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ROBUST SAMPLED-DATA H_∞ OUTPUT FEEDBACK CONTROL OF ACTIVE SUSPENSION SYSTEM

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Abstract. *This paper concerns with the problem of robust sampled-data H_∞ control for active suspension systems subjected to saturated control input. By using the nature of the sector nonlinear condition, the saturation of the systems' control input is modeled, which is commonly occurred in the practical systems due to the actuator composed by elastic elements. A passenger dynamic is considered in modeling the suspension system wherein the uncertainties aroused from uncertain passengers are assumed to be a polytopic type. Output feedback control strategy is adopted by constructing an output matrix since the measurements of body acceleration and body deflection are unavailable. The controller design is then cast into a convex optimization problem with linear matrix inequalities (LMIs) constraints. An application example is given to illustrate the effectiveness of the developed controller design scheme.*
Keywords: Sampled-data, Suspension control performance, Saturation

1. **Introduction.** Vehicle suspension system has attracted much attention in the past decades since it plays an important role in ride comfort, vehicle safety, road damage minimisation and overall vehicle performance [1, 2, 3, 4]. The main objective of the suspension control system is to improve the ride comfort, and keep the suspension stroke within an acceptable level through isolating from road noise, bumps, and vibrations, etc. However, these requirements are generally at odds, the problem of tuning suspension system then involves finding a right compromise. In general, passive suspension which fully depends on traditional springs and dampers only satisfies some essential requirements, whereas semi-active/active suspension systems allow designers to balance those objectives along with the road profile by using an external power supply. In recent years, many studies have been focused on H_∞ semi-active/active suspension control [5, 6, 7, 8, 9, 10, 11].

Most of the literature available on control input saturation focuses on limiting the control input voltage to guarantee the suspension system safety by using a method of contractive invariant ellipsoid [8, 12, 13]. Few authors have dealt with the problem of nonlinear saturation of the electro-hydraulic mechanism, which is a common occurrence in practical systems, which motives us to do further researches in this study.

The human body dynamic is introduced in modelling process [13, 14] of the active suspension control system to reflect the reality more closely; however, the following two aspects should be considered for designers: 1) Not all the state variables are available, such as the deflection and velocity of passengers. Apparently, the state feedback control strategy cannot be used any more; 2) The system becomes uncertain due to the changeable passengers. The observer-based control method is adopted in [15] where the uncertainty item A_θ is modeled in the observer; however, it is unreasonable due to the unknown parameter used in the observer. Therefore, how to utilize the available measurements reasonably to meet the performance requirements of the suspension system is a meaningful work, which is another motivation of our present study.

The remainder of the paper is organized as follows. The problem formulation is given in Section 2. The controller design according to the technical indices of suspension control performances is presented in Section 3. Section 4 provides the design results and simulations. Finally, the study's findings are summarized in Section 5.

Notation: $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$ is the set of real $n \times m$ matrices; I is the identity matrix of appropriate dimensions; $\|\cdot\|$ stands for the Euclidean vector norm or spectral norm as appropriate. The notation $X > 0$ (respectively, $X < 0$), for $X \in \mathbb{R}^{n\times n}$ means that the matrix X is a real symmetric positive definite (respectively, negative definite). The asterisk $*$ in a matrix is used to denote term that is induced by symmetry.

2. **Problem Formulation.** To develop a more precise model for the active suspension system, biodynamic model established by Wei and Griffin [16] is considered. A 3-DOF quarter-car vertical suspension model is shown in Figure 1, where the human body is separated by two parts according to biodynamic responses, that is, the buttocks and legs part and the upper part, which are interconnected by a spring and a damper. As shown in Figure 1, m_{h1} is the masse of the upper part of a seated man and m_{h2} is the mass of the buttocks and legs together with the seat cushion; $m_s^{'}$ is the sprung mass, which represents the car chassis; m_u is the unsprung mass, which represents the wheel assembly; z_h, z_s and z_u are the displacements of the corresponding masses; k_h and c_h stand for the damping and stiffness of the components inside human body, respectively; k_s and c_s represent the suspension stiffness and damping coefficient, respectively, and k_t and c_t model the tyre stiffness and damping, respectively.

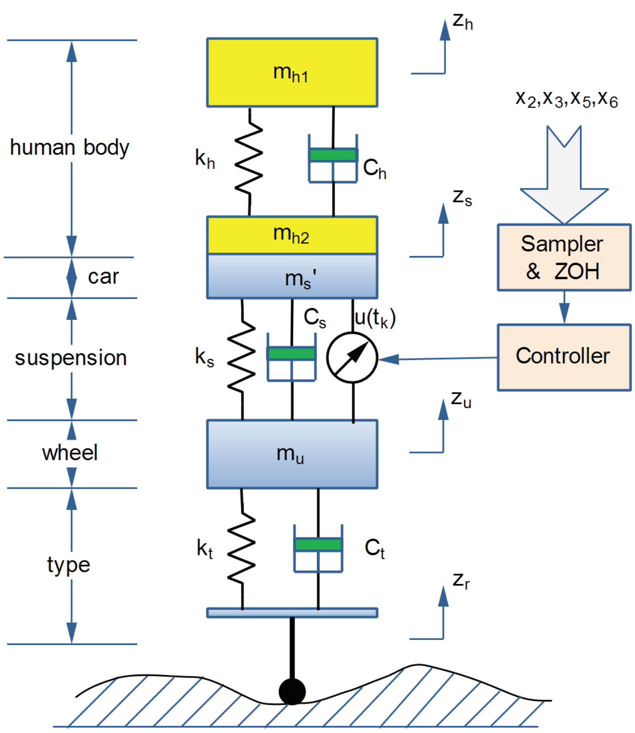

FIGURE 1. Vibration model of seat suspension system

Dynamic equations of the active seat suspension system with consideration of human bodies are expressed as

$$\begin{aligned} m_u\ddot{z}_u &= k_s(z_s - z_u) + c_s(\dot{z}_s - \dot{z}_u) - k_t(z_u - z_r) - c_t(z_u - z_r) + u \\ m_s\ddot{z}_s &= k_h(z_h - z_s) + c_h(\dot{z}_h - \dot{z}_s) - k_s(z_s - z_u) - c_s(\dot{z}_s - \dot{z}_u) - u \\ m_{h_1}\ddot{z}_h &= -k_h(z_h - z_s) - c_h(\dot{z}_h - \dot{z}_s) \end{aligned} \tag{1}$$

where $m_s = m_{h2} + m_s^{'}$.

Define the state variables as follows:

$$\begin{array}{ll} x_1(t) = z_h(t) - z_s(t), & \text{human body deflection} \\ x_2(t) = z_s(t) - z_u(t), & \text{suspension deflection} \\ x_3(t) = z_u(t) - z_r(t), & \text{tyre deflection} \\ x_4(t) = \dot{z}_h(t), & \text{human bodies velocity} \\ x_5(t) = \dot{z}_s(t), & \text{sprung mass velocity} \\ x_6(t) = \dot{z}_u(t), & \text{unsprung mass velocity} \end{array}$$

Then the state-space equation of the active seat suspension model is given by

$$\dot{x}(t) = Ax(t) + B\sigma(u(t)) + B_\omega\omega(t) \tag{2}$$

with consideration of the actuator saturation, where the state vector $x(t) = [x_1(t)\ x_2(t)\ x_3(t)\ x_4(t)\ x_5(t)\ x_6(t)]^T$, and the road disturbance $\omega(t) = \dot{z}_r(t)$, $\sigma(u(t_k)) = \text{sign}(u(t))\min\{u_{\max}, |u(t)|\}$ denotes the control input with a sector nonlinear saturation, and then the system matrices in Equation (2) can be obtained by

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{k_h}{m_{h_1}} & 0 & 0 & -\frac{c_h}{m_{h_1}} & \frac{c_h}{m_{h_1}} & 0 \\ \frac{k_h}{m_s} & -\frac{k_s}{m_s} & 0 & \frac{c_h}{m_s} & -\frac{c_s+c_h}{m_s} & \frac{c_s}{m_s} \\ 0 & \frac{k_s}{m_u} & -\frac{k_t}{m_u} & 0 & \frac{c_s}{m_u} & -\frac{c_s+c_t}{m_u} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1}{m_s} & \frac{1}{m_u} \end{bmatrix}^T$$

$$B_\omega = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & -\frac{c_t}{m_u} \end{bmatrix}^T$$

3. **Robust Controller Design.** In this section, we aim to design a controller such that the active seat suspension system meets the following performances:

(1) Body acceleration z_1 satisfies a prescribed level of H_∞ performance for road disturbance, which is one of the most important performance indices of active suspension system and wildly used for ride comfort. From (2), one can obtain

$$z_1(t) = C_1x(t) \tag{3}$$

where $C_1 = \left[-\frac{k_h}{m_{h_1}}\ 0\ 0\ -\frac{c_h}{m_{h_1}}\ \frac{c_h}{m_{h_1}}\ 0\right]$, then this performance can be expressed by

$$\|z_1\|_2 < \gamma^2\|\omega\|_2 \tag{4}$$

(2) The maximum allowable stroke of the suspension z_{su} should be taken into account since it is related to the ride comfort and the safety of the vehicles structure. The requirement can be described by

$$|z_{su}(t)| = |z_2(t)| \leq z_{2\max} \tag{5}$$

where $z_2(t) = C_2x(t)$ and $C_2 = [0\ I\ 0\ 0\ 0\ 0]$.

Since the body deflection $x_1(t)$ and the body velocity $x_4(t)$, in practice, are unmeasurable, the state feedback for the suspension system with consideration of body dynamics is not suitable any more. Here, we select some measurable states $x_2(t)$, $x_3(t)$, $x_5(t)$, $x_6(t)$ of the suspension system in (2) as the system's output. The the output $y(t)$ is then expressed by

$$y(t) = Cx(t) \tag{6}$$

where $C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

So far, the robust suspension controller of the system (2) can be designed by a way of output feedback control, by which the complicated designing process and computation can be avoided. It follows

$$u(t) = Ky(t) \tag{7}$$

where K is the controller gain to be designed.

Remark 3.1. *The suspension deflection and tyre deflection can be easily measured by using suitable displacement transducer; the sprung and the wheel velocities in vertical direction can be obtained by integrating the sprung and wheel acceleration signal respectively, which can be measured by accelerometer straightforwardly.*

It should be point out that the mass of the passenger is uncertain due to the difference among the passengers. Then the dynamic (2) with consideration of the uncertainties can be further expressed by

$$\dot{x}(t) = A(\alpha)x(t) + B(\alpha)\sigma(u(t)) + B_\omega\omega(t) \tag{8}$$

$$z_1(t) = C_1(\alpha)x(t) \tag{9}$$

where the matrices $A(\alpha)$, $B(\alpha)$ and $C_1(\alpha)$ are constrained within the polytope $\Im$ given by

$$\Im \triangleq \left\{ \Theta(\alpha) | \Theta(\alpha) = \sum_{i=1}^{r} \alpha_i\Theta_i; \sum_{i=1}^{r} \alpha_i = 1, \alpha_i \geq 1 \right\} \tag{10}$$

and $\Theta_i \triangleq (A_i, B_i, C_{1i})$, r is the number of polytope vertices, and $\alpha = \{\alpha_1, \cdots, \alpha_r\}^T$ is the polytope coordinate vector.

Remark 3.2. *Polytope-based method is used to describe the uncertainty arouse from the different passengers in suspension system in this study, which is better than the one in* [13] *expressed by*

$$A(t) = A + \Delta A(t), \quad \Delta A(t) = LF(t)E$$

where L and E are known constant real matrices of appropriate dimensions, and $F(t)$ is an unknown matrix function with Lebesgue-measurable elements satisfying $F^T(t)F(t) \leq I$, because the uncertainty aroused from the different passengers does not vary over time when the passenger is certain.

It is assumed that the measurable state variables of the suspension system are sampled and hold at instant t_k. Then the suspension system can be rewritten as

$$\dot{x}(t) = A(\alpha)x(t) + B(\alpha)\sigma(u(t_k)) + B_\omega\omega(t), \quad t_k \leq t \leq t_{k+1} \tag{11}$$

Defining $d(t) = t - t_k$, we have

$$t_k = t - (t - t_k) = t - d(t), \quad t_k \le t \le t_{k+1} \tag{12}$$

It leads to

$$u(t_k) = KCx(t - d(t)) \quad t_k \le t \le t_{k+1} \tag{13}$$

The following definition will be used to in deriving results.

Definition 3.1. [17] *A nonlinearity $\psi : \mathbb{R}^m \mapsto \mathbb{R}^m$ is said to satisfy a sector condition if*

$$(\psi(v) - L_1 v)^T (\psi(v) - L_2 v) \le 0, \ \ \forall v \in \mathbb{R}^r \tag{14}$$

for some real matrices $L_1, L_2 \in \mathbb{R}^{r\times r}$, where $L = L_2 - L_1$ is a positive-definite symmetric matrix. In this case, we say that belongs to the sector $[L_1, L_2]$.

Assume there exist the diagonal matrices H_1 and H_2 such that $0 \le H_1 < I \le H_2$, then the saturation function $\sigma(u(t_k))$ in Equation (11) can be written as

$$\sigma(u(t_k)) = H_1 u(t_k) + \psi(u(t_k)) \tag{15}$$

where $\psi(u(t_k))$ is a nonlinear vector-valued function which satisfies a sector condition with $L_1 = 0$ and $L_2 = H$, in which $H = H_2 - H_1$, i.e., $\psi(u(t_k))$ satisfies:

$$\psi^T (u(t_k))(\psi(u(t_k) - Hu(t_k))) \le 0 \tag{16}$$

Then for presentation convenience, the closed-loop system can be expressed as

$$\dot{x}(t) = \mathcal{A}(\alpha)\xi(t) \tag{17}$$

where $\mathcal{A}(\alpha) = [A(\alpha) \ B(\alpha)H_1KC \ 0 \ B(\alpha) \ B_\omega]$, and $\xi(t) = [x^T(t) \ x^T(t - d(t)) \ x^T(t - h) \ \psi^T(u(t_k)) \ \omega^T(t)]^T$.

Before providing the solution to the problem of active seat suspension control, we recall the following useful lemmas.

Lemma 3.1. [18] *For any constant matrix $R \in \mathbb{R}^n, R > 0$, scalar $0 \le d(t) \le h$, and vector function $\dot{x} : [-h \ 0] \to \mathbb{R}^n$ such that the following integration is well defined, then it holds that:*

$$-h\int_{t-h}^{0} \dot{x}^T(t)R\dot{x}(t) \le \begin{bmatrix} x(t) \\ x(t-d(t)) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} -R & * & * \\ R & -2R & * \\ 0 & R & -R \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-d(t)) \\ x(t-h) \end{bmatrix} \tag{18}$$

Lemma 3.2. [17, 19] *Let $Y_0(\xi), Y_1(\xi(t)), \cdots, Y_p(\xi(t))$ be quadratic functions of $\xi(t) \in \mathbb{R}^n$*

$$Y_i(\xi(t)) = \xi(t)^T T_i \xi(t), \ \ i = 0, 1, \cdots, p, \tag{19}$$

with $T_i = T_i^T$. Then, the implication

$$Y_0(\xi(t)) \le 0, \cdots, Y_p(\xi(t)) \le 0 \Longrightarrow Y_0(\xi(t)) \le 0 \tag{20}$$

holds if there exist $\kappa_1, \cdots, \kappa_p > 0$ such that

$$T_0 - \sum_{i=1}^{p} \kappa_i^{-1} T_i \le 0 \tag{21}$$

Theorem 3.1. *The sampled-data active suspension system (11) with a consideration of actuator saturation is said to be asymptotically stable and meets the suspension performance (4)-(5), if there exist $P > 0$, $Q > 0$, and $R > 0$ such that*

$$\begin{bmatrix} \Phi(\alpha) & * & * \\ P\bar{\mathcal{A}}(\alpha) & -PR^{-1}P & * \\ \mathcal{C}_1(\alpha) & 0 & -I \end{bmatrix} < 0 \tag{22}$$

$$\begin{bmatrix} -\nu P & * \\ C_2 & -I \end{bmatrix} < 0 \tag{23}$$

where

$$\Phi(\alpha) = \begin{bmatrix} PA(\alpha) + A(\alpha)^T P + Q - R & * & * & * & * \\ C^T K^T H_1^T B(\alpha)^T P + R & -2R & * & * & * \\ 0 & R & -Q - R & * & * \\ B(\alpha)^T P & \kappa HKC & 0 & -\kappa I & * \\ B_\omega P & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}$$

$$\bar{\mathcal{A}}(\alpha) = \begin{bmatrix} hA(\alpha) & hB(\alpha)H_1KC & 0 & B(\alpha) & hB_\omega \end{bmatrix}$$

$$\mathcal{C}_1(\alpha) = \begin{bmatrix} C_1(\alpha) & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof: Choose a Lyapunov functional candidate for the system (17) as

$$V(t) = x^T(t)Px(t) + \int_{t-h}^{t} x^T(s)Qx(s)ds + h\int_{-h}^{0}\int_{t-h}^{t} \dot{x}^T(v)R\dot{x}(v)dvds \tag{24}$$

Then we have

$$\begin{aligned} \dot{V}(t) = & 2x^T P\mathcal{A}(\alpha)\xi(t) + x^T(t)Qx(t) - x^T(t-h)Qx(t-h) \\ & + h^2\dot{x}^T(t)R\dot{x}(t) - h\int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)ds \end{aligned} \tag{25}$$

From Lemma 3.1, we have

$$\begin{aligned} z_1^T(t)z_1(t) - \gamma^2\omega^T(t)\omega(t) + \dot{V}(t) \le & \xi^T(t)[\Phi_0(\alpha) + h^2\mathcal{A}^T(\alpha)R\mathcal{A}(\alpha)]\xi(t) \\ & + z_1^T(t)z_1(t) - \gamma^2\omega^T(t)\omega(t) \end{aligned} \tag{26}$$

where

$$\Phi_0(\alpha) = \begin{bmatrix} PA(\alpha) + A(\alpha)^T P + Q - R & * & * & * & * \\ C^T K^T H_1^T B(\alpha)^T P + R & -2R & * & * & * \\ 0 & R & -Q - R & * & * \\ B(\alpha)^T P & 0 & 0 & 0 & * \\ B_\omega P & 0 & 0 & 0 & 0 \end{bmatrix}$$

Recalling Equation (16), we have

$$\psi^T(u(t_k))(\psi(u(t_k) - HKCx(t - d(t)))) \le 0 \tag{27}$$

which can be written as

$$\xi^T(t)\Phi_1(\alpha)\xi(t) \le 0 \tag{28}$$

where

$$\begin{aligned} \Phi_1(\alpha) &= M * N(\alpha) \\ M &= \begin{bmatrix} 0 & 0 & 0 & I & 0 \end{bmatrix}^T \\ N(\alpha) &= \begin{bmatrix} 0 & -\kappa HKC & 0 & \kappa I & 0 \end{bmatrix} \end{aligned}$$

where $\kappa > 0$. Based on Schur complement and Lemma 3.2, it can be seen that Equation (22) is a sufficient condition to guarantee Equation (27) and

$$z_1^T(t)z_1(t) - \gamma^2\omega^T(t)\omega(t) + \dot{V}(t) < 0 \tag{29}$$

Under zero initial conditions, integrating both side of Equation (29) yields

$$0 < V(t) < \int_0^t [\omega^T(t)\omega(t) - z_1^T(t)z_1(t)]dt \tag{30}$$

from which $\|z_1(t)\|_2 < \gamma\|\omega(t)\|_2$ can be easily obtained for all nonzero $\omega(t) \in L_2[0,\infty)$, H_∞ performance is then established.

Defining $inf\{(V(0) + \int_0^t[\omega^T(t)\omega(t) - z_1^T(t)z_1(t)]dt\} = \vartheta$, where $\vartheta = \nu^{-1}z_{2\max}^2(\nu > 0)$, we can obtain

$$x^T(t)Px(t) < V(t) < V(0) + \int_0^t [\omega^T(t)\omega(t) - z_1^T(t)z_1(t)]dt \leq \vartheta \tag{31}$$

from Equation (29) and Equation (24).

Recalling the 2nd performance of the active suspension system listed in Section 2, we have

$$\|x^T(t)C_2^TC_2x(t)\|_2 < z_{2\max}^2 \tag{32}$$

Note that

$$\|x^T(t)C_2^TC_2x(t)\|_2 = \|x^T(t)P^{\frac{1}{2}}P^{-\frac{1}{2}}C_2^TC_2P^{-\frac{1}{2}}P^{\frac{1}{2}}x(t)\|_2 \tag{33}$$

Combining it with (31), it leads to

$$\|x^T(t)C_2^TC_2x(t)\|_2 < \vartheta\lambda_{\max}\left(P^{-\frac{1}{2}}C_2^TC_2P^{-\frac{1}{2}}\right) \tag{34}$$

where $\lambda(\cdot)$ represents maximal eigenvalue. Obviously,

$$P^{-\frac{1}{2}}C_2^TC_2P^{-\frac{1}{2}} < \nu I \tag{35}$$

guarantees Equation (32) holds. By Schur complement, one can easily know that Equation (35) is equivalent to Equation (23).

Theorem 3.1 gives the conditions to meet the design requirements for the active seat suspension system under the saturated control input. However, the controller gain cannot be obtained directly due to some nonlinear items existing in Equation (22) and Equation (23). Next we will give a method to search the controller gain by a set of tractable LMIs.

Theorem 3.2. *For given scalars ρ, γ, ν and h, the sampled-data closed-loop suspension system (11) with consideration of the saturated control input is asymptotically stable and meets the active suspension performance (4)-(5), if there exist $X > 0$, $W > 0$, $\tilde{Q} > 0$, $\tilde{R} > 0$, and $\kappa > 0$ such that*

$$\begin{bmatrix} \tilde{\Phi}_i & * & * \\ \tilde{\mathcal{A}}_i & -2\rho^2X + \rho\tilde{R} & * \\ \tilde{\mathcal{C}}_{1i} & 0 & -I \end{bmatrix} < 0 \tag{36}$$

$$\begin{bmatrix} -\nu X & * \\ C_2X & -I \end{bmatrix} < 0 \tag{37}$$

$$CX = WC \tag{38}$$

Moreover, the controller parameter is given by $K = YW^{-1}$, *where*

$$\tilde{\Phi}_i = \begin{bmatrix} A_iX + XA_i^T + \tilde{Q} - \tilde{R} & * & * & * & * \\ C^TY^TH_1^TB_i^T + \tilde{R} & -2\tilde{R} & * & * & * \\ 0 & R & -\tilde{Q} - \tilde{R} & * & * \\ \kappa^{-1}B_i^T & HYC & 0 & -\kappa^{-1}I & * \\ B_\omega & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}$$
$$\tilde{\mathcal{A}}_i = \begin{bmatrix} hA_iX & hB_iH_1YC & 0 & hB_i & hB_\omega \end{bmatrix}$$
$$\tilde{\mathcal{C}}_{1i} = \begin{bmatrix} C_{1i}X & 0 & 0 & 0 & 0 \end{bmatrix}$$

Proof: Recalling the definition of $A(\alpha)$, $B(\alpha)$ and $C_1(\alpha)$, from Equation (22) we have

$$\Gamma_i = \begin{bmatrix} \hat{\Phi}_i & * & * \\ P\bar{\mathcal{A}}_i & -PR^{-1}P & * \\ \mathcal{C}_{1i} & 0 & -I \end{bmatrix} < 0, \quad i \in \mathscr{R} \tag{39}$$

where

$$\hat{\Phi}_i = \begin{bmatrix} PA_i + A_i^TP + Q - R & * & * & * & * \\ C^TK^TH_1^TB_i^TP + R & -2R & * & * & * \\ 0 & R & -Q - R & * & * \\ B(\alpha)^TP & \kappa HKC & 0 & -\kappa I & * \\ B_\omega P & 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}$$
$$\bar{\mathcal{A}}_i = \begin{bmatrix} hA_i & hB_iH_1KC & 0 & hB_i & hB_\omega \end{bmatrix}$$
$$\mathcal{C}_{1i} = \begin{bmatrix} C_{1i} & 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that

$$-PR^{-1}P \leq -2\rho^2P + \rho^2R \tag{40}$$

It follows that

$$\hat{\Gamma}_i < 0, \quad i \in \mathscr{R} \tag{41}$$

where $\hat{\Gamma}_i$ is a substitution by replacing the item $-PR^{-1}P$ in Equation (39) with $-2\rho^2P + \rho^2R$.

Defining $P^{-1} = X$, $XQX = \tilde{Q}$, $XRX = \tilde{R}$, $KW = Y$, $J_1 = \text{diag}\{X, X, X, \kappa^{-1}I, I, X, I\}$ and $J_2 = \text{diag}\{X, I\}$, pre- and post-multiplying (41), (23) and their transposes, respectively, together with Equation (38) we can obtain Equations (36) and (37) hold. This completes the proof.

It is observed from Theorem 3.2 that Equations (36) and (37) are feasible problem of LMIs; however, the equality constrain in Equation (38) is difficult to deal with. Now we introduce the following algorithm to address this problem.

It is noted that Equation (38) is equivalent to

$$\text{trace}\left[(CX - WC)^T(CX - WC)\right] = 0 \tag{42}$$

It can be converted to the following optimization problem by using Schur complement

$$\begin{cases} \begin{bmatrix} -\varepsilon I & * \\ WC - CX & -I \end{bmatrix} < 0 \\ \varepsilon \to 0 \end{cases} \tag{43}$$

where the scalar ε is a small enough positive. Then the controller gain can be obtained by (36), (37) and (43).

Remark 3.3. *In* [20, 21]*, the output matrix C is assumed to be full rank; however, in this study, C is not a square matrix. Although some intelligent optimization algorithm can be used to find feasible solutions on continuous-time system with SOF control design, for the sake of technical simplicity, we take the above algorithm to tackle this problem.*

4. **Application Example.** The nominal values of the quarter-car model [8, 22] are listed in Table 1. The proposed approach will be applied to design the controller for the active suspension system.

The uncertainties of human body in Equation (11) can be expressed as

$$m_{h1} = 43.4(1+\theta), \quad m_{h2} = 7.8(1+\theta)$$

where $|\theta| \le 0.1$.

The following 3 different road profiles are considered to illustrate the effectiveness of our proposed method.

a. Shock (see Figure 2(a)). Shock (Single bump) are discrete events of relatively short duration and high intensity, for example, an isolated bump or pothole in an otherwise smooth road surface. Such a disturbance can be described as [8]

$$z_r(t) = \begin{cases} \frac{A}{2}\left(1-\cos\left(\frac{2\pi V_0}{l}t\right)\right), & 0 \le t \le \frac{l}{V_0} \\ 0, & t > \frac{l}{V_0} \end{cases}$$

where A is the height of the bump and l is the length of the bump. Here we choose $A = 0.06$m, $l = 5$m, $V_0 = 4.5$(km/h).

b. Zero-mean white noise as a disturbance $\omega(t)$ (see Figure 2(b)). It represents a case of rough road profile.

c. Superposition of multi-nonlinear functions with different frequency (see Figure 2(c)). It can be described as

$$z_r(t) = 0.02\sin 2\pi t + 0.001\sin 10\pi t + 0.001\sin 16\pi t$$

by which one can analysis the influence on the suspension system with various frequency.

Table 1. Nominal value of the quarter-car model

Model parameters	Symbol	Values	Unit
Body mass	$m_{h1}+m_{h2}$	43.4+7.8	Kg
Body damping rate	c_h	1485	Ns/m
Body stiffness	k_h	44130	N/m
Sprung mass	m_s	972.2	Kg
Suspension stiffness	k_s	42719.6	N/m
Suspension damping rate	c_s	1095	Ns/m
Wheel assembly mass	m_u	113.6	Kg
Tyre stiffness	k_t	101115	N/m
Tyre damping	c_t	14.6	Ns/m

From the technical requirements of the active suspension system stated in Section 2, the following technical parameters are taken: $\gamma = 8$, $H = 1$, $H_1 = 0.5$, $u_{\max} = 1600$N, $z_{2\max} = 0.03$m and $h = 10$ms. From Theorem 3.2 together with its corresponding algorithm, we can obtain the controller gain $K = 10^5 \times [1.3110 \;\; 0.2109 \;\; 0.2233 \;\; -0.0399]$.

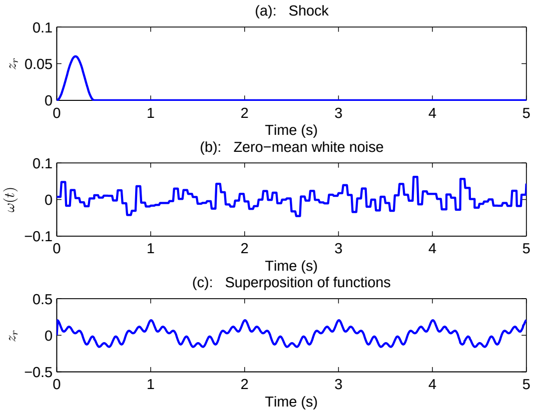

FIGURE 2. Three tyres of road profile

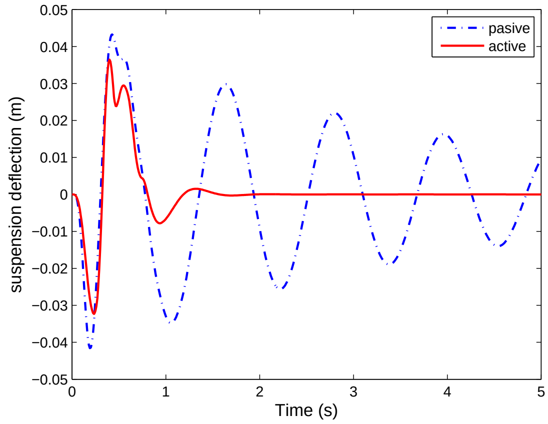

FIGURE 3. Suspension deflection under bump excitation

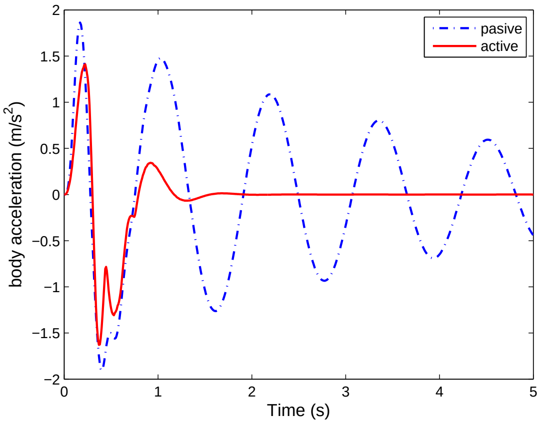

FIGURE 4. Body acceleration under bump excitation

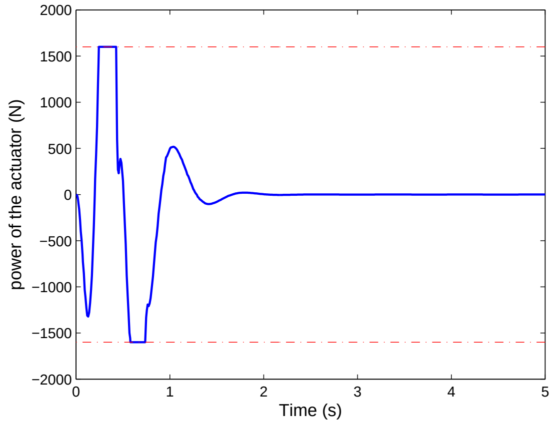

FIGURE 5. Control force under bump excitation

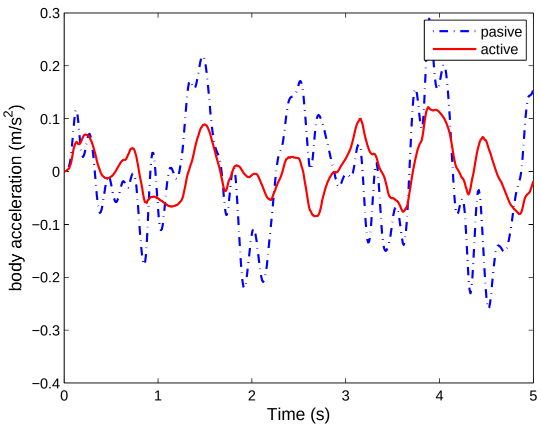

FIGURE 6. Body acceleration under white noise disturbance

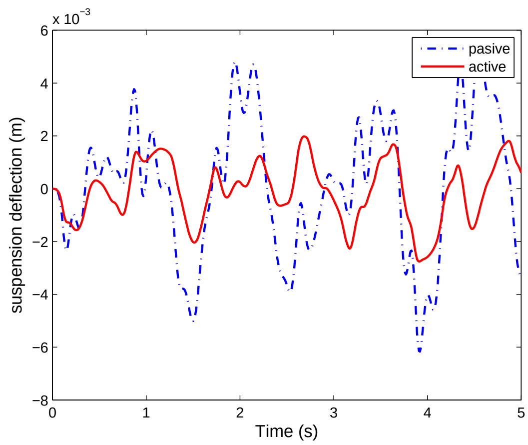

FIGURE 7. Suspension deflection under white noise disturbance

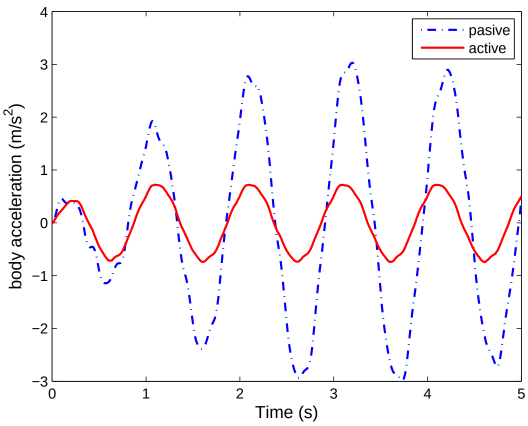

FIGURE 8. Body acceleration under superposition of functions

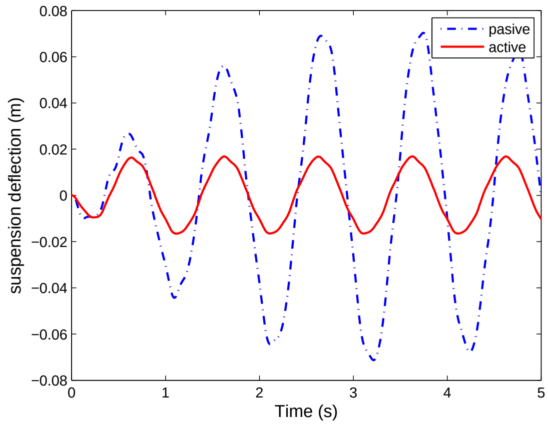

FIGURE 9. Suspension deflection superposition of functions

Next we will evaluate the control quantities from the following aspects: 1) Body acceleration $z_1(t)$; 2) Suspension deflection $z_2(t)$. The responses of the active suspension control system under isolated bump are plotted in Figures 4 and 5, from which it can be seen that the better performance can be got in comparison with the passive mode wherein no actuator is used to provide the active force. Under the disturbance of shock case, the response of body acceleration converges to zero quickly with attenuation ratio around 6:1, settle time around 1.5s and little overshot by using the designed controller subjected to the nonlinear saturation shown in Figure 5. From Figure 4, one can see that the ride comfort is greatly improved, and the safety can be guaranteed from Figure 3.

Figures 6 and 7 demonstrate the effectiveness of the designed controller for the suspension system under white noise disturbance from ground. The responses of body acceleration and suspension deflection are shown in Figures 8 and 9, which illustrate the proposed method can also lead to good control performances under the road profiles with different frequencies, especially in 4-8Hz frequency range which is regarded as more sensitive frequency range to human bodies in the vertical direction according to ISO-2631. From Figures 6-9, one can be seen that the output feedback controller constrained by nonlinear sector saturation can also meet the active suspension design requirements.

5. **Conclusion.** This paper has investigated sampled-data robust H_∞ control of the active suspension system. Based on the requirements of the control performance, such as ride comfort, good handling, and bounded control force, the output feedback control strategy is proposed due to some unavailable physical variables. The controller design is then cast into a convex optimization problem with LMI constraints. Simulation results indicate the designed controller can meet the requirements of the active suspension system against different disturbances.

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