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A MATHEMATICAL MODEL OF ELASTIC CURVE FOR SIMPLY SUPPORTED BEAMS SUBJECTED TO A CONCENTRATED LOAD TAKING INTO ACCOUNT THE SHEAR DEFORMATIONS

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Abstract. *This paper presents a mathematical model of elastic curve for simply supported beams subjected to a concentrated load located anywhere along length of beam considering the bending and shear deformations, i.e., the equation of displacements and also the equation of slope for tangents to the elastic curve are presented. The traditional model of elastic curve used for simply supported beams subjected to a concentrated load does not consider the shear deformations. Also a comparison is made between the traditional model and the proposed model with respect to the maximum displacement of beam to observe the differences. Besides the effectiveness and accuracy of the model developed in this paper, a significant advantage is that the displacements and slopes are calculated at any length of the beam using the mathematical formulae.*
Keywords: Elastic curve, Simply supported beams, Concentrated load, Maximum displacement, Bending and shear deformations

1. **Introduction.** Structural analysis is the study of structures such as discrete systems. The theory of the structures is essentially based on the fundamentals of mechanics with which are formulated the different structural elements. The laws or rules that define the balance and continuity of a structure can be expressed in different ways, including: partial differential equations of continuous medium three-dimensional space; ordinary differential equations, or various theories of beams, or simply the algebraic equations for a discrete structure [1].

Structural analysis can be addressed using three main approaches [2]: a) the tensorial formulation (Newtonian mechanics and vectorial), b) the formulation based on the principles of virtual work, c) the formulation based on classical mechanics [3,4].

In regards to the conventional techniques of structural analysis of beams and rigid frames to obtain the displacements and slopes of the tangents to the elastic curve, the common practice considers only the bending deformations [5,6].

Recently, a method of structural analysis for statically indeterminate beams and rigid frames has been developed, and the method takes into account the bending deformations and shear to generate a system of equations in function of rotations and displacements [7-9]. Also a moment-distribution method to consider a new variable was presented, and this variable is the shear deformation [10]. A mathematical model to obtain the fixed-end moments of a beam subjected to a uniformly distributed load and also to a triangularly distributed load taking into account the shear deformations was developed [11,12]. A mathematical model of elastic curve for simply supported beams subjected to a uniformly distributed load taking into account the shear deformations was presented [13].

This paper presents a mathematical model of elastic curve for simply supported beams subjected to a concentrated load located anywhere along length of beam taking into account the bending and shear deformations. Also a comparison is made between the traditional model and the proposed model with respect to the maximum displacement of beam to observe the differences.

2. **Proposed Mathematical Model.** A deformed structure member is presented in Figure 1, and it shows the difference between the Timoshenko theory and the Euler-Bernoulli theory: in first "θ_z" and "dy/dx" do not coincide necessarily, while in the second these are equal [7-13].

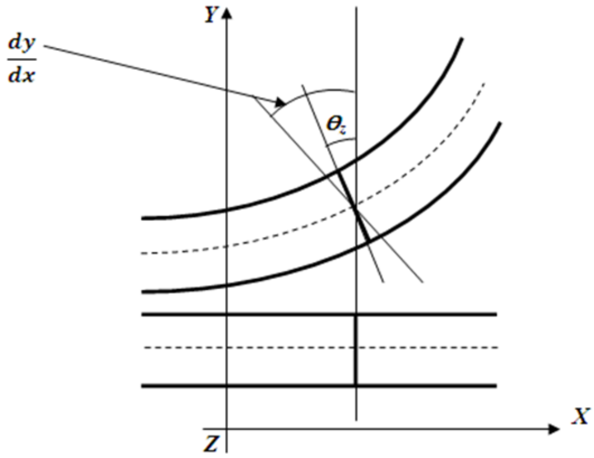

FIGURE 1. Deformation of a structure member

The main difference between the Euler-Bernoulli theory and the Timoshenko theory is that in the first the relative rotation of the section is approximated by the derivative of vertical displacement; this is an approximation valid only for long members in relation to the dimensions of cross section, and then it happens that due to the fact that shear deformations are negligible in comparison with the deformations caused by moment. On the Timoshenko theory, which considers the deformation due to shear, and is valid for short members and long members, the equation of the elastic curve is given by the complex system of equations:

$$G\left(\frac{dy}{dx} - \theta_z\right) = \frac{V_y}{A_s} \tag{1}$$

$$E\left(\frac{d\theta_z}{dx}\right) = \frac{M_z}{I_z} \tag{2}$$

where: G = shear modulus, dy/dx = total rotation around axis "Z", θ_z = rotation around axis "Z", due to the bending, V_y = shear force in direction "Y", A_s = shear area, $d\theta_z/dx = d^2y/dx^2$, E = modulus of elasticity, M_z = bending moment around axis "Z", I_z = moment of inertia around axis "Z".

Deriving Equation (1) and substituting into Equation (2), it is arrived at the equation of the elastic curve including the effect of shear stress:

$$\frac{d^2y}{dx^2} = \frac{1}{GA_s}\frac{dV_y}{dx} + \frac{M_z}{EI_z} \tag{3}$$

Equation (3) is integrated to obtain the rotation in anywhere:

$$\frac{dy}{dx} = \frac{V_y}{GA_s} + \int \frac{M_z}{EI_z} dx \tag{4}$$

Figure 2 shows the beam "AB" subjected to a concentrated load located anywhere along the length of the beam "P" and simply supported at their ends.

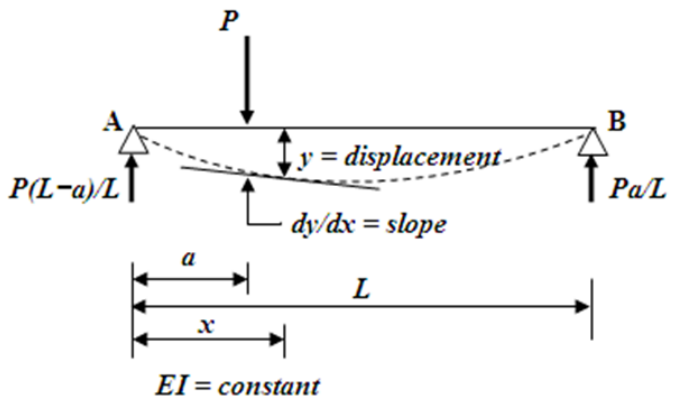

FIGURE 2. Beam "AB" subjected to a concentrated load

The beam of Figure 2 is analyzed to find shear force and moment anywhere of the beam on axis "x" is:

To $0 \leq x \leq a$:

$$V_x = -\frac{P(L-a)}{L} \tag{5}$$

$$M_x = \frac{P(L-a)x}{L} \tag{6}$$

To $a \leq x \leq L$:

$$V_x = \frac{Pa}{L} \tag{7}$$

$$M_x = \frac{Pa(L-x)}{L} \tag{8}$$

where: L = beam length, $V_x = V_y$, and $M_x = M_z$.

We analyze for $0 \leq x \leq a$.

By substitution of Equations (5) and (6) into Equation (4), we obtain:

$$\frac{dy}{dx} = -\frac{P(L-a)}{GA_sL} + \frac{P(L-a)}{EI_zL} \int (x)dx \tag{9}$$

These two equations must be integrated independently and not in the combined form, because the boundary conditions for the slope related to the bending deflections and shear are different in where the concentrated load is applied.

- Shear deformations:

$$\frac{dy}{dx} = -\frac{P(L-a)}{GA_sL} \tag{10}$$

The slope is the same in the segment $0 \leq x \leq a$. Therefore, $dy/dx = \theta_{s1}$:

$$\theta_{s1} = -\frac{P(L-a)}{GA_sL} \tag{11}$$

Equation (10) is integrated to obtain the displacements as follows:

$$y_{s1} = -\frac{P(L-a)}{GA_sL}x + C_1 \tag{12}$$

The boundary conditions are considered; when $x = 0$ and $y = 0$, we obtain $C_1 = 0$. Then Equation (12) is presented:

$$y_{s1} = -\frac{P(L-a)}{GA_sL}x \tag{13}$$

- Bending deformations:

$$\frac{dy}{dx} = \frac{P(L-a)}{EI_zL}\int(x)dx \tag{14}$$

Equation (14) is integrated to find the rotations:

$$\frac{dy}{dx} = \theta_{f1} = \frac{P(L-a)}{EI_zL}\left(\frac{x^2}{2} + C_2\right) \tag{15}$$

Substituting $x = a$, into Equation (15) to obtain the rotation $dy/dx = \theta_{fa1}$, where the concentrated load is localized, this is:

$$\theta_{fa1} = \frac{P(L-a)}{EI_zL}\left(\frac{a^2}{2} + C_2\right) \tag{16}$$

Equation (15) is integrated to find the displacements as follows:

$$y_{f1} = \frac{P(L-a)}{EI_zL}\left(\frac{x^3}{6} + C_2x + C_3\right) \tag{17}$$

The boundary conditions are considered; when $x = 0$ and $y = 0$, we obtain $C_3 = 0$. Then Equation (17) is presented:

$$y_{f1} = \frac{P(L-a)}{EI_zL}\left(\frac{x^3}{6} + C_2x\right) \tag{18}$$

Substituting $x = a$, into Equation (18) to find the displacement $y = y_{fa1}$, where the concentrated load is localized, this is:

$$y_{fa1} = \frac{P(L-a)}{EI_zL}\left(\frac{a^3}{6} + C_2a\right) \tag{19}$$

We analyze for $a \le x \le L$.

Equations (7) and (8) are substituted into Equation (4), and this is presented:

$$\frac{dy}{dx} = \frac{Pa}{GA_sL} + \frac{Pa}{EI_zL}\int(L-x)dx \tag{20}$$

- Shear deformations:

$$\frac{dy}{dx} = \frac{Pa}{GA_sL} \tag{21}$$

The slope is constant in the segment $a \le x \le L$. Therefore, $dy/dx = \theta_{s2}$:

$$\theta_{s2} = \frac{Pa}{GA_sL} \tag{22}$$

Equation (21) is integrated to obtain the displacements as follows:

$$y_{s2} = \frac{Pa}{GA_sL}x + C_4 \tag{23}$$

The boundary conditions are considered; when $x = L$ and $y = 0$, we obtain $C_4 = -Pa/GA_s$. Then Equation (23) is presented:

$$y_{s2} = \frac{Pa}{GA_s L}x - \frac{Pa}{GA_s} \tag{24}$$

- Bending deformations:

$$\frac{dy}{dx} = \frac{Pa}{EI_z L}\int (L - x)dx \tag{25}$$

Equation (25) is integrated to find the rotations:

$$\frac{dy}{dx} = \theta_{f2} = \frac{Pa}{EI_z L}\left(Lx - \frac{x^2}{2} + C_5\right) \tag{26}$$

Substituting $x = a$, into Equation (26) to find the rotation $dy/dx = \theta_{fa2}$, where the concentrated load is localized, this is:

$$\theta_{fa2} = \frac{Pa}{EI_z L}\left(La - \frac{a^2}{2} + C_5\right) \tag{27}$$

Equation (26) is integrated to obtain the displacements, because there are unknown conditions for rotations; this is as follows:

$$y_{f2} = \frac{Pa}{EI_z L}\left(\frac{Lx^2}{2} - \frac{x^3}{6} + C_5 x + C_6\right) \tag{28}$$

The boundary conditions are considered; when $x = L$ and $y = 0$, we obtain $C_6 = -L^3/3 - C_5 L$. Then Equation (28) is presented:

$$y_{f2} = \frac{Pa}{EI_z L}\left[\frac{Lx^2}{2} - \frac{x^3}{6} - \frac{L^3}{3} - C_5(L - x)\right] \tag{29}$$

Substituting $x = a$, into Equation (29) to find the displacement $y = y_{fa2}$, where the concentrated load is localized, this is:

$$y_{fa2} = \frac{Pa}{EI_z L}\left[\frac{La^2}{2} - \frac{a^3}{6} - \frac{L^3}{3} - C_5(L - a)\right] \tag{30}$$

Equations (16) and (27) are equalized, because bending rotations must be equal at the point $x = a$, where the load is applied to find the constant "C_5" in function of "C_2"; this is:

$$C_5 = C_2\frac{(L - a)}{a} - \frac{aL}{2} \tag{31}$$

Equations (19) and (30) are equalized, because bending displacements must be equal at the point $x = a$, where the load is applied and subsequently the Equation (31) is substituted in this equation to find the constant "C_2"; this is:

$$C_2 = \frac{a(a - 2L)}{6} \tag{32}$$

Equation (32) is substituted into Equation (31) to find the constant "C_5":

$$C_5 = -\frac{(a^2 + 2L^2)}{6} \tag{33}$$

Equation (32) is substituted into Equation (15) to obtain the bending rotations anywhere of the segment $0 \le x \le a$:

$$\theta_{f1} = \frac{P(L - a)}{EI_z L}\left[\frac{x^2}{2} + \frac{a(a - 2L)}{6}\right] \tag{34}$$

Total rotation "θ_{t1}"anywhere of the segment $0 \leq x \leq a$, this is:

$$\theta_{t1} = \theta_{s1} + \theta_{f1} \tag{35}$$

Substituting Equations (11) and (34) into Equation (35) is:

$$\theta_{t1} = -\frac{P(L-a)}{GA_sL} + \frac{P(L-a)}{EI_zL}\left[\frac{x^2}{2} + \frac{a(a-2L)}{6}\right] \tag{36}$$

And if we substituted [7-13]:

$$\varnothing = \frac{12EI_z}{GA_sL^2} \tag{37}$$

where "G" is obtained as follows:

$$G = \frac{E}{2(1+\nu)} \tag{38}$$

where: $\varnothing$ is form factor, and ν is Poisson's ratio.

Then, Equation (37) is substituted into Equation (36); this is:

$$\theta_{t1} = -\frac{P(L-a)L}{12EI_z}\left\{\varnothing - \frac{12}{L^2}\left[\frac{x^2}{2} + \frac{a(a-2L)}{6}\right]\right\} \tag{39}$$

If shear deformations are neglected ($\varnothing = 0$) into Equation (39), the rotation is presented [14-17]:

$$\theta_{t1} = \frac{P(L-a)}{EI_zL}\left[\frac{x^2}{2} + \frac{a(a-2L)}{6}\right] \tag{40}$$

Equation (32) is substituted into Equation (18) to obtain the bending displacements anywhere of the segment $0 \leq x \leq a$:

$$y_{f1} = \frac{P(L-a)}{EI_zL}\left\{\frac{x^3}{6} + \left[\frac{a(a-2L)}{6}\right]x\right\} \tag{41}$$

Total displacement "y_{t1}" anywhere of the segment $0 \leq x \leq a$, this is:

$$y_{t1} = y_{s1} + y_{f1} \tag{42}$$

Substituting Equations (13) and (41) into Equation (42) is:

$$y_{t1} = -\frac{P(L-a)}{GA_sL}x + \frac{P(L-a)}{EI_zL}\left\{\frac{x^3}{6} + \left[\frac{a(a-2L)}{6}\right]x\right\} \tag{43}$$

Now substituting the form factor of Equation (37) into Equation (43) is presented:

$$y_{t1} = -\frac{P(L-a)L}{12EI_z}\left\{\varnothing x - \frac{12}{L^2}\left[\frac{x^3}{6} + \frac{a(a-2L)x}{6}\right]\right\} \tag{44}$$

Therefore, Equation (44) is the elastic curve of beam subjected to a concentrated load localized anywhere of the member considering the bending deformations and shear of the segment $0 \leq x \leq a$.

If shear deformations are neglected ($\varnothing = 0$) into Equation (44), the displacement is presented [14-17]:

$$y_{t1} = \frac{P(L-a)}{EI_zL}\left[\frac{x^3}{6} + \frac{a(a-2L)x}{6}\right] \tag{45}$$

Equation (33) is substituted into Equation (26) to obtain the bending rotations anywhere of the segment $a \leq x \leq L$:

$$\theta_{f2} = \frac{Pa}{EI_zL}\left[Lx - \frac{x^2}{2} - \frac{(a^2+2L^2)}{6}\right] \tag{46}$$

Total rotation "θ_{t2}" anywhere of the segment $a \le x \le L$, this is:

$$\theta_{t2} = \theta_{s2} + \theta_{f2} \tag{47}$$

Substituting Equations (22) and (46) into Equation (47) is:

$$\theta_{t2} = \frac{Pa}{GA_sL} + \frac{Pa}{EI_zL}\left[Lx - \frac{x^2}{2} - \frac{(a^2+2L^2)}{6}\right] \tag{48}$$

Substituting the form factor of Equation (37) into Equation (48) is presented:

$$\theta_{t2} = \frac{PaL}{12EI_z}\left\{\varnothing + \frac{12}{L^2}\left[Lx - \frac{x^2}{2} - \frac{(a^2+2L^2)}{6}\right]\right\} \tag{49}$$

If shear deformations are neglected ($\varnothing = 0$) into Equation (49), the rotation is presented [17]:

$$\theta_{t2} = \frac{Pa}{EI_zL}\left[Lx - \frac{x^2}{2} - \frac{(a^2+2L^2)}{6}\right] \tag{50}$$

Equation (33) is substituted into Equation (29) to obtain the bending displacements anywhere of the segment $a \le x \le L$:

$$y_{f2} = \frac{Pa}{EI_zL}\left\{\frac{Lx^2}{2} - \frac{x^3}{6} - \frac{L^3}{3} + \frac{(a^2+2L^2)}{6}(L-x)\right\} \tag{51}$$

Total displacement "y_{t2}" anywhere of the segment $a \le x \le L$, this is:

$$y_{t2} = y_{s2} + y_{f2} \tag{52}$$

Substituting Equations (24) and (51) into Equation (52) is:

$$y_{t2} = \frac{Pa}{GA_sL}x - \frac{Pa}{GA_s} + \frac{Pa}{EI_zL}\left\{\frac{Lx^2}{2} - \frac{x^3}{6} - \frac{L^3}{3} + \frac{(a^2+2L^2)}{6}(L-x)\right\} \tag{53}$$

Now substituting the form factor of Equation (37) into Equation (53) is presented:

$$y_{t2} = \frac{PaL}{12EI_z}\left\{\varnothing x - \varnothing L + \frac{12}{L^2}\left[\frac{Lx^2}{2} - \frac{x^3}{6} - \frac{L^3}{3} + \frac{(a^2+2L^2)}{6}(L-x)\right]\right\} \tag{54}$$

Therefore, Equation (54) is the elastic curve of beam subjected to a concentrated load localized anywhere of the member considering the bending deformations and shear of the segment $a \le x \le L$.

If shear deformations are neglected ($\varnothing = 0$) into Equation (54), the displacement is presented [17]:

$$y_{t2} = \frac{Pa}{EI_zL}\left[\frac{Lx^2}{2} - \frac{x^3}{6} - \frac{L^3}{3} + \frac{(a^2+2L^2)}{6}(L-x)\right] \tag{55}$$

2.1. **Rotation angles in supports.** The rotation angles at the ends of the beam are obtained by substituting $x = 0$ into Equation (39) and $x = L$ into Equation (49):

$$\theta_A = -\frac{P(L-a)L}{12EI_z}\left\{\varnothing - \frac{12}{L^2}\left[\frac{a(a-2L)}{6}\right]\right\} \tag{56}$$

$$\theta_B = \frac{PaL}{12EI_z}\left\{\varnothing + \frac{12}{L^2}\left[\frac{L^2}{2} - \frac{(a^2+2L^2)}{6}\right]\right\} \tag{57}$$

If shear deformations are neglected ($\varnothing = 0$) into Equations (56) and (57), the rotations at the supports are [17]:

$$\theta_A = \frac{Pa(L-a)(a-2L)}{6EI_zL} \tag{58}$$

$$\theta_B = \frac{Pa\left(L^2 - a^2\right)}{6EI_zL} \tag{59}$$

The rotation angles are functions of the position of the load and reach their maximum values when it is located near the midpoint of the beam. In this case of rotation angle "θ_A", the maximum value of the angle is obtained by differentiating Equation (56) with respect to "a":

$$\frac{d\theta_A}{da} = \frac{PL}{12EI_z}\text{Ø} - \frac{P}{6EI_zL}\left(3a^2 - 6aL + 2L^2\right) \tag{60}$$

Subsequently Equation (60) is made equal to zero to obtain the position of the load as follows:

$$a = \frac{\left(3 - \sqrt{3 + 1.5\text{Ø}}\right)L}{3} \tag{61}$$

If shear deformations are neglected (Ø = 0) into Equation (61), the position of the load is presented [17]:

$$a = \frac{\left(3 - \sqrt{3}\right)L}{3} \tag{62}$$

Substituting Equation (61) into Equation (56) is presented:

$$\theta_{A\max} = -\frac{PL^2\left(\text{Ø} + 2\right)^{3/2}}{18\sqrt{6}EI_z} \tag{63}$$

If shear deformations are neglected (Ø = 0) into Equation (63), the maximum rotation is shown [17]:

$$\theta_{A\max} = -\frac{\sqrt{3}PL^2}{27EI_z} \tag{64}$$

2.2. **Maximum deflection in beam.** The maximum deflection in beam "$y_{\max}$" occurs where the deflection curve has a horizontal tangent. If the load is to the right of the midpoint, i.e., if $a > L - a$, this point is on the left side of the load. We can locate this point, equaling the slope given by the Equation (39) to zero and solving for the distance "x", and now we denote "x_1". In this way we obtain the following equation:

$$x_1 = \sqrt{\frac{a(2L - a)}{3} + \frac{\text{Ø}L^2}{6}} \tag{65}$$

If shear deformations are neglected (Ø = 0) into Equation (65), the deflection curve has a horizontal tangent [17]:

$$x_1 = \sqrt{\frac{a(2L - a)}{3}} \tag{66}$$

Substituting Equation (65) into Equation (44) is determined the maximum deflection:

$$y_{\max} = -\frac{P(L - a)\left[L(4a + \text{Ø}L) - 2a^2\right]^{3/2}}{18\sqrt{6}EI_zL} \tag{67}$$

If shear deformations are neglected (Ø = 0) into Equation (67), the maximum deflection is shown [17]:

$$y_{\max} = -\frac{P(L - a)\left(2aL - a^2\right)^{3/2}}{9\sqrt{3}EI_zL} \tag{68}$$

Now, if the load is to the left of the midpoint, i.e., if $a < L - a$, this point is on the right side of the load. We can locate this point, equaling the slope given by the Equation

(49) to zero and solving for the distance "x", and now we denote "x_1". In this way we obtain the following equation:

$$x_1 = L - \sqrt{\frac{(L^2 - a^2)}{3} + \frac{\varnothing L^2}{6}} \tag{69}$$

If shear deformations are neglected ($\varnothing = 0$) into Equation (69), the deflection curve has a horizontal tangent [17]:

$$x_1 = L - \sqrt{\frac{(L^2 - a^2)}{3}} \tag{70}$$

Substituting Equation (69) into Equation (54) is determined the maximum deflection:

$$y_{\max} = -\frac{Pa\left[L^2(\varnothing + 2) - 2a^2\right]^{3/2}}{18\sqrt{6}EI_zL} \tag{71}$$

If shear deformations are neglected ($\varnothing = 0$) into Equation (71), the maximum deflection is shown [17]:

$$y_{\max} = -\frac{Pa\left(L^2 - a^2\right)^{3/2}}{9\sqrt{3}EI_zL} \tag{72}$$

2.3. **Special case (the load is located in the center of the beam).** An important special case occurs when the load "P" acts on the midpoint of the beam ($a = 0.5L$). Then, we obtain the following results with Equations (39), (49), (44), (54), (56) and (57), respectively. And substituting $x = 0.5L$ into Equation (44) or (54) is found maximum deflection:

$$\theta_{t1} = -\frac{P\left[L^2(2\varnothing + 3) - 12x^2\right]}{48EI_z} \tag{73}$$

$$\theta_{t2} = \frac{P\left[24Lx - L^2(9 - 2\varnothing) - 12x^2\right]}{48EI_z} \tag{74}$$

$$y_{t1} = -\frac{Px\left[L^2(2\varnothing + 3) - 4x^2\right]}{48EI_z} \tag{75}$$

$$y_{t2} = -\frac{P\left[4x^3 - 12Lx^2 + L^2x(9 - 2\varnothing) + L^3(2\varnothing - 1)\right]}{48EI_z} \tag{76}$$

$$\theta_A = -\frac{PL^2(2\varnothing + 3)}{48EI_z} \tag{77}$$

$$\theta_B = \frac{PL^2(2\varnothing + 3)}{48EI_z} \tag{78}$$

$$y_{\max} = -\frac{PL^3}{48EI_z}(1 + \varnothing) \tag{79}$$

If shear deformations are neglected ($\varnothing = 0$) into Equations (73), (74), (75), (76), (77), (78) and (79) [17]:

$$\theta_{t1} = -\frac{P\left(L^2 - 4x^2\right)}{16EI_z} \tag{80}$$

$$\theta_{t2} = \frac{P\left(8Lx - 3L^2 - 4x^2\right)}{16EI_z} \tag{81}$$

$$y_{t1} = -\frac{Px\left(3L^2 - 4x^2\right)}{48EI_z} \tag{82}$$

$$y_{t2} = -\frac{P\left(4x^3 - 12Lx^2 + 9L^2x - L^3\right)}{48EI_z} \tag{83}$$

$$\theta_A = -\frac{PL^2}{16EI_z} \tag{84}$$

$$\theta_B = \frac{PL^2}{16EI_z} \tag{85}$$

$$y_{\max} = -\frac{PL^3}{48EI_z} \tag{86}$$

3. **Application.** A steel beam is presented to obtain: 1) The maximum rotations in support "A" and where the load is localized to produce the maximum rotation; 2) The maximum deflections and its localization, when the load is applied in $a = 0.25L$, $a = 0.50L$ and $a = 0.75L$. By the traditional model (bending deformations are considered) and the proposed model (bending deformations and shear are considered), the used beam profile is W24X94 and the beam length varies from 3.00 to 10.00m; the profile properties are:

$P = 49.05$ kN
$E = 20019.6$ kN/cm^2
$A = 173.12$ cm^2
$A_c = 78.25$ cm^2
$I = 105469$ cm^4
$\nu = 0.32$

The shear modulus by Equation (38) is obtained:

$$G = \frac{E}{2(1+\nu)} = \frac{20019.6}{2(1+0.32)} = 7583.182 \text{ kN/cm}^2$$

Equation (37) is used to find the form factor. The maximum rotation in support "A" by Equation (58) is obtained, and by Equation (62) is found where the load is localized for traditional model, and for proposed model by Equation (56) is found the maximum rotation in support "A" and by Equation (61) is obtained where the load is localized. The maximum deflection by Equation (72) is obtained, and its localization by Equation (70) is found for traditional model and for proposed model by Equation (71) is found and its localization by Equation (69) is obtained, when $a < L - a$. When the load is located in the center of the beam, the maximum deflection by Equation (86) is found for traditional model and for proposed model by Equation (79) is obtained. The maximum deflection by Equation (68) is obtained and its localization by Equation (66) is found for traditional model, and for proposed model by Equation (67) is found and its localization by Equation (65) is obtained, when $a > L - a$.

Table 1 presents the results of maximum rotations and Figure 3 shows the behavior of the maximum rotations with respect to length of the beam for the traditional model and the proposed model. Table 2 shows the results of maximum displacements and Figure 4 presents the behavior of both models for $a = 0.25L$. Table 3 presents the maximum displacements and Figure 5 shows the behavior of the two models for $a = 0.5L$. Table 4 shows the maximum displacements and Figure 4 presents the behavior of both models for $a = 0.75L$.

4. **Results.** Table 1 shows the maximum rotations in support "A", the proposed model (the bending and shear deformations are considered) is greater with respect to traditional model (the bending deformations are considered) in all cases, the largest difference exists in $L = 3.00$ m of 38.06% and the smallest difference occurs in $L = 10.00$ m of 3.22%, and where the load is localized to produce the maximum rotation, and the traditional model is greater with respect to proposed model in all cases, the largest difference exists in $L = 3.00$ m of 18.12% and the smallest difference occurs in $L = 10.00$ m of 1.47%.

TABLE 1. The maximum rotations "$\theta_{A\max}$"

L (m)	a (m)			$\theta_{A\max} \times 10^4$ (rad)		
	TM	PM	TM/PM	TM	PM	PM/TM
3	1.2679	1.0734	1.1812	−1.34	−1.85	1.3806
4	1.6906	1.5413	1.0969	−2.38	−2.88	1.2101
5	2.1132	1.9925	1.0606	−3.73	−4.21	1.1287
6	2.5359	2.4347	1.0416	−5.36	−5.85	1.0914
7	2.9585	2.8715	1.0303	−7.30	−7.78	1.0658
8	3.3812	3.3048	1.0231	−9.54	−10.02	1.0503
9	3.8038	3.7358	1.0182	−12.07	−12.55	1.0398
10	4.2265	4.1652	1.0147	−14.90	−15.38	1.0322

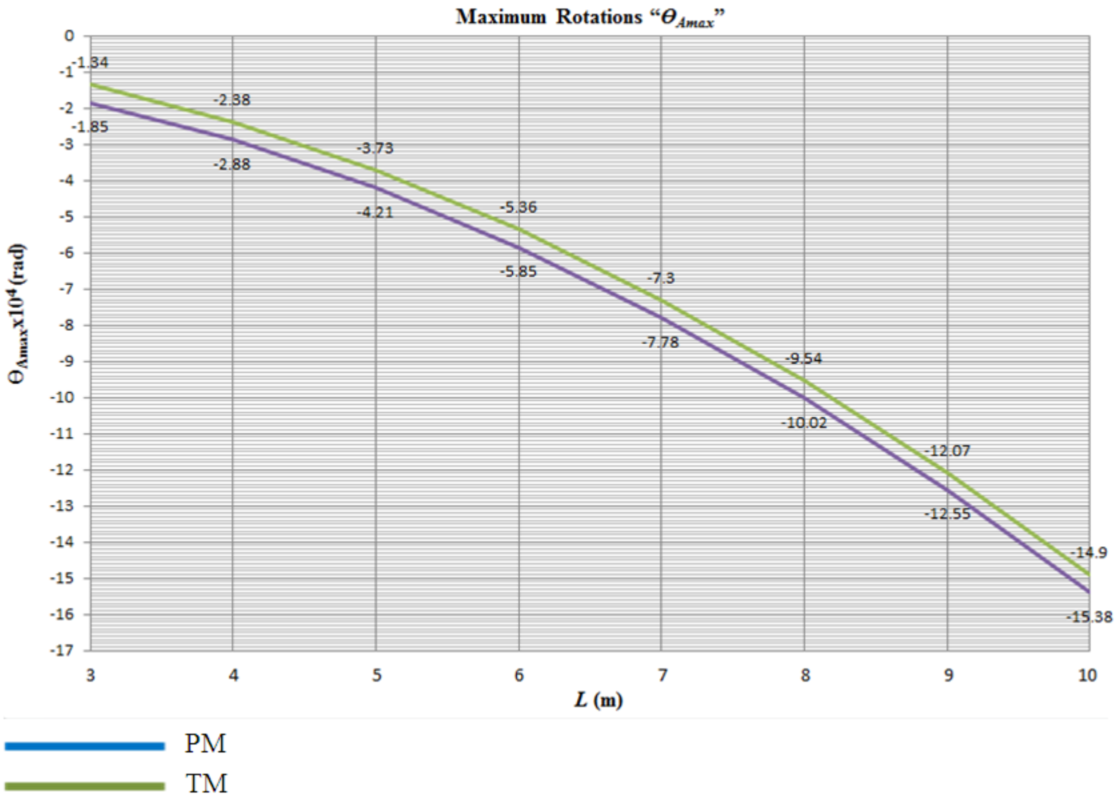

FIGURE 3. Maximum rotations "$\theta_{A\max}$"

Table 2 presents the maximum deflections, when $a = 0.25L$, the proposed model is greater with respect to traditional model in all cases, the largest difference exists in $L = 3.00$ m of 40.66% and the smallest difference occurs in $L = 10.00$ m of 3.43% and the distance "x_1" is greater than the traditional model with respect to proposed model in all cases, the largest difference exists in $L = 3.00$ m of 17.83% and the smallest difference occurs in $L = 10.00$ m of 1.46%.

Table 3 shows the maximum deflections, when $a = 0.5L$, the proposed model is greater with respect to traditional model in all cases, the largest difference exists in $L = 3.00$ m of 47.33% and the smallest difference occurs in $L = 10.00$ m of 4.26% and the distance "x_1" is equal for two models.

TABLE 2. The maximum deflection "$y_{\max}$" for $a = 0.25L$

L (m)	x_1 (m)			$y_{\max}$ (cm)		
	TM	PM	TM/PM	TM	PM	PM/TM
3	1.3229	1.1227	1.1783	−0.0091	−0.0128	1.4066
4	1.7639	1.6101	1.0955	−0.0216	−0.0264	1.2222
5	2.2049	2.0804	1.0598	−0.0423	−0.0482	1.1395
6	2.6459	2.5414	1.0411	−0.0730	−0.0801	1.0973
7	3.0869	2.9970	1.0300	−0.1160	−0.1242	1.0707
8	3.5279	3.4490	1.0229	−0.1732	−0.1825	1.0537
9	3.9688	3.8986	1.0180	−0.2465	−0.2570	1.0426
10	4.4098	4.3465	1.0146	−0.3382	−0.3498	1.0343

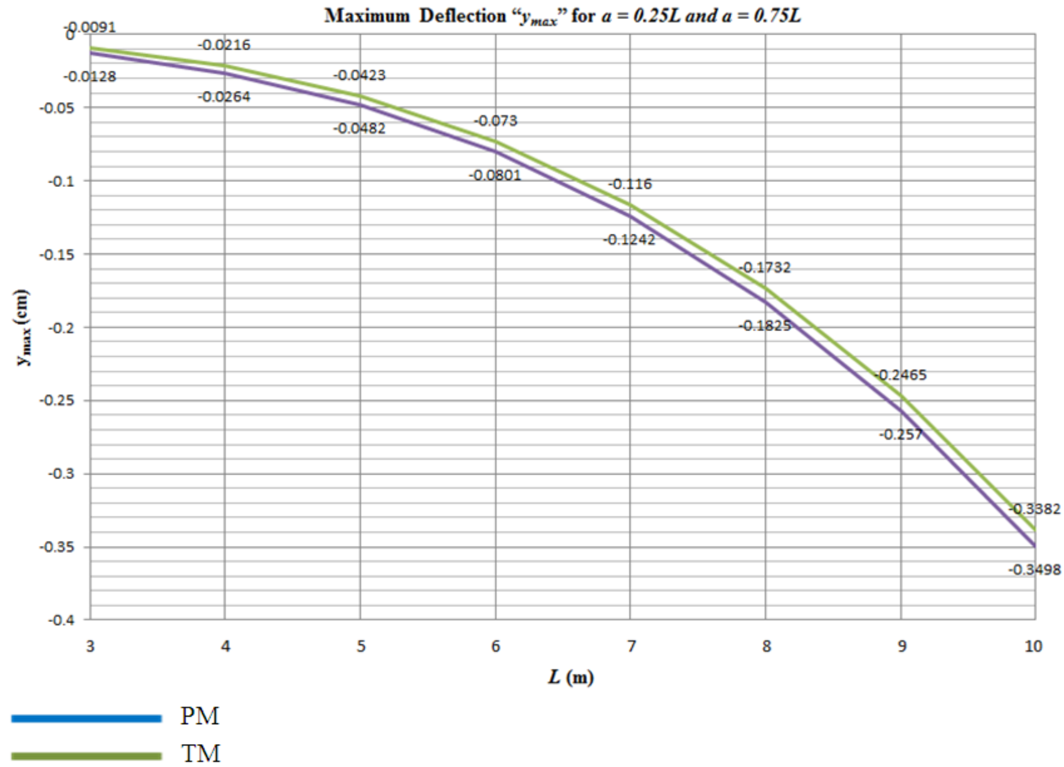

FIGURE 4. Maximum deflection "$y_{\max}$" for $a = 0.25L$ and $a = 0.75L$

Table 4 presents the maximum deflections, when $a = 0.75L$, the proposed model is greater with respect to traditional model in all cases, the largest difference exists in $L = 3.00$ m of 40.66% and the smallest difference occurs in $L = 10.00$ m of 3.43% and the distance "x_1" is greater in the proposed model with respect to traditional model in all cases, the largest difference exists in $L = 3.00$ m of 11.94% and the smallest difference occurs in $L = 10.00$ m of 1.13%.

5. **Conclusions.** This paper presents a mathematical model of elastic curve for simply supported beams subjected to a concentrated load located at anywhere along length of the beam taking into account the bending deformations and shear. The mathematical technique presented in this research is very adequate to obtain the deflections in anywhere

TABLE 3. The maximum deflection "$y_{\max}$" for $a = 0.50L$

L (m)	x_1 (m)			$y_{\max}$ (cm)		
	TM	PM	TM/PM	TM	PM	PM/TM
3	1.5000	1.5000	1.0000	−0.0131	−0.0193	1.4733
4	2.0000	2.0000	1.0000	−0.0310	−0.0392	1.2645
5	2.5000	2.5000	1.0000	−0.0605	−0.0708	1.1702
6	3.0000	3.0000	1.0000	−0.1045	−0.1169	1.1187
7	3.5000	3.5000	1.0000	−0.1660	−0.1805	1.0873
8	4.0000	4.0000	1.0000	−0.2478	−0.2643	1.0666
9	4.5000	4.5000	1.0000	−0.3528	−0.3714	1.0527
10	5.0000	5.0000	1.0000	−0.4840	−0.5046	1.0426

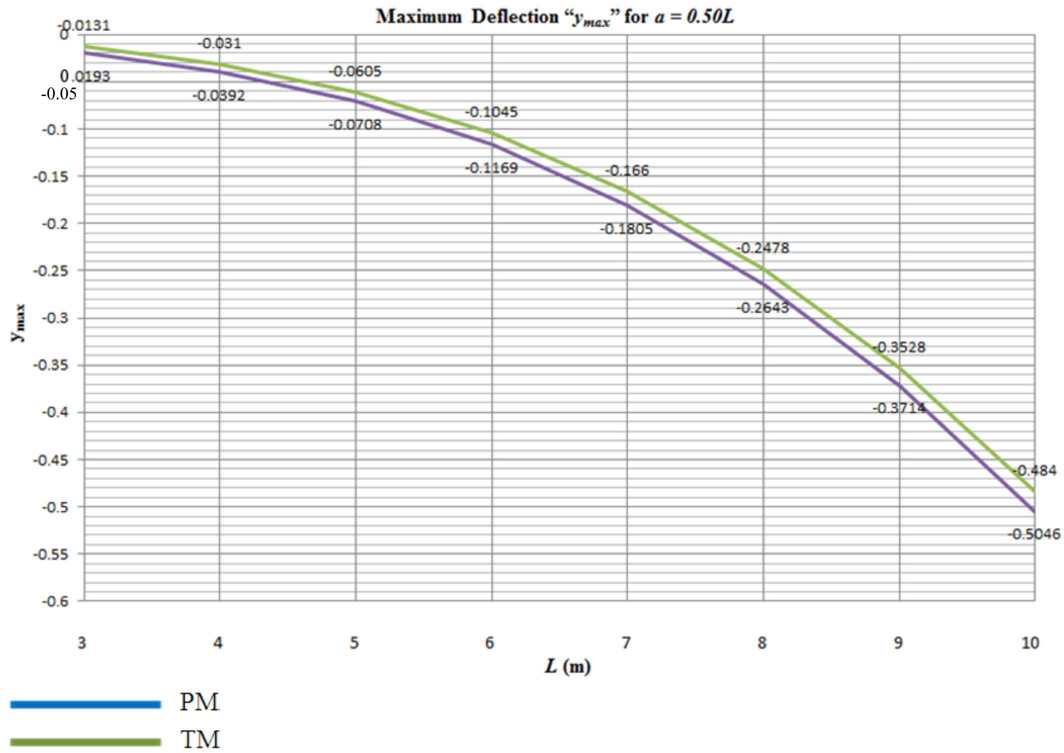

FIGURE 5. Maximum deflection "$y_{\max}$" for $a = 0.50L$

of beam subjected to a concentrated load, since it presents the mathematical equation of elastic curve.

The maximum deflections by the traditional model (bending deformations are considered) are lower in all cases, with respect to the proposed model (bending deformations and shear are considered). This condition implies that we must take into account the maximum deflections permitted by building regulations, because in some situations it could be presented which does not meet the standards set by these codes.

In any type of structure, the shear forces and bending moments are present; therefore, the bending and shear deformations appear. Then, the proposed model (the bending and shear deformations are considered) is more appropriate for structural analysis and is also more suited to the actual conditions with respect to the traditional model (bending deformations are considered).

Table 4. The maximum deflection "$y_{\max}$" for $a = 0.75L$

L (m)	x_1 (m)			$y_{\max}$ (cm)		
	TM	PM	TM/PM	TM	PM	PM/TM
3	1.6771	1.8773	1.1194	−0.0091	−0.0128	1.4066
4	2.2361	2.3899	1.0688	−0.0216	−0.0264	1.2222
5	2.7951	2.9196	1.0445	−0.0423	−0.0482	1.1395
6	3.3541	3.4586	1.0312	−0.0730	−0.0801	1.0973
7	3.9131	4.0030	1.0230	−0.1160	−0.1242	1.0707
8	4.4721	4.5510	1.0176	−0.1732	−0.1825	1.0537
9	5.0312	5.1014	1.0140	−0.2465	−0.2570	1.0426
10	5.5902	5.6535	1.0113	−0.3382	−0.3498	1.0343

The mathematical model presented in this paper is applied only for simply supported beams subjected to a concentrated load located at anywhere along the length of the beam. The suggestions for future research may be: 1) when is presented another type of load; 2) when the cross-section of the beam is variable.

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