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DYNAMICS FOR A STOCHASTIC TWO-SPECIES COMPETITIVE MODEL OF PLANKTON ALLEOPATHY

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Abstract. *This paper deals with a stochastic two-species competitive model of plankton alleopathy. Some very verifiable criteria on the uniformly weakly persistent in the mean almost surely (a.s.) and extinction for each species are obtained. Moreover, we also prove that there is a stationary distribution to this system and it has the ergodic property. Finally, some sufficient conditions for global asymptotic stability of the positive solution are established. An example is given to illustrate our main theoretical findings. Our results are new and complement previously known results.*
Keywords: Stochastic competitive model, Permanence, Extinction, Plankton alleopathy, Stationary distribution, Global asymptotic stability

1. **Introduction.** It is well known that the dynamical behavior of predator-prey models plays an important role in ecology and mathematical biology. In recent years, a lot of predator-prey models have attracted much attention due to its theoretical and practical significance. Many results on various predator-prey models are reported (see [1-3]). In 1974 and 1996, Maynard [4] and Chattopadhyay [5] considered the following system

$$\begin{cases} \dot{x}_1(t) = x_1(t)\left[K_1 - \alpha_1 x_1(t) - \beta_{12} x_2(t) - \gamma_1 x_1(t) x_2(t)\right], \\ \dot{x}_2(t) = x_2(t)\left[K_2 - \alpha_2 x_2(t) - \beta_{21} x_1(t) - \gamma_2 x_1(t) x_2(t)\right], \end{cases} \tag{1}$$

where $x_1(t)$ and $x_2(t)$ denote the population densities (number of cells per liter) of two competing species; K_1, K_2 are the rates of cell proliferation per hour; α_1, α_2 are the rates of intra-specific competition of first and second species, respectively; β_{12}, β_{21} are the rates of inter specific competition of first and second species respectively and K_1/α_1, K_2/α_2 are environmental carrying capacities (representing number of cells per liter). γ_1 and γ_2 are the rates of toxic inhibition of the first species by the second and vice versa, respectively. The units of α_1, α_2, β_{12} and β_{21} are per hour per cell and the unit of time is hour. α_1, α_2, β_{12}, β_{21}, γ_1 and γ_2 are positive constants.

Considering that discrete time models governed by difference equations are more appropriate to describe the dynamics relationship among populations than continuous ones and discrete time models can also provide efficient models of continuous ones for numerical simulations, Wu and Zhang [6] established the following two-species competitive discrete-time system of plankton allelopathy

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} x_1 + \delta x_1\left(K_1 - \alpha_1 x_1 - \beta_{12} x_2 - \gamma_1 x_1 x_2\right) \\ x_2 + \delta x_2\left(K_2 - \alpha_2 x_2 - \beta_{21} x_1 - \gamma_2 x_1 x_2\right) \end{bmatrix}. \tag{2}$$

Applying the center manifold theorem and bifurcation theory, Wu and Zhang [6] investigated the flip bifurcation of system (2). Moreover, numerical simulations display interesting dynamical behaviors (including period-doubling orbits and chaotic sets) for the system (2).

In the real world, the coefficients of system are not unchanged constants owing to the variation of environment. Moreover, species live in a real fluctuating medium, and human exploitation activities might result in the duration of abrupt changes, Xu et al. [7] investigated the following competitive system with impulsive perturbations

$$\begin{cases} \left.\begin{aligned} \dot{x}_1(t) = x_1(t)\left[K_1(t) - \alpha_1(t)x_1(t) - \beta_{12}(t)x_2(t) - \gamma_1(t)x_1(t)x_2(t)\right], \\ \dot{x}_2(t) = x_2(t)\left[K_2(t) - \alpha_2(t)x_2(t) - \beta_{21}(t)x_1(t) - \gamma_2(t)x_1(t)x_2(t)\right], \end{aligned}\right\} t \neq t_k, \\ \left.\begin{aligned} x_1(t_k^+) = (1+\gamma_{1k})x_1(t_k), \\ x_2(t_k^+) = (1+\gamma_{2k})x_2(t_k), \end{aligned}\right\} t = t_k, k \in N, \end{cases} \tag{3}$$

where $x_1(0^+) = x_1(0) > 0$, $x_2(0^+) = x_2(0) > 0$ and N is the set of positive integers, all the coefficients $K_i(t)$, $\alpha_i(t)$, $\gamma_i(t)$ $(i = 1, 2)$, $\beta_{12}(t)$, $\beta_{21}(t)$ are all continuous almost periodic functions which are bounded above and below by positive constants, $\gamma_{1k} > -1$ and $\gamma_{2k} > -1$ are constants and $0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$ are impulse points with $\lim_{k\to+\infty} t_k = +\infty$. The jump conditions reflect the possibility of impulsive effects on two species. From the viewpoint of biology, $\gamma_{ik} > 0$ implies that the perturbations may stand for stocking and $\gamma_{ik} < 0$ the perturbations stand for harvesting.

It shall be pointed out that population dynamics is inevitably affected by the environmental white noise which is an important component in an ecosystem (see [8-10]). Thus, environmental perturbations should not be neglected. Thus, we think that it is important to investigate the effect of environmental noises [11,12]. In 1994, Mao [13] had revealed important effect of noise: it can stabilize a system in some cases.

To the best of our knowledge, there are not many papers considering the stochastic two-species competitive model of plankton alleopathy. In model (1), we assume that the environmental noises mainly affect the rate of cell proliferation a_i with $K_i \to K_i + \sigma_i dB_i(t)$, where $B_i(t)$ stands for a standard Brownian montion defined on a complete probability space (Ω, F, P) and σ_i^2 is the intensity of the noise, $i = 1, 2$. Then we obtain the following stochastic two-species competitive model of plankton alleopathy

$$\begin{cases} dx_1(t) = x_1(t)\left[K_1 - \alpha_1 x_1(t) - \beta_{12}x_2(t) - \gamma_1 x_1(t)x_2(t)\right]dt + \sigma_1 x_1(t)dB_1(t), \\ dx_2(t) = x_2(t)\left[K_2 - \alpha_2 x_2(t) - \beta_{21}x_1(t) - \gamma_2 x_1(t)x_2(t)\right]dt + \sigma_2 x_2(t)dB_2(t). \end{cases} \tag{4}$$

In this paper, we make an attempt to discuss the dynamics of system (4). We will find how the noise affects the population models. The rest of this paper is arranged as follows. In Section 2, we present some notations, definitions and lemmas. In Section 3, we establish some sufficient criteria on the main result on the uniformly weakly persistent in the mean almost surely (a.s.) and extinction for each species. In Section 4, we consider the stationary distribution and ergodic property of model (4). In Section 5, some sufficient conditions for global asymptotic stability of the positive solution are established. In Section 6, examples together with the numerical simulations are given to verify the validity of the main theoretical analysis. In Section 7, we make a conclusion.

2. **Notations and Preliminaries.** For simplicity, we use the notations as follows.

$$R_+^2 = \left\{a = (a_1, a_2) \in R^2 \middle| a_i > 0,\ i = 1, 2\right\}, \quad b_i = K_i - 0.5\sigma_i^2,\ i = 1, 2.$$

$$\langle f(t)\rangle = t^{-1}\int_0^t f(s)ds, \quad \langle f\rangle^* = \lim_{t\to+\infty}\sup t^{-1}\int_0^t f(s)ds,$$

$$\langle f\rangle_* = \lim_{t\to+\infty}\inf t^{-1}\int_0^t f(s)ds.$$

Lemma 2.1. [14] *Suppose that* $z(t) \in C(\Omega \times [0, +\infty), R_+)$.
(i) If there exist two constants T *and* λ_0 *such that* $\ln z(t) \leq \lambda t - \lambda_0 \int_0^t z(s)ds + \sum_{i=1}^2 \sigma_i B_i(t)$ *for all* $t \geq T$, *where* σ_i, $i = 1, 2$, *are constants, then*

$$\begin{cases} \lim_{t\to+\infty} \sup\langle z(t)\rangle \leq \dfrac{\lambda}{\lambda_0} \ a.s., & if\ \lambda \geq 0; \\ \lim_{t\to+\infty} z(t) = 0 \ a.s., & if\ \lambda < 0. \end{cases}$$

(ii) If there exist three constants T, λ *and* λ_0 *such that* $\ln z(t) \geq \lambda t - \lambda_0 \int_0^t z(s)ds + \sum_{i=1}^2 \alpha_i B_i(t)$ *for all* $t \geq T$, *then* $\lim_{t\to+\infty} \sup\langle z(t)\rangle \geq \frac{\lambda}{\lambda_0}$ *a.s.*

Lemma 2.2. [15] *For any given initial value* $x(0) = (x_1(0), x_2(0)) \in R_+^2$, *system (4) has a unique positive solution* $x = (x_1(t), x_2(t))$ *on* $t \geq 0$ *a.s. and the solution satisfies*

$$\lim_{t\to+\infty} \sup \frac{\ln x_i(t)}{\ln t} \leq 1 \ a.s., \ i = 1, 2. \tag{5}$$

Proof: The proof is rather standard and hence is omitted (see e.g., [15]).

Definition 2.1. $x(t)$ *is said to be persistent in the mean if* $\langle x\rangle_* > 0$.

Definition 2.2. *Population* x *is said to go to extinction if for any initial value* $x(0) = x_0 > 0$, *we have* $\lim_{t\to+\infty} x(t; 0, x_0) = 0$. *Population* x *is said to be uniformly weakly persistent in the mean if there are constants* $\beta > 0$ *and* $M > 0$ *such that for any initial value* $x_0 > 0$, *we have* $M \geq \limsup_{t\to+\infty}\langle x(t; 0, x_0)\rangle \geq \beta$.

3. **Persistence and Extinction.** The property of permanence and extinction plays an important role in population dynamics since it means the long time survival or disappearance for each species. In this section, we will establish some sufficient conditions on the uniformly weakly persistent in the mean almost surely (a.s.) and extinction for each species.

Theorem 3.1. *For system (4), the following assertions hold.*
(i) If $b_i < 0$ $(i = 1, 2)$, *then* x_i $(i = 1, 2)$ *goes to extinction almost surely (a.s.), i.e.,* $\lim_{t\to+\infty} x_i(t) = 0$ *a.s.,* $i = 1, 2$.
(ii) If $b_1 > 0$, $b_2 < 0$, *then* x_2 *goes to extinction almost surely (a.s.) and*

$$\lim_{t\to+\infty} t^{-1} \int_0^t x_1(s)ds = \frac{b_1}{\alpha_1}, \ a.s.$$

(iii) If $b_i > 0$ $(i = 1, 2)$, *then* x_i $(i = 1, 2)$ *will be uniformly weakly persistent in the mean almost surely (a.s.), i.e.,* $\langle x_1(t)\rangle^* \leq \frac{b_i}{\alpha_i}$, *a.s.*
(iv) If $b_1 > 0$ *and* $\alpha_1 b_2 - \beta_{21} b_1 < 0$, *then* x_2 *goes to extinction almost surely (a.s.) and* x_1 *will be uniformly weakly persistent in the mean almost surely (a.s.), i.e.,* $\langle x_1(t)\rangle^* \leq \frac{b_1}{\alpha_1}$, *a.s.*
(v) If $b_2 > 0$ *and* $\alpha_2 b_1 - \beta_{12} b_2 < 0$, *then* x_1 *goes to extinction almost surely (a.s.) and* x_2 *will be uniformly weakly persistent in the mean almost surely (a.s.), i.e.,* $\langle x_2(t)\rangle^* \leq \frac{b_2}{\alpha_2}$, *a.s.*

Proof: Applying Itô's formula to system (4), we have

$$\begin{cases} d\ln x_1(t) = [b_1 - \alpha_1 x_1(t) - \beta_{12} x_2(t) - \gamma_1 x_1(t) x_2(t)]\, dt + \sigma_1 dB_1(t), \\ d\ln x_1(t) = [b_2 - \alpha_2 x_2(t) - \beta_{21} x_1(t) - \gamma_2 x_1(t) x_2(t)]\, dt + \sigma_2 dB_2(t). \end{cases} \tag{6}$$

Integrating and then dividing by t yields

$$\begin{cases} t^{-1}\ln\dfrac{x_1(t)}{x_1(0)} = b_1 - \alpha_1\langle x_1(t)\rangle - \beta_{12}\langle x_2(t)\rangle - \gamma_1\langle x_1(t)x_2(t)\rangle + \dfrac{\sigma_1 B_1(t)}{t}, \\ t^{-1}\ln\dfrac{x_2(t)}{x_2(0)} = b_2 - \alpha_2\langle x_2(t)\rangle - \beta_{21}\langle x_1(t)\rangle - \gamma_2\langle x_1(t)x_2(t)\rangle + \dfrac{\sigma_2 B_2(t)}{t}. \end{cases} \tag{7}$$

Now we prove (i). It follows from (7) that

$$\begin{cases} t^{-1}\ln\dfrac{x_1(t)}{x_1(0)} \le b_1 + \dfrac{\sigma_1 B_1(t)}{t}, \\ t^{-1}\ln\dfrac{x_2(t)}{x_2(0)} \le b_2 + \dfrac{\sigma_2 B_2(t)}{t}. \end{cases} \tag{8}$$

Note that

$$\lim_{t\to+\infty}\frac{B_i(t)}{t} = 0,\ \ a.s.$$

and $b_i < 0$ $(i = 1, 2)$, we have

$$\lim_{t\to+\infty} x_i(t) = 0,\ \ a.s. \tag{9}$$

Next we prove (ii). It follows from (8) that

$$t^{-1}\ln\frac{x_1(t)}{x_1(0)} \le b_1 - \alpha_1\langle x_1(t)\rangle + \frac{\sigma_1 B_1(t)}{t}. \tag{10}$$

In view of $b_1 > 0$ and Lemma 2.1, we have

$$\langle x_1(t)\rangle^* \le \frac{b_1}{\alpha_1},\ \ a.s. \tag{11}$$

On the other hand, in view of (7), we have

$$t^{-1}\ln\frac{x_2(t)}{x_2(0)} \le b_2 + \frac{\sigma_2 B_2(t)}{t}. \tag{12}$$

Note that

$$\lim_{t\to+\infty}\frac{B_2(t)}{t} = 0,\ \ a.s.$$

and $b_2 < 0$, we have

$$\lim_{t\to+\infty} x_2(t) = 0,\ \ a.s. \tag{13}$$

According to (7), (11) and (13), we get

$$\begin{aligned} t^{-1}\ln\frac{x_1(t)}{x_1(0)} &\ge b_1 - \varepsilon - \alpha_1\langle x_1(t)\rangle - \beta_{12}\langle x_2(t)\rangle^* - \gamma_1\langle x_1(t)x_2(t)\rangle^* + \frac{\sigma_1 B_1(t)}{t} \\ &\ge b_1 - \varepsilon - \alpha_1\langle x_1(t)\rangle + \frac{\sigma_1 B_1(t)}{t} \end{aligned} \tag{14}$$

for sufficiently large t and arbitrary $\varepsilon > 0$. In view of Lemma 2.1, one can obtain

$$\langle x_1(t)\rangle_* \ge \frac{b_1 - \varepsilon}{\alpha_1},\ \ a.s. \tag{15}$$

It follows from the arbitrariness of ε that

$$\langle x_1(t)\rangle_* \ge \frac{b_1}{\alpha_1},\ \ a.s. \tag{16}$$

From (11) and (16), we have

$$\lim_{t\to+\infty} t^{-1}\int_0^t x_1(s)ds = \lim_{t\to+\infty}\langle x_1(t)\rangle = \frac{b_1}{\alpha_1},\ \ a.s. \tag{17}$$

In the sequel, we prove (iii). It follows from (7) that

$$t^{-1}\ln\frac{x_i(t)}{x_i(0)} \le b_i - \alpha_1\langle x_i(t)\rangle + \frac{\sigma_i B_i(t)}{t},\ i=1,2. \tag{18}$$

In view of $b_i > 0$ and Lemma 2.1, we have

$$\langle x_i(t)\rangle^* \le \frac{b_i}{\alpha_i},\ a.s.,\ i=1,2. \tag{19}$$

We prove (iv). By (7), we have

$$\begin{cases} -\beta_{21}t^{-1}\ln\dfrac{x_1(t)}{x_1(0)} = -\beta_{21}\left[b_1 - \alpha_1\langle x_1(t)\rangle - \beta_{12}\langle x_2(t)\rangle - \gamma_1\langle x_1(t)x_2(t)\rangle + \dfrac{\sigma_1 B_1(t)}{t}\right], \\ \alpha_1 t^{-1}\ln\dfrac{x_2(t)}{x_2(0)} = \alpha_1\left[b_2 - \alpha_2\langle x_2(t)\rangle - \beta_{21}\langle x_1(t)\rangle - \gamma_2\langle x_1(t)x_2(t)\rangle + \dfrac{\sigma_2 B_2(t)}{t}\right], \end{cases} \tag{20}$$

which leads to

$$\begin{aligned} &\alpha_1 t^{-1}\ln\frac{x_2(t)}{x_2(0)} - \beta_{21}t^{-1}\ln\frac{x_1(t)}{x_1(0)} \\ =\ &(\alpha_1 b_2 - \beta_{21}b_1) - (\alpha_1\alpha_2 - \beta_{12}\beta_{21})\langle x_2(t)\rangle \\ &- (\alpha_1\gamma_2 - \beta_{21}\gamma_1)\langle x_1(t)x_2(t)\rangle \\ &+ t^{-1}[\alpha_1\sigma_2 B_2(t) - \beta_{21}\sigma_1 B_1(t)]. \end{aligned} \tag{21}$$

In view of (5), we have $[\ln x_1(t)/t] \le 0$. Substituting this inequality into (21), we get

$$\begin{aligned} &\alpha_1 t^{-1}\ln\frac{x_2(t)}{x_2(0)} \\ \le\ &(\alpha_1 b_2 - \beta_{21}b_1) + \varepsilon - (\alpha_1\alpha_2 - \beta_{12}\beta_{21})\langle x_2(t)\rangle - (\alpha_1\gamma_2 - \beta_{21}\gamma_1)\langle x_1(t)x_2(t)\rangle \\ &+ t^{-1}[\alpha_1\sigma_2 B_2(t) - \beta_{21}\sigma_1 B_1(t)] \\ \le\ &(\alpha_1 b_2 - \beta_{21}b_1) + \varepsilon - (\alpha_1\alpha_2 - \beta_{12}\beta_{21})\langle x_2(t)\rangle \\ &+ t^{-1}[\alpha_1\sigma_2 B_2(t) - \beta_{21}\sigma_1 B_1(t)] \end{aligned} \tag{22}$$

for sufficiently large t. By the condition $\alpha_1 b_2 - \beta_{21}b_1 < 0$, we can choose ε sufficiently small such that $\alpha_1 b_2 - \beta_{21}b_1 + \varepsilon < 0$. In view of Lemma 2.1, we have

$$\lim_{t\to+\infty} x_2(t) = 0,\ a.s. \tag{23}$$

Substituting (23) into the first equation of (7), we get

$$t^{-1}\ln\frac{x_1(t)}{x_1(0)} \le b_1 - \alpha_1\langle x_1(t)\rangle + \frac{\sigma_1 B_1(t)}{t}. \tag{24}$$

In view of $b_1 > 0$ and Lemma 2.1, we have

$$\langle x_1(t)\rangle^* \le \frac{b_1}{\alpha_1}. \tag{25}$$

Finally we prove (v). By (7), we have

$$\begin{cases} \alpha_2 t^{-1}\ln\dfrac{x_1(t)}{x_1(0)} = \alpha_2\left[b_1 - \alpha_1\langle x_1(t)\rangle - \beta_{12}\langle x_2(t)\rangle - \gamma_1\langle x_1(t)x_2(t)\rangle + \dfrac{\sigma_1 B_1(t)}{t}\right], \\ -\beta_{12}t^{-1}\ln\dfrac{x_2(t)}{x_2(0)} = -\beta_{12}\left[b_2 - \alpha_2\langle x_2(t)\rangle - \beta_{21}\langle x_1(t)\rangle - \gamma_2\langle x_1(t)x_2(t)\rangle + \dfrac{\sigma_2 B_2(t)}{t}\right], \end{cases} \tag{26}$$

which leads to

$$\begin{aligned}
&\alpha_2 t^{-1}\ln\frac{x_1(t)}{x_1(0)} - \beta_{12}t^{-1}\ln\frac{x_2(t)}{x_2(0)} \\
&= (\alpha_2 b_1 - \beta_{12}b_2) - (\alpha_1\alpha_2 - \beta_{12}\beta_{21})\langle x_1(t)\rangle - (\alpha_2\gamma_1 - \beta_{12}\gamma_2)\langle x_1(t)x_2(t)\rangle \\
&\quad + t^{-1}[\alpha_2\sigma_1 B_1(t) - \beta_{12}\sigma_2 B_2(t)]. \qquad (27)
\end{aligned}$$

In view of (5), we have $[\ln x_2(t)/t] \le 0$. Substituting this inequality into (27), we get

$$\begin{aligned}
\alpha_1 t^{-1}\ln\frac{x_1(t)}{x_1(0)} &\le (\alpha_2 b_1 - \beta_{12}b_2) + \varepsilon - (\alpha_1\alpha_2 - \beta_{12}\beta_{21})\langle x_1(t)\rangle \\
&\quad - (\alpha_2\gamma_1 - \beta_{12}\gamma_2)\langle x_1(t)x_2(t)\rangle + t^{-1}[\alpha_2\sigma_1 B_1(t) - \beta_{12}\sigma_2 B_2(t)] \\
&\le (\alpha_2 b_1 - \beta_{12}b_2) + \varepsilon - (\alpha_1\alpha_2 - \beta_{12}\beta_{21})\langle x_1(t)\rangle \\
&\quad + t^{-1}[\alpha_2\sigma_1 B_1(t) - \beta_{12}\sigma_2 B_2(t)]. \qquad (28)
\end{aligned}$$

for sufficiently large t. By the condition $\alpha_2 b_1 - \beta_{12}b_2 < 0$, we can choose ε sufficiently small such that $\alpha_2 b_1 - \beta_{12}b_2 + \varepsilon < 0$. In view of Lemma 2.1, we have

$$\lim_{t\to+\infty} x_1(t) = 0, \ a.s. \qquad (29)$$

Substituting (29) into the second equation of (7), we get

$$t^{-1}\ln\frac{x_2(t)}{x_2(0)} \le b_2 - \alpha_2\langle x_2(t)\rangle + \frac{\sigma_2 B_2(t)}{t}. \qquad (30)$$

In view of $b_2 > 0$ and Lemma 2.1, we have

$$\langle x_2(t)\rangle^* \le \frac{b_2}{\alpha_2}. \qquad (31)$$

The proof of Theorem 2.1 is completed.

4. **Stationary Distribution and Ergodicity.**

Based on Section 3, we will consider another interesting topic on stationary distribution and ergodicity.

Let $X(t)$ be a homogeneous Markov process in E^l (E^l denotes Euclidean l-space) satisfying the following stochastic differential equation:

$$dX(t) = b(X)dt + \sum_{m=1}^{k}\beta_m(X)dB_m(t). \qquad (32)$$

The diffusion matrix is $\bar{A}(x) = (\bar{a}_{ij}(x))$, $\bar{a}_{ij} = \sum_{m=1}^{k}\beta_m^{(i)}(x)\beta_m^{(j)}(x)$.

Assumption 1. There is a bounded domain $U \subset E^l$ with regular boundary Γ, which has the properties that

(H1) In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $\bar{A}(x)$ is bounded away from zero.

(H2) If $x \in E^L \setminus U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} E_x\tau < +\infty$ for every compact subset $K \in E^l$.

Lemma 4.1. [16] *If Assumption 1 holds, then the Markov process $X(t)$ has a stationary distribution $\mu(.)$. Let $f(.)$ be a function integrable with respect to the measure $\mu(.)$. Then*

$$P\left\{\lim_{t\to+\infty}\frac{1}{t}\int_0^t f(X(s))ds = \int_{E^l} f(x)\mu(dx)\right\} = 1.$$

Remark 4.1. *To verify (H1), it is sufficient to show that H is uniformly elliptical in U, where $Hu = b(x)u_x + 0.5trace\left(\bar{A}(x)u_{xx}\right)$, i.e., there exists a positive number c such that*

$$\sum_{i,j=1}^{k} \bar{a}_{ij}(x)\theta_i\theta_j > c|\theta|^2, \quad x \in U, \quad \theta \in R^k.$$

In detail, one can see [17,18]. To verify (H2), it is sufficient to prove that there exist a neighborhood U and a non-negative C^2-function $V(x)$ such that for any $x \in E^L \setminus U$, $LV(x) < 0$ (see [19]).

Remark 4.2. *The diffusion matrix of (4) is*

$$\bar{A}(x) = \begin{bmatrix} \sigma_1^2 x_1^2 & 0 \\ 0 & \sigma_2^2 x_2^2 \end{bmatrix}.$$

Lemma 4.2. [20] *For any initial value $x(0) \in R_+^2$ and $p > 0$, there is a constant $K = K(p) > 0$ such that the solution $x(t)$ of (4) satisfies*

$$\lim_{t\to+\infty} \sup E\left[x_i^p(t)\right] \le K, \ i = 1, 2.$$

According to Lemma 4.2, there exists a $T > 0$ such that $E\left[x_i^p(t)\right] \le 2K$ for $t \ge T$. Note that $E[x_i(t)]$ is continuous, then there is a constant $K_* > 0$ such that $E\left[x_i^p(t)\right] \le K_*$ for $0 \le t < T$. Define $L + \max\{2k, K_*\}$. Then

$$E\left[x_i^p(t)\right] \le L = L(p), \quad t \ge 0, \quad p > 0, \ i = 1, 2. \tag{33}$$

Theorem 4.1. *Let (x_1^*, x_2^*) be a positive solution of the following system*

$$\begin{cases} \alpha_1 x_1 + \beta_{12} x_2 + \gamma_1 x_1 x_2 = K_1, \\ \alpha_2 x_2 + \beta_{21} x_1 + \gamma_2 x_1 x_2 = K_2. \end{cases}$$

If the following conditions

$$\begin{aligned} &\varrho_1 = \alpha_1 - 0.5\left(\beta_{12} + 2\gamma_1 x_2^* + \gamma_1 x_1^* + \beta_{21} + \gamma_2 x_2^*\right) > 0, \\ &\varrho_2 = \alpha_2 - 0.5\left(\beta_{21} + 2\gamma_2 x_1^* + \gamma_2 x_2^* + \beta_{12} + \gamma_1 x_1^*\right) > 0, \\ &0.5\sum_{i=1}^{2} \sigma_i^2 x_i^* < \min_{i=1,2}\left\{\varrho_i (x_i^*)^2\right\} \end{aligned}$$

hold, then there is a stationary distribution $\mu(.)$ for system (4) and it has the ergodic property

$$P\left\{\lim_{t\to+\infty} \frac{1}{t}\int_0^t x_i(s)ds = \int_{R_+^2} z_i \mu(dz_1, dz_2, dz_3)\right\} = 1, \ i = 1, 2.$$

Proof: Define

$$V(x) = \sum_{i=1}^{2}\left[x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*}\right]. \tag{34}$$

Applying Itô's formula we have

$$dV(x) = LV(x)dt + \sum_{i=1}^{2}\left(x_i - x_i^*\right)\sigma_i dB_i(t), \tag{35}$$

where

$$\begin{aligned} LV(x) &= (x_1 - x_1^*)\left[K_1 - \alpha_1 x_1 - \beta_{12} x_2 - \gamma_1 x_1 x_2\right] + 0.5\sigma_1^2 x_1^* \\ &\quad + (x_2 - x_2^*)\left[K_2 - \alpha_2 x_2 - \beta_{21} x_1 - \gamma_2 x_1 x_2\right] + 0.5\sigma_2^2 x_2^* \\ &= -(x_1 - x_1^*)\left[\alpha_1(x_1 - x_1^*) - \beta_{12}(x_2 - x_2^*) - \gamma_1(x_1 x_2 - x_1^* x_2^*)\right] + 0.5\sigma_1^2 x_1^* \end{aligned}$$

$$
\begin{aligned}
&\quad -(x_2-x_2^*)\left[\alpha_2(x_2-x_2^*)-\beta_{21}(x_1-x_1^*)-\gamma_2(x_1x_2-x_1^*x_2^*)\right]+0.5\sigma_2^2x_2^*\\
&\le -\left[\alpha_1-0.5(\beta_{12}+2\gamma_1x_2^*+\gamma_1x_1^*+\beta_{21}+\gamma_2x_2^*)\right](x_1-x_1^*)^2\\
&\quad -\left[\alpha_2-0.5(\beta_{21}+2\gamma_2x_1^*+\gamma_2x_2^*+\beta_{12}+\gamma_1x_1^*)\right]\left(x_2-x_2^2\right)^2+0.5\sum_{i=1}^{2}\alpha_i^2x_i^*\\
&= -\sum_{i=1}^{2}\varrho_i(x_i-x_i^*)^2+0.5\sum_{i=1}^{2}\sigma_i^2x_i^*. \qquad (36)
\end{aligned}
$$

Note that $\varrho_i>0$ and $0.5\sum_{i=1}^{2}\sigma_i^2x_i^*<\min_{i=1,2}\{\varrho_i(x_i^*)^2\}$, then the ellipsoid

$$
-\sum_{i=1}^{2}\varrho_i(x_i-x_i^*)^2+0.5\sum_{i=1}^{2}\sigma_i^2x_i^*=0 \qquad (37)
$$

lies entirely in R_+^2. We can take U to be a neighborhood of the ellipsoid with $\bar U\subseteq R_+^2$, then for $x\in R_+^2\setminus U$, we obtain $LV(x)<0$. Namely, (H2) holds. On the other hand, there exists a $c>0$ such that

$$
\sum_{i=1}^{2}\bar a_{ij}(x)\theta_i\theta_j=\sum_{i=1}^{2}\sigma_i^2x_i^2\theta_i^2>c|\theta|^2 \qquad (38)
$$

for $x\in\bar U$ and $\theta\in R^2$. That is to say, (H1) holds. Thus, (4) has a stationary distribution $\mu(.)$ and it is ergodic. In view of the ergodic property, for $M>0$, we have

$$
\lim_{t\to+\infty}t^{-1}\int_0^t[x_i(s)\wedge M]ds=\int_{R_+^2}[z_i\wedge M]\mu(dz_1,dz_2,dz_3),\ a.s. \qquad (39)
$$

It follows from the dominated convergence theorem and (33) that

$$
E\left[\lim_{t\to+\infty}t^{-1}\int_0^t[x_i(s)\wedge M]ds\right]=\lim_{t\to+\infty}t^{-1}\int_0^t[x_i(s)\wedge M]ds\le L. \qquad (40)
$$

In view of (39), we have

$$
\int_{R_+^2}[z_i\wedge M]\mu(dz_1,dz_2,dz_3)\le L. \qquad (41)
$$

Let $M\to+\infty$, then $\int_{R_+^2}z_i\mu(dz_1,dz_2,dz_3)\le L$. Thus, the function $f(x)=x$ is integrable with respect to the measure $\mu(.)$. It follows from Lemma 4.1 that the desired assertion holds.

5. **Global Asymptotic Stability.** In this section, we give sufficient conditions of global asymptotic stability.

Definition 5.1. *Let $x(t)$ and $y(t)$ be two arbitrary solutions of (4) with initial values $x(0)\in R_+^2$ and $y(0)\in R_+^2$, respectively. If for every $1\le i\le 2$, $\lim_{t\to+\infty}|x_i(t)-y_i(t)|=0$, a.s., then we say (4) is globally asymptotically stable (or globally attractive).*

Lemma 5.1. [21] *Suppose that an n-dimensional stochastic process $X(t)$ on $t\ge 0$ satisfies the condition*

$$
E|X(t)-X(s)|^{\lambda_1}\le c|t-s|^{1+\lambda_2},\ \ 0\le s,t<\infty,
$$

for some positive constants λ_1, λ_2, and c. Then there exists a continuous modification $\tilde X(t)$ of $X(t)$ which has the property that for every $\gamma\in(0,\lambda_2/\lambda_1)$ there is a positive *random variable $h(\omega)$ such that*

$$P\left\{\omega : \sup_{0<|t-s|<h(\omega),\, 0\le s,t<\infty} \frac{|\tilde{X}(t,\omega)-X(t,\omega)|}{|t-s|^{\gamma}} \le \frac{2}{1-2^{-\gamma}}\right\} = 1.$$

In other words, almost every sample path of $\tilde{X}(t)$ is locally but uniformly Hölder continuous with exponent γ.

Lemma 5.2. *Let $x(t)$ be a positive solution of (4), then almost every sample path of $x_i(t)$ $(i = 1, 2)$ is uniformly continuous.*

Proof: The proof is similar to that of Lemma 15 in [20]. Here we omit it.

Lemma 5.3. [22] *Let f be a non-negative function defined on $[0, +\infty)$ such that f is integrable and is uniformly continuous. Then $\lim_{t\to+\infty} f(t) = 0$.*

Theorem 5.1. *If there exist positive constants c_1 and c_2 such that*

$$(A)\quad \begin{cases} c_1\alpha_1 > c_1\gamma_1 x_2^* + c_2\beta_{21} + c_2\gamma_2 x_2^*, \\ c_2\alpha_2 > c_2\gamma_2 x_2^* + c_1\beta_{12} + c_1\gamma_1 x_2^*, \end{cases}$$

then (4) is globally asymptotically stable.

Proof: Let $x(t)$ and $y(t)$ be two solutions of (4) with initial values $x(0) \in R_+^2$ and $y(0) \in R_+^2$, respectively. Define

$$V(t) = \sum_{i=1}^{2} c_i |\ln x_i(t) - \ln y_i(t)|. \tag{42}$$

Calculating the right differential $d^+V(t)$, we have

$$\begin{aligned} d^+V(t) &= \sum_{i=1}^{2} c_i \mathrm{sgn}(x_i(t)-y_i(t))d(\ln x_i(t) - \ln y_i(t)) \\ &= -c_1\mathrm{sgn}(x_1(t)-y_1(t))[\alpha_1(x_1(t)-y_1(t)) + \beta_{12}(x_2(t)-y_2(t)) + \gamma_1(x_1(t)x_2(t) \\ &\quad -y_1(t)y_2(t))]dt - c_2\mathrm{sgn}(x_2(t)-y_2(t))[\alpha_2(x_2(t)-y_2(t)) \\ &\quad + \beta_{21}(x_1(t)-y_1(t)) + \gamma_2(x_1(t)x_2(t)-y_1(t)y_2(t))]dt \\ &\le -\left[c_1\alpha_1 - c_1\gamma_1x_2^* - c_2\beta_{21} - c_2\gamma_2x_2^*\right]|x_1(t)-y_1(t)|dt \\ &\quad -\left[c_2\alpha_2 - c_2\gamma_2x_2^* - c_1\beta_{12} - c_1\gamma_1x_2^*\right]|x_2(t)-y_2(t)|dt. \end{aligned} \tag{43}$$

Then

$$\begin{aligned} V(t) \le V(0) &- \int_0^t \left[c_1\alpha_1 - c_1\gamma_1x_2^* - c_2\beta_{21} - c_2\gamma_2x_2^*\right]|x_1(s)-y_1(s)|ds \\ &- \int_0^t \left[c_2\alpha_2 - c_2\gamma_2x_2^* - c_1\beta_{12} - c_1\gamma_1x_2^*\right]|x_2(s)-y_2(s)|ds. \end{aligned} \tag{44}$$

Thus

$$\begin{aligned} &V(t) + \int_0^t [c_1\alpha_1 - c_1\gamma_1x_2^* - c_2\beta_{21} - c_2\gamma_2x_2^*]|x_1(s)-y_1(s)|ds \\ &+ \int_0^t [c_2\alpha_2 - c_2\gamma_2x_2^* - c_1\beta_{12} - c_1\gamma_1x_2^*]|x_2(s)-y_2(s)|ds \le V(0) < \infty. \end{aligned} \tag{45}$$

Note that $V(t) \ge 0$, it follows from (A) that $|x_i(t)-y_i(t)|$ is integrable. In view of Lemma 5.2 and Lemma 5.3 that the required assertion holds.

Remark 5.1. *Although there are many papers that deal with the stochastic models, there are only few papers that consider the stationary distribution and ergodicity. In addition, the results on the global asymptotic stability are useful to estimate the risk of extinction of species for the competitive model. From this viewpoint, we think that our results are completely new.*

6. **Examples.** In this section, we give examples to illustrate our main results. Consider the following two stochastic two-species competitive models of plankton alleopathy.

Example 6.1.

$$\begin{cases} dx_1(t) = x_1(t)\left[4 - 0.5x_1(t) - 0.03x_2(t) - 0.02x_1(t)x_2(t)\right]dt + \sigma_1 x_1(t)dB_1(t), \\ dx_2(t) = x_2(t)\left[3 - 0.3x_2(t) - 0.04x_1(t) - 0.05x_1(t)x_2(t)\right]dt + \sigma_2 x_2(t)dB_2(t). \end{cases} \tag{46}$$

Corresponding to system (4), we have $K_1 = 4$, $K_2 = 3$, $\alpha_1 = 0.5$, $\beta_{12} = 0.03$, $\gamma_1 = 0.02$, $\alpha_2 = 0.3$, $\beta_{21} = 0.04$, $\gamma_2 = 0.05$. Using Milstein's method [23], we get the following discrete equation of (46)

$$\begin{cases} x_1^{(k+1)} = x_1^{(k)} + x_1^{(k)}\left[4 - 0.5x_1^{(k)} - 0.03x_2^{(k)} - 0.02x_1^{(k)}x_2^{(k)}\right]\triangle t \\ \qquad\qquad + \sigma_1 x_1^{(k)}\sqrt{\triangle t}\xi^{(k)} + \dfrac{\sigma_1^2}{2}x_1^{(k)}\left(\left(\xi^{(k)}\right)^2\triangle t - \triangle t\right), \\ x_2^{(k+1)} = x_2^{(k)} + x_2^{(k)}\left[3 - 0.3x_2^{(k)} - 0.04x_1^{(k)} - 0.05x_1^{(k)}x_2^{(k)}\right]\triangle t \\ \qquad\qquad + \sigma_2 x_2^{(k)}\sqrt{\triangle t}\eta^{(k)} + \dfrac{\sigma_2^2}{2}x_2^{(k)}\left(\left(\eta^{(k)}\right)^2\triangle t - \triangle t\right). \end{cases} \tag{47}$$

We can obtain $x_1^* = 2.5$, $x_2^* = 1.3$. Let $\sigma_1^2 = 0.3$, $\sigma_2^2 = 0.4$, $c_1 = 112$, $c_2 = 102$. It is easy to check that all the conditions in Theorem 5.1 are fulfilled. Hence we can conclude that then system (46) is globally attractive. From Figure 1, we know that two species in the community can coexist.

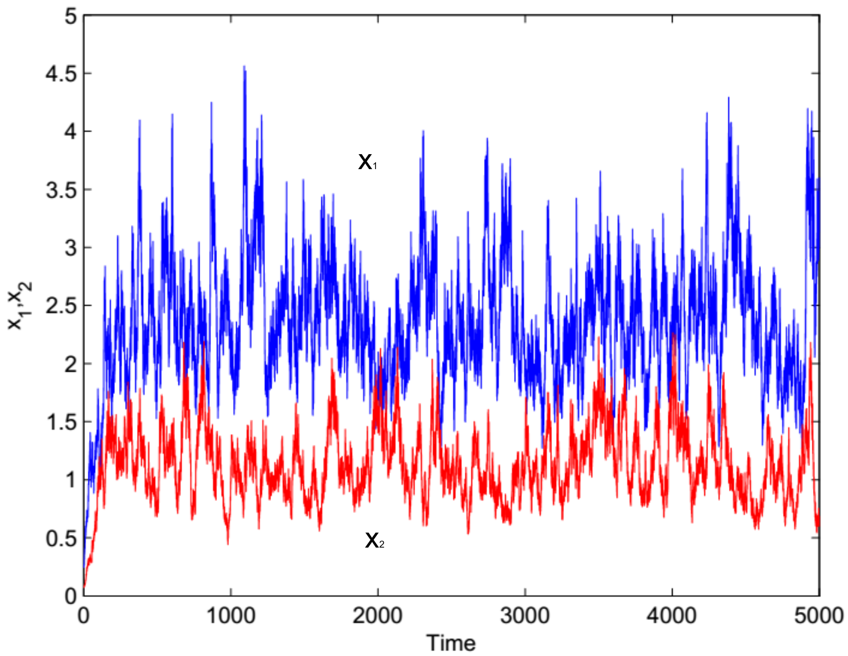

FIGURE 1. Time response of state variables x_1 and x_2

Example 6.2.

$$\begin{cases} dx_1(t) = x_1(t)\left[5 - 0.3x_1(t) - 0.01x_2(t) - 0.03x_1(t)x_2(t)\right]dt + \sigma_1 x_1(t)dB_1(t), \\ dx_2(t) = x_2(t)\left[2 - 0.1x_2(t) - 0.01x_1(t) - 0.02x_1(t)x_2(t)\right]dt + \sigma_2 x_2(t)dB_2(t). \end{cases} \tag{48}$$

Corresponding to system (4), we have $K_1 = 5$, $K_2 = 2$, $\alpha_1 = 0.3$, $\beta_{12} = 0.01$, $\gamma_1 = 0.03$, $\alpha_2 = 0.1$, $\beta_{21} = 0.01$, $\gamma_2 = 0.02$. Using Milstein's method [23], we get the following discrete equation of (48)

$$\begin{cases} x_1^{(k+1)} = x_1^{(k)} + x_1^{(k)}\left[5 - 0.3x_1^{(k)} - 0.01x_2^{(k)} - 0.03x_1^{(k)}x_2^{(k)}\right]\triangle t \\ \qquad + \sigma_1 x_1^{(k)}\sqrt{\triangle t}\xi^{(k)} + \dfrac{\sigma_1^2}{2}x_1^{(k)}\left(\left(\xi^{(k)}\right)^2\triangle t - \triangle t\right), \\ x_2^{(k+1)} = x_2^{(k)} + x_2^{(k)}\left[23 - 0.1x_2^{(k)} - 0.01x_1^{(k)} - 0.02x_1^{(k)}x_2^{(k)}\right]\triangle t \\ \qquad + \sigma_2 x_2^{(k)}\sqrt{\triangle t}\eta^{(k)} + \dfrac{\sigma_2^2}{2}x_2^{(k)}\left(\left(\eta^{(k)}\right)^2\triangle t - \triangle t\right). \end{cases} \tag{49}$$

We can obtain $x_1^* = 3.1$, $x_2^* = 1.1$. Let $\sigma_1^2 = 0.25$, $\sigma_2^2 = 0.38$, $c_1 = 101$, $c_2 = 99$. It is easy to check that all the conditions in Theorem 5.1 are fulfilled. Hence we can conclude that then system (48) is globally attractive. From Figure 2, we know that two species in the community can coexist.

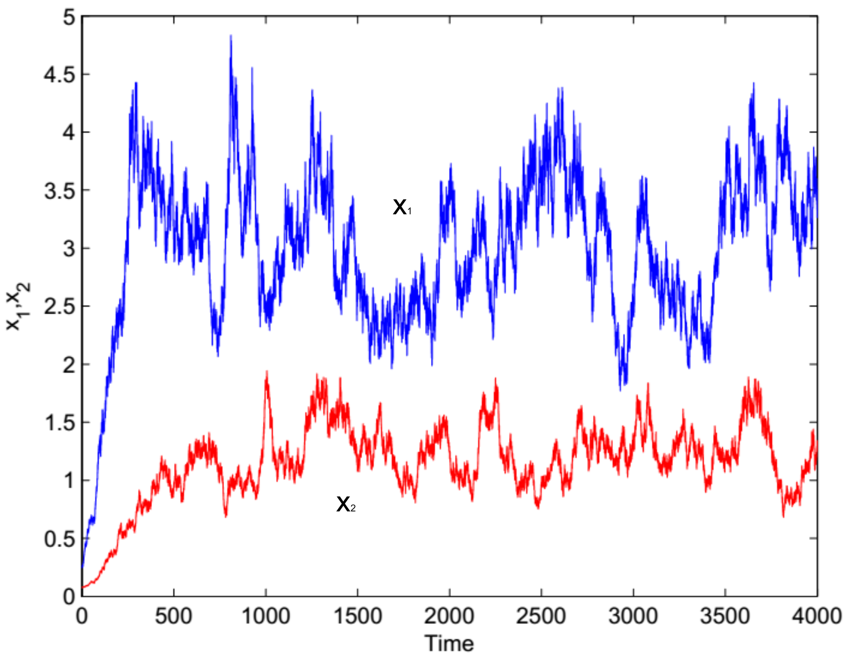

FIGURE 2. Time response of state variables x_1 and x_2

Example 6.3.

$$\begin{cases} dx_1(t) = x_1(t)\left[3 - 0.2x_1(t) - 0.05x_2(t) - 0.01x_1(t)x_2(t)\right]dt + \sigma_1 x_1(t)dB_1(t), \\ dx_2(t) = x_2(t)\left[5 - 0.5x_2(t) - 0.06x_1(t) - 0.06x_1(t)x_2(t)\right]dt + \sigma_2 x_2(t)dB_2(t). \end{cases} \tag{50}$$

Corresponding to system (4), we have $K_1 = 3$, $K_2 = 5$, $\alpha_1 = 0.2$, $\beta_{12} = 0.05$, $\gamma_1 = 0.01$, $\alpha_2 = 0.5$, $\beta_{21} = 0.06$, $\gamma_2 = 0.06$. Using Milstein's method [23], we get the following discrete equation of (50)

$$\begin{cases} x_1^{(k+1)} = x_1^{(k)} + x_1^{(k)}\left[3 - 0.2x_1^{(k)} - 0.05x_2^{(k)} - 0.01x_1^{(k)}x_2^{(k)}\right]\triangle t \\ \qquad + \sigma_1 x_1^{(k)}\sqrt{\triangle t}\xi^{(k)} + \dfrac{\sigma_1^2}{2}x_1^{(k)}\left(\left(\xi^{(k)}\right)^2\triangle t - \triangle t\right), \\ x_2^{(k+1)} = x_2^{(k)} + x_2^{(k)}\left[5 - 0.5x_2^{(k)} - 0.06x_1^{(k)} - 0.06x_1^{(k)}x_2^{(k)}\right]\triangle t \\ \qquad + \sigma_2 x_2^{(k)}\sqrt{\triangle t}\eta^{(k)} + \dfrac{\sigma_2^2}{2}x_2^{(k)}\left(\left(\eta^{(k)}\right)^2\triangle t - \triangle t\right). \end{cases} \tag{51}$$

We can obtain $x_1^* = 2.5$, $x_2^* = 1.3$. Let $\sigma_1^2 = 0.24$, $\sigma_2^2 = 0.33$, $c_1 = 122$, $c_2 = 79$. It is easy to check that all the conditions in Theorem 5.1 are fulfilled. Hence, we can conclude that then system (50) is globally attractive. From Figure 3, we know that two species in the community can coexist.

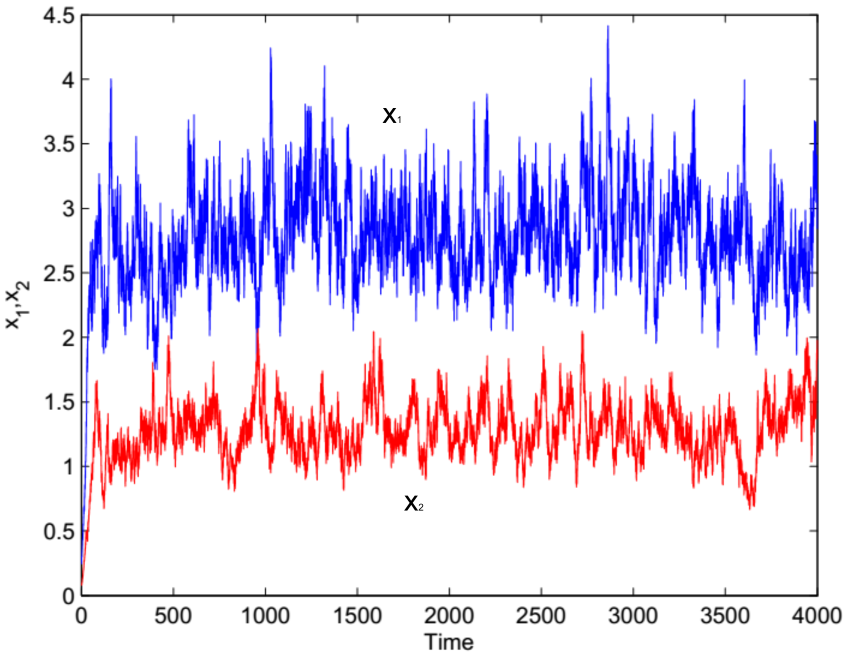

FIGURE 3. Time response of state variables x_1 and x_2

7. **Conclusions.** In this paper, we are concerned with the dynamical properties of a stochastic two-species competitive model of plankton alleopathy. We have established the sufficient conditions on the uniformly weakly persistent in the mean almost surely (a.s.) and extinction for each species. We prove that there is a stationary distribution to this system and it has the ergodic property. Moreover, the sufficient conditions for global asymptotic stability of the positive solution of this system are obtained. There are still many interesting and challenging questions that need investigate. In this paper, we only discuss the rates of cell proliferation per hour K_i $(i = 1, 2)$ are stochastic; for other parameters, for example, $\alpha_i(t)$ $(i = 1, 2)$ are stochastic, they are not considered. This aspect will be our future work.

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