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JOINT ESTIMATION OF PARAMETERS, STATES AND TIME DELAY BASED ON SINGULAR PENCIL MODEL

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Abstract. *A new estimation method using singular pencil models for parameters, states and time delay is presented in this paper. The method describes a discrete system with time delay by using singular pencil models. By means of the observable canonical form, it is very easy to separate the states with the parameters which contain the time delay information. Thus, joint estimation of the parameters, states and time delay has become feasible by an ordinary Kalman filter for noise known and an extended Kalman filter for noise unknown. Simulation results exhibit the validity of the method.*
Keywords: Singular pencil (SP), Joint estimation, Observable canonical form, Time delay

1. **Introduction.** Singular pencil model (SPM), as a unified representation of state space and polynomial input-output models, has broad applications in control (predictive control [1], self-tuning control [2]), fault diagnosis [3] and other aspects [4]. It also has much potential for analysis, optimization and identification of different systems, linear or nonlinear, proper or improper [5].

The problem of joint estimation of parameters and states has received great attention in recent years. Different estimation methods have been successfully utilized in linear or nonlinear systems [6-17,24,25]. For example, Kabouris and Georgakakos used the linearized maximum likelihood (LMI) method to simultaneously estimate the activated sludge states and parameters from noisy process measurements [11,12]. Tulsyan et al. considered simultaneous state-parameter estimation problem for nonlinear stochastic systems by using Bayesian approach [13]. Rajaraman et al. proposed an observer design methodology for a class of nonlinear systems by means of augmented states and Kharitonov's stability theorem [14]. Sliding mode technologies have also been utilized in the states and parameters estimation problem. By introducing an extended state vector, Basin and Rodriguez-Ramirez transformed the original problem into a sliding mode mean-square and mean-module filtering issue [15]. Rao et al. solved the 'sensorless' control problem for induction motors by designing two observers which were connected to a first- and second-order sliding mode [16]. Although many achievements have been made in joint estimation of parameters and states in recent years, full consideration of the joint estimation of parameters, states and time delay with singular pencil models has not been reported in the literature. In fact, joint estimation of parameters, states and time delay has a good prospect in the situation that the time delay cannot be easily measured, especially for the coupled MIMO systems in which only some of the inputs are with time delay. This motivates us to develop an estimation technique, taking account of the time delay information. In this paper, a new estimation method of parameters, states and time delay based on singular pencil models is presented. By using the canonical form, the states and the parameters which contain the time delay information can be easily separated. Thus, the simultaneous estimation of parameters, states and time delay has become feasible. On-line joint estimation of parameters, states and time delay has a broad prospect of application because of its linearity.

The main contributions of the paper are as follows:

(a) Using singular pencil models, clearly describe a multi-input and multi-output (MIMO) system with multi time delay. Although the state augmentation technologies as paper [14,15] shows are utilized in this paper, the singular pencil model based estimation methods are different tools which can express distinct formulas with uniform format. So the MIMO system with multi time delay can be easily presented with singular pencil model.

(b) Using the observable canonical form, separate the states with the parameters containing the time delay information for joint estimation. We first established the connections between the states and the process output in present time. Then the state variables in the next time are described by the current process input and the current states. Therefore, the states, parameters and time delay can be connected with input-output data by using the observable canonical form which can be easily transformed into the singular pencil models for estimation.

(c) Jointly estimate the states, parameters and time delay with singular pencil models for noise known and noise unknown. This part of contribution is an extension of the work of Chen et al. [4] in which only the estimation of states and parameters are presented. We fully considered the states, parameters and time delay information in different situations such as single-input single-output system, and multi-input multi-output system with multi time delays and made the corresponding derivation.

(d) Edit Chen's algorithm [4] of the extended Kalman filter, and make the recursive rate improve much better.

The paper is organized in seven sections. In Section 2, an introduction to the singular pencil models is given. In Section 3, the description of systems with time delay in the form of singular pencil models is developed. In Section 4, joint estimation of parameters, states and time delay by using Kalman filter is described, and in Section 5, useful transformations have been discussed. In Section 6, simulations are given, and in Section 7, the conclusions are drawn.

2. **Preliminary.** Singular pencil model [17-19] is a unified expression of dynamic systems with the form

$$\begin{bmatrix} E - FD & G \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0 \tag{1}$$

where $x \in \mathcal{R}^n$ is the internal variable vector which also refers to an auxiliary variable vector or a state variable vector, and $w \in \mathcal{R}^{p+m}$ is the external variable vector. E, F are $(n+p) \times n$ matrices, G is $(n+p) \times (p+m)$ matrix and D is a linear operator. This form of model is sometimes also referred to as a generalized state space model, a descriptor system and other addresses. D can either be the Laplace operator s for continuous systems or the forward shift operator z for discrete systems.

Here are the two commonly used representations which can be formulated by singular pencil models. One is the state-space model [1]

$$\begin{bmatrix} A-Iz & 0 & B \\ C & -I & D \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ u_k \end{bmatrix} = 0 \tag{2}$$

where u_k is the input vector, y_k is the output vector, and (A, B, C, D) are the state-space matrices.

The other one is the polynomial input-output model of the form [5]

$$\left[\begin{array}{cccc|c|c} -Iz & & & & -A_n & B_n \\ I & \ddots & & & \vdots & \vdots \\ & & \ddots & -Iz & \vdots & \vdots \\ & & & I & -A_0 & B_0 \end{array}\right] \begin{bmatrix} x_k \\ y_k \\ u_k \end{bmatrix} = 0 \tag{3}$$

which can be used to represent the class of auto-regressive moving average (ARMA) models of the form [2]

$$\sum_{i=0}^{n} A_i z^{n-i} y_k = \sum_{i=0}^{m} B_i z^{m-i} u_k \tag{4}$$

where A_i and B_i are matrices and z is the forward shift operator.

In addition, SP models can also be used to nonlinear systems, optimization, identification, controller design and other aspects because of their characteristics of unification, simplicity and generality [2]. Readers can refer to literature [1-5,17-19] and references therein.

3. **Singular Pencil Models with Time-Delay.** In this section, a new estimation method of parameters, states and time delay based on singular pencil model and observable canonical form [20] together with the core idea of the method is presented. The main contribution of the method is as follows.

First, the parameters, states and time-delay information are included in an ARMA model by extending the state. Second, the ARMA models are transformed into the observable canonical forms which establish the connections between the input-output and the states to be estimated. Then the state and parameters are separated by using the SPM. Thus, the states, parameters and time-delay can be simultaneously estimated.

Let us consider single-input single-output (SISO) and multi-input multi-output (MIMO) system separately.

3.1. **SISO system with time delay.** The SISO system with time delay can be described in discrete form

$$A\left(z^{-1}\right) y(k) = z^{-d} B\left(z^{-1}\right) u(k) + C\left(z^{-1}\right) e(k) \tag{5}$$

where $d = [\tau/T_0]$, d are steps of delay, τ is delay time, T_0 is sample time, and $e(k)$ is assumed to be a zero mean white Gaussian random noise sequence with covariance R ($R > 0$).

$$A\left(z^{-1}\right) = 1 + a_1 z^{-1} + \cdots + a_{n-1} z^{-(n-1)} + a_n z^{-n} \tag{6a}$$

$$B\left(z^{-1}\right) = b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{n-1} z^{-(n-1)} + b_n z^{-n} \tag{6b}$$

$$C\left(z^{-1}\right) = 1 + c_1 z^{-1} + \cdots + c_{n-1} z^{-(n-1)} + c_n z^{-n} \tag{6c}$$

Extending the states $x \in \mathcal{R}^{n+d}$, the system with time delay can also be written in the following discrete form

$$\tilde{A}\left(z^{-1}\right) y(k) = \tilde{B}\left(z^{-1}\right) u(k) + \tilde{C}\left(z^{-1}\right) e(k) \tag{7}$$

where

$$\tilde{A}\left(z^{-1}\right) = 1 + a_1 z^{-1} + \cdots + a_{n-1} z^{-(n-1)} + a_n z^{-n} + a_{n+1} z^{-(n+1)} + \cdots + a_{n+d-1} z^{-(n+d-1)} + a_{n+d} z^{-(n+d)} \tag{8a}$$

$$\tilde{B}\left(z^{-1}\right) = b_1 z^{-1} + b_2 z^{-2} + \cdots + b_{n-1} z^{-(n-1)} + b_n z^{-n} + b_{n+1} z^{-(n+1)} + \cdots + b_{n+d-1} z^{-(n+d-1)} + b_{n+d} z^{-(n+d)} \tag{8b}$$

$$\tilde{C}\left(z^{-1}\right) = 1 + c_1 z^{-1} + \cdots + c_{n-1} z^{-(n-1)} + c_n z^{-n} + c_{n+1} z^{-(n+1)} + \cdots + c_{n+d-1} z^{-(n+d-1)} + c_{n+d} z^{-(n+d)} \tag{8c}$$

From the above Equations (5) and (7), if delay does exist in the system, some items of b_i in Equation (8b) must be equal to zero. For example, $b_1 = 0$, $b_2 = 0$, $\ldots$, $b_d = 0$, $b_{d+1} \neq 0$ then can be extracted from Equation (8b). Therefore, Equation (7) can be modified in the following form

$$\tilde{A}\left(z^{-1}\right) y(k) = z^{-d} \tilde{\tilde{B}}\left(z^{-1}\right) u(k) + \tilde{C}\left(z^{-1}\right) e(k) \tag{9}$$

where

$$\tilde{\tilde{B}}\left(z^{-1}\right) = b_{d+1} z^{-1} + b_{d+2} z^{-2} + \cdots + b_{n+d-1} z^{-(n-1)} + b_{n+d} z^{-n} \tag{10}$$

If $b_{d+1} = b_1$, $b_{d+2} = b_2$,$\ldots$,$b_{n+d-1} = b_{n-1}$, $b_{n+d} = b_n$, $a_{n+1} = a_{n+2} = \ldots = a_{n+d} = 0$, $c_{n+1} = c_{n+2} = \ldots = c_{n+d} = 0$, then Equation (9) becomes Equation (5). Look at the b_i in Formula (8b) from left to right, i.e., in the incremental order of i, the numbers of coefficient before the first non-zero one are the steps of delay. Therefore, the time delay τ can be estimated by $\tau = d^* T_0$, in which T_0 is sample time.

Equation (7) can be written in observable canonical form (11a) (11b).

$$\begin{bmatrix} x_{n+d}(k+1) \\ x_{n+d-1}(k+1) \\ \vdots \\ x_{n+1}(k+1) \\ x_n(k+1) \\ x_{n-1}(k+1) \\ \vdots \\ x_2(k+1) \\ x_1(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -a_{n+d} \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & -a_{n+d-1} \\ \vdots & \ddots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 & 0 & \cdots & 0 & -a_{n+1} \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & -a_n \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \ddots & 0 & -a_2 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_{n+d}(k) \\ x_{n+d-1}(k) \\ \vdots \\ x_{n+1}(k) \\ x_n(k) \\ x_{n-1}(k) \\ \vdots \\ x_2(k) \\ x_1(k) \end{bmatrix} + \begin{bmatrix} b_{n+d} \\ b_{n+d-1} \\ \vdots \\ b_{n+1} \\ b_n \\ b_{n-1} \\ \vdots \\ b_2 \\ b_1 \end{bmatrix} u(k) + \begin{bmatrix} c_{n+d} \\ c_{n+d-1} \\ \vdots \\ c_{n+1} \\ c_n \\ c_{n-1} \\ \vdots \\ c_2 \\ c_1 \end{bmatrix} e(k) \tag{11a}$$

$$y(k) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{n+d}(k) \\ x_{n+d-1}(k) \\ \vdots \\ x_{n+1}(k) \\ x_n(k) \\ x_{n-1}(k) \\ \vdots \\ x_2(k) \\ x_1(k) \end{bmatrix} + e(k) \tag{11b}$$

From Equations (11a) and (11b), it can be seen that $y(k)$ is the function of $x_1(k)$ and $e(k)$, and $x_i(k+1)$ is the function of $x_1(k)$, and $e(k)$, i.e.,

$$y(k) = f(x_1(k), e(k)), \quad x_i(k+1) = f(x_1(k), u(k), e(k))$$

Thus, $x_i(k+1)$ is the function of $y(k)$, $u(k)$ and $e(k)$, i.e.,

$$x_i(k+1) = f(y(k), u(k), e(k)), \quad i = 1, 2, \ldots, (n+d)$$

where, $f((\bullet))$ is the function of $(\bullet)$.

Then Equations (11a) and (11b) can be modified in the form of a singular pencil model as follows

$$\left[\begin{array}{ccccccccc|c|c|c} -z & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & -a_{n+d} & b_{n+d} & c_{n+d} \\ 1 & -z & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & -a_{n+d-1} & b_{n+d-1} & c_{n+d-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & -z & 0 & 0 & \cdots & 0 & 0 & -a_{n+1} & b_{n+1} & c_{n+1} \\ 0 & 0 & \cdots & 1 & -z & 0 & \cdots & 0 & 0 & -a_n & b_n & c_n \\ 0 & 0 & \cdots & 0 & 1 & -z & \cdots & 0 & 0 & -a_{n-1} & b_{n-1} & c_{n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \ddots & -z & 0 & -a_2 & b_2 & c_2 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & -z & -a_1 & b_1 & c_1 \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & -1 & 0 & 1 \end{array}\right] \begin{bmatrix} x_{n+d}(k) \\ x_{n+d-1}(k) \\ \vdots \\ x_{n+1}(k) \\ x_n(k) \\ x_{n-1}(k) \\ \vdots \\ x_2(k) \\ x_1(k) \\ y(k) \\ u(k) \\ e(k) \end{bmatrix} = 0 \tag{12}$$

From Equation (12), the parameters and the states have been separated. Then, the parameters, the states and the time delay can be simultaneously identified by an ordinary Kalman filter for exact noise statistics or an extended Kalman filter for arbitrary noise statistics. If the parameters have no relations with the system, the corresponding items in identification will be equal to or approximate to zero (see Section 4).

3.2. **MIMO system with time delays.** For a clear description, consider a 2-input and 2-output system with time delays. u_1 has two steps of delay. u_2 has one step of delay. The system can be described in auto-regressive moving average (ARMAX) model.

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a_{111} & a_{112} \\ a_{211} & a_{212} \end{bmatrix} z^{-1} + \begin{bmatrix} a_{121} & a_{122} \\ a_{221} & a_{222} \end{bmatrix} z^{-2} + \begin{bmatrix} a_{131} & a_{132} \\ a_{231} & a_{232} \end{bmatrix} z^{-3} \right.$$
$$\left. + \begin{bmatrix} a_{141} & a_{142} \\ a_{241} & a_{242} \end{bmatrix} z^{-4} \right\} \begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix}$$

$$
\begin{aligned}
&= \left\{ \begin{bmatrix} b_{111} & b_{112} \\ b_{211} & b_{212} \end{bmatrix} z^{-1} + \begin{bmatrix} b_{121} & b_{122} \\ b_{221} & b_{222} \end{bmatrix} z^{-2} + \begin{bmatrix} b_{131} & b_{132} \\ b_{231} & b_{232} \end{bmatrix} z^{-3} \right. \\
&\quad \left. + \begin{bmatrix} b_{141} & b_{142} \\ b_{241} & b_{242} \end{bmatrix} z^{-4} \right\} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} + \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} c_{111} & c_{112} \\ c_{211} & c_{212} \end{bmatrix} z^{-1} \right. \\
&\quad \left. + \begin{bmatrix} c_{121} & c_{122} \\ c_{221} & c_{222} \end{bmatrix} z^{-2} + \begin{bmatrix} c_{131} & c_{132} \\ c_{231} & c_{232} \end{bmatrix} z^{-3} + \begin{bmatrix} c_{141} & c_{142} \\ c_{241} & c_{242} \end{bmatrix} z^{-4} \right\} \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix}
\end{aligned} \tag{13}
$$

where $a_{131} = a_{132} = a_{231} = a_{232} = a_{141} = a_{142} = a_{241} = a_{242} = 0$, $b_{111} = b_{112} = b_{211} = b_{212} = b_{121} = b_{221} = b_{142} = b_{242} = 0$, $c_{131} = c_{132} = c_{231} = c_{232} = c_{141} = c_{142} = c_{241} = c_{242} = 0$.

Equation (13) can be rewritten in observable canonical form as follows

$$
\begin{aligned}
\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \\ x_5(k+1) \\ x_6(k+1) \\ x_7(k+1) \\ x_8(k+1) \end{bmatrix} &= \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & -a_{141} & 0 & 0 & 0 & -a_{142} \\ 1 & 0 & 0 & -a_{131} & 0 & 0 & 0 & -a_{132} \\ 0 & 1 & 0 & -a_{121} & 0 & 0 & 0 & -a_{122} \\ 0 & 0 & 1 & -a_{111} & 0 & 0 & 0 & -a_{112} \\ \hline 0 & 0 & 0 & -a_{241} & 0 & 0 & 0 & -a_{242} \\ 0 & 0 & 0 & -a_{231} & 1 & 0 & 0 & -a_{232} \\ 0 & 0 & 0 & -a_{221} & 0 & 1 & 0 & -a_{222} \\ 0 & 0 & 0 & -a_{211} & 0 & 0 & 1 & -a_{212} \end{array} \right] \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \\ x_6(k) \\ x_7(k) \\ x_8(k) \end{bmatrix} \\
&\quad + \left[\begin{array}{c|c} b_{141} & b_{142} \\ b_{131} & b_{132} \\ b_{121} & b_{122} \\ b_{111} & b_{112} \\ \hline b_{241} & b_{242} \\ b_{231} & b_{232} \\ b_{221} & b_{222} \\ b_{211} & b_{212} \end{array} \right] \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} + \left[\begin{array}{c|c} c_{141}+a_{141} & c_{142}+a_{142} \\ c_{131}+a_{131} & c_{132}+a_{132} \\ c_{121}+a_{121} & c_{122}+a_{122} \\ c_{111}+a_{111} & c_{112}+a_{112} \\ \hline c_{241}+a_{241} & c_{242}+a_{242} \\ c_{231}+a_{231} & c_{232}+a_{232} \\ c_{221}+a_{221} & c_{222}+a_{222} \\ c_{211}+a_{211} & c_{212}+a_{212} \end{array} \right] \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix}
\end{aligned} \tag{14a}
$$

$$
\begin{bmatrix} y_1(k) \\ y_2(k) \end{bmatrix} = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \\ x_6(k) \\ x_7(k) \\ x_8(k) \end{bmatrix} + \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array} \right] \begin{bmatrix} e_1(k) \\ e_2(k) \end{bmatrix} \tag{14b}
$$

From Equations (14a) and (14b), it can be seen

$$
\begin{aligned}
&y_1(k) = f(x_4(k), e_1(k)), \ y_2(k) = f(x_8(k), e_2(k)), \\
&x_1(k+1) = f(x_{4,8}(k), u_{1,2}(k), e_{1,2}(k)), \ x_2(k+1) = f(x_{1,4,8}(k), u_{1,2}(k), e_{1,2}(k)), \\
&x_3(k+1) = f(x_{2,4,8}(k), u_{1,2}(k), e_{1,2}(k)), \ x_4(k+1) = f(x_{3,4,8}(k), u_{1,2}(k), e_{1,2}(k)), \\
&x_5(k+1) = f(x_{4,8}(k), u_{1,2}(k), e_{1,2}(k)), \ x_6(k+1) = f(x_{4,5,8}(k), u_{1,2}(k), e_{1,2}(k)), \\
&x_7(k+1) = f(x_{4,6,8}(k), u_{1,2}(k), e_{1,2}(k)), x_8(k+1) = f(x_{4,7,8}(k), u_{1,2}(k), e_{1,2}(k)).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
x_{1,2,3,4}(k+1) &= f(y_{1,2}(k), u_{1,2}(k), e_{1,2}(k)), \\
x_{5,6,7,8}(k+1) &= f(y_{1,2}(k), u_{1,2}(k), e_{1,2}(k)).
\end{aligned}
$$

where, $f((\bullet))$ is the function of $(\bullet)$.

Then the parameters and the states can be separated and they can be estimated based on the SP model by using input-output data:

$$\left[\begin{array}{cccc:cccc|cc|cc|cc}
-z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_{141} & -a_{142} & b_{141} & b_{142} & c_{141} & c_{142} \\
1 & -z & 0 & 0 & 0 & 0 & 0 & 0 & -a_{131} & -a_{132} & b_{131} & b_{132} & c_{131} & c_{132} \\
0 & 1 & -z & 0 & 0 & 0 & 0 & 0 & -a_{121} & -a_{122} & b_{121} & b_{122} & c_{121} & c_{122} \\
0 & 0 & 1 & -z & 0 & 0 & 0 & 0 & -a_{111} & -a_{112} & b_{111} & b_{112} & c_{111} & c_{112} \\
\hdashline
0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & -a_{241} & -a_{242} & b_{241} & b_{242} & c_{241} & c_{242} \\
0 & 0 & 0 & 0 & 1 & -z & 0 & 0 & -a_{231} & -a_{232} & b_{231} & b_{232} & c_{231} & c_{232} \\
0 & 0 & 0 & 0 & 0 & 1 & -z & 0 & -a_{221} & -a_{222} & b_{221} & b_{222} & c_{221} & c_{222} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -z & -a_{211} & -a_{212} & b_{211} & b_{212} & c_{211} & c_{212} \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1
\end{array}\right]
\begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \\ x_6(k) \\ x_7(k) \\ x_8(k) \\ y_1(k) \\ y_2(k) \\ u_1(k) \\ u_2(k) \\ e_1(k) \\ e_2(k) \end{bmatrix} = 0 \tag{15}$$

Note that in Equations (11a) and (12), the parameters $a_{n+1}, \ldots, a_{n+d}, c_{n+1}, \ldots, c_{n+d}$ are definitely equal to zero and in Equations (13), (14a) and (15), the parameters a_{141}, a_{131}, a_{241}, a_{231}, a_{142}, a_{132}, a_{242}, a_{232}, c_{141}, c_{131}, c_{241}, c_{231}, c_{142}, c_{132}, c_{242}, c_{232}, are certainly equal to zero, too. They do not need to be estimated anymore. This can make the matrix smaller and easier to calculate.

4. **Joint Estimation of Parameters, States and Time Delays.** Kalman filter, whether the ordinary algorithm (OKF) or the extended algorithm (EKF) is not a new thing. Readers can find the detailed presentations in [20]. In this section, we will give some examples to discuss how to estimate the parameters, states and time delays step by step by using the OKF and EKF following the work of Chen et al. [4]. To meet the demand of joint estimation of parameters, states and time delays, we have done some necessary alters and derivations.

4.1. **Noise known.** For joint estimation of states, parameters and time delay, $w_k = \left[y_k^T, u_k^T\right]^T$ is assumed to be known. Using a column vector r to contain all the parameters a_i, b_i in A_*, A_0, B_*, B_0, Equation (3) can be written as

$$\begin{cases} x_{k+1} = E_* x_k + G_* w_k = E_* x_k + \tilde{G}_*(w_k)\, r \\ 0 = E_0 x_k + G_0 w_k = E_0 x_k + \tilde{G}_0(w_k)\, r - J w_k \end{cases} \tag{16}$$

where $J = [I_p | 0_{p\times m}]$, $w_k = \left[y_k^T, u_k^T\right]^T$, $\tilde{G}_*(w_k)\, r = [-A_* | B_*]\, w_k$, $\tilde{G}_0(w_k)\, r = [-A_0 | B_0]\, w_k$ $+ J w_k$, and the matrices $\tilde{G}_*(w_k)$ and $\tilde{G}_0(w_k)$ are established with the measurement y_k and u_k according to the known sequence of the corresponding parameters a_i, b_i in r. The nuclear idea, for the joint estimation of the parameters, states and time delay, is to separate the parameters to be identified a_i, b_i into the column vector r and to build the matrices $\tilde{G}_*(w_k)$ and $\tilde{G}_0(w_k)$ with the measurement y_k and u_k. If random equation error perturbations are considered, Equation (16) can be modified as follows

$$\begin{cases} x_{k+1} = E_* x_k + \tilde{G}_*(w_k)\, r + C_* e_k \\ 0 = E_0 x_k + \tilde{G}_0(w_k)\, r - J w_k + e_k \end{cases} \tag{17}$$

where e_k is a zero mean white Gaussian random noise sequence, and C_* is a matrix of noise parameters.

If the system parameters are invariant, i.e., $r = r_k = r_{k+1}$, let the length of r be l, and then the state space equations are

$$\begin{cases} s_{k+1} = F_k s_k + \begin{bmatrix} C_* e_k \\ 0 \end{bmatrix} \\ y_k = H_k s_k + e_k \end{cases} \tag{18}$$

The corresponding parameters in Equation (18) and the recursive equations of ordinary Kalman filter are defined in Table 1.

TABLE 1. Parameters definition and recursive algorithm of Kalman filters

	Ordinary Kalman filter [21]	Extended Kalman filter [22]
s_k	$\begin{bmatrix} x_k^T & r_k^T \end{bmatrix}^T$	$\begin{bmatrix} x_k^T & r_k^T & \eta_k^T \end{bmatrix}^T$
F_k	$\begin{bmatrix} E_* & \tilde{G}_*(w_k) \\ 0 & I_l \end{bmatrix}$	$\begin{bmatrix} E_* & \tilde{G}_*(w_k) & \tilde{C}_*(e_k) \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$
H_k	$\begin{bmatrix} E_0 & \tilde{G}_0(w_k) \end{bmatrix}$	$\begin{bmatrix} E_0 & \tilde{G}_0(w_k) & 0 \end{bmatrix}$
K_k	$\left(F_k P_k H_k^T + S\right)\left(H_k P_k H_k^T + R\right)^{-1}$	$\Xi_k P_k H_k^T \left(H_k P_k H_k^T + R_k\right)^{-1}$
P_{k+1}	$F_k P_k F_k^T + Q - K_k\left(H_k P_k H_k^T + R\right) K_k^T$	$\Xi_k P_k \Xi_k^T - K_k\left(H_k P_k H_k^T + R_k\right) K_k^T$
$\hat{s}_{k+1}$	$F_k \hat{s}_k + K_k(y_k - H_k \hat{s}_k)$	$F_k \hat{s}_k + K_k(y_k - H_k \hat{s}_k)$
S	$\begin{bmatrix} C_* R \\ 0 \end{bmatrix}$	–
Q	$\begin{bmatrix} C_* R C_*^T & 0 \\ 0 & 0 \end{bmatrix} \geq 0$	–
Ξ_k	–	$\begin{bmatrix} E_* - C_* E_0 & \tilde{G}_*(w_k) - C_* \tilde{G}_0(w_k) & \tilde{C}_*(e_k) \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$

1) For SISO System with Time Delay:

$$x_k = \begin{bmatrix} x_{n+d}(k) & x_{n+d-1}(k) & \cdots & x_{n+1}(k) & x_n(k) & x_{n-1}(k) & \cdots & x_2(k) & x_1(k) \end{bmatrix}^T \tag{19}$$

$$r_k = \begin{bmatrix} a_n & a_{n-1} & \cdots & a_2 & a_1 & b_{n+d} & b_{n+d-1} & \cdots & b_{n+1} & b_n & b_{n-1} & \cdots & b_2 & b_1 \end{bmatrix}^T \tag{20}$$

where e_k is assumed to be a zero mean white Gaussian random noise sequence with covariance R ($R > 0$). Equation (18) formulates a state space form of linear stochastic system with state vector s_k. For the singular pencil model with time delay, if the matrix C_* and the covariance matrix R of e_k are assumed to be known, then (18) can be written specifically.

$$E_* = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \ddots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{(n+d)\times(n+d)} \tag{21}$$

Measurement matrix

$$\tilde{G}_*(w_k) = \begin{bmatrix} Y & U \end{bmatrix}_{(n+d)\times(2n+d)} \tag{22}$$

where Y, U are a matrix of $(n+d)\times n$ and $(n+d)\times(n+d)$, respectively.

Y is the output matrix. Because the states to be estimated are indirectly contacting with the output, they can be established of the relationships by Equation (21). So the corresponding part in Equation (22a) is equal to zero.

$$Y = \begin{bmatrix} 0_{d\times n} \\ -y(k)\cdot I_{n\times n} \end{bmatrix}_{(n+d)\times n} \tag{22a}$$

U is the input matrix with elements input $u(k)$ in the diagonal positions.

$$U = diag\left(u(k), u(k), \cdots, u(k), u(k)\right) \tag{22b}$$

C_* is the noise parameters matrix

$$C_* = \begin{bmatrix} 0 & \cdots & 0 & c_n & \cdots & c_1 \end{bmatrix}^T \tag{23}$$

Singular pencil model can describe the regular output equation with a uniform expression where

$$E_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{1\times(n+d)} \tag{24}$$

and

$$\tilde{G}_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{1\times(2n+d)} \tag{25}$$

2) For MIMO System with Time Delays:

Consider a two-input two-output system owning Equations (13), (14) or (15). Define eight state variables

$$x_k = \left[x_8(k), x_7(k), \cdots, x_2(k), x_1(k)\right]^T \tag{26}$$

and the parameters matrix to be estimated

$$\begin{aligned} r_k = [&a_{121}, a_{111}, a_{221}, a_{211}, a_{122}, a_{112}, a_{222}, a_{212}, b_{141}, b_{131}, b_{121}, b_{111}, b_{241}, b_{231}, b_{221}, b_{211}, \\ &b_{142}, b_{132}, b_{122}, b_{112}, b_{242}, b_{232}, b_{222}, b_{212}]^T \end{aligned} \tag{27}$$

The matrix E_* with the following form establishes relationships between the eight states variables.

$$E_* = \left[\begin{array}{cccc:cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \tag{28}$$

Construct the measurement matrix with input output variables

$$\tilde{G}_*(w_k) = \left[\begin{array}{cccc} Y_1 & Y_2 & U_1 & U_2 \end{array}\right]_{8\times 24} \tag{29}$$

where Y_1, Y_2, U_1, U_2 are a 8×4, 8×4, 8×8 and 8×8 matrix, respectively.

Y_1 is the output matrix which is a partitioned form

$$Y_1 = \left[\begin{array}{c:c} \begin{array}{c} 0_{2\times 2} \\ -y_1(k) \cdot I_{2\times 2} \end{array} & 0_{4\times 2} \\ \hdashline 0_{4\times 2} & \begin{array}{c} 0_{2\times 2} \\ -y_1(k) \cdot I_{2\times 2} \end{array} \end{array}\right]_{8\times 4} \tag{29a}$$

Output matrix Y_2 can be built in the same way

$$Y_2 = \left[\begin{array}{c:c} \begin{array}{c} 0_{2\times 2} \\ -y_2(k) \cdot I_{2\times 2} \end{array} & 0_{4\times 2} \\ \hdashline 0_{4\times 2} & \begin{array}{c} 0_{2\times 2} \\ -y_2(k) \cdot I_{2\times 2} \end{array} \end{array}\right]_{8\times 4} \tag{29b}$$

U_1 and U_2 are input matrix which are diagonal forms

$$U_1 = diag\left(u_1(k), u_1(k), \cdots, u_1(k), u_1(k)\right)_{8\times 8} \tag{29c}$$

$$U_2 = diag\left(u_2(k), u_2(k), \cdots, u_2(k), u_2(k)\right)_{8\times 8} \tag{29d}$$

C_* is the noise parameters matrix

$$C_* = \left[\begin{array}{cccc:cccc} 0 & 0 & c_{121} & c_{111} & 0 & 0 & c_{221} & c_{211} \\ 0 & 0 & c_{122} & c_{112} & 0 & 0 & c_{222} & c_{212} \end{array}\right]^T \tag{30}$$

The output equation can be rewritten by singular pencil model with

$$E_0 = \left[\begin{array}{cccc:cccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{31}$$

and

$$\tilde{G}_0 = \left[\begin{array}{ccccc} 0 & 0 & \cdots & 0 & 0 \\ \hdashline 0 & 0 & \cdots & 0 & 0 \end{array}\right]_{2\times 24} \tag{32}$$

Then, one can use an ordinary Kalman filter to estimate the augmented state vector s_k, i.e., states, parameters and time delay of the system, in the minimum error covariance sense by applying the recursive equations in Table 1.

4.2. **Noise unknown.** An algorithm for the simultaneous estimation of the state variable x_k, system parameters r_k and noise parameters in C_* and R was derived by Chen et al. [4]. The algorithm can be implemented in the system with time delay. By editing some items of the algorithm, the recursive rate can be improved.

Considering Equation (17), let $\tilde{C}_*(e_k)$ be a row blocks matrix of the form

$$\tilde{C}_*(e_k) = \begin{bmatrix} E_1 & E_2 & \cdots & E_m \end{bmatrix} \tag{33}$$

where, E_i is a $p \times ((n+d) \times n)$ block diagonal matrix, i.e.,

$$E_i = \left[\begin{array}{c:c:c} \begin{matrix} 0_{d\times n} \\ e_i(k) \cdot I_{n\times n} \end{matrix} & & \\ \hdashline & \ddots & \\ \hdashline & & \begin{matrix} 0_{d\times n} \\ e_i(k) \cdot I_{n\times n} \end{matrix} \end{array}\right]$$

Applying the commutative law of multiplication, one gets

$$C_* e_k = \tilde{C}_*(e_k)\,\eta_k \tag{34}$$

where η_k is a column vector including the non-zero parameters c_i in C_*. With Equation (34), Equation (17) is formulated by

$$\begin{cases} s_{k+1} = F_k s_k \\ y_k = H_k s_k + e_k \end{cases} \tag{35}$$

In this case, e_k is unmeasurable but can be calculated by

$$\hat{e}_k = y_k - H_k \hat{s}_k \tag{36}$$

where $\hat{x}_k$, $\hat{s}_k$ are the values of x_k and s_k to be estimated at time k.

Applying the extended Kalman filter to the system with time delay, s_k can be estimated, that is to say, the states, parameters and time delay can be estimated under the condition of unknown parameters of C_*. The recursive extended Kalman filter algorithm is shown in Table 1, where

$$\hat{R}_{k+1} = \hat{R}_k + \frac{1}{k+1}\left(\hat{e}_k \hat{e}_k^T - \hat{R}_k\right) \tag{37}$$

Since $\hat{R}_{k+1} = \hat{R}_k = R$ is in the steady state, the recursive rate can be improved by multiplying the item $1/(k+1)$, a number greater than 1, that is

$$\hat{R}_{k+1} = \hat{R}_k + \frac{\lambda}{k+1}\left(\hat{e}_k \hat{e}_k^T - \hat{R}_k\right), \quad \lambda > 1 \tag{38}$$

From the equation above, it can be seen

$$\lim_{k\to\infty} \hat{R}_{k+1} = \hat{R}_k = R \tag{39}$$

So Equation (37) and Equation (38) are equivalent at the steady state. From Table 2, the simulation result of an SISO system with time delay can be seen clearly. When $\lambda = 1$, $k = 4000$, the covariance matrix $R = 8.0837$. However, when $\lambda = 8$, $k = 1000$, the covariance matrix R can be equal to 1.1283. The recursive rate has been improved dramatically. From Table 3, an MIMO example, it draws the same conclusion.

Please pay close attention. λ may not be too large, otherwise it will cause the algorithm to not convergent. In experience, $1 < \lambda < 10$.

5. **Useful Transformations.** For a state-space equation of an MIMO system, if it is completely observable, then it can be transformed into an observable canonical form through the following steps.

1) Step I: Transform the state-space equation into a row companion form $\left(A_0^{(1)}, \tilde{B}_0^{(1)}, \tilde{C}_0^{(1)}\right)$, and acquire the structure parameters n_1, n_2, $\dots$, n_m of the observable canonical model.

$$\sum_{i=1}^{m} n_i = n$$

2) Step II: Construct the linear transformation matrix T_0, and make T_0^{-1} as the following form

$$\begin{aligned} T_0^{-1} = \Big[& e_{n_1}, A_0^{(1)} e_{n_1}, \cdots, \left(A_0^{(1)}\right)^{n_1-1} e_{n_1}, e_{n_1+n_2}, A_0^{(1)} e_{n_1+n_2}, \cdots, \left(A_0^{(1)}\right)^{n_2-1} e_{n_1+n_2}, \\ & e_{n_1+n_2+\cdots+n_m}, A_0^{(1)} e_{n_1+n_2+\cdots+n_m}, \cdots, \left(A_0^{(1)}\right)^{n_m-1} e_{n_1+n_2+\cdots+n_m} \Big] \end{aligned} \tag{40}$$

3) Step III: Construct the observable canonical form

$$\begin{cases} \tilde{A}_0^{(2)} = T_0 A_0^{(1)} T_0^{-1} \\ \tilde{B}_0^{(2)} = T_0 \tilde{B}_0^{(1)} \\ \tilde{C}_0^{(2)} = \tilde{C}_0^{(1)} T_0^{-1} \end{cases} \tag{41}$$

If $\tilde{C}_0^{(2)}$ has not been transformed into observable canonical form, it must do another transformation for output vector $z(k)$ and noise vector $\omega(k)$, i.e., let $\tilde{z}\left(k\right) = Q_0 z\left(k\right)$ and $\tilde{\omega}\left(k\right) = Q_0 \omega\left(k\right)$, where Q_0 satisfies the following equation

$$\tilde{C}_0^{(2)} = Q_0 C_0^{(1)} T_0^{-1}$$

For the definition of row companion form and the detailed transformations, readers can refer to [20,23].

6. **Simulations.** The pseudo random bit sequence (PRBS) signals are chosen as input signals $u\left(k\right)$ to sufficiently excite the systems with time delay. Two examples, each representing a typical condition, are selected to exhibit the validity of the method. In order to show if the estimated system is close to the simulated system, the comparisons of step responses of the simulated and estimated systems are given for an SISO system and an MIMO system respectively, without considering the noise.

6.1. **Example 1.** Consider an SISO system with time delay for noise parameters known and noise parameters unknown.

$$\left(1 + a_1 z^{-1} + a_2 z^{-2}\right) y\left(k\right) = z^{-d}\left(b_3 z^{-1} + b_4 z^{-2}\right) u\left(k\right) + \left(1 + c_1 z^{-1} + c_2 z^{-2}\right) e\left(k\right)$$

where $a_1 = -0.2$, $a_2 = -0.24$, $b_3 = 1$, $b_4 = 2$, $d = 2$, $c_1 = 1.5$, $c_2 = 0.75$.

In order to sufficiently explain the method, two extra states are selected. The estimation results of parameters, states and time delay are presented in Table 2. The dynamic processes of the estimated states and parameters of the SISO system are shown in Figure 1, and the comparison of step responses of the simulated and estimated systems is shown in Figure 2.

TABLE 2. Estimation results of the parameters, states and time delay for example 1 (SISO)

	exact noise statistics	arbitrary noise statistics	
k	1000	1000	4000
λ	–	8	1
x_4	-30.0065	-30.0425	-30.0151
x_3	14.9707	14.9770	-45.0676
x_2	-20.5220	-23.8254	-40.2507
x_1	-40.0262	-48.3994	-30.4510
$a_2\,(-0.24)$	-0.2396	-0.2404	-0.2397
$a_1\,(-0.20)$	-0.1995	-0.1991	-0.1993
$b_4\,(2.00)$	2.0004	2.0028	2.0010
$b_3\,(1.00)$	1.0024	1.0044	1.0035
$b_2\,(0.00)$	0.0035	0.0020	0.0024
$b_1\,(0.00)$	0.0024	0.0007	-0.0001
$c_2\,(0.75)$	known	0.7357	0.4813
$c_1\,(1.50)$	known	1.4801	1.2540
R_0	1.0217	0.9934	0.9844
R_{1000}	–	1.1283	–
R_{4000}	–	–	8.0837

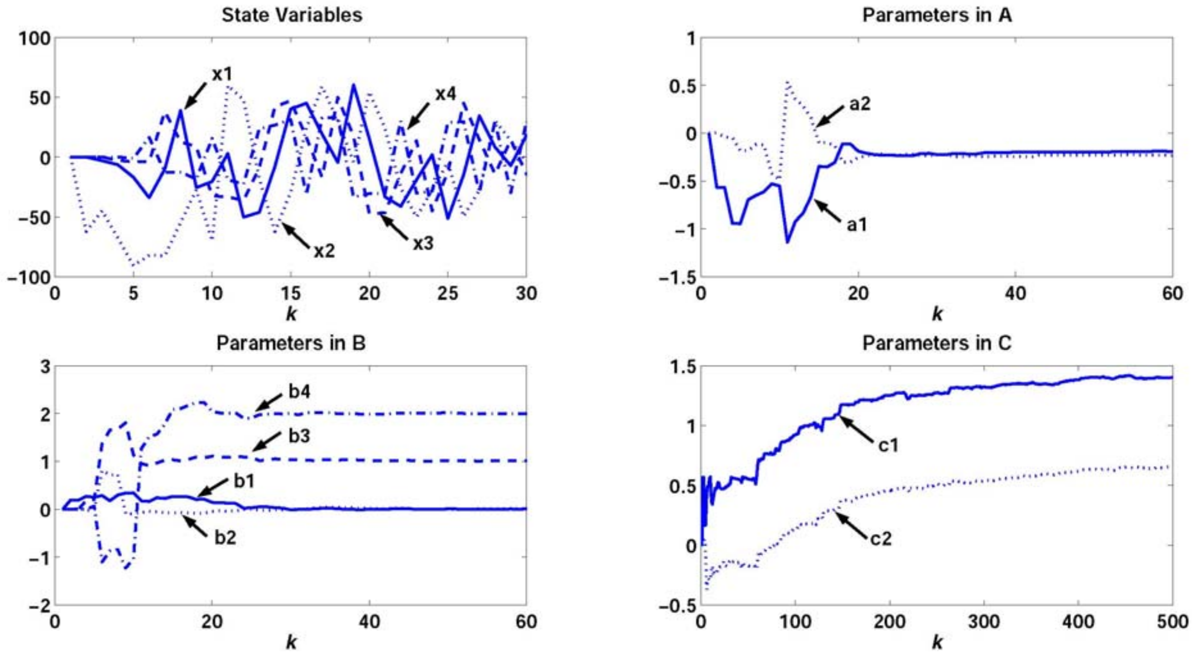

FIGURE 1. The estimated states and parameters of an SISO system

6.2. **Example 2.** Consider a two-input two-output system with time delays under the different conditions of noise parameters known and noise parameters unknown. u_1 has two steps of delay and u_2 has one step of delay. The system owns the same equation as Equation (13) or Equation (14).

$$\begin{bmatrix} a_{111} & a_{112} \\ a_{211} & a_{212} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1.5 \end{bmatrix}, \begin{bmatrix} a_{121} & a_{122} \\ a_{221} & a_{222} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0.25 & 1.0 \end{bmatrix},$$

$$\begin{bmatrix} a_{131} & a_{132} \\ a_{231} & a_{232} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} a_{141} & a_{142} \\ a_{241} & a_{242} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} b_{111} & b_{112} \\ b_{211} & b_{212} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} b_{121} & b_{122} \\ b_{221} & b_{222} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} b_{131} & b_{132} \\ b_{231} & b_{232} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} b_{141} & b_{142} \\ b_{241} & b_{242} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} c_{111} & c_{112} \\ c_{211} & c_{212} \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 1 & 0.5 \end{bmatrix}, \begin{bmatrix} c_{121} & c_{122} \\ c_{221} & c_{222} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0.5 & 0.67 \end{bmatrix},$$

$$\begin{bmatrix} c_{131} & c_{132} \\ c_{231} & c_{232} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} c_{141} & c_{142} \\ c_{241} & c_{242} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The estimation results of parameters, states and time delay are presented in Table 3. We refer the system from inputs to y_1 as Subsystem 1 and the system from inputs to y_2 as Subsystem 2. The dynamic processes of the estimated states and parameters of the MIMO system are shown in Figure 3, and the comparison of step responses of the simulated and estimated subsystems is shown in Figure 4.

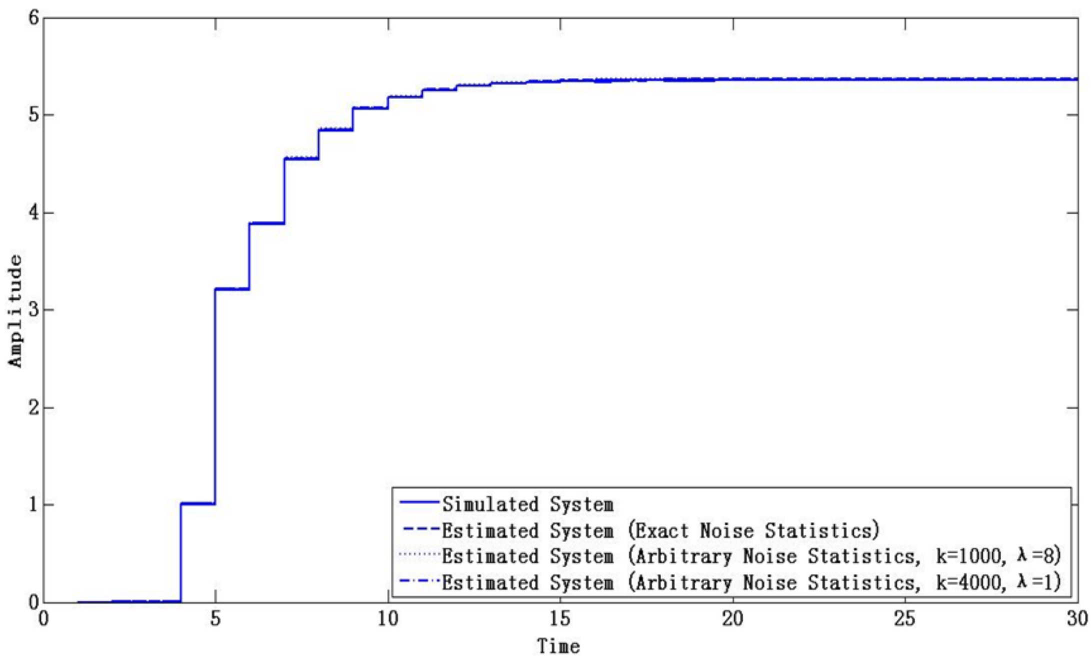

Figure 2. Comparison of step responses of the simulated and estimated systems (SISO)

From the estimated results in Table 2 and Table 3, it can be seen that the estimation method based on SP models presented in this paper is very effective. The estimation results display not only the higher recursive precision but also the higher recursive rate.

7. **Conclusions.** The systems with time delay can be described with singular pencil models. By using observable canonical form, it is very easy to construct the systems with SP models so as to separate the parameters, time delay with the states and exchange the positions of parameters and input-output variables. Thus, the parameters, states and time delay can be simultaneously estimated with an ordinary Kalman filter for noise known and extended Kalman filter for noise unknown.

TABLE 3. Estimation results of the parameters, states and time delay for example 2 (MIMO)

	exact noise statistics	arbitrary noise statistics	
k	1000	1000	4000
λ	–	8	1
x_8	0.0950	1.8396	7.8305
x_7	−15.1200	−16.9794	−7.0481
x_6	14.9467	15.3045	−50.3144
x_5	3.0631	−1.2706	67.7708
x_4	15.0592	26.6706	25.9516
x_3	−15.1453	−26.7442	25.9594
x_2	−4.4237	−0.6872	−69.2990
x_1	27.6641	32.8543	−7.0967
a_{121} (0.00)	−0.0018	0.0035	0.0050
a_{111} (0.00)	−0.0084	−0.1162	−0.5202
a_{221} (0.25)	0.2509	0.2510	0.2509
a_{211} (0.00)	−0.0064	−0.7795	−0.7281
a_{122} (0.00)	0.0032	0.1233	0.5308
a_{112} (1.00)	−1.0036	−0.9945	−0.9924
a_{222} (1.00)	1.0094	1.7821	1.7302
a_{212} (1.50)	1.5017	1.5011	1.5013
b_{141} (0.00)	−0.0054	−0.1198	−0.5217
b_{131} (1.00)	1.0036	1.0003	0.9973
b_{121} (0.00)	0.0040	0.0027	0.0025
b_{111} (0.00)	0.0003	0.0038	0.0012
b_{241} (−1.00)	−1.0058	−1.7778	−1.7294
b_{231} (0.00)	0.0012	0.0006	−0.0004
b_{221} (0.00)	−0.0027	0.0048	−0.0010
b_{211} (0.00)	−0.0003	0.0035	0.0009
b_{142} (0.00)	−0.0009	−0.0028	0.0003
b_{132} (0.00)	−0.0019	0.0090	0.0054
b_{122} (0.00)	−0.0030	0.0099	−0.0001
b_{112} (0.00)	−0.0058	0.0008	0.0012
b_{242} (0.00)	0.0019	−0.0002	0.0007
b_{232} (0.00)	0.0045	0.0043	0.0001
b_{222} (1.00)	0.9965	1.0075	0.9996
b_{212} (0.00)	−0.0013	0.0050	−0.0005
c_{121} (0.00)	known	−0.0022	−0.2998
c_{111} (1.00)	known	0.5401	0.1332
c_{221} (0.50)	known	−0.2785	−0.1896
c_{211} (1.00)	known	0.0478	0.1019
c_{122} (0.00)	known	−0.0711	−0.1684
c_{112} (0.50)	known	0.3292	0.3501
c_{222} (0.67)	known	0.2497	0.2532
c_{212} (0.50)	known	0.3377	0.4002
R_0	1.0193, −0.0174 −0.0174, 0.9243	0.9670, 0.0293 0.0293, 1.0982	1.0041, −0.0136 −0.0136, 1.0343
R_{1000}	–	1.5813, 0.1341 0.1341, 1.1873	–
R_{4000}	–	–	6.0445, −1.4429 −1.4429, 2.9427

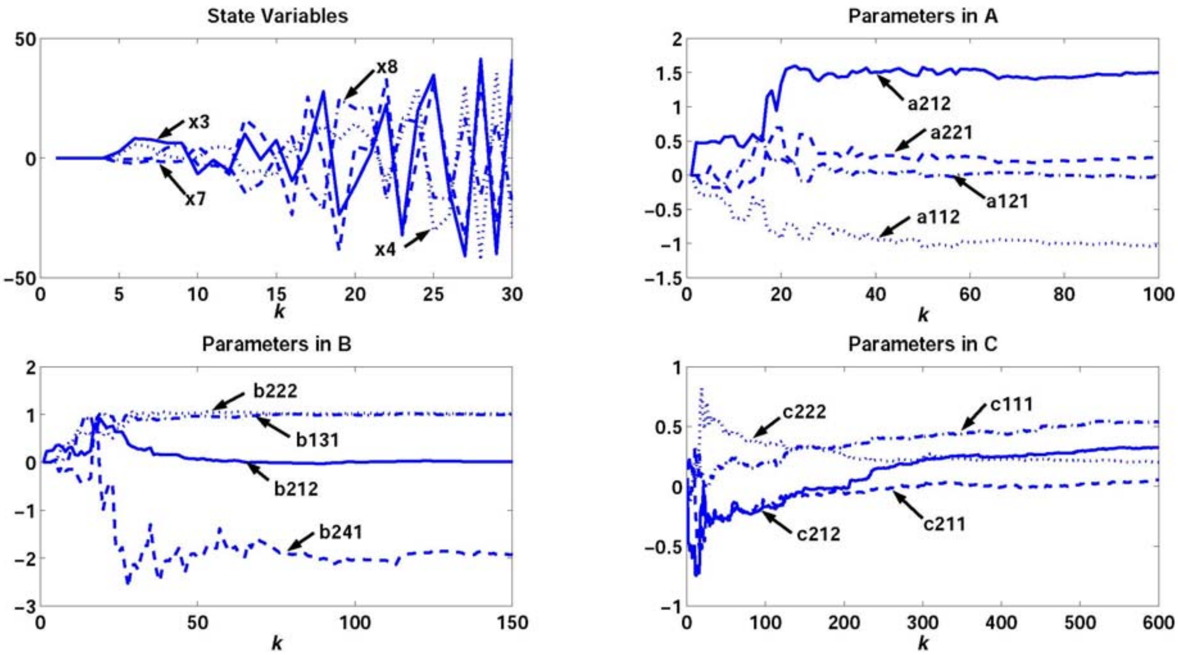

FIGURE 3. The estimated states and parameters of an MIMO system

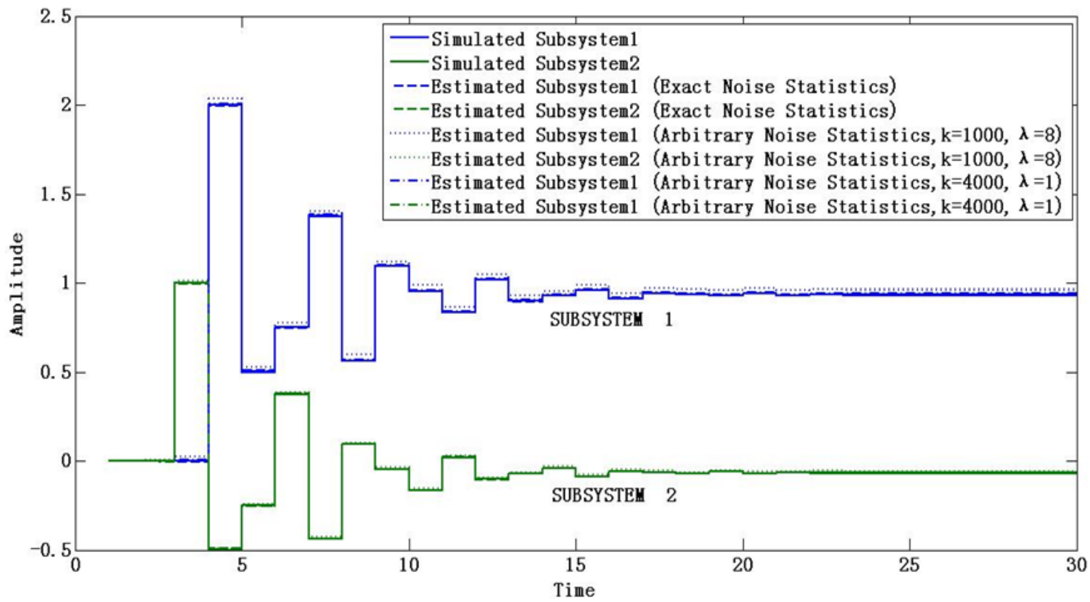

FIGURE 4. Comparison of step responses of the simulated and estimated subsystems (MIMO)

The recursive rate has been improved by editing some items of Chen's algorithm [4]. Simulation results exhibit the validity of the method. It also easily implements the linear control theory because of its linearity.

From the papers published in recent years, the estimation with addicted input-output data [24] and the estimation of the dynamic delay [25] instead of the static delay are full of challenge. Our future work will focus on these problems.

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