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CONSTRAINED STATE ESTIMATOR FOR NONLINEAR DISCRETE-TIME DYNAMICAL SYSTEMS WITH UNCERTAIN PARAMETERS

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Abstract. *This paper deals with the problem of state estimation for nonlinear stochastic discrete-time dynamical systems with state constraints and uncertain parameters. The nonlinear system and the output measurements are assumed to be corrupted by zero mean white Gaussian noise. Moreover, the system parameters are assumed to be uncertain and the states of the system are assumed to be subject to nonlinear constraints due to physical or safety considerations. The proposed estimator is based on the extended Kalman filter, the active set method, and the multiple projection approach. The proposed estimator is different from other estimators in which a recursive approach is used to satisfy the violated constraints of nonlinear uncertain systems. To illustrate the effectiveness and simplicity of the proposed estimator, a numerical example and a fed-batch penicillin fermentation process are presented. Simulation results show that the proposed estimator leads to a good state estimator while satisfying the state constraints.*

Keywords: Constrained state estimator, Uncertain nonlinear system, Extended Kalman filter, Active set method, Multiple projection approach, Penicillin fermentation process

1. **Introduction.** The Kalman filter (KF) has been widely used for solving estimation problems in many fields of science and engineering. As well known, the KF is the optimal minimum variance filter for linear Gaussian models and it is the optimal linear filter for linear non-Gaussian models [1]. A number of filters were also designed for state estimation of nonlinear systems. Examples of those filters are the moving horizon filter [2], the particle filter [3], the ensemble Kalman filter [4], the unscented Kalman filter (UKF) [5], and the extended Kalman filter (EKF) [6]. The aforementioned filters are not optimal for nonlinear systems and many factors may affect their accuracy and stability [7-9] such as strong nonlinearities, modeling uncertainties, disturbances, bad initializations, time-delays, state equality and/or inequality constraints.

Incorporating of these constraints into the state estimate may result in improved performance. Therefore, several techniques were proposed in the literature to handle these constraints [10]. Some of these techniques have been developed for linear systems with linear equality constraints such as the model reduction technique [11], the constrained KF [12], the equality constraints as noise-free measurements [13], and the Lagrange multipliers approach as a special case of the projection approach [6,14,15]. Indeed, the technique developed in [11] eliminates the physical meaning of states while the technique developed in [13] may lead to a singular covariance matrix.

Other techniques have also been developed for linear systems with inequality constraints as well as for nonlinear systems with equality and/or inequality constraints. Examples of

these techniques are the KF with inequality constraints [16], the moving horizon estimation technique [17], the pseudo-measurement approach [18], the smoothly constrained KF approach [19], the UKF approach [20-23], the projection approach [24,25], the recursive estimator with predictor-corrector optimization method [26], the EKF approach [27,28], and the active set method [29] using the multiple projection approach [30-33].

In this paper, we address the problem of state estimation of nonlinear discrete-time dynamical systems with nonlinear equality and/or inequality constraints and uncertain parameters. The EKF, the active set method, and the multiple projection approach [31-33] are used to develop a recursive estimator. More precisely, the EKF is first modified to deal with the problem of uncertain parameters. Then, the modified EKF is used to estimate the states of the system. If the imposed constraints on the system states are satisfied, we proceed to the next sampling instant. Otherwise, the violated constraints are identified and treated as additional measurements with zero measurement noise. Following the decomposition procedure and the multiple projection approach, the estimator is updated using the assumed additional measurements. This process is repeated till the imposed constraints are satisfied.

The proposed estimator is different from other estimators [10] that are based on the solution of a quadratic optimization problem at the sampling instant where the constraints are violated. Indeed, the proposed estimator is designed based on the multiple projection approach to satisfy the violated constraints of nonlinear uncertain systems in a recursive way. The estimation results of this estimator are consistent, in the sense that all the estimated states and the covariance matrices are updated to comply with the imposed constraints. This is not always the case for the constrained state estimator based on the solution of a quadratic optimization problem, since the results depend on the chosen weighting matrix of the cost function which may lead to inconsistence results. For example, if one of the state variables is subject to an upper bound constraint, and the weighting matrix is diagonal, the result of the quadratic optimization technique is equivalent to just clipping this state without affecting the other states of the system. Moreover, with the proposed approach, the constrained state estimates and the associated covariance matrix obtained at any sampling instant are used to estimate the state vector at the next sampling instant. In fact, this is not the case when using the quadratic optimization approach.

The rest of the paper is organized as follows. The motivation of our research is presented in Section 2. In Section 3, the constrained estimation problem of stochastic nonlinear discrete-time dynamical systems with uncertain parameters is formulated. The dynamics of the constrained state estimator with uncertain parameters is presented in Section 4. Sections 5 and 6 are devoted, respectively, to the presentation of the algorithm of the proposed estimator and the simulation results. The paper is concluded in Section 7.

2. **Motivation of Our Research.**

Most mathematical models of real systems, physical processes, and man made systems are inherently nonlinear. Examples of these are biological systems [34,35], chemical, and biochemical processes [36], microelectronic industry [37], satellite dynamics [38], etc. These systems are also stochastic by their nature since their inputs and outputs are usually corrupted by noise signals. Moreover, the parameters of these system are uncertain due to the impact of the environment (e.g., heat changes, humidity) on the nominal values of their components. In addition, many of these systems are subject to a set of equality and/or inequality constraints. These constraints are either due to physical limitations or safety requirements. See for example penicillin fermentation process [36] and active queue management (AQM) routers supporting transmission control

protocol (TCP) flows [39]. Therefore, we have to deal with such constrained stochastic nonlinear systems with uncertain parameters.

On the other hand, different techniques were proposed in the literature to solve constrained deterministic control problems. Interested reader can refer to [40-46] and the references therein. However, in almost all practical applications it is either very expensive or sometimes physically impossible to measure all the state variables of the system. To implement control strategies based on state feedback, it is required to reconstruct the unmeasured states from the available input and output data. Accordingly, the development of a state estimation algorithm to reconstruct the states of constrained dynamical systems becomes mandatory to deal with these systems. In this paper, we are interested in the state estimation problem of constrained stochastic nonlinear discrete-time dynamical systems with uncertain parameters.

A state estimation problem with uncertain parameters is usually solved in the literature by treating the parameters as static states, i.e., the random parameter vector at any sampling instant equals the random parameter vector at the next sampling instant with known mean value (nominal value) and covariance matrix at $k = 0$. This approach increases the dimensionality of the system, and hence the computational burden. More importantly, although it gives some results if the time horizon is short, for long time horizon the estimator is unstable since the augmented system is neither observable nor detectable. Therefore, our interest in the present research work is to develop a state estimator for constrained stochastic nonlinear discrete-time systems with uncertain parameters. This estimator, beside leading to good estimation results, has to take into consideration the statistical information of the uncertain parameters and avoids the undesirable features of the estimators proposed in the literature.

3. **Problem Formulation.**

In this section, we describe the constrained state estimation problem of stochastic nonlinear discrete-time dynamical systems with uncertain parameters. It is assumed that the state equations of the system, the output measurements and the imposed set of inequality constraints are nonlinear with uncertain parameters.

Consider the following class of nonlinear discrete-time system:

$$x_{k+1} = f_k(x_k, u_k, b_k) + w_k \tag{1}$$

$$y_{k+1} = h_{k+1}(x_{k+1}, c_{k+1}) + v_{k+1} \tag{2}$$

$$\underline{\zeta} \leq g_{k+1}(x_{k+1}, \alpha_{k+1}) \leq \bar{\zeta} \tag{3}$$

where $x_k \in R^n$ (the subscript k is the discrete time instant) is the state vector with initial condition x_0, $u_k \in R^q$ is the input vector and $y_k \in R^m$ is the output vector. The random vectors $w_k \in R^n$ and $v_k \in R^m$ are the input and output noise, respectively. The vectors $b_k \in R^\ell$, $c_k \in R^\iota$ and $\alpha_k \in R^\kappa$ are, respectively, the parameters of the state equations, the output measurements and the constraints, for which the means represent the nominal values of these parameters. The vectors $f_k \in R^n : R^n \to R^n$, $h_{k+1} \in R^m : R^n \to R^m$ and $g_{k+1} \in R^r : R^n \to R^r$ are nonlinear functions of the states and the parameters. The vector function g_{k+1} is physically restricted by the known lower and upper vectors $\underline{\zeta}$ and $\bar{\zeta}$, respectively. We made the following assumption throughout the paper.

Assumption 3.1.

a) The random vectors in (1)-(3) have the following normal distributions $x_0 \sim N(0, P_0)$, $w_k \sim N(0, Q)$, $v_k \sim N(0, R)$, $b_k \sim N(\bar{b}, \Gamma)$, $c_k \sim N(\bar{c}, U)$ and $\alpha_k \sim N(\bar{\alpha}, \Psi)$.

b) In the system (1)-(3) it is assumed that the vectors x_k, y_k, b_k, c_{k+1}, α_{k+1}, w_k and v_{k+1} satisfying the following statistical properties: $E\{x_k w_j^T\} = 0 \ \forall k \leq j$, $E\{x_k v_j^T\} = 0$

$\forall k, j$, $E\{y_k v_j^T\} = 0$ $\forall k < j$, $E\{x_k b_j^T\} = 0$ $\forall k \leq j$, $E\{x_k c_j^T\} = 0$ $\forall k, j$, $E\{y_k c_j^T\} = 0$ $\forall k < j$, $E\{b_k c_j^T\} = 0$ $\forall k, j$, $E\{x_k \alpha_j^T\} = 0$ $\forall k, j$ and $E\{y_k \alpha_j^T\} = 0$ $\forall k, j$.

Our objective is to get a state estimate $\hat{x}_{k+1|k+1}$ of the systems (1)-(3) such that the following inequality constraints are satisfied:

$$\underline{\zeta} \leq g_{k+1}(\hat{x}_{k+1|k+1}, \bar{\alpha}) \leq \bar{\zeta} \tag{4}$$

Remark 3.1. *In general, for nonlinear dynamical systems, the probability distribution functions are non-Gaussian. In order to evaluate the first and second moments of the estimator using the well-known conditional expected properties of Gaussian distribution [1], the EKF gives an approximation of the optimal estimate through the linearization of the nonlinear dynamics around the last state estimate.*

4. **Derivation of the Constrained Estimator.** In this section, we will first present the dynamics of the EKF for nonlinear discrete-time dynamical systems with uncertain parameters. Then, the constrained estimator will be derived using the multiple projection approach.

4.1. **The dynamics of the EKF with uncertain parameters.** Recall the nonlinear discrete-time dynamical system described by Equations (1) and (2). Also, assume that the last filtered state estimate and its associated covariance matrix are given by $\hat{x}_{k|k}$ and $P_{k|k}$, respectively. Moreover, we assume that the conditional probability density function $p(x_k|Y_k)$, $Y_k = [y_1^T, y_2^T, \ldots, y_k^T]^T$, is approximated by Gaussian distribution with conditional mean $\hat{x}_{k|k}$ and covariance matrix $P_{k|k}$.

Let F_k, B_k, H_{k+1} and D_{k+1} be the Jacobean matrices of f_k and h_{k+1} defined by:

$$\begin{aligned} F_k &= \frac{\partial f_k}{\partial x_k}\bigg|_{\hat{x}_{k|k},\bar{b}}, \quad B_k = \frac{\partial f_k}{\partial b_k}\bigg|_{\hat{x}_{k|k},\bar{b}} \\ H_{k+1} &= \frac{\partial h_{k+1}}{\partial x_{k+1}}\bigg|_{\hat{x}_{k+1|k},\bar{c}}, \quad D_{k+1} = \frac{\partial h_{k+1}}{\partial c_{k+1}}\bigg|_{\hat{x}_{k+1|k},\bar{c}} \end{aligned} \tag{5}$$

The following gives the dynamics of the EKF using the multiple projection approach.

a) The predicted estimator of the states of the nonlinear discrete-time dynamical system (1)-(2) and the associated covariance matrix are given by:

$$\hat{x}_{k+1|k} \cong f_k(\hat{x}_{k|k}, u_k, \bar{b}) \tag{6}$$

$$P_{k+1|k} \cong F_k P_{k|k} F_k^T + B_k \Gamma B_k^T + Q \tag{7}$$

$$\hat{y}_{k+1|k} \cong h_{k+1}(\hat{x}_{k+1|k}, \bar{c}) \tag{8}$$

b) The filtered estimate of the states and its associated covariance matrix are such that:

$$\hat{x}_{k+1|k+1} \cong \hat{x}_{k+1|k} + K1_{k+1}[y_{k+1} - \hat{y}_{k+1|k}] \tag{9}$$

$$P_{k+1|k+1} \cong [I - K1_{k+1} H_{k+1}] P_{k+1|k} \tag{10}$$

with

$$K1_{k+1} \cong P_{k+1|k} H_{k+1}^T [H_{k+1} P_{k+1|k} H_{k+1}^T + D_{k+1} U D_{k+1}^T + R]^{-1} \tag{11}$$

$$\hat{y}_{k+1|k+1} \cong h_{k+1}(\hat{x}_{k+1|k+1}, \bar{c}) \tag{12}$$

Proof:

a) Linearizing (1) around $\hat{x}_{k|k}$ and $\bar{b}$ while neglecting higher order terms, we obtain:

$$x_{k+1} \cong f_k\left(\hat{x}_{k|k}, u_k, \bar{b}\right) + F_k[x_k - \hat{x}_{k|k}] + B_k(b_k - \bar{b}) + w_k \tag{13}$$

Taking the conditional expectation of (13) knowing Y_k, yields:

$$\hat{x}_{k+1|k} \cong E\{[f_k\left(\hat{x}_{k|k}, u_k, \bar{b}\right) + F_k[x_k - \hat{x}_{k|k}] + B_k(b_k - \bar{b}) + w_k]|Y_k\} \tag{14}$$

where $\hat{x}_{k|k} = E\{x_k|Y_k\}$, $\hat{x}_{k+1|k} = E\{x_{k+1}|Y_k\}$.

Under Assumption 3.1, it can be shown that the predicted estimate (14) leads to (6).

To get the covariance matrix (7), defining the errors $\tilde{x}_{k|k}$, $\tilde{x}_{k+1|k}$ and $\tilde{b}_k$ as:

$$\begin{aligned} \tilde{x}_{k|k} &= x_k - \hat{x}_{k|k} \\ \tilde{x}_{k+1|k} &= x_{k+1} - \hat{x}_{k+1|k} \\ \tilde{b}_k &= b_k - \bar{b} \end{aligned} \tag{15}$$

Using (6), (13) and (15), we have:

$$\tilde{x}_{k+1|k} \cong F_k \tilde{x}_{k|k} + B_k \tilde{b}_k + w_k \tag{16}$$

The covariance matrix of Equation (16) is given by:

$$P_{k+1|k} = E\{\tilde{x}_{k+1|k} \tilde{x}^T_{k+1|k}\} \tag{17}$$

Using (16) in (17) and taking into consideration Assumption 3.1, it can be shown that the covariance matrix (17) leads to (7).

Now, to get Equation (8), defining the error $\tilde{c}_{k+1}$ as $\tilde{c}_{k+1} = c_{k+1} - \bar{c}$. Then, linearizing (2) around $\hat{x}_{k+1|k}$ and $\bar{c}$ while neglecting higher order terms, we obtain:

$$y_{k+1} \cong h_{k+1}\left(\hat{x}_{k+1|k}, \bar{c}\right) + H_{k+1}\tilde{x}_{k+1|k} + D_{k+1}\tilde{c}_{k+1} + v_{k+1} \tag{18}$$

The conditional expectation of (18) knowing Y_k is given by:

$$\hat{y}_{k+1|k} = E\{y_{k+1}|Y_k\} \tag{19}$$

Using (18) in (19) and taking into consideration Assumption 3.1, it can be shown that (19) leads to (8).

b) To get the filtered estimate (9), we assume that we receive the system measurement y_{k+1} given by (2) at the sampling instant $k+1$. Hence, the predicted estimate $\hat{x}_{k+1|k}$ will be updated taking into consideration the new measurements.

Using (8) and (18) in $\tilde{y}_{k+1|k} =: y_{k+1} - \hat{y}_{k+1|k}$, we obtain:

$$\tilde{y}_{k+1|k} \cong H_{k+1}\tilde{x}_{k+1|k} + D_{k+1}\tilde{c}_{k+1} + v_{k+1} \tag{20}$$

The filtered estimate is given by:

$$\hat{x}_{k+1|k+1} = E\{x_{k+1}|Y^k, y_{k+1}\} \tag{21}$$

Using the properties of conditional expectation of Gaussian random variables, we can write (21) as:

$$\begin{aligned} \hat{x}_{k+1|k+1} &= E\{x_{k+1}|Y_k, y_{k+1}\} \\ &\cong E\{x_{k+1}|Y_k, \tilde{y}_{k+1|k}\} \\ &\cong E\{x_{k+1}|Y_k\} + E\{x_{k+1}|\tilde{y}_{k+1|k}\} \\ &\cong \hat{x}_{k+1|k} + K1_{k+1}\tilde{y}_{k+1|k} \end{aligned} \tag{22}$$

where

$$\begin{aligned} K1_{k+1} &\cong P_{x_{k+1}\tilde{y}_{k+1|k}} P^{-1}_{\tilde{y}_{k+1|k}\tilde{y}_{k+1|k}} \\ &\cong P_{\tilde{x}_{k+1|k}\tilde{y}_{k+1|k}} P^{-1}_{\tilde{y}_{k+1|k}\tilde{y}_{k+1|k}} \\ &\cong E\{\tilde{x}_{k+1|k}\tilde{y}^T_{k+1|k}\}[E\{\tilde{y}_{k+1|k}\tilde{y}^T_{k+1|k}\}]^{-1} \end{aligned} \tag{23}$$

Using (20) in (23) and taking into consideration Assumption 3.1, it can be shown that (22) and (23) lead to (9) and (11), respectively.

The estimation error of the filtered estimate is defined by:

$$\tilde{x}_{k+1|k+1} = x_{k+1} - \hat{x}_{k+1|k+1}$$

Using (15) and (22), $\tilde{x}_{k+1|k+1}$ is such that:

$$\tilde{x}_{k+1|k+1} = \tilde{x}_{k+1|k} - K1_{k+1}\tilde{y}_{k+1|k} \tag{24}$$

The covariance matrix of the filtered estimation error (24) is given by:

$$P_{k+1|k+1} = E\{\tilde{x}_{k+1|k+1}\tilde{x}^T_{k+1|k+1}\} \tag{25}$$

Using (20) and (24) in (25) and taking into consideration Assumption 3.1, it can be shown that (25) leads to (10).

4.2. **Derivation of the constrained estimator.** It is required to compute $\hat{x}_{k+1|k+1}$ which satisfies the imposed system constraints (4).

Let $\hat{x}1_{k+1|k+1}$ and $P1_{k+1|k+1}$ be the filtered estimate of the states and its associated covariance matrix, respectively, before imposing the constraints. Then, (9) can be written as:

$$\hat{x}1_{k+1|k+1} \cong \hat{x}_{k+1|k} + K1_{k+1}[y_{k+1} - \hat{y}_{k+1|k}] \tag{26}$$

with

$$K1_{k+1} \cong P_{k+1|k}H^T_{k+1}[H_{k+1}P_{k+1|k}H^T_{k+1} + D_{k+1}UD^T_{k+1} + R]^{-1} \tag{27}$$

and (11) can be written as:

$$P1_{k+1|k+1} \cong [I - K1_{k+1}H_{k+1}]P_{k+1|k} \tag{28}$$

Consider the following two cases.

Case 1:

If $\hat{x}1_{k+1|k+1}$ satisfies the system constraints (4). Then, we let

$$\hat{x}_{k+1|k+1} = \hat{x}1_{k+1|k+1} \tag{29}$$

$$P_{k+1|k+1} = P1_{k+1|k+1} \tag{30}$$

and proceed to the next sampling instant.

Case 2:

If l constraints of the r imposed constraints (4) are violated by $\hat{x}1_{k+1|k+1}$, then the violated constraints have to be saturated at the corresponding upper or lower bounds which have been violated. Thus,

$$d_{k+1} = \theta_{k+1} \tag{31}$$

where $d_{k+1} \in R^l : R^n \to R^l$ is a vector function containing the l violated constraints from (4). The vector $\theta_{k+1} \in R^l$ contains the corresponding upper or lower bounds which have been violated in (4).

In the sequel, we will denote L_{k+1} and M_{k+1} as the Jacobean matrices of d_{k+1} defined by:

$$L_{k+1} = \frac{\partial d_{k+1}}{\partial x_{k+1}}\Big|_{\hat{x}1_{k+1|k+1},\bar{\alpha}}, \quad M_{k+1} = \frac{\partial d_{k+1}}{\partial \alpha_{k+1}}\Big|_{\hat{x}1_{k+1|k+1},\bar{\alpha}} \tag{32}$$

The following gives the dynamics of the constrained estimator if some of the constraints imposed on the states of the system are not satisfied.

If some of the constraints imposed on the states of the system are not satisfied, the updated filtered estimate and its associated covariance matrix are computed such that:

$$\hat{x}_{k+1|k+1} \cong \hat{x}1_{k+1|k+1} + K2_{k+1}[\theta_{k+1} - \hat{\theta}_{k+1|k+1}] \tag{33}$$

$$P_{k+1|k+1} \cong [I - K2_{k+1}D_{k+1}]P1_{k+1|k+1} \tag{34}$$

with

$$\hat{\theta}_{k+1|k+1} \cong d_{k+1}(\hat{x}1_{k+1|k+1}, \bar{\alpha}) \tag{35}$$

$$K2_{k+1} \cong P1_{k+1|k+1}\, L^T_{k+1}[L_{k+1}P1_{k+1|k+1}L^T_{k+1} + M_{k+1}\Psi M^T_{k+1} + S]^{-1} \tag{36}$$

where $\hat{x}1_{k+1|k+1}$ and $P1_{k+1|k+1}$ are given by Equations (26) and (28), respectively. $S \in R^{l\times l}$ is a positive definite matrix used to compensate for the errors resulting from Taylor series expansion as well as to avoid matrix singularity in (36) during the implementation of the algorithm.

The above constrained estimator is derived as follows.

Let $\tilde{\alpha}_{k+1} = \alpha_{k+1} - \bar{\alpha}$, then linearizing Equation (31) around $\hat{x}1_{k+1|k+1}$ and $\bar{\alpha}$, we obtain:

$$\theta_{k+1} \cong d_{k+1}(\hat{x}1_{k+1|k+1}, \bar{\alpha}) + L_{k+1}\tilde{x}1_{k+1|k+1} + M_{k+1}\tilde{\alpha}_{k+1} + s_{k+1} \tag{37}$$

where L_{k+1} and M_{k+1} are given by (32) and s_{k+1} is the higher order terms represented by a zero mean Gaussian random variable with covariance matrix S.

Equation (37) is now considered as a new received measurement at the sampling instant $k+1$. Therefore, using the multiple projection theorem presented in the previous subsection, the updated filtered estimator is given by:

$$\hat{x}_{k+1|k+1} = E\{x_{k+1}|Y_k, y_{k+1}, \theta_{k+1}\} \tag{38}$$

which implies that:

$$\hat{x}_{k+1|k+1} \cong E\{x_{k+1}|Y_k, \tilde{y}_{k+1|k}\} + E\{x_{k+1}|\tilde{\theta}_{k+1|k+1}\} \tag{39}$$

where

$$\tilde{\theta}_{k+1|k+1} = \theta_{k+1} - \hat{\theta}_{k+1|k+1} \tag{40}$$

with

$$\begin{aligned} \hat{\theta}_{k+1|k+1} &\cong E\{\theta_{k+1}|Y_k, \tilde{y}_{k+1|k}\} \\ &\cong E\{d_{k+1}(x_{k+1}, \alpha_{k+1})|Y_k, \tilde{y}_{k+1|k}\} \\ &\cong d_{k+1}(\hat{x}1_{k+1|k+1}, \bar{\alpha}) \end{aligned} \tag{41}$$

It is important to note that (41) is obtained from (37) because Y_k, $\tilde{\alpha}_{k+1}$ and s_{k+1} are assumed to be mutually independent.

Using (37) and (41) in (40), we obtain:

$$\tilde{\theta}_{k+1|k+1} \cong L_{k+1}\tilde{x}1_{k+1|k+1} + M_{k+1}\tilde{\alpha}_{k+1} + s_{k+1} \tag{42}$$

Since the first term in the right hand side of (39) equals $\hat{x}1_{k+1|k+1}$ (see (21)),

$$\begin{aligned} \hat{x}_{k+1|k+1} &\cong \hat{x}1_{k+1|k+1} + E\{x_{k+1}|\tilde{\theta}_{k+1|k+1}\} \\ &\cong \hat{x}1_{k+1|k+1} + K2_{k+1}\tilde{\theta}_{k+1|k+1} \\ &\cong \hat{x}1_{k+1|k+1} + K2_{k+1}[\theta_{k+1} - \hat{\theta}_{k+1|k+1}] \end{aligned} \tag{43}$$

where the gain matrix $K2_{k+1}$ in (43) is such:

$$\begin{aligned} K2_{k+1} &\cong P_{x_{k+1}\tilde{\theta}_{k+1|k+1}} P^{-1}_{\tilde{\theta}_{k+1|k+1}\tilde{\theta}_{k+1|k+1}} \\ &\cong P_{\tilde{x}1_{k+1|k+1}\tilde{\theta}_{k+1|k+1}} P^{-1}_{\tilde{\theta}_{k+1|k+1}\tilde{\theta}_{k+1|k+1}} \\ &\cong E\{\tilde{x}1_{k+1|k+1}\tilde{\theta}^T_{k+1|k+1}\}[E\{\tilde{\theta}_{k+1|k+1}\tilde{\theta}^T_{k+1|k+1}\}]^{-1} \end{aligned} \tag{44}$$

Using (42) in (44) and taking into consideration Assumption 3.1, it can be shown that (43) and (44) lead to (33) and (36), respectively.

Moreover, the covariance matrix $P_{k+1|k+1}$ is such:

$$P_{k+1|k+1} = E\{\tilde{x}_{k+1|k+1}\tilde{x}^T_{k+1|k+1}\} \tag{45}$$

with,

$$\begin{aligned} \tilde{x}_{k+1|k+1} &= x_{k+1} - \hat{x}_{k+1|k+1} \\ &\cong \tilde{x}1_{k+1|k+1} - K2_{k+1}\tilde{\theta}_{k+1|k+1} \end{aligned} \tag{46}$$

Thus,

$$P_{k+1|k+1} = E\{(\tilde{x}1_{k+1|k+1} - K2_{k+1}\tilde{\theta}_{k+1|k+1})(\tilde{x}1_{k+1|k+1} - K2_{k+1}\tilde{\theta}_{k+1|k+1})^T\} \tag{47}$$

Using (42) in (47) and taking into consideration Assumption 3.1, it can be shown that (47) leads to (34).

5. **Algorithm.**
Step 1: Initialize the value of $\hat{x}_{0|0}$, $P_{0|0} = P_0$, Q, R, S, Γ, U, Ψ and $k = 0$.
Step 2: Calculate F_k, B_k, H_{k+1} and D_{k+1} using (5).
Step 3: Calculate $\hat{x}_{k+1|k}$, $P_{k+1|k}$ and $\hat{y}_{k+1|k}$ using (6)-(8), respectively.
Step 4: Calculate $K1_{k+1}$, $\hat{x}1_{k+1|k+1}$ and $P1_{k+1|k+1}$ using (27), (26) and (28), respectively.
Step 5: If all the constraints (4) are satisfied, then
Go to Step 6
Else
Identify the violated constraints and formulate Equation (31).
Calculate L_{k+1} and M_{k+1}, $\hat{\theta}_{k+1|k+1}$, $K2_{k+1|k+1}$, $\hat{x}_{k+1|k+1}$ and $P_{k+1|k+1}$ using (32), (35), (36), (33) and (34), respectively.
Assign $\hat{x}1_{k+1|k+1} = \hat{x}_{k+1|k+1}$ and $P1_{k+1|k+1} = P_{k+1|k+1}$.
Go to Step 5
Step 6: Assign $x_{k+1|k+1} = \hat{x}1_{k+1|k+1}$ and $P_{k+1|k+1} = P1_{k+1|k+1}$.
Set $k = k + 1$.
If $k = k_f$ (k_f is the last sampling instant), then
Stop
Else
Go to Step 2

Remark 5.1.
a) The above algorithm is recursive in the sense that if a subset of the constraints is violated at any sampling time, it has to be saturated to the violated bounds, and the corresponding equality constraints are used to update the estimator using the multiple projection approach.
b) One cannot guarantee the satisfaction of the violated constraints through one iteration of Step 5. Therefore, one has to repeat Step 5 till the satisfaction of all the constraints.
c) It is important to note that, we do not need to know a priori the number of violated constraints at any sampling time in order to apply the multiple projection approach.
d) The update procedure of Step 5 in the above algorithm is equivalent to the maximum likelihood update (see Appendix A).
e) The rate of convergence of the proposed update estimator is as stated for Gauss-Newton method [47,48] (see Appendix B).

6. **Simulation Results.** In this section, two examples are given to illustrate effectiveness and efficiency of the proposed constrained state estimator for nonlinear stochastic discrete-time systems with uncertain parameters. The first example is a numerical one with uncertain parameters, while the second example deals with the practical problem of a fed-batch penicillin fermentation process.

Example 6.1. *Let us consider a discrete system represented by the following nonlinear difference equations:*

$$\begin{aligned} x_{1_{k+1}} &= 0.9x_{1_k} + b_k x_{2_k} + w_{1_k} \\ x_{2_{k+1}} &= 0.2x_{1_k} + 0.1x_{2_k} - 0.1x_{2_k}^2 + w_{2_k} \\ y_{k+1} &= c_{k+1}x_{1_{k+1}} + v_{k+1} \end{aligned} \tag{48}$$

with initial condition vector $x_0 = [0.5 \ \ -0.5]^T$ *and constraint* $x_{1_k}^2 + x_{2_k}^2 \leq 2$. *The states and the output of the system are assumed to be corrupted by a white Gaussian random*

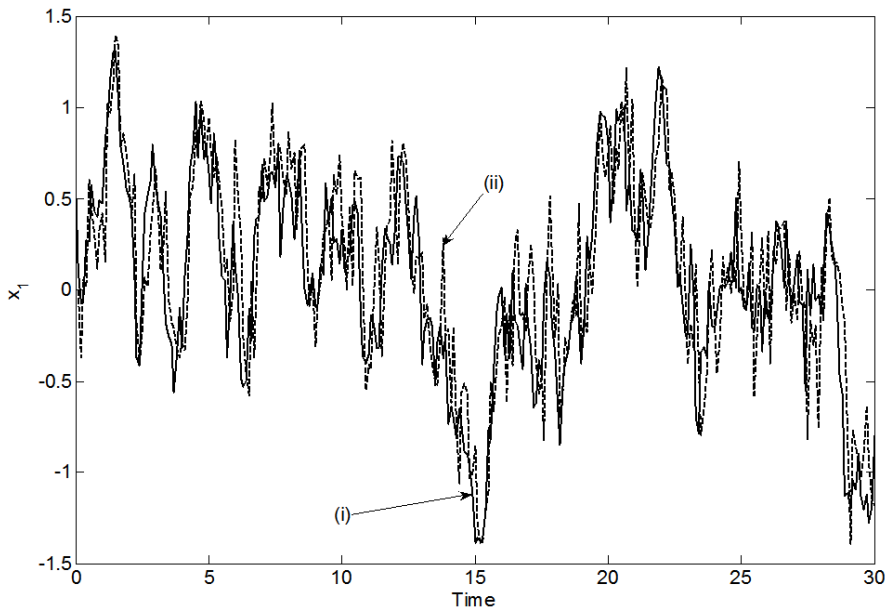

FIGURE 1. State x_1 (i) and its estimate $\hat{x}_1$ (ii)

noise sequences w_{i_k} $(i = 1, 2)$ *and* v_k*, respectively. The random parameters in (48) have the following normal distributions:* $b_k \sim N(0.1, 0.03)$ *and* $c_k \sim N(1, 0.1)$*. For simulation purpose, the initial condition of the estimator is given by:* $\hat{x}_{0|0} = [0 \;\; 0]^T$*. The different covariance matrices are such that:* $P_{0|0} = I_2$*,* $Q = 0.5I_2$*,* $R = U = \Gamma = 0.1$*,* $S = 0.01$ *and* $\Psi = 0$*. The simulation results presented in Figures 1-3 show the two states and the constraint, respectively. In these figures, the solid lines* $(-)$ *represent the actual values of the states* x_1 *and* x_2 *as well as the actual value of the imposed constraint, while the dashed lines* $(- - -)$ *represent the corresponding estimated values of the states* x_1 *and* x_2*. From these figures it is clear that the estimated states follow the actual ones while the imposed constraint is strictly satisfied.*

Example 6.2. *In this subsection, the proposed estimator is applied to the mathematical model of penicillin fermentation process [36]. The dynamics of penicillin fermentation process are complicated due to the complexity of cellular metabolism and the parameters varying with the environmental conditions. Penicillin can be substantially produced with bioreactor technology. In this case, nutrients also called substrates are continuously supplied to the bioreactor to sustain cell growth. Then, the unconsumed nutrients, biomass, and penicillin products are continuously removing from the bioreactor to maintain the process volume constant.*

The discrete-time version of the continuous-time nonlinear model of penicillin fermentation process [36] is given by the following difference equations:

$$X_{k+1} = X_k + T_s \left[\frac{\mu_m S_k}{(1/K_i)S_k^2 + S_k + K_m} X_k - \frac{u_k}{V_k} X_k \right] + w_{1_k}$$

$$S_{k+1} = S_k + T_s \left[\frac{-\mu_m S_k}{((1/K_i)S_k^2 + S_k + K_m)Y_X} X_k - \frac{\nu}{Y_P} X_k + \frac{u_k}{V_k}(S_{in_k} - S_k) \right] + w_{2_k}$$

$$F_{k+1} = F_k + T_s \left[\nu X_k - \frac{u_k}{V_k} F_k \right] + w_{3_k}$$

$$
\begin{aligned}
V_{k+1} &= V_k + T_s u_k + w_{4_k} \\
y_{1_{k+1}} &= S_{k+1} + v_{1_k} \\
y_{2_{k+1}} &= V_{k+1} + v_{2_k}
\end{aligned}
\tag{49}
$$

where X, S, F and V denote, respectively, the concentration of biomass, the concentration of substrate, the concentration of product and the process volume. The measurements of the process are the concentration of substrate S and the process volume V, while the

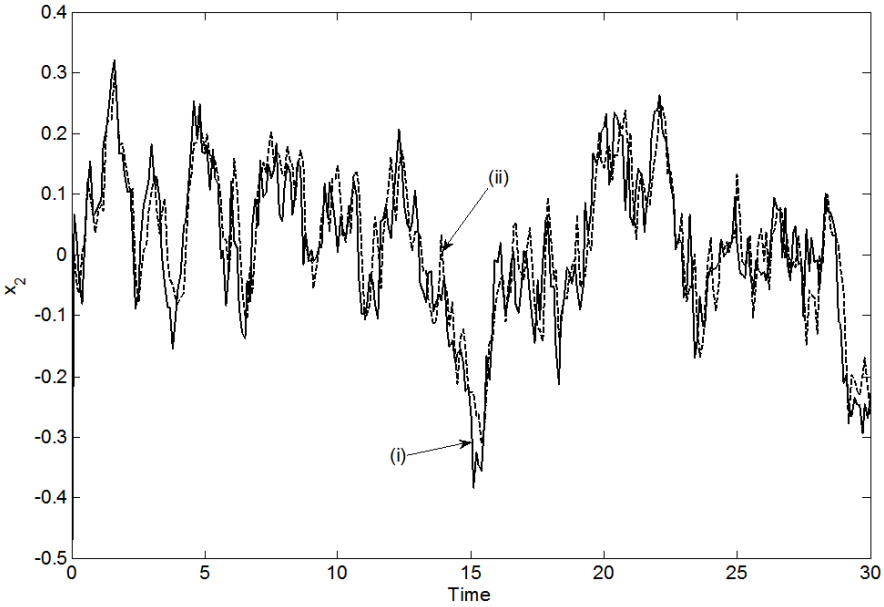

FIGURE 2. State x_2 (i) and its estimate $\hat{x}_2$ (ii)

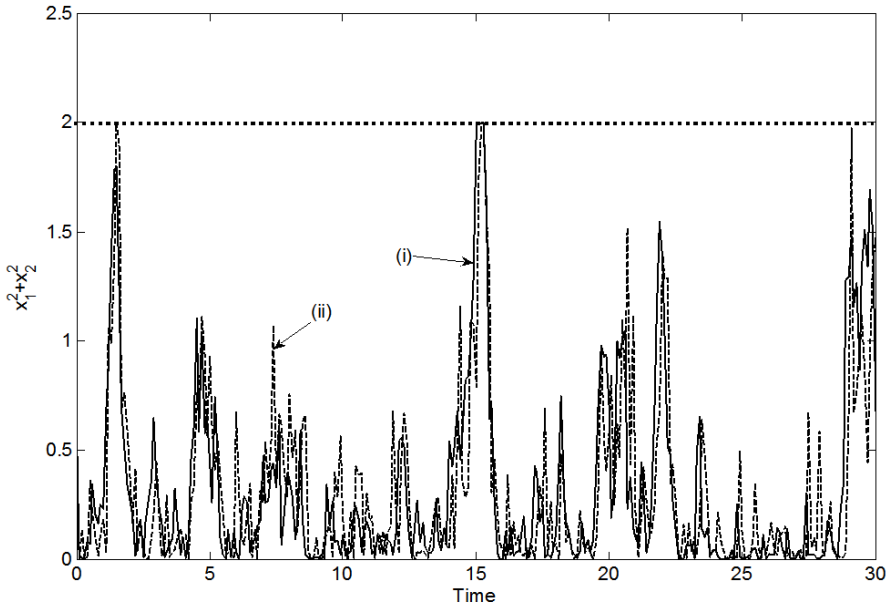

FIGURE 3. Constraint $x_1^2 + x_2^2$ (i) with its estimate $\hat{x}_1^2 + \hat{x}_2^2$ (ii) and upper limit (dotted line)

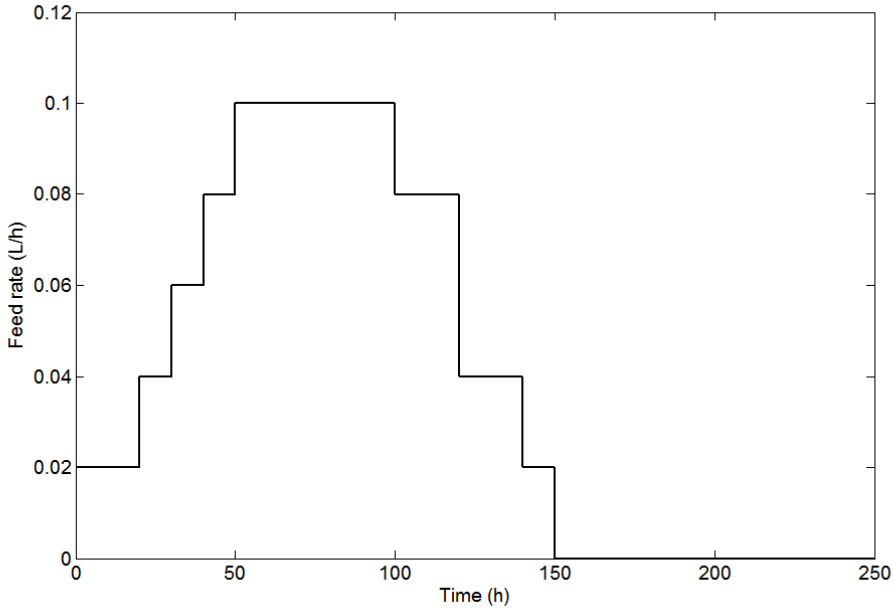

FIGURE 4. Feed rate u vs. time

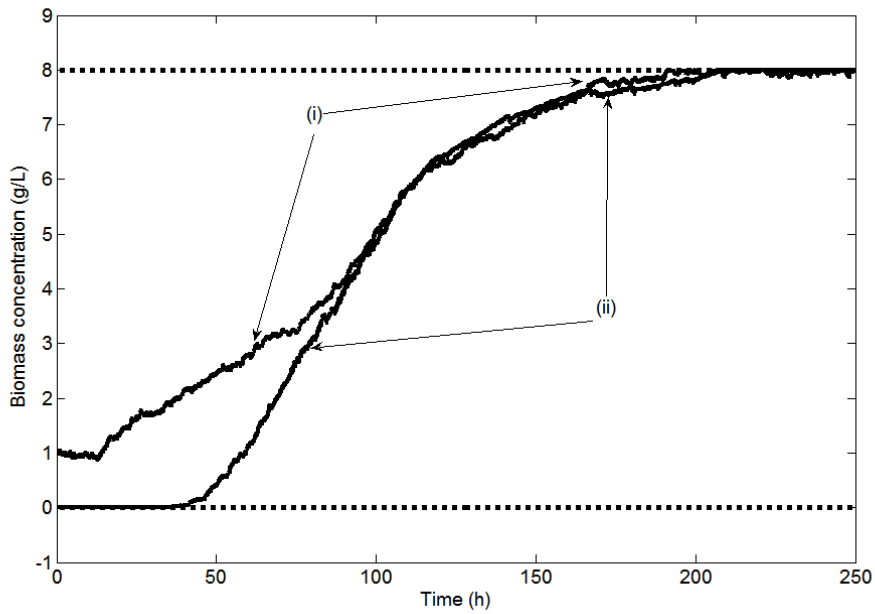

FIGURE 5. Constraints (dotted lines), biomass concentration (i) and its estimate (ii)

inputs to the process are the feed rate u and the concentration of inlet substrate S_{in}. The states and outputs of the system are assumed to be corrupted by a white Gaussian random noise sequences w_{i_k} $(i = 1, \ldots, 4)$ and v_{i_k} $(i = 1, 2)$, respectively. The model parameters are: the kinetic parameters μ_m (L/h), K_i (g/L), K_m (g/L) and ν (L/h); and the yield coefficients Y_X $(g[X]/g[S])$ and Y_P $(g[P]/g[S])$. These parameters are assumed to be random with the following normal distributions: $\mu_m \sim N(0.02, 0.001)$, $K_i \sim N(5, 0.25)$, $K_m \sim N(0.05, 0.0025)$, $\nu \sim N(0.004, 0.0002)$, $Y_X \sim N(0.5, 0.025)$ and $Y_P \sim N(1.2, 0.06)$.

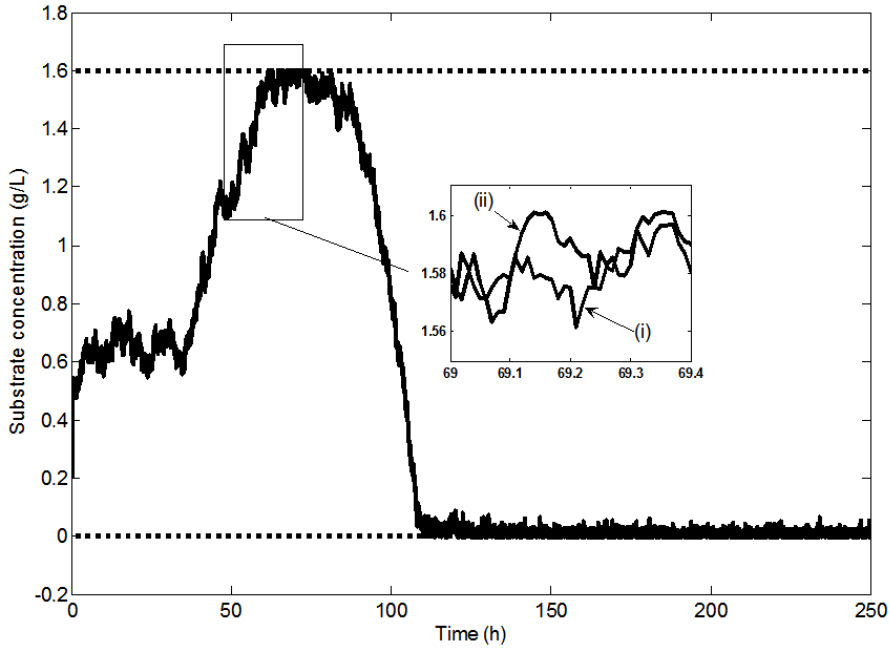

FIGURE 6. Constraints (dotted lines), substrate concentration (i) and its estimate (ii)

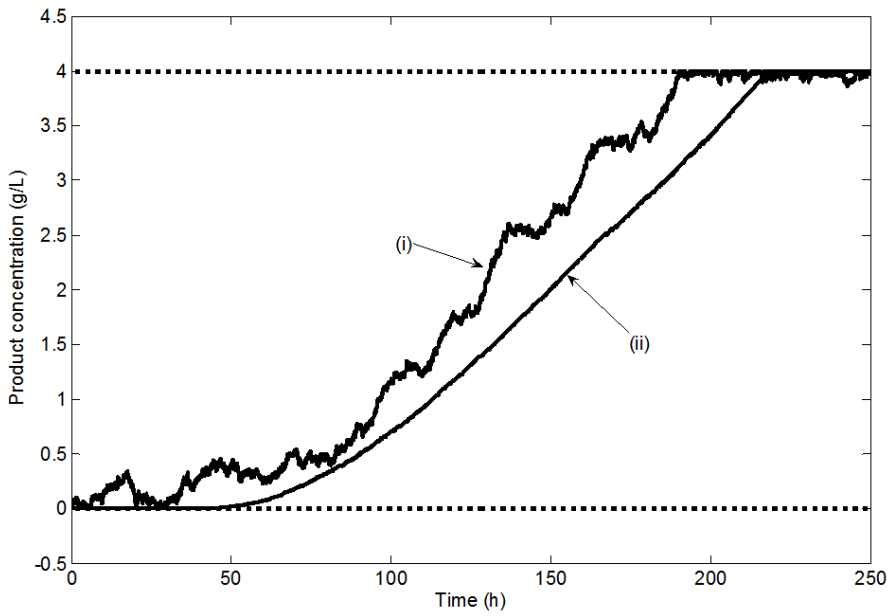

FIGURE 7. Constraints (dotted lines), product concentration (i) and its estimate (ii)

The sampling time is $T_s = 0.01\ h$ and the initial conditions are $X_0 = 1$, $S_0 = 0.5$, $F_0 = 0$, and $V_0 = 150$. The inputs are $S_{in} = 200$ and u which is given by Figure 4.

Due to the physical constraints, the concentrations of biomass, substrate and product are restricted by $0 \leq X \leq 8$, $0 \leq S \leq 1.6$ and $0 \leq P \leq 4$. Also, the volume of the

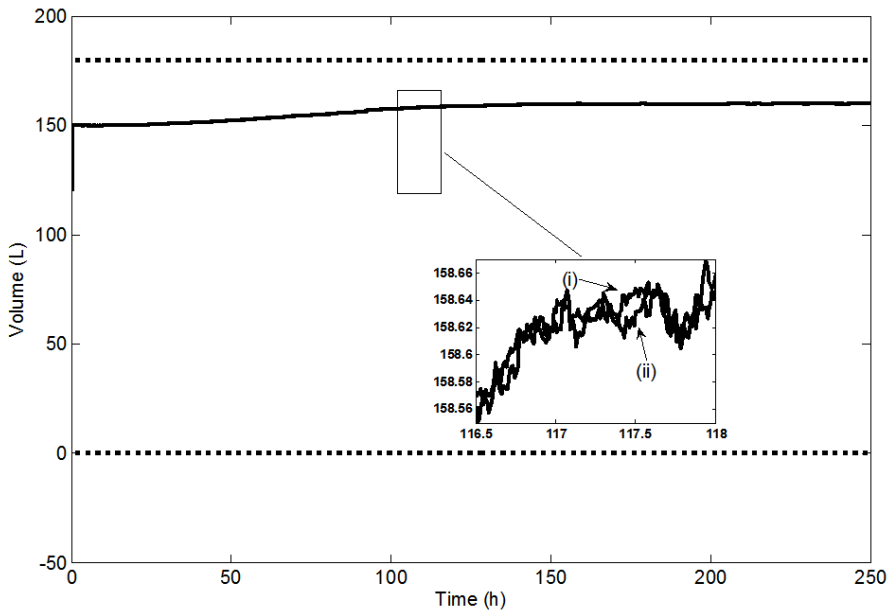

FIGURE 8. Constraints (dotted lines), process volume (i) and its estimate (ii)

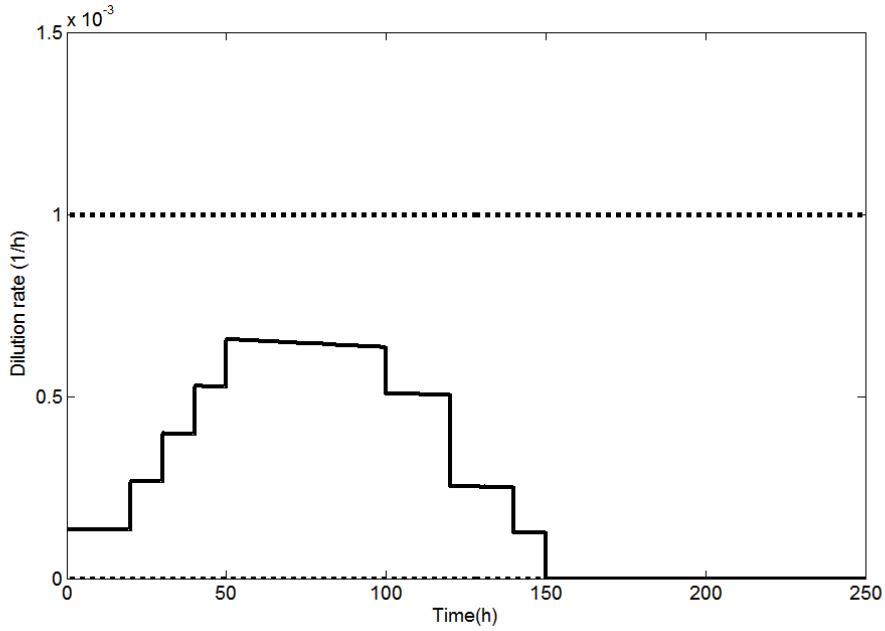

FIGURE 9. Dilution rate D

fermentation process is bounded by $0 \leq V \leq 180$ and the dilution rate $D_k = u_k/V_k$ is restricted by $0 \leq D \leq 0.001$.

For simulation purpose, the covariance matrices are defined by $P_{0|0} = 0.05I_4$ (where I_n is $n \times n$ identity matrix), $Q = 0.05I_4$, $R = 0.5I_2$, $S = 0.85I_l$, $\Gamma = 0.01I_6$, $U = 0$, and $\Psi = 0$. The simulation results presented in Figures 5-9 show the concentration of biomass, the concentration of substrate, the concentration of product, the process volume and the dilution rate, respectively. It is clear from Figures 5-9 that the estimates of the states satisfy the imposed system bounds.

By applying the approach presented in the literature in which the uncertain parameters $\{\mu_m, K_i, K_m, v, Y_x, Y_p\}$ are treated as static states, i.e., $\beta_{k+1} = \beta_k$ with the prior at $k = 0$ being $N(\bar{\beta}, \sum)$ where $\bar{\beta}$ is the vector of the mean or nominal values of the parameters, and $\sum$ is its covariance matrix, we ended with an unobservable undetectable augmented system. As a result, this will lead to unstable state estimator of the system.

7. **Conclusion.** In this paper, a robust state estimator is developed for nonlinear stochastic discrete-time dynamical systems with uncertain parameters. The system model as well as the measured output vector is assumed to be corrupted by zero mean white Gaussian noise. The statistical data of the parameters as well as the input and output noise vectors are assumed to be known. The developed approach is based on the EKF and the active set method, in which multiple projection approach is used to derive the state estimator which satisfies the imposed system constraints. Two illustrative examples are presented to show the effectiveness of the developed approach. Simulation results show that the proposed estimator gives good estimate results of the states, while satisfying system constraints.

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Appendix A. The Maximum Likelihood Update.

Since the update procedure is applied only at the sampling instants at which some of the constraints are violated, it is independent of the dynamics of the system [49]. Therefore, the update estimator is equivalent to the static estimation problem in which we correct the current state estimator using the assumed new received subset of measurements in which the interval constraint is enforced. To simplify our notation, and without introducing any ambiguity, the subscript indicates the time is omitted from the variables.

Linearizing Equation (31) around $\bar{\alpha}$, we obtain:

$$\theta \cong d(x, \bar{\alpha}) + M\tilde{\alpha} + s \tag{50}$$

where $M = \frac{\partial d}{\partial \alpha}|\bar{\alpha}$, $\tilde{\alpha} = \alpha - \bar{\alpha}$ and s is the higher order terms represented by a zero mean Gaussian random variable with covariance matrix S.

Assumption A.1.

a) At the ith iteration, $\hat{x}_i$ is a random vector with the normal distribution $\hat{x}_i \sim N(x, P_i)$.
b) The vector $\hat{x}_{i+1}$ is the estimate we are looking for, i.e., $x = \hat{x}_{i+1}$. Also, the vector $\hat{x}_{i+1}$ is close enough to $\hat{x}_i$

Remark A.1.

c) Assumption A.1: a) means the random vector θ given by (50) has the normal distribution $\theta \sim N(d(x, \bar{\alpha}), \bar{S})$ where $\bar{S} = M\Psi M^T + S$.
d) Assumption A.1: b) can be used to replace $d(x, \bar{\alpha})$ by its first order approximation, i.e., $d(\hat{x}_{i+1}, \bar{\alpha}) = d(\hat{x}_i, \bar{\alpha}) + L(\hat{x}_i)\tilde{x}_i$ where $L(\hat{x}_i) = \frac{\partial d(x,\bar{\alpha})}{\partial x}|_{x=\hat{x}_i}$ and $\tilde{x}_i = \hat{x}_{i+1} - \hat{x}_i$.

Defining the random vector $\eta_i = \begin{bmatrix}\theta^T & \hat{x}_i^T\end{bmatrix}^T$ and notice that its normal distribution is given by $\eta_i \sim N(\gamma(x), \bar{Q}_i)$ with $\gamma(x) = \begin{bmatrix}d^T(x, \bar{\alpha}) & x^T\end{bmatrix}^T$ and $\bar{Q}_i = \begin{bmatrix}\bar{S}_i & 0 \\ 0 & P_i\end{bmatrix}$.

Using Assumption A.1 and Remark A.1, the normal distribution of η_i can be written as $\eta_i \sim N(\gamma(\hat{x}_{i+1}), \bar{Q}_i)$ where

$$\gamma(\hat{x}_{i+1}) = \begin{bmatrix}(d(\hat{x}_i, \bar{\alpha}) + L(\hat{x}_i)\tilde{x}_i)^T & \hat{x}_{i+1}^T\end{bmatrix}^T.$$

Defining $\Omega_i = [\tilde{\theta}_i^T \;\; 0]^T$, $\tilde{\theta}_i = \theta - d(\hat{x}_i, \bar{\alpha})$ and $\Pi_i = \begin{bmatrix}L^T(\hat{x}_i) & I_n\end{bmatrix}^T$.

Consider the following likelihood function:

$$\begin{aligned}\Gamma(\tilde{x}_i) &= \left(1\Big/\sqrt{(2\pi)^{n+\iota}\left|\bar{Q}_i\right|}\right)\exp\left\{-\frac{1}{2}(\eta_i-\gamma(\hat{x}_{i+1}))^T\bar{Q}_i^{-1}(\eta_i-\gamma(\hat{x}_{i+1}))\right\}\\ &= \left(1\Big/\sqrt{(2\pi)^{n+\iota}\left|\bar{Q}_i\right|}\right)\exp\left\{-\frac{1}{2}(\Omega_i-\Pi_i\tilde{x}_i)^T\bar{Q}_i^{-1}(\Omega_i-\Pi_i\tilde{x}_i)\right\}\end{aligned} \tag{51}$$

Notice that maximizing Γ is equivalent to minimizing the quadratic function:

$$\begin{aligned}&\min_{\tilde{x}_i}\{(\Omega_i-\Pi_i\tilde{x}_i)^T\bar{Q}_i^{-1}(\Omega_i-\Pi_i\tilde{x}_i)\}\Leftrightarrow\\ &\Pi_i^T\bar{Q}_i^{-1}(\Omega_i-\Pi_i\tilde{x}_i)=0\Leftrightarrow\\ &\tilde{x}_i=(\Pi_i^T\bar{Q}_i^{-1}\Pi_i)^{-1}\Pi_i^T\bar{Q}_i^{-1}\Omega_i\\ &\quad=(L^T(\hat{x}_i)\bar{S}_i^{-1}L(\hat{x}_i)+P_i^{-1})^{-1}L^T(\hat{x}_i)\bar{S}_i^{-1}\tilde{\theta}_i\end{aligned} \tag{52}$$

Using the matrix inversion lemma $(D+CA^{-1}B)^{-1}CA^{-1}=D^{-1}C(A+BD^{-1}C)^{-1}$, Equation (52) can be written as:

$$\begin{aligned}\hat{x}_{i+1}&=\hat{x}_i+P_iL^T(\hat{x}_i)(L(\hat{x}_i)P_iL^T(\hat{x}_i)+\bar{S}_i)^{-1}\tilde{\theta}_i\\ &=\hat{x}_i+K_i\tilde{\theta}_i\end{aligned} \tag{53}$$

Using (52), the covariance matrix P_{i+1} is such that:

$$\begin{aligned}P_{i+1}&\triangleq E\{(\hat{x}_{i+1}-\hat{x}_i)(\hat{x}_{i+1}-\hat{x}_i)^T\}\\ &=E\{\tilde{x}_i\tilde{x}_i^T\}\\ &=(\Pi_i^T\bar{Q}_i^{-1}\Pi_i)^{-1}\\ &=(L(\hat{x}_i)^T\bar{S}_i^{-1}L(\hat{x}_i)+P_i^{-1})^{-1}\end{aligned} \tag{54}$$

Using the matrix inversion lemma $(A+BD^{-1}C)^{-1}=A^{-1}-A^{-1}B(D+CA^{-1}B)^{-1}CA^{-1}$, Equation (54) can be written as:

$$P_{i+1}=(I-K_iL(\hat{x}_i))P_i \tag{55}$$

From (53) and (55), it is clear that the proposed estimator given by (33)-(36) is equivalent to the maximum likelihood update.

Appendix B. The Nonlinear Update Estimator is a Gauss-Newton Method.

Consider the following quadratic function

$$f=\frac{1}{2}\begin{bmatrix}\theta-d(\hat{x}_{i+1},\bar{\alpha})\\ \hat{x}_{i+1}-\hat{x}_i\end{bmatrix}^T\bar{Q}_i^{-1}\begin{bmatrix}\theta-d(\hat{x}_{i+1},\bar{\alpha})\\ \hat{x}_{i+1}-\hat{x}_i\end{bmatrix} \tag{56}$$

Notice that $\theta-d(\hat{x}_{i+1},\bar{\alpha})=\tilde{\theta}_i-L(\hat{x}_i)\tilde{x}_i$ and $\tilde{x}_i=\hat{x}_{i+1}-\hat{x}_i$. The Gauss-Newton method [36] for minimizing (56) is based on the successive application of Newton method to find the roots of $\frac{\partial f}{\partial\tilde{x}_i}=0$, which is equivalent to:

$$\begin{aligned}&-L^T(\hat{x}_i)\bar{S}_i^{-1}\tilde{\theta}_i+(L^T(\hat{x}_i)\bar{S}_i^{-1}L(\hat{x}_i)+P_i^{-1})\tilde{x}_i=0\Leftrightarrow\\ &\hat{x}_{i+1}=\hat{x}_i+(L^T(\hat{x}_i)\bar{S}_i^{-1}L(\hat{x}_i)+P_i^{-1})^{-1}L^T(\hat{x}_i)\bar{S}_i^{-1}\tilde{\theta}_i\end{aligned} \tag{57}$$

Following (52) to (53), we get

$$\hat{x}_{i+1}=\hat{x}_i+K_i\tilde{\theta}_i \tag{58}$$

Therefore, the sequence of iterates generated by Gauss-Newton method is identical to the proposed update procedure. Hence, the rate of convergence of the proposed nonlinear update estimator is as stated for Gauss-Newton method.