

NEW INVESTIGATIONS INTO THE MFS FOR 2D INVERSE LAPLACE PROBLEMS

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ABSTRACT. *The method of fundamental solutions (MFS) is one of the boundary-type meshless collocation methods for the solution of boundary value problems. By using the fundamental solutions of governing equations, the solution to the partial differential equation can be obtained based on the boundary points. In this paper, we investigate the applications of the MFS together with the conditional number analysis to solve elliptic problems with only partially accessible boundary conditions. The effective condition number is used to investigate the ill-conditioned interpolation system. A innovative simple algorithm is proposed to solve such inverse problems. Compared with traditional results, our numerical research shows that the new algorithm of the MFS is accurate, computationally efficient and stable for inverse problems with arbitrary geometry.*

Keywords: Method of fundamental solutions, Boundary condition, Effective condition number, Laplace equation

1. Introduction. Recent years have witnessed a research boom in the method of fundamental solutions (MFS) in dealing with science and engineering problems [1]. It can be viewed as a variant of the boundary element method which uses the fundamental solution of governing equation.

The applications of MFS for direct problems can be found in literature [2-4] and references therein. For inverse problems, the MFS has been successfully applied to solve inverse heat conduction problems [5-10]. Various Cauchy problems are also considered by the MFS [11-17]. Young et al. [18] investigated the applications of the MFS together with the conditional number analysis to solve inverse problems involving under-specified and/or over-specified boundary conditions. Hon and Li [19] extended the application of the MFS to determine an unknown free boundary of a Cauchy problem of parabolic-type equation from measured Dirichlet and Neumann data with noises. Jin and Marin [20] used the MFS to deal with inverse source problems associated with the steady-state heat conduction. It is noted in [20] that the inverse source problem is transformed into a fourth-order partial differential equation. Based on the MFS, Wang et al. [21] proposed a new general scheme for inverse source identification problems. A comprehensive survey is made in the MFS solution of inverse problems [22, 23]. The MFS has also been considered to solve inverse Stefan and static problems [24-28]. It is pointed that all previous literature assumes that the accessible boundary should be over-specified.

The purpose of the present paper is to apply the MFS for the direct solution of some inverse problems associated with the Laplace equation. The MFS is employed to discretize the equation, and the effective condition number is used to analyze the ill-conditioned resulting matrix system. Interesting results for several numerical examples are presented to show that additional boundary data and regularization techniques are not necessary for inverse Laplace problems. The structure of the paper is as follows. In Section 2, we briefly describe the inverse Laplace problem. The key idea of the MFS is described in Section 3. Followed by Section 4, the effective condition number is introduced to scale the interpolation system. Section 5 presents two benchmark numerical examples on piecewise smooth domain and irregular smooth domain. Section 6 concludes this study with some remarks.

2. Problem Description. Many steady-state engineering problems, such as ground water and heat conduction in an isotropic homogeneous media, can be mathematically described as

$$\nabla^2 u(X) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad X = (x, y) \in \Omega, \quad (1)$$

where u is the water head or a field potential to be solved, X is the spatial coordinate of the problem and Ω is an open bounded physical domain.

For direct problems, certain boundary conditions should be given simultaneously on the whole physical boundary $\partial\Omega$. As is known to all, it is not always possible to get boundary conditions from a closed boundary for many practical problems. More specifically, only partial boundary conditions are accessible on $\Gamma \subset \partial\Omega$. We consider the following certain boundary condition

$$Bu(X) = \bar{u}(X), \quad X \in \Gamma, \quad (2)$$

where B is a linear operator defined on the accessible boundary Γ . $\bar{u}(X)$ is the prescribed boundary data at point X . We aim to determine the boundary condition on the rest part of under-specified physical boundary $\Gamma_0 = \partial\Omega - \Gamma$.

For the above inverse problem, previous literature assumes that the other types of boundary conditions are added on the accessible boundary Γ , i.e., the accessible boundary Γ is over-specified. Here, we state that additional boundary condition is not necessary for this kind of inverse problem. Traditional research concludes that it does not guarantee the uniqueness of such inverse problem. However, our conclusion will be supported by numerical investigations in Section 5.

3. The MFS. The MFS is first proposed by Kupradze and Aleksidze in 1964. It is a versatile meshless numerical method to solve well-posed as well as ill-posed problems. The fundamental theory of the MFS is to decompose the solutions of the boundary value problems by the superposition of corresponding fundamental solutions with proper source intensities. By the method of collocation through known augmented data on the boundary, one can get the unknown parameters to be determined.

The general mathematical formulation of the MFS can be expressed by

$$u_M(X) = \sum_{i=1}^M \lambda_i F(r_i), \quad (3)$$

where M is the total source point number, and λ_i ($i = 1, \dots, M$) are the unknown coefficients that are used to determine the values of the field variable. $F(\cdot)$ is the fundamental solution or the free space Greens function of the Laplacian operator. $r = \|X - Y\|$ is

the Euclidean norm distance between two points X and Y . The fundamental solution of two-dimensional Laplace equation is written as

$$F(\cdot) = \frac{1}{2\pi} \log(r). \quad (4)$$

To avoid the singularity of the fundamental solution $F(\cdot)$ at origin, the prescribed source points are usually introduced on a ‘pseudo-boundary’ outside the computational domain. The ‘pseudo-boundary’ can be a contour geometrically similar to the boundary contour of the region under consideration or a circular contour [29]. For convenience, this paper will consider the circular contour case.

The unknown coefficients λ_i ($i = 1, \dots, M$) are determined by the collocation method using the known boundary data. By collocating Equation (3) on the prescribed boundary data (2) at N collocation points, we are able to obtain a system of linear algebraic equations. More specifically, we have

$$\sum_{j=1}^M \lambda_j F(r_{ij}) = \bar{u}(X_i), \quad X_i \in \Gamma, \quad (5)$$

where i and j are the index of collocation points on Γ and source points on ‘pseudo-boundary’ Γ' , respectively.

The above equations can be written in the following matrix system

$$F\lambda = b, \quad (6)$$

where $F = F(r_{ij})$ is an $N \times M$ interpolation matrix composed by the values of fundamental solutions for prescribed source and collocation points. The column matrix b can be obtained from the known boundary conditions. In order to get a square matrix F_{NN} , we choose equal numbers of collocation and source points.

Due to the global interpolation approach, the MFS produces a dense matrix system when a large number of boundary collocation points are used. This will lead to the ill-conditioned interpolation matrix F_{NN} .

4. Condition of Interpolation System. The traditional condition number is considered to be a ratio of the largest and smallest singular values of the interpolation matrix F . It only depends on the distribution and number of source and collocation points. In practical applications, the boundary data b may be disturbed by some noise. Thus, the whole interpolation system may be different. Clearly, we should not rely solely on the traditional condition number to scale all practical ill-conditioned MFS systems.

As an alternative measurement index, the effective condition number (ECN) is introduced to investigate the condition of the whole matrix system [30, 31]. This will be briefly introduced in the subsequent content. Other types of effective condition number can be found in [32-34] and references therein.

Based on the singular value decomposition, the matrix F can be decomposed into $F = UDV^T$. Here, $U = [u_1, u_2, \dots, u_N]$ and $V = [v_1, v_2, \dots, v_N]$ denote orthogonal matrices, while the vectors u_i and v_i are the left and right singular vectors of F , respectively. $D = \text{Diag}(\sigma_1, \sigma_2, \dots, \sigma_N)$ and σ_i , $1 \leq i \leq N$ are the singular values of F .

To express the ECN, we consider a perturbed matrix system $F(\lambda + \Delta\lambda) = b + \Delta b$. We can get

$$b = \sum_{i=1}^N \alpha_i u_i, \quad \Delta b = \sum_{i=1}^N \Delta \alpha_i u_i. \quad (7)$$

Let $\alpha = (b_1, \dots, b_N)^T = U^*b$ and $\Delta\alpha = (\Delta b_1, \dots, \Delta b_N)^T = U^*\Delta b$. The solution can be expressed in terms of the inverse of F , i.e., $\lambda = F^{-1}b := VD^{-1}U^Tb$, $\Delta\alpha =$

$F^{-1}\Delta b$. Assume $p \leq N$ is the largest integer such that $\sigma_p > 0$. We have $D^{-1} = \text{Diag}(\sigma_1^{-1}, \dots, \sigma_p^{-1}, 0, \dots, 0)$.

Since U is orthogonal, we have

$$\|\lambda\| = \sqrt{\sum_{i=1}^N \left(\frac{\alpha_i}{\sigma_i}\right)^2}, \quad \|\Delta\lambda\| = \sqrt{\sum_{i=1}^N \left(\frac{\Delta\alpha_i}{\sigma_i}\right)^2} \leq \frac{\|\Delta b\|}{\sigma_N}. \quad (8)$$

For the traditional condition number $\text{Cond}(F)$, we get

$$\frac{\|\Delta\lambda\|}{\|\lambda\|} \leq \text{Cond}(F) \frac{\|\Delta b\|}{\|b\|}. \quad (9)$$

We can get the ECN by substituting (8) into the inequality (9)

$$\text{ECN}(F, b) = \frac{\|b\|}{\sigma_N \sqrt{\left(\frac{\beta_1}{\sigma_1}\right)^2 + \dots + \left(\frac{\beta_N}{\sigma_N}\right)^2}}. \quad (10)$$

It is a new error bound for (6) and can be considered as an alternative replacement to traditional condition number.

5. Numerical Examples. To support our conclusion, two benchmark examples are considered in the following part. The first example is based on a piecewise smooth domain and the second one on an irregular smooth domain. In order to investigate the effect of noisy boundary data to the MFS, the following random noise is added to the boundary conditions

$$u = \bar{u}(1 + \delta). \quad (11)$$

Here, $\delta = \varepsilon \times \text{Rand}$ and ε is the level of noise in the boundary data. The MATLAB random number generator ‘Rand’ is used to produce random numbers in $[0, 1]$. Differences between traditional and current algorithms are shown in Table 1.

TABLE 1. Differences between traditional and current algorithms

Algorithm	Traditional	Current
Differences	over-specified boundary	normal boundary
	Two step	One step

5.1. Piecewise smooth domain. We consider a piecewise smooth domain $\Omega = \{(x, y) | 0 \leq x, y \leq 1\}$, whose geometry configuration is shown in Figure 1. Only partially boundary condition is accessible on Γ (The solid line with ‘.’ in Figure 1). We aim to recover the boundary data in the rest part (The broken line in Figure 1). The following Dirichlet boundary condition is considered

$$u(x, y) = e^x \cos y, \quad (x, y) \in \Gamma. \quad (12)$$

Since there exist investigations on the choice of source points [29, 35], we choose the source points on the fictitious boundary $\Gamma' = \{(r, \theta) | r = 5, -\pi/4 \leq \theta < 5\pi/4\}$ (The solid line with ‘*’ in Figure 1). For convenience, all collocation points and source points are evenly distributed in this paper. However, our conclusion holds for scattered distributed points.

In this example, we fix the boundary point number $N = 16$. For boundary conditions with/without noise, Figure 2 shows the numerical results of Dirichlet boundary condition

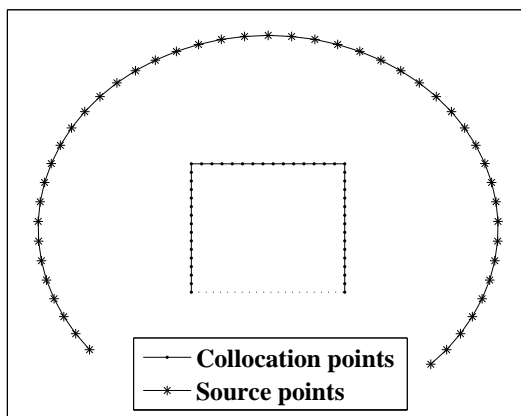
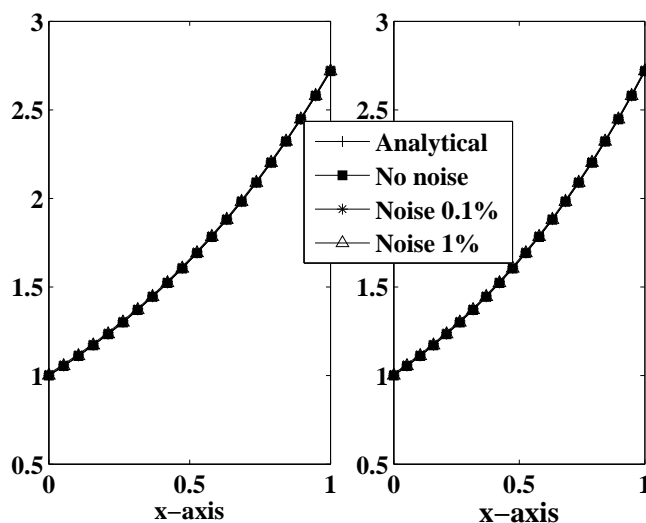


FIGURE 1. Schematic diagram of piecewise smooth domain

FIGURE 2. Dirichlet boundary condition on under-specified x -axis for traditional algorithm (Left) and our algorithm (Right)

on the under-specified x -axis for traditional algorithm and our algorithm. From it we see that numerical results are in good agreement with analytical solutions for noisy and non-noisy boundary conditions for both traditional algorithm and our algorithm. Similar results can be seen for the rest investigations, so the comparisons are not presented in the following parts.

Corresponding to Figure 2, we illustrate the errors between analytical and numerical results in Figure 3. The absolute errors between analytical and numerical results without noise are around 0.001. The average absolute error (AAE) is 1.47×10^{-4} . For added noise 0.1%, the errors are similar, i.e., the absolute errors between analytical and numerical results with noise 0.1% are also around 0.001. The corresponding AAE is 8.03×10^{-5} . However, for more noise 1% added, the errors increase to around 0.009. Although the AAE is 1.35×10^{-3} , the largest absolute error is 0.01.

This example shows that even with only partially accessible boundary conditions, we can also get accurate results. More importantly, no regularization techniques are used for

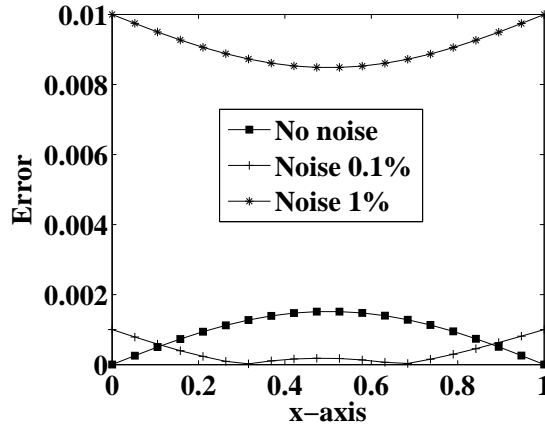


FIGURE 3. Errors between analytical and numerical results on under-specified x -axis

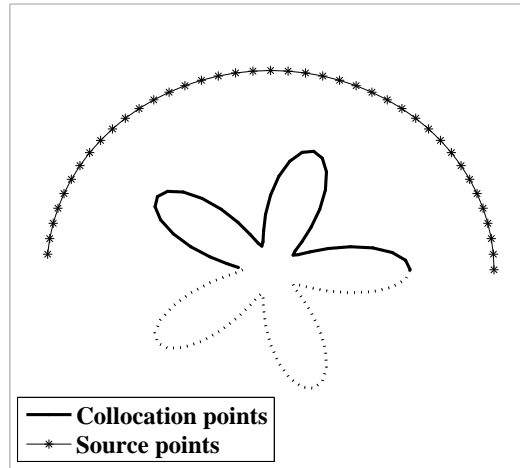


FIGURE 4. Schematic diagram of smooth domain

non-noisy boundary conditions and boundary conditions with small noise levels. However, for larger noise levels, regularization methods are also needed. All in all, we conclude that there is no need to use additional boundary data which is used in traditional procedures.

5.2. Smooth domain. Here, we consider an irregular smooth domain $\Omega = \{(r, \theta) | 0 \leq \theta < 2\pi, 0 \leq r \leq 0.3(1.5 + \cos(5\theta))\}$, whose schematic diagram is shown in Figure 4. Only partially boundary condition Γ is accessible (The solid line with ‘.’ in Figure 4). We aim to recover the boundary data in the rest part (The broken line in Figure 4). The Dirichlet boundary condition is given by

$$u(x, y) = x^2 - y^2, \quad (x, y) \in \Gamma. \quad (13)$$

The source points are located on the fictitious boundary $\Gamma' = \{(r, \theta) | r = 5, 0 \leq \theta < \pi\}$ (The solid line with ‘*’ in Figure 4).

5.2.1. Noisy boundary. For this case, the boundary point number is $N = 16$ and half accessible boundary $\Gamma = \{(r, \theta) | 0 \leq \theta < \pi, r = 0.3(1.5 + \cos(5\theta))\}$. Figure 5 reveals the numerical results of Dirichlet boundary condition on the under-specified x -axis for

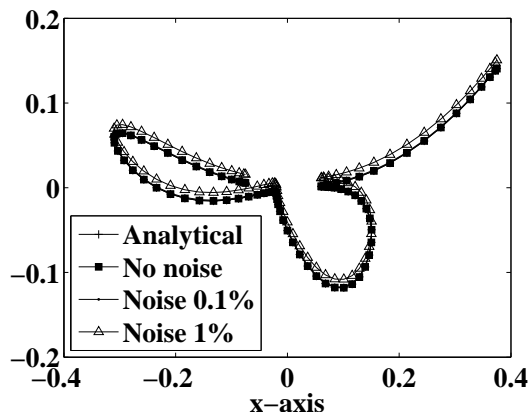


FIGURE 5. Dirichlet boundary condition

boundary conditions with/without noise. From it we see the numerical results are in good agreement with analytical solutions for noisy-free boundary condition and boundary conditions with noise 0.1%. As more noise added, the numerical results are obviously different from analytical results. This is different from the results shown in Figure 2. The reason for this phenomenon may contribute to the size of accessible boundary. It will be investigated in the subsequent subsection. Although not presented, similar errors can be achieved as shown in Figure 3. The previous conclusion still holds in this case, i.e., there is no need to use additional boundary data in solving this kind of inverse problems.

We note that if fixed noise levels are used, the numerical results are less sensitive to noise levels. Also, it is also an ideal case for practical problems. Due to the limitation of space, the numerical results are omitted here.

5.2.2. *Influence of accessible boundary.* We consider the effect of accessible boundary

$$\Gamma = \{(r, \theta) | r = 0.3(1.5 + \cos(5\theta)), 0 \leq \theta < \alpha\} \quad (14)$$

on the accuracy of numerical results, where the polar angle $\alpha \in [\pi/8, 3\pi/2]$. For fixed boundary point number $N = 16$, Figure 6 shows the sensitivity of AAM versus the measurement of the accessible boundary Γ with angle step $\pi/8$. As the accessible boundary increases, the AAM becomes small for $\alpha \leq 9\pi/8$. After $\alpha > 9\pi/8$, convergence curve of the AAM oscillates around 10^{-5} . This may be partially due to the ill-conditioned interpolation system.

The corresponding traditional condition number (Cond) and ECN variation curves versus the measurement of the accessible boundary Γ with angle step $\pi/8$ are shown in Figure 7. From it we can see the ECN is a superior criterion to the Cond. More specifically, the ECN increases with the AAM decreases. This is not obvious for the Cond case. Also, we find that the ECN is far smaller than the Cond.

In [18], it is stated that a necessary condition for the inverse problem given by Equations (1) and (2) to be identifiable is that $\text{measure}(\Gamma) > \text{measure} \Gamma_0$. The results in Figure 7 show that accurate numerical results can be got even for smaller accessible boundary. For example, the AAM is smaller than 10^{-4} for accessible boundary with $\alpha = \pi/8$.

Unless otherwise stated, we fix half accessible boundary $\Gamma = \{(r, \theta) | r = 0.3(1.5 + \cos(5\theta)), 0 \leq \theta < \pi\}$ in the following discussions.

5.2.3. *Noisy boundary analysis.* For various noisy levels, the relationship among the traditional condition number (Cond), ECN and AAE are shown in Table 1. Since the coefficient matrix only relates to the positioning of the boundary points, the Cond remains the same

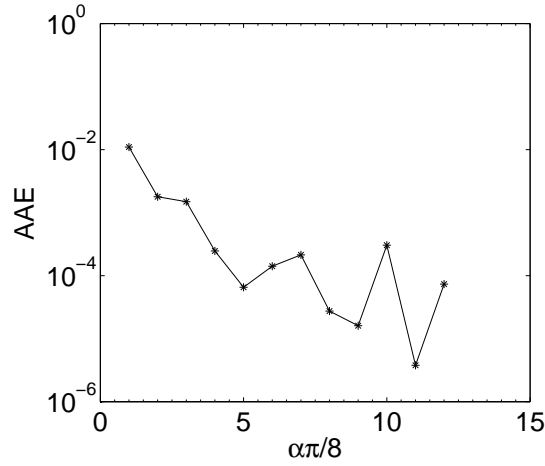


FIGURE 6. AAM versus the variation of accessible boundary data

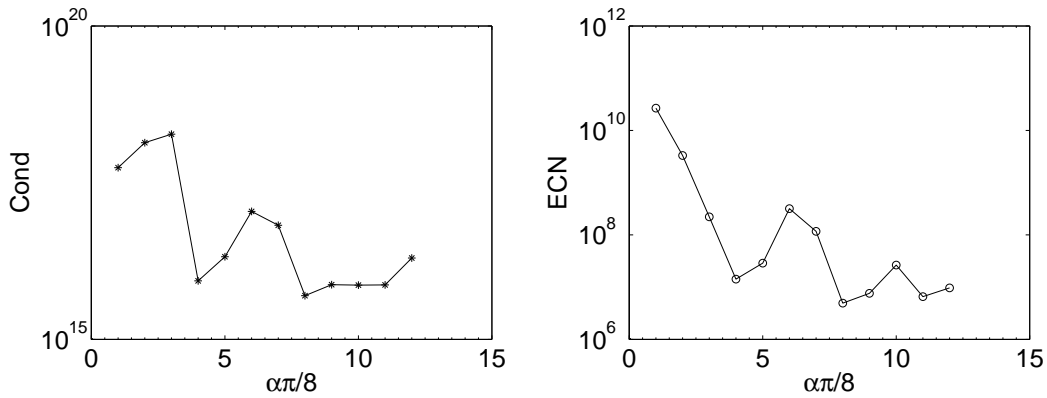


FIGURE 7. Cond (Left) and ECN (Right) against the variation of accessible boundary data

TABLE 2. Effect of noise level

Noise δ	0	0.00001	0.0001	0.001
Cond	2.60×10^{17}	2.60×10^{17}	2.60×10^{17}	2.60×10^{17}
ECN	4.83×10^7	1.53×10^6	1.18×10^5	1.18×10^4
AAE	5.10×10^{-5}	2.44×10^{-3}	3.67×10^{-2}	1.71×10^{-1}

for various noise levels. Differently, the ECN is related on the positioning of the boundary points and the Dirichlet boundary data as well as the imposed noise levels. This conclusion is supported by data in Table 2.

Once a tiny amount of noise $\delta = 0.00001$ is added, the ECN drops from order 10^7 to 10^6 . And the corresponding AAE increases from order 10^{-5} to 10^{-3} . The data in Table 2 strongly support the relation $ECN = O(AAE^{-1})$. This relationship is same with direct problems [30]. More interestingly, the small ECN ($ECN = 1.18 \times 10^4$) found in Table 2 suggests that the numerical solutions ($AAE = 1.71 \times 10^{-1}$) are not trustworthy. In this case, one should turn to other ways instead of applying MFS directly.

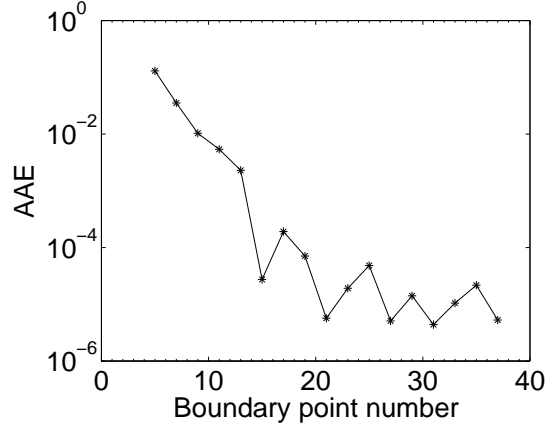


FIGURE 8. AAM versus boundary point numbers

TABLE 3. Condition numbers for Section 5.2.4

N	Cond	ECN	AAE
5	1.17×10^5	1.44×10^1	1.29×10^{-1}
9	3.06×10^9	3.55×10^3	1.03×10^{-2}
13	1.68×10^{13}	2.40×10^5	2.29×10^{-3}
15	4.98×10^{15}	4.94×10^6	2.73×10^{-5}
19	1.32×10^{18}	1.43×10^8	7.07×10^{-5}
23	1.62×10^{18}	1.33×10^8	1.92×10^{-5}
25	1.20×10^{19}	1.03×10^8	4.81×10^{-5}
29	5.30×10^{18}	4.47×10^8	1.41×10^{-5}
33	8.01×10^{18}	3.22×10^8	1.05×10^{-5}
35	1.62×10^{19}	5.09×10^8	2.17×10^{-5}
37	2.03×10^{18}	2.83×10^8	5.28×10^{-6}

5.2.4. *Convergence analysis.* As is known to all, more boundary data will lead to more accurate numerical results. Figure 8 displays AAE versus boundary point numbers for noisy-free boundary conditions. From it we observe that convergence curve of the MFS is smooth for boundary point number $N \leq 15$. After the boundary point number $N > 15$, convergence curve of the MFS has oscillatory phenomenon. For $N > 20$, there is almost no decrease in the AAE. This phenomenon is similar with the direct problems which may partially contribute to the ill-conditioned interpolation system [36, 37].

Corresponding to Figure 8, Table 3 gives the Cond, ECN and AAE versus boundary point numbers. It is shown that the Cond and ECN increase with the increasing boundary point numbers, while the AAE decreases. For boundary point number $N \leq 15$, we observe that the Cond and ECN are inversely proportional to the AAE. From this point, the Cond and ECN can be used to determine the optimal boundary point number with the MFS to get the best numerical accuracy. After $N > 15$ the relationship between the ECN and the AAE is inconclusive. The reason for this case may be partially contribute to the ill-conditioned interpolation system. Similar to Figure 7, we find the Cond is much larger than the corresponding ECN.

5.3. **Discussions.** From the foregoing numerical investigations, we can conclude that the MFS can be directly applied to simulate inverse problems with noisy and non-noisy

boundary conditions. Only Dirichlet boundary condition is used and there is no need to use additional boundary conditions. It is computationally efficient, stable with respect to arbitrary geometry. Furthermore, the approximation of the solution and its derivative on the entire solution domain are available by simple and direct function evaluation.

Although this paper considers only the inverse Laplace problems, the method can be similarly extended to the other types of inverse problems in higher dimensions.

6. Conclusions. In this paper, the MFS is used to solve the inverse problems associated with the Laplace equation under both smooth and piecewise smooth geometry. The condition number analysis of interpolation system is also considered. Numerical solutions obtained by the proposed analysis closely agree with the corresponding analytical solutions over the under-specified boundaries. Even for the existence of various noise levels, it is accurate without considering regularization techniques. The proposed numerical examples have proved the capability of the MFS together with the condition number analysis to easily handle the inverse Laplace problems under boundary conditions with/without noise levels.

However, a more systematic study on the accuracy and condition number for the MFS deserves further investigation.

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