\le\sum_{j=1}^{k-2}\bar{a}_{kj2}\left(z_j^2+z_j^{\frac{2\sigma}{r_jp_1\cdots p_{j-1}}}\right)+\sum_{j=1}^{k}\bar{b}_{kj2}\left(z_j^2(t-\tau_j(t))+z_j^{\frac{2\sigma}{r_jp_1\cdots p_{j-1}}}(t-\tau_j(t))\right)\\
&+\bar{a}_{k,k-1,3}\left(z_{k-1}^2+z_{k-1}^{\frac{2\sigma}{r_{k-1}p_1\cdots p_{k-2}}}\right)+\sum_{j=1}^{k-1}\bar{c}_{kj2}\left(z_j^2(t-\tau_{j+1}(t))+z_j^{\frac{2\sigma}{r_jp_1\cdots p_{j-1}}}(t-\tau_{j+1}(t))\right)\\
&+\lambda_{k3}(\bar{a}_{k12},\cdots,\bar{a}_{k,k-1,3},\bar{b}_{k12},\cdots,\bar{b}_{kk2},\bar{c}_{k12},\cdots,\bar{c}_{k,k-1,2},\bar{h}_{k-2})\left(z_k^2+z_k^{\frac{2\sigma}{r_kp_1\cdots p_{k-1}}}\right)
\end{aligned}\tag{37}
$$

where $\bar{a}_{k12},\cdots,\bar{a}_{k,k-2,2}, \bar{a}_{k,k-1,3}, \bar{b}_{k12},\cdots,\bar{b}_{kk2}, \bar{c}_{k12},\cdots,\bar{c}_{k,k-1,2}>0$ and $\lambda_{k3}(\bar{a}_{k12},\cdots,\bar{a}_{k,k-2,2},\bar{a}_{k,k-1,3},\bar{b}_{k12},\cdots,\bar{b}_{kk2},\bar{c}_{k12},\cdots,\bar{c}_{k,k-1,2},\bar{h}_{k-2})>0$ are some constants to be designed. Substituting (28)-(31) and (37) into (27), we get

$$\begin{aligned}\dot{V}_k \le & -\sum_{j=1}^{k-1} a_{kj}\left(z_j^2+z_j^{\frac{2\sigma}{r_jp_1\cdots p_{j-1}}}\right)+\left(z_k^{2-\frac{1}{p_1\cdots p_{k-1}}}+\operatorname{sgn}(z_k)|z_k|^{\frac{2\sigma-r_{k+1}p_k}{r_kp_1\cdots p_{k-1}}}\right)\alpha_k^{p_k}\\ &+\left(\lambda_k(\bar{a}_{k11},\cdots,\bar{c}_{k,k-1,2},\bar{h}_{k-1})+\frac{b_k}{1-\delta_k}+\frac{c_k}{1-\delta_{k+1}}\right)\cdot\left(z_k^2+z_k^{\frac{2\sigma}{r_kp_1\cdots p_{k-1}}}\right)\\ &-\sum_{j=1}^{k} b_{kj}\left(z_j^2(t-\tau_j(t))+z_j^{\frac{2\sigma}{r_jp_1\cdots p_{j-1}}}(t-\tau_j(t))\right)\\ &-\sum_{j=1}^{k} c_{kj}\left(z_j^2(t-\tau_{j+1}(t))+z_j^{\frac{2\sigma}{r_jp_1\cdots p_{j-1}}}(t-\tau_{j+1}(t))\right)\\ &+\left(\operatorname{sgn}(z_k)|z_k|^{\frac{2\sigma-r_{k+1}p_k}{r_kp_1\cdots p_{k-1}}}+z_k^{2-\frac{1}{p_1\cdots p_{k-1}}}\right)(x_{k+1}^{p_k}-\alpha_k^{p_k}),\end{aligned}\tag{38}$$

and constants satisfy
$a_{kj}=\begin{cases} a_{k-1,j}-\bar{a}_{kj1}-\bar{a}_{kj2}>0, & j=1,\cdots,k-2\\ a_{k-1,j}-\bar{a}_{k,k-1,1}-\bar{a}_{k,k-1,2}-\bar{a}_{k,k-1,3}>0, & j=k-1\end{cases}$,
$b_{kj}=\begin{cases} b_{k-1,j}-\bar{b}_{kj1}-\bar{b}_{kj2}>0, & j=1,\cdots,k-1\\ b_k-\bar{b}_{kk1}-\bar{b}_{kk2}>0, & j=k\end{cases}$, $c_{kj}=\begin{cases} c_{k-1,j}-\bar{c}_{kj1}-\bar{c}_{kj2}>0,\\ j=1,\cdots,k-1\\ c_k>0, j=k\end{cases}$,
$\lambda_k(\bar{a}_{k11},\cdots,\bar{c}_{k,k-1,2},\bar{h}_{k-1})=\lambda_{k1}+\lambda_{k2}(\bar{a}_{k11},\cdots,\bar{c}_{k,k-1,1},\bar{h}_{k-1})+\lambda_{k3}(\bar{a}_{k12},\cdots,\bar{c}_{k,k-1,2},\bar{h}_{k-2})$.
Choosing the virtual controller

$$\alpha_k(z_k)=-h_k^{\frac{1}{p_1\cdots p_k}}\left(z_k+\operatorname{sgn}(z_k)|z_k|^{\frac{r_{k+1}p_k}{r_k}}\right)^{\frac{1}{p_1\cdots p_k}},\tag{39}$$

where $h_k=2^{p_1\cdots p_{k-1}-1}\left(a_{kk}+\lambda_k(\bar{a}_{k11},\cdots,\bar{c}_{k,k-1,2},\bar{h}_{k-1})+\frac{b_k}{1-\delta_k}+\frac{c_k}{1-\delta_{k+1}}\right)^{p_1\cdots p_{k-1}}$, $a_{kk}>0$, and noticing from Lemmas 2.1, 2.2, 2.5 that

$$\begin{aligned}&-h_k^{\frac{1}{p_1\cdots p_{k-1}}}\left(z_k+\operatorname{sgn}(z_k)|z_k|^{\frac{r_{k+1}p_k}{r_k}}\right)^{\frac{1}{p_1\cdots p_{k-1}}}\cdot\left(z_k^{2-\frac{1}{p_1\cdots p_{k-1}}}+\operatorname{sgn}(z_k)|z_k|^{\frac{2\sigma-r_{k+1}p_k}{r_kp_1\cdots p_{k-1}}}\right)\\ &\le-\left(a_{kk}+\lambda_k(\bar{a}_{k11},\cdots,\bar{c}_{k,k-1,2},\bar{h}_{k-1})+\frac{b_k}{1-\delta_k}+\frac{c_k}{1-\delta_{k+1}}\right)\left(z_k^2+z_k^{\frac{2\sigma}{r_kp_1\cdots p_{k-1}}}\right)\end{aligned}\tag{40}$$

we obtain (15). □