International Journal of Innovative Computing, Information and Control
Volume 12, Number 1, February 2016
ICIC International ©2016 ISSN 1349-4198
pp. 275–291

APPROACHES TO MULTIPLE ATTRIBUTE GROUP DECISION MAKING WITH MULTI-GRANULARITY LINGUISTIC TERM SETS

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Received July 2015; revised November 2015

Abstract. *This paper mainly focuses on dealing with multi-attribute group decision making with linguistic 2-tuples assessed in multi-granularity linguistic term sets. Based on weighted average aggregation operators, we first develop an operator to integrate and normalize linguistic 2-tuples under different multiple granularity linguistic term sets. Two systematical approaches are then put forward for tackling multi-attribute group decision making problems, where different multi-granularity linguistic term sets are used to express decision makers' assessments. The proposed approaches are illustrated by an adapted numerical example and a practical example with respect to the recommendation of undergraduate students for graduate admission.*

Keywords: Linguistic 2-tuple, Operator, Aggregation, Multiple attribute group decision making, Multi-granularity linguistic term sets

1. Introduction.

In many multi-attribute group decision making cases, decision makers often have to assess a set of alternatives over several attributes, and their common goal is to select the best alternative(s). There are two forms of values for decision makers expressing their opinions: quantitative and qualitative. In the situations where the given information is too imprecise or unquantifiable, it is rather difficult for a decision maker to provide his/her preferences with quantitative values. For this reason, giving assessments with qualitative values such as linguistic values is appropriate. Since the concept of linguistic variable was introduced by Zadeh [1], it has been extensively applied to group decision-making. For example, when evaluating the character of a person, linguistic terms or labels such as "good", "very good", "medium" and "bad" can be used. However, the linguistic terms can only express discrete information. When the given information is continuous and may not be suitable for any one of the terms, information loss and imprecision will certainly emerge. To break through this limitation, Herrera and Martínez [2] introduced the notion of linguistic 2-tuple, which is constituted by a linguistic value and a crisp number lying in $[-0.5, 0.5)$. Therefore, this paper primarily focuses on solving group decision making problems with linguistic 2-tuples.

In fact, since different decision makers possess different knowledge, educational backgrounds and experiences, they may employ different linguistic term sets to evaluate alternatives. This is one kind of multi-attribute group decision making based on multi-granularity linguistic term sets. Among the related articles published recently, Andrés et al. [3] gave a definition to multi-granular linguistic evaluation framework, in which the assessments provided by different decision makers were in different linguistic scales. Cai et al. [4] proposed a distance method between multiple granularity linguistic values and applied it to a multi-attribute group decision making case on medical diagnosis.

Herrera et al. [5] presented a fusion approach for managing multi-granularity linguistic information and applied it to a multiple information sources decision-making. Besides, Chen and Ben-Arieh [6] put forward a method which allowed experts to use different scales of linguistic terms and proposed a conversion function to make the information uniform. Zhu and Hipel [7] considered a problem of group target decision making with multi-granularity linguistic term sets and put forward a method based on target distance. Chuu [8] established a fusion approach to deal with group decision making with linguistic information assessed in different linguistic label sets, as well as to promote the evaluation process of the supply chain flexibility. Zhang and Guo [9] presented an approach to handle multiple attribute multi-granularity linguistic group decision making problems in which the weights of attributes are incompletely given. Also, they made an application of the approach to an investment problem. Based on deviation and the technique for order preference by similarity to ideal solution (TOPSIS) method, Liu et al. [10] gave a method to deal with multi-attribute group decision-making with multi-granularity linguistic term sets and illustrated an example about an investment in a high-tech project. Dong et al. [11] proposed a group decision making model based on consensus by using linguistic 2-tuples with multi-granularity linguistic term sets. In the above listed research works ([3-11]), different linguistic term sets are adopted to assess alternatives over attributes for various decision makers, but each decision maker employs the same linguistic term set to provide his/her assessments. In some group decision making problems, attribute assessment values are required to be provided by decision makers under the same linguistic term set, but various multi-granularity linguistic term sets are used to express assessments for different attributes. This paper focuses on these two types of multi-attribute group decision making problems with multiple granularity linguistic term sets.

Aggregation is an important issue to be discussed in multiple attribute group decision making. Until now, a lot of linguistic aggregation operators have been developed by different researchers [2,12-21]. Herrera and Martínez [2] put forward a 2-tuple weighted average operator and a 2-tuple ordered weighted average operator to synthesize linguistic 2-tuples. In [12], Wei introduced a series of novel operators for the aggregation of linguistic 2-tuples and made an application of the proposed operators to deal with multiple attribute group decision problems. Xu and Huang [13] developed several operators such as 2-tuple linguistic geometric average (TGA) operator and 2-tuple linguistic ordered weighted geometric average (TOWGA) operator, and employed them to collect linguistic information. In addition, Wei [14] put forward three generalized operators for integrating linguistic 2-tuples. Also, he developed a method for tackling multi-attribute group decision making problems, and applied this method to solving an investment problem. In this paper, based on transformation functions between crisp numbers and linguistic 2-tuples, we present the weighted average aggregation operators developed by Herrera and Martínez [2] and Chen and Tai [15] in a more intuitive way, to synthesize 2-tuple linguistic values which are evaluated in one linguistic term set. Also, their properties are studied. We introduce a new aggregation operator called multi-granularity 2-tuple linguistic weighted average (MG2TLWA) operator to integrate and normalize linguistic 2-tuples based on different linguistic term sets. The paper put forward two approaches for tackling multi-attribute group decision making problems. A numerical example including a comparative analysis about an investment problem and a practical example with respect to the recommendation of undergraduate students for graduate admission are provided to demonstrate the validity and advantage of our proposed methods.

The remaining parts are set out as follows. Section 2 reviews some basic concepts related to linguistic values, linguistic 2-tuples and transformation functions between linguistic 2-tuples and crisp values. In Section 3, an MG2TLWA operator is developed for the collection of linguistic 2-tuples assessed in different linguistic term sets, and two approaches are proposed for group decision-making with different kinds of multi-granularity linguistic term sets. In Section 4, an adapted example is presented and the alternative ranking result is compared with an existing method. Also, a practical example about the recommendation of undergraduate students for graduate admission is given to demonstrate how to apply one of our approaches in detail in Section 4. Finally, conclusions are drawn in Section 5.

2. **Preliminaries.** Let $S = \{s_u | u = 0, 1, \ldots, g\}$ be a linguistic term set, whose cardinality is $g + 1$. The term s_u represents a possible value of one linguistic variable. For instance, a linguistic term set of 5 labels can be given as:

$$S = \{s_0 = \textit{very poor}, s_1 = \textit{poor}, s_2 = \textit{medium}, s_3 = \textit{good}, s_4 = \textit{very good}\}.$$

Usually, the following properties are required to be satisfied for S [2]:

1) $s_u < s_t$, if and only if $u < t$;
2) There is a negation operator: $Neg(s_u) = s_t$, so that $t = g - u$;
3) $\min(s_u, s_t) = s_u$, if and only if $s_u \leq s_t$;
4) $\max(s_u, s_t) = s_u$, if and only if $s_u \geq s_t$.

However, it is evident that the labels in a linguistic term set can only represent discrete information. In order to express continuous information with linguistic values, Herrera and Martínez [2] defined the linguistic 2-tuple as follows.

Definition 2.1 ([2])**.** *Let β be the result of an aggregation of the indexes of a set of labels assessed in a linguistic term set $S = \{s_u | u = 0, 1, \ldots, g\}$. Let $u = round(\beta)$ and $\alpha = \beta - u$ be two values, respectively satisfying $u \in [0, g]$ and $\alpha \in [-0.5, 0.5)$, and then α is called a symbolic translation.*

From this point of view, a linguistic representation model which represents the linguistic information is introduced by Herrera and Martínez [2] below.

Definition 2.2 ([2])**.** *A linguistic 2-tuple is denoted as (s_u, α), where s_u is the uth element of the predefined linguistic term set $S = \{s_u | u = 0, 1, \ldots, g\}$ and represents the linguistic label of the decision information; α is a crisp number indicating the deviation of the original result β from the closest linguistic label u.*

Theorem 2.1. *For two linguistic 2-tuples (s_u, α_1) and (s_t, α_2), their comparison method is given below.*

1) If $u < t$, then $(s_u, \alpha_1) < (s_t, \alpha_2)$;
2) If $u = t$, then there exist three cases:
a. If $\alpha_1 < \alpha_2$, then $(s_u, \alpha_1) < (s_t, \alpha_2)$;
b. If $\alpha_1 = \alpha_2$, then $(s_u, \alpha_1) = (s_t, \alpha_2)$;
c. If $\alpha_1 > \alpha_2$, then $(s_u, \alpha_1) > (s_t, \alpha_2)$.

Herrera and Martínez [2] and Chen and Tai [15] established the following transform relation between a linguistic 2-tuple and a crisp number.

Definition 2.3 ([2])**.** *Let $S = \{s_u | u = 0, 1, \ldots, g\}$ be a linguistic term set, $\beta \in [0, g]$ be the result of a symbolic aggregation operation, and a 2-tuple (s_u, α) representing the identical information to β can be generated with:*

$$\Delta : [0, g] \to S \times [-0.5, 0.5)$$
$$\Delta(\beta) = (s_u, \alpha), \ \textit{with} \left\{ \begin{array}{ll} s_u, & u = round(\beta) \\ \alpha = \beta - u, & \alpha \in [-0.5, 0.5) \end{array} \right. \tag{1}$$

where round is the common rounding operation and α is the value of symbolic translation.

Conversely, based on a linguistic 2-tuple (s_u, α), a corresponding numerical value β can be obtained by a converse function defined below.

Definition 2.4 ([2])**.** *Let $S = \{s_u | u = 0, 1, \ldots, g\}$ be a linguistic term set, (s_u, α) be a 2-tuple, then there is always a function Δ^{-1}, which can return a 2-tuple to its numerical value β where $\beta \in [0, g]$, expressed as:*

$$\begin{array}{l} \Delta^{-1} : S \times [-0.5, 0.5) \to [0, g] \\ \Delta^{-1}(s_u, \alpha) = u + \alpha = \beta \end{array} \tag{2}$$

Obviously, converting a linguistic term into a linguistic 2-tuple can be accomplished by adding a value 0 as symbolic translation. That is,

$$\Delta(s_u) = (s_u, 0), \ s_u \in S \tag{3}$$

For instance, given a linguistic term set $S = \{s_u | u = 0, 1, 2, 3, 4\}$, and three linguistic 2-tuples $(s_1, 0), (s_2, 0.3)$ and $(s_4, -0.5)$, then by the functions aforementioned, one can obtain:

$$\Delta^{-1}(s_1, 0) = 1, \ \Delta^{-1}(s_2, 0.3) = 2.3, \ \Delta^{-1}(s_4, -0.5) = 3.5$$

Conversely, $\Delta(1) = (s_1, 0)$, $\Delta(2.3) = (s_2, 0.3)$, $\Delta(3.5) = (s_4, -0.5)$.

However, the numerical values obtained from the above operation need to be further normalized. In the following, we recall a generalized translation function developed in [15].

Definition 2.5 ([15])**.** *Let $S = \{s_u | u = 0, 1, \ldots, g\}$ be a linguistic term set, $\beta \in [0, 1]$ be the result of a symbolic aggregation operation, and then the following function is utilized to convert a crisp value β into a linguistic 2-tuple (s_u, α):*

$$\begin{array}{l} \Delta : [0, 1] \to S \times [-0.5/g, 0.5/g) \\ \Delta(\beta) = (s_u, \alpha), \ with \left\{ \begin{array}{ll} s_u, & u = round(\beta \bullet g) \\ \alpha = \beta - \frac{u}{g}, & \alpha \in [-0.5/g, 0.5/g) \end{array} \right. \end{array} \tag{4}$$

On the contrary, a linguistic 2-tuple (s_u, α) can be translated into a crisp value via the function defined below.

Definition 2.6 ([15])**.** *Let $S = \{s_u | u = 0, 1, \ldots, g\}$ be a linguistic term set, (s_u, α) be a linguistic 2-tuple, and a converse function Δ^{-1} is defined to return an equivalent number β from (s_u, α) as*

$$\begin{array}{l} \Delta^{-1} : S \times [-0.5/g, 0.5/g) \to [0, 1] \\ \Delta^{-1}(s_u, \alpha) = \frac{u}{g} + \alpha = \beta \end{array} \tag{5}$$

Remark 2.1. *Differently, the value of β defined by **Definition 2.5** and **Definition 2.6** lies in $[0, 1]$, while the value of β defined by **Definition 2.3** and **Definition 2.4** is in the range of $[0, g]$. This distinction reflects the intrinsic difference between **Definition 2.3** and **Definition 2.5**, as well as **Definition 2.4** and **Definition 2.6**. In other words, the values obtained by **Definition 2.5** and **Definition 2.6** are standardized, which makes it more convenient to make comparison between linguistic 2-tuples evaluated with different linguistic term sets.*

3. **Group Decision-Making with Multi-Granularity Linguistic Term Sets.** This section primarily addresses how to deal with multiple attribute group decision making with multi-granularity linguistic term sets. Firstly, by learning the aggregation operators developed by Herrera and Martínez [2] and Chen and Tai [15], we give two simplified aggregation methods for synthesizing linguistic 2-tuples whose linguistic values are denoted by the same linguistic label set. Moreover, their properties are discussed. Then, based

on the second method, we develop an operator for aggregating 2-tuple linguistic values which are evaluated in different linguistic scales. Two approaches for dealing with group decision-making with multi-granularity linguistic term sets are developed on the basis of the presented operators.

According to the weighted average aggregation operator proposed by Herrera and Martínez [2], we give a simplified and intuitive aggregation method as follows.

Definition 3.1. *Let $S = \{s_u | u = 0, 1, \ldots, g\}$ be a linguistic term set, and $V = \{v = (s_\varepsilon, \alpha) | \alpha \in [-0.5, 0.5), s_\varepsilon \in \{s_0, s_1, \ldots, s_g\}\}$ be a set of linguistic 2-tuples. Then a 2-tuple linguistic weighted average (2TLWA) operator of dimension m is a function $2TLWA : (V)^m \to V$, defined by*

$$\begin{aligned} &2TLWA(v_1, v_2, \ldots, v_m) = 2TLWA\left((s_{\varepsilon_1}, \alpha_1), (s_{\varepsilon_2}, \alpha_2), \ldots, (s_{\varepsilon_m}, \alpha_m)\right) = (s_\varepsilon, \alpha) \\ &\varepsilon = round\left(\sum_{p=1}^m \omega_p(\varepsilon_p + \alpha_p)\right),\ \alpha = \left(\sum_{p=1}^m \omega_p(\varepsilon_p + \alpha_p)\right) - \varepsilon \end{aligned} \tag{6}$$

where $\varpi = (\omega_1, \omega_2, \ldots, \omega_m)^T$ is an associated weighting vector with $\sum_{p=1}^m \omega_p = 1$ and $\omega_p \in [0, 1]$.

Clearly, the value of α lies in $[-0.5, 0.5)$. If $\omega_p = 1/m$ for all $p = 1, 2, \ldots, m$, then the 2TLWA operator can be reduced to a 2-tuple linguistic average (2TLA) operator:

$$\begin{aligned} &2TLA(v_1, v_2, \ldots, v_m) = 2TLA\left((s_{\varepsilon_1}, \alpha_1), (s_{\varepsilon_2}, \alpha_2), \ldots, (s_{\varepsilon_m}, \alpha_m)\right) = (s_\varepsilon, \alpha) \\ &\varepsilon = round\left(\frac{\sum_{p=1}^m (\varepsilon_p + \alpha_p)}{m}\right),\ \alpha = \frac{\sum_{p=1}^m (\varepsilon_p + \alpha_p)}{m} - \varepsilon \end{aligned} \tag{7}$$

Equation (7) is intrinsically identical to the arithmetic mean in [2]. However, Equation (7) is more intuitive to be understood.

Bottom on **Definition 3.1**, we get several properties of the 2TLWA operator.

Theorem 3.1. *If all $v_p = (s_{\varepsilon_p}, \alpha_p)$ $(p = 1, 2, \ldots, m)$ are identical, i.e., $v_p = (s_{\varepsilon_p}, \alpha_p) = (s_{\varepsilon_*}, \alpha_*)$ for $p = 1, 2, \ldots, m$, then*

$$2TLWA\left(v_1, v_2, \ldots, v_m\right) = (s_{\varepsilon_*}, \alpha_*) \tag{8}$$

Proof: If $(s_{\varepsilon_p}, \alpha_p) = (s_{\varepsilon_*}, \alpha_*)$, then

$$\begin{aligned} &2TLWA(v_1, v_2, \ldots, v_m) \\ &= 2TLWA\left((s_{\varepsilon_1}, \alpha_1), (s_{\varepsilon_2}, \alpha_2), \ldots, (s_{\varepsilon_m}, \alpha_m)\right) \\ &= \left(s_{round\left(\sum_{p=1}^m \omega_p(\varepsilon_p + \alpha_p)\right)}, \sum_{p=1}^m \omega_p(\varepsilon_p + \alpha_p) - round\left(\sum_{p=1}^m \omega_p(\varepsilon_p + \alpha_p)\right)\right) \\ &= \left(s_{round(\varepsilon_* + \alpha_*)}, (\varepsilon_* + \alpha_*) - round\left(\varepsilon_* + \alpha_*\right)\right) \\ &= (s_{\varepsilon_*}, \alpha_*) \end{aligned}$$

Thus, the proof of **Theorem 3.1** is completed.

It is easy to prove the following two theorems.

Theorem 3.2. *If $v_p = (s_{\varepsilon_p}, \alpha_p) \le v'_p = (s_{\varepsilon'_p}, \alpha'_p)$ for all $p = 1, 2, \ldots, m$, then*

$$\begin{aligned} &2TLWA\left((s_{\varepsilon_1}, \alpha_1), (s_{\varepsilon_2}, \alpha_2), \ldots, (s_{\varepsilon_m}, \alpha_m)\right) \\ &\le 2TLWA\left((s_{\varepsilon'_1}, \alpha'_1), (s_{\varepsilon'_2}, \alpha'_2), \ldots, (s_{\varepsilon'_m}, \alpha'_m)\right) \end{aligned} \tag{9}$$

Theorem 3.3. *Let $v_p = (s_{\varepsilon_p}, \alpha_p)$ $(p = 1, 2, \ldots, m)$ be m linguistic 2-tuples, then*

$$\min_{1 \le p \le m} \left\{(s_{\varepsilon_p}, \alpha_p)\right\} \le 2TLWA\left((s_{\varepsilon_1}, \alpha_1), (s_{\varepsilon_2}, \alpha_2), \ldots, (s_{\varepsilon_m}, \alpha_m)\right) \le \max_{1 \le p \le m} \left\{(s_{\varepsilon_p}, \alpha_p)\right\} \tag{10}$$

The above three theorems indicate idempotency, monotonicity and boundary of the 2TLWA operator, respectively.

In the following, we present a weighted aggregation method which has been proposed in [15], in a more intuitive way.

Definition 3.2. *Let $S = \{s_u | u = 0, 1, \ldots, g\}$ be a linguistic term set, and $\tilde{V} = \{\tilde{v} = (s_{\tilde{\varepsilon}}, \tilde{\alpha}) | \tilde{\alpha} \in [-0.5/g, 0.5/g), s_{\tilde{\varepsilon}} \in \{s_0, s_1, \ldots, s_g\}\}$ be a set of linguistic 2-tuples. Then a generalized 2-tuple linguistic weighted average (G2TLWA) operator of dimension m is a function: $G2TLWA : \left(\tilde{V}\right)^m \to \tilde{V}$, expressed with*

$$\begin{array}{l} G2TLWA(\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_m) = G2TLWA\left((s_{\tilde{\varepsilon}_1}, \tilde{\alpha}_1), (s_{\tilde{\varepsilon}_2}, \tilde{\alpha}_2), \ldots, (s_{\tilde{\varepsilon}_m}, \tilde{\alpha}_m)\right) = (s_{\tilde{\varepsilon}}, \tilde{\alpha}) \\ \tilde{\varepsilon} = round\left(g \bullet \left(\sum_{p=1}^m \omega_p \left(\frac{\tilde{\varepsilon}_p}{g} + \tilde{\alpha}_p\right)\right)\right), \tilde{\alpha} = \left(\sum_{p=1}^m \omega_p \left(\frac{\tilde{\varepsilon}_p}{g} + \tilde{\alpha}_p\right)\right) - \frac{\tilde{\varepsilon}}{g} \end{array} \tag{11}$$

where $\varpi = (\omega_1, \omega_2, \ldots, \omega_m)^T$ is an associated weighting vector with $\sum_{p=1}^m \omega_p = 1$ and $\omega_p \in [0, 1]$.

Likewise, if $\omega_p = 1/m$ for all $p = 1, 2, \ldots, m$, then the G2TLWA operator can be reduced to a generalized 2-tuple linguistic average (G2TLA) operator:

$$\begin{array}{l} G2TLA(\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_m) = G2TLA\left((s_{\tilde{\varepsilon}_1}, \tilde{\alpha}_1), (s_{\tilde{\varepsilon}_2}, \tilde{\alpha}_2), \ldots, (s_{\tilde{\varepsilon}_m}, \tilde{\alpha}_m)\right) = (s_{\tilde{\varepsilon}}, \tilde{\alpha}) \\ \tilde{\varepsilon} = round\left(g \bullet \left(\frac{\sum_{p=1}^m \left(\frac{\tilde{\varepsilon}_p}{g} + \tilde{\alpha}_p\right)}{m}\right)\right), \ \tilde{\alpha} = \left(\frac{\sum_{p=1}^m \left(\frac{\tilde{\varepsilon}_p}{g} + \tilde{\alpha}_p\right)}{m}\right) - \frac{\tilde{\varepsilon}}{g} \end{array} \tag{12}$$

On the basis of **Definition 3.2**, we conclude the properties of the G2TLWA operator in the following theorem.

Theorem 3.4. *Let $\tilde{v}_p = (s_{\tilde{\varepsilon}_p}, \tilde{\alpha}_p) \in \tilde{V}$ $(p = 1, 2, \ldots, m)$ be m linguistic 2-tuples associated with a weighting vector $\varpi = (\omega_1, \omega_2, \ldots, \omega_m)^T$ with $\sum_{p=1}^m \omega_p = 1$ and $\omega_p \in [0, 1]$, and then we have:*

(1) Boundary:

$$\min_{1 \le p \le m} \left\{(s_{\tilde{\varepsilon}_p}, \tilde{\alpha}_p)\right\} \le G2TLWA\left((s_{\tilde{\varepsilon}_1}, \tilde{\alpha}_1), (s_{\tilde{\varepsilon}_2}, \tilde{\alpha}_2), \ldots, (s_{\tilde{\varepsilon}_m}, \tilde{\alpha}_m)\right) \le \max_{1 \le p \le m} \left\{(s_{\tilde{\varepsilon}_p}, \tilde{\alpha}_p)\right\}.$$

(2) Idempotency: If $(s_{\tilde{\varepsilon}_p}, \tilde{\alpha}_p) = (s_{\tilde{\varepsilon}_*}, \tilde{\alpha}_*)$ for all $p = 1, 2, \ldots, q$, then

$$G2TLWA\left((s_{\tilde{\varepsilon}_1}, \tilde{\alpha}_1), (s_{\tilde{\varepsilon}_2}, \tilde{\alpha}_2), \ldots, (s_{\tilde{\varepsilon}_m}, \tilde{\alpha}_m)\right) = (s_{\tilde{\varepsilon}_*}, \tilde{\alpha}_*).$$

(3) Monotonicity: If $v_p = (s_{\tilde{\varepsilon}_p}, \tilde{\alpha}_p) \le v'_p = (s_{\tilde{\varepsilon}'_p}, \tilde{\alpha}'_p)$ for all $p = 1, 2, \ldots, m$, then

$$\begin{array}{c} G2TLWA\left((s_{\tilde{\varepsilon}_1}, \tilde{\alpha}_1), (s_{\tilde{\varepsilon}_2}, \tilde{\alpha}_2), \ldots, (s_{\tilde{\varepsilon}_m}, \tilde{\alpha}_m)\right) \\ \le G2TLWA\left((s_{\tilde{\varepsilon}'_1}, \tilde{\alpha}'_1), (s_{\tilde{\varepsilon}'_2}, \tilde{\alpha}'_2), \ldots, (s_{\tilde{\varepsilon}'_m}, \tilde{\alpha}'_m)\right). \end{array}$$

Proof: Since $v_p = (s_{\tilde{\varepsilon}_p}, \tilde{\alpha}_p) \le v'_p = (s_{\tilde{\varepsilon}'_p}, \tilde{\alpha}'_p)$, then

$$\begin{array}{l} \frac{\tilde{\varepsilon}_p}{g} + \tilde{\alpha}_p \le \frac{\tilde{\varepsilon}'_p}{g} + \tilde{\alpha}'_p \\ \Rightarrow \omega_p \left(\frac{\tilde{\varepsilon}_p}{g} + \tilde{\alpha}_p\right) \le \omega_p \left(\frac{\tilde{\varepsilon}'_p}{g} + \tilde{\alpha}'_p\right) \\ \Rightarrow \sum_{p=1}^m \omega_p \left(\frac{\tilde{\varepsilon}_p}{g} + \tilde{\alpha}_p\right) \le \sum_{p=1}^m \omega_p \left(\frac{\tilde{\varepsilon}'_p}{g} + \tilde{\alpha}'_p\right) \\ \Rightarrow G2TLWA\left((s_{\tilde{\varepsilon}_1}, \tilde{\alpha}_1), (s_{\tilde{\varepsilon}_2}, \tilde{\alpha}_2), \ldots, (s_{\tilde{\varepsilon}_m}, \tilde{\alpha}_m)\right) \\ \quad \le G2TLWA\left(\left(s'_{\tilde{\varepsilon}_1}, \tilde{\alpha}'_1\right), \left(s'_{\tilde{\varepsilon}_2}, \tilde{\alpha}'_2\right), \ldots, \left(s'_{\tilde{\varepsilon}_m}, \tilde{\alpha}'_m\right)\right) \end{array}$$

Since $(s_{\tilde{\varepsilon}_p}, \tilde{\alpha}_p) = (s_{\tilde{\varepsilon}_*}, \tilde{\alpha}_*)$ for all $p = 1, 2, \ldots, m$, then

$$\begin{array}{l} \sum_{p=1}^m \omega_p \left(\frac{\tilde{\varepsilon}_p}{g} + \tilde{\alpha}_p\right) = \sum_{p=1}^m \frac{1}{m} \left(\frac{\tilde{\varepsilon}_*}{g} + \tilde{\alpha}_*\right) = \left(\frac{\tilde{\varepsilon}_*}{g} + \tilde{\alpha}_*\right) \\ \Rightarrow G2TLWA\left((s_{\tilde{\varepsilon}_1}, \tilde{\alpha}_1), (s_{\tilde{\varepsilon}_2}, \tilde{\alpha}_2), \ldots, (s_{\tilde{\varepsilon}_m}, \tilde{\alpha}_m)\right) \\ \quad = G2TLWA\left((s_{\tilde{\varepsilon}_*}, \tilde{\alpha}_*), (s_{\tilde{\varepsilon}_*}, \tilde{\alpha}_*), \ldots, (s_{\tilde{\varepsilon}_*}, \tilde{\alpha}_*)\right) = (s_{\tilde{\varepsilon}_*}, \tilde{\alpha}_*) \end{array}$$

By combining the above two conclusions, one can obtain the following inequalities. If $\min\limits_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\}\le(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\le\max\limits_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\}$, then

$$\begin{aligned}&G2TLWA\left(\min_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\},\min_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\},\ldots,\min_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\}\right)\\ \le\ &G2TLWA\left((s_{\tilde{\varepsilon}_1},\tilde{\alpha}_1),(s_{\tilde{\varepsilon}_2},\tilde{\alpha}_2),\ldots,(s_{\tilde{\varepsilon}_m},\tilde{\alpha}_m)\right)\\ \le\ &G2TLWA\left(\max_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\},\max_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\},\ldots,\max_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\}\right)\\ \Rightarrow\ &\min_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\}\le G2TLWA\left((s_{\tilde{\varepsilon}_1},\tilde{\alpha}_1),(s_{\tilde{\varepsilon}_2},\tilde{\alpha}_2),\ldots,(s_{\tilde{\varepsilon}_m},\tilde{\alpha}_m)\right)\le\max_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\}\end{aligned}$$

where the number of $(s_{\tilde{\varepsilon}_*},\tilde{\alpha}_*)$ in $G2TLWA\left((s_{\tilde{\varepsilon}_*},\tilde{\alpha}_*),(s_{\tilde{\varepsilon}_*},\tilde{\alpha}_*),\ldots,(s_{\tilde{\varepsilon}_*},\tilde{\alpha}_*)\right)$, the number of $\min\limits_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\}$ in $G2TLWA\left(\min\limits_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\},\min\limits_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\},\ldots,\min\limits_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\}\right)$ as well as that of $\max\limits_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\}$ in $G2TLWA\left(\max\limits_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\},\max\limits_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\},\ldots,\max\limits_{1\le p\le m}\left\{(s_{\tilde{\varepsilon}_p},\tilde{\alpha}_p)\right\}\right)$ is m.

The aggregation methods in Equation (6) and Equation (11) are based on linguistic 2-tuples with the same linguistic term set. Indeed, due to the difference among attributes, using different linguistic term sets is more appropriate to evaluate attribute values of alternatives. To handle a group decision-making problem with multi-granularity linguistic term sets, two important things are required to be simultaneously achieved: normalization and aggregation. On the basis of the previously developed G2TLWA operator, a multi-granularity 2-tuple linguistic weighted average (MG2TLWA) operator is introduced below.

Definition 3.3. *Let* $S^{(p)}=\left\{s_u^{(p)}|u=0,1,\ldots g^p\right\}$ *be a linguistic term set and* $V^{(p)}=\left\{v^{(p)}=\left(s_\eta^{(p)},\gamma\right)|\gamma\in[-0.5/g^p,0.5/g^p),s_\eta^{(p)}\in\left\{s_0^{(p)},s_1^{(p)},\ldots,s_{g^p}^{(p)}\right\}\right\}$ *be a set of linguistic 2-tuples for all* $p=1,2,\ldots,m$. *An MG2TLWA operator of dimension* m *is a function with the following form:*

$$\begin{aligned}&MG2TLWA:\left(V_1^{(1)}\times V_2^{(2)}\times\ldots\times V_m^{(m)}\right)\to V^{(*)}\\ &\qquad MG2TLWA\left(v_1^{(1)},v_2^{(2)},\ldots,v_m^{(m)}\right)\\ &\quad=MG2TLWA\left(\left(s_{\eta_1}^{(1)},\gamma_1\right),\left(s_{\eta_2}^{(2)},\gamma_2\right),\ldots,\left(s_{\eta_m}^{(m)},\gamma_m\right)\right)=\left(s_{\eta_*}^{(*)},\gamma_*\right)\\ &\eta_*=round\left(g^*\bullet\left(\sum_{p=1}^m\omega_p\left(\frac{\eta_p}{g^p}+\gamma_p\right)\right)\right),\gamma_*=\left(\sum_{p=1}^m\omega_p\left(\frac{\eta_p}{g^p}+\gamma_p\right)\right)-\frac{\eta_*}{g^*}\end{aligned}\tag{13}$$

where $V^{(*)}=\left\{v^{(*)}=\left(s_{\eta_*}^{(*)},\gamma_*\right)|\gamma_*\in[-0.5/g^*,0.5/g^*),s_{\eta_*}^{(*)}\in S^{(*)}=\left\{s_0^{(*)},s_1^{(*)},\ldots,s_{g^*}^{(*)}\right\}\right\}$ *is a set of linguistic 2-tuples and* $S^{(*)}$ *is the linguistic term set with the cardinality* $g^*=\max\{g^1,g^2,\ldots,g^m\}$, *and* $\varpi=(\omega_1,\omega_2,\ldots,\omega_m)^T$ *is the associated weighting vector of the given 2-tuples satisfying* $\sum_{p=1}^m\omega_p=1$ *and* $\omega_p\in[0,1]$.

Remark 3.1. *In **Definition 3.3**, the predefined linguistic term sets are different from the perspective of granularity. It should be noted that if all 2-tuples are given under a linguistic term set, then the MG2TLWA operator can also be employed to aggregate them into a group linguistic 2-tuple, and is degenerated to the G2TLWA operator depicted in Definition 3.2. In a word, this aggregation operator is suitable for integrating linguistic 2-tuples denoted by one linguistic term set as well as multi-granularity linguistic term sets. That is to say, no matter* g^p $(p=1,2,\ldots,m)$ *in* $S^{(p)}$ *are identical or partially* *equal or entirely different, the MG2TLWA operator is always applicable for aggregating multi-granularity linguistic 2-tuples.*

Remark 3.2. *Obviously, this method is an extension of the G2TLWA operator, and it should be noticed that the values of* $\left(\frac{\tilde{\varepsilon}_p}{g^p} + \tilde{\alpha}_p\right)$ *are all no less than 0 and no more than 1, which indicates that the values are normalized. However, the collective result obtained from the 2TLWA operator lies in* $[0, g]$. *When* g *changes, then the range of the linguistic value in a 2-tuple changes, such that 2-tuples expressed with different linguistic term sets should be further normalized. Therefore, the 2TLWA operator cannot be directly extended to tackle group decision making problems with linguistic information evaluated in multi-granularity linguistic term sets.*

By studying the MG2TLWA operator, we have the following properties.

Theorem 3.5 (Idempotency). *If all* $\left(s_{\eta_p}^{(p)}, \gamma_p\right)$ $(p=1, 2, \ldots, m)$ *are identical, i.e.,* $\left(s_{\eta_p}^{(p)}, \gamma_p\right) = \left(s_{\eta_{**}}^{(**)}, \gamma_{**}\right)$ *for* $p = 1, 2, \ldots, m$, *then*

$$MG2TLWA\left(\left(s_{\eta_1}^{(1)}, \gamma_1\right), \left(s_{\eta_2}^{(2)}, \gamma_2\right), \ldots, \left(s_{\eta_m}^{(m)}, \gamma_m\right)\right) = \left(s_{\eta_{**}}^{(**)}, \gamma_{**}\right) \tag{14}$$

Proof: If $\left(s_{\eta_p}^{(p)}, \gamma_p\right) = \left(s_{\eta_{**}}^{(**)}, \gamma_{**}\right)$, then

$$\begin{aligned}
&MG2TLWA\left(v_1^{(1)}, v_2^{(2)}, \ldots, v_m^{(m)}\right) \\
&= MG2TLWA\left(\left(s_{\eta_1}^{(1)}, \gamma_1\right), \left(s_{\eta_2}^{(2)}, \gamma_2\right), \ldots, \left(s_{\eta_m}^{(m)}, \gamma_m\right)\right) \\
&= \left(s^{(**)}_{round\left(g^{**}\bullet\left(\sum_{p=1}^m \omega_p\left(\frac{\eta_{**}}{g^{**}}+\gamma_{**}\right)\right)\right)}, \left(\sum_{p=1}^m \omega_p\left(\frac{\eta_{**}}{g^{**}} + \gamma_{**}\right)\right) - \frac{round\left(g^{**}\bullet\left(\sum_{p=1}^m \omega_p\left(\frac{\eta_{**}}{g^{**}}+\gamma_{**}\right)\right)\right)}{g^{**}}\right) \\
&= \left(s^{(**)}_{round(\eta_{**}+\gamma_{**}\bullet g^{**})}, \left(\frac{\eta_{**}}{g^{**}} + \gamma_{**}\right) - \frac{round(\eta_{**}+\gamma_{**}\bullet g^{**})}{g^{**}}\right) = \left(s_{\eta_{**}}^{(**)}, \gamma_{**}\right)
\end{aligned}$$

Therefore, **Theorem 3.5** is completely proved.

Theorem 3.6 (Monotonicity). *If* $\left(s_{\eta_p}^{(p)}, \gamma_p\right) \le \left(s_{\eta'_p}^{(p)}, \gamma'_p\right)$ *for all* $p = 1, 2, \ldots, m$, *then*

$$\begin{aligned}
&MG2TLWA\left(\left(s_{\eta_1}^{(1)}, \gamma_1\right), \left(s_{\eta_2}^{(2)}, \gamma_2\right), \ldots, \left(s_{\eta_m}^{(m)}, \gamma_m\right)\right) \\
&\le MG2TLWA\left(\left(s_{\eta'_1}^{(1)}, \gamma'_1\right), \left(s_{\eta'_2}^{(2)}, \gamma'_2\right), \ldots, \left(s_{\eta'_m}^{(m)}, \gamma'_m\right)\right)
\end{aligned} \tag{15}$$

Proof: As $\left(s_{\eta_p}^{(p)}, \gamma_p\right) \le \left(s_{\eta'_p}^{(p)}, \gamma'_p\right)$, we have

$$\begin{aligned}
&\tfrac{\eta_p}{g^p} + \gamma_p \le \tfrac{\eta'_p}{g^p} + \gamma'_p \Rightarrow \omega_p\left(\tfrac{\eta_p}{g^p} + \gamma_p\right) \le \omega_p\left(\tfrac{\eta'_p}{g^p} + \gamma'_p\right) \\
&\Rightarrow \textstyle\sum_{p=1}^m \omega_p\left(\tfrac{\eta_p}{g^p} + \gamma_p\right) \le \sum_{p=1}^m \omega_p\left(\tfrac{\eta'_p}{g^p} + \gamma'_p\right) \\
&\Rightarrow MG2TLWA\left(\left(s_{\eta_1}^{(1)}, \gamma_1\right), \left(s_{\eta_2}^{(2)}, \gamma_2\right), \ldots, \left(s_{\eta_m}^{(m)}, \gamma_m\right)\right) \\
&\le MG2TLWA\left(\left(s_{\eta'_1}^{(1)}, \gamma'_1\right), \left(s_{\eta'_2}^{(2)}, \gamma'_2\right), \ldots, \left(s_{\eta'_m}^{(m)}, \gamma'_m\right)\right)
\end{aligned}$$

Thus, Equation (15) is proved.

If $\omega_p = 1/m$ for all $p = 1, 2, \ldots, m$, this operator is reduced to a multi-granularity 2-tuple linguistic average (MG2TLA) operator, which can be considered as an extension of the above-stated G2TLA operator.

$$\begin{aligned} & MG2TLA : \left(V_1^{(1)} \times V_2^{(2)} \times \ldots \times V_m^{(m)}\right) \to V^{(*)} \\ & \quad MG2TLA\left(v_1^{(1)}, v_2^{(2)}, \ldots, v_m^{(m)}\right) \\ & \quad = MG2TLA\left(\left(s_{\eta_1}^{(1)}, \gamma_1\right), \left(s_{\eta_2}^{(2)}, \gamma_2\right), \ldots, \left(s_{\eta_m}^{(m)}, \gamma_m\right)\right) = \left(s_{\eta_*}^{(*)}, \gamma_*\right) \\ & \eta_* = round\left(g^* \bullet \left(\frac{\sum_{p=1}^m \left(\frac{\eta_p}{g^p} + \gamma_p\right)}{m}\right)\right), \ \gamma_* = \left(\frac{\sum_{p=1}^m \left(\frac{\eta_p}{g^p} + \gamma_p\right)}{m}\right) - \frac{\eta_*}{g^*} \end{aligned} \tag{16}$$

Considering a multiple attribute group decision making problem with multi-granularity linguistic assessments, let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives, where x_i $(i = 1, 2, \ldots, n)$ represents the ith alternative; $A = \{a_1, a_2, \ldots, a_m\}$ be a collection of attributes, in which a_j $(j = 1, 2, \ldots, m)$ stands for the jth attribute. The weighting vector of the attributes is $W = (w_1, w_2, \ldots, w_m)^T$. Suppose $D = \{d^1, d^2, \ldots, d^q\}$ is a fixed set of q decision makers, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_q)^T$ is a weight vector of decision makers. Based on different linguistic term sets $S^{(j)} = \left\{s_0^{(j)}, s_1^{(j)}, \ldots, s_{g^j}^{(j)}\right\}$ $(j = 1, 2, \ldots, m)$, for various attributes, where $g^j + 1$ is the granularity of the set $S^{(j)}$, each decision maker provides his/her judgments on each alternative with respect to different attributes.

Based on the above presented operators, a systematical procedure is developed for tackling multi-attribute group decision making problems with multi-granularity linguistic term sets as follows.

Procedure 1:

Step 1. According to different linguistic term sets, convert the linguistic values in decision matrices provided by decision makers into linguistic 2-tuples through Equation (3), and yield 2-tuple linguistic matrices denoted by $R^k = \left(r_{ij}^k\right)_{n\times m} = \left(\left(s_{\eta_{kij}}^{(j)}, \gamma_{kij}\right)\right)_{n\times m}$, $k = 1, 2, \ldots, q$.

Step 2. Plug the weights of decision makers into the G2TLWA operator to aggregate individual linguistic 2-tuples into a collective linguistic 2-tuple, i.e.,

$$\begin{aligned} & G2TLWA\left(r_{ij}^1, r_{ij}^2, \ldots, r_{ij}^q\right) \\ & = G2TLWA\left(\left(s_{\eta_{1ij}}^{(j)}, \gamma_{1ij}\right), \left(s_{\eta_{2ij}}^{(j)}, \gamma_{2ij}\right), \ldots, \left(s_{\eta_{qij}}^{(j)}, \gamma_{qij}\right)\right) = \left(s_{\eta_{ij}}^{(j)}, \gamma_{ij}\right) \\ & \eta_{ij} = round\left(g^j \bullet \left(\sum_{k=1}^q \lambda_k \left(\frac{\eta_{kij}}{g^j} + \gamma_{kij}\right)\right)\right), \ \gamma_{ij} = \left(\sum_{k=1}^q \lambda_k \left(\frac{\eta_{kij}}{g^j} + \gamma_{kij}\right)\right) - \frac{\eta_{ij}}{g^j} \end{aligned} \tag{17}$$

Thus, we obtain a collective linguistic 2-tuple decision matrix

$$R = (r_{ij})_{n\times m} = \left(\left(s_{\eta_{ij}}^{(j)}, \gamma_{ij}\right)\right)_{n\times m}$$

It is noted that the collective linguistic 2-tuples in the decision matrix R may be based on different multi-granularity linguistic term sets.

Step 3. Employ the MG2TLWA operator to integrate the linguistic 2-tuples in the ith row of the collective linguistic 2-tuple linguistic decision matrix R into a comprehensive value denoted by $C_{x_i} = \left(s_{\eta_i}^{(*)}, \gamma_i\right)$ for the alternative x_i $(i = 1, 2, \ldots, n)$, i.e.,

$$\begin{aligned} & C_{x_i} = \left(s_{\eta_i}^{(*)}, \gamma_i\right) = MG2TLWA\left(r_{i1}, r_{i2}, \ldots, r_{im}\right) \\ & \quad = MG2TLWA\left(\left(s_{\eta_{i1}}^{(1)}, \gamma_{i1}\right), \left(s_{\eta_{i2}}^{(2)}, \gamma_{i2}\right), \ldots, \left(s_{\eta_{im}}^{(m)}, \gamma_{im}\right)\right) \\ & \eta_i = round\left(g^* \bullet \left(\sum_{j=1}^m w_j \left(\frac{\eta_{ij}}{g^j} + \gamma_{ij}\right)\right)\right), \ \gamma_i = \left(\sum_{j=1}^m w_j \left(\frac{\eta_{ij}}{g^j} + \gamma_{ij}\right)\right) - \frac{\eta_i}{g^*} \end{aligned} \tag{18}$$

where $g^* = \max\{g^1, g^2, \ldots, g^m\}$ and $s_{\varepsilon_i}^{(*)} \in S^{(*)} = \left\{s_0^{(*)}, s_1^{(*)}, \ldots, s_{g^*}^{(*)}\right\}$.
Step 4. Based on the comparison rules of linguistic 2-tuples given in **Theorem 2.1**, rank the scores of the alternatives and select the best one(s).
Step 5. End.

In the above procedure, the G2TLWA operator is used to aggregate individual linguistic 2-tuples of each attribute which is based on the same linguistic term set, while the MG2TLWA operator is utilized to integrate group linguistic 2-tuples of attributes which are based on different linguistic term sets.

In some group decision making problems, different decision makers employ various linguistic term sets, but each decision maker chooses the same linguistic term set to provide his/her assessments on the alternatives over all attributes. Assume that the linguistic term set $S^{(k)} = \left\{s_0^{(k)}, s_1^{(k)}, \ldots, s_{g^k}^{(k)}\right\}$ is used to express assessments of the decision maker d^k for $k = 1, 2, \ldots, q$. In this case, the following procedure is developed to solve multi-attribute group decision making problems.

Procedure 2:
Step 1. Employ Equation (3) to obtain 2-tuple linguistic decision matrices expressed as $R^k = \left(r_{ij}^k\right)_{n\times m} = \left(\left(s_{\eta_{kij}}^{(k)}, \gamma_{kij}\right)\right)_{n\times m}$ for $k = 1, 2, \ldots, q$.
Step 2. Plug the weights of decision makers into the MG2TLWA operator to aggregate individual linguistic 2-tuples into a group linguistic 2-tuple, i.e.,

$$\begin{aligned} &MG2TLWA\left(r_{ij}^1, r_{ij}^2, \ldots, r_{ij}^q\right) \\ &= MG2TLWA\left(\left(s_{\eta_{1ij}}^{(1)}, \gamma_{1ij}\right), \left(s_{\eta_{2ij}}^{(2)}, \gamma_{2ij}\right), \ldots, \left(s_{\eta_{qij}}^{(q)}, \gamma_{qij}\right)\right) = \left(s_{\eta_{ij}}^{(*)}, \gamma_{ij}\right) \\ \eta_{ij} &= round\left(g^* \bullet \left(\sum\nolimits_{k=1}^{q} \lambda_k \left(\frac{\eta_{kij}}{g^k} + \gamma_{kij}\right)\right)\right), \ \gamma_{ij} = \left(\sum\nolimits_{k=1}^{q} \lambda_k \left(\frac{\eta_{kij}}{g^k} + \gamma_{kij}\right)\right) - \frac{\eta_{ij}}{g^*} \end{aligned} \tag{19}$$

where $g^* = \max\{g^1, g^2, \ldots, g^q\}$ and $s_{\eta_{ij}}^{(*)} \in S^{(*)} = \left\{s_0^{(*)}, s_1^{(*)}, \ldots, s_{g^*}^{(*)}\right\}$. Then, we obtain a group linguistic 2-tuple decision matrix $R = (r_{ij})_{n\times m} = \left(\left(s_{\eta_{ij}}^{(*)}, \gamma_{ij}\right)\right)_{n\times m}$. It should be noted that the obtained linguistic 2-tuples in R are based on the same linguistic term set.
Step 3. Apply the G2TLWA operator to aggregate the linguistic 2-tuples in the ith row of the elements in the decision matrix R into a comprehensive score denoted by $C_{x_i} = \left(s_{\eta_i}^{(*)}, \gamma_i\right)$ for each alternative x_i $(i = 1, 2, \ldots, n)$, i.e.,

$$\begin{aligned} C_{x_i} &= \left(s_{\eta_i}^{(*)}, \gamma_i\right) = G2TLWA\left(r_{i1}, r_{i2}, \ldots, r_{im}\right) \\ &= G2TLWA\left(\left(s_{\eta_{i1}}^{(*)}, \gamma_{i1}\right), \left(s_{\eta_{i2}}^{(*)}, \gamma_{i2}\right), \ldots, \left(s_{\eta_{im}}^{(*)}, \gamma_{im}\right)\right) \\ \eta_i &= round\left(g^* \bullet \left(\sum\nolimits_{j=1}^{m} w_j \left(\tfrac{\eta_{ij}}{g^*} + \gamma_{ij}\right)\right)\right), \ \gamma_i = \left(\sum\nolimits_{j=1}^{m} w_j \left(\tfrac{\eta_{ij}}{g^*} + \gamma_{ij}\right)\right) - \tfrac{\eta_i}{g^*} \end{aligned} \tag{20}$$

Step 4. Rank the scores $C_{x_i} = \left(s_{\eta_i}^{(*)}, \gamma_i\right)$ $(i = 1, 2, \ldots, n)$ as per the comparison rules of linguistic 2-tuples described in **Theorem 2.1**.
Step 5. End.

In this procedure, the MG2TLWA operator is employed to aggregate individual attribute linguistic 2-tuples, which are denoted by different linguistic term sets, while the G2TLWA operator is used to fuse group linguistic 2-tuples of attributes, which are based on the same linguistic term set.

4. **Numerical Examples.** This section provides two numerical examples to verify our procedures. The first one is an adapted example including a comparative analysis, and the second one is a practical example with respect to the recommendation of undergraduate students for graduate admission.

4.1. **An illustrative example and comparative study.** Consider the following numerical example which has been investigated in [10].

A company wants to invest in a high-tech project in the best of four companies $X = \{x_1, x_2, x_3, x_4\}$. Five attributes: sales (a_1), management (a_2), production (a_3), technology (a_4), financing (a_5), are considered to assess the alternatives. According to Step 3 of the approach in [10], the weights of attributes with reference to different decision makers are separately generated as $W_1 = (0.2155, 0.2205, 0.1103, 0.1647, 0.2890)^T$, $W_2 = (0.1604, 0.2099, 0.2611, 0.2611, 0.1074)^T$ and $W_3 = (0.1983, 0.0720, 0.1752, 0.2273, 0.3272)^T$. To conduct a comparative study with the approach in [10], the attribute weighting vector is determined as $W = (0.1914, 0.1675, 0.1822, 0.2177, 0.2412)^T$, which is generated by the average of W_1, W_2 and W_3.

Suppose three decision makers d^q ($q = 1, 2, 3$) participate in this multiple attribute group decision making process and their importance weight vector is $\lambda' = (0.3219, 0.3185, 0.3596)^T$, which is the normalized form of that given in [10]. Each decision maker d^q uses the following multi-granularity linguistic term set $S^{(q)}$ to furnish his/her assessments on the five attributes for each alternative. The original assessments are given as decision matrices in Tables 1-3.

$$S^{(1)} = \left\{s_0^{(1)} = very\ poor, s_1^{(1)} = poor, s_2^{(1)} = fair, s_3^{(1)} = good, s_4^{(1)} = very\ good\right\}$$

$$S^{(2)} = \left\{s_0^{(2)} = very\ poor, s_1^{(2)} = poor, s_2^{(2)} = slightly\ poor, s_3^{(2)} = fair,\right.$$
$$\left. s_4^{(2)} = slightly\ good, s_5^{(2)} = good, s_6^{(2)} = very\ good\right\}$$

$$S^{(3)} = \left\{s_0^{(3)} = extremly\ poor, s_1^{(3)} = very\ poor, s_2^{(3)} = poor, s_3^{(3)} = slightly\ poor,\right.$$
$$s_4^{(3)} = fair, s_5^{(3)} = slightly\ good, s_6^{(3)} = good, s_7^{(3)} = very\ good,$$
$$\left. s_8^{(3)} = extremely\ good\right\}$$

TABLE 1. The decision matrix given by expert d_1 based on $S^{(1)}$

	a_1	a_2	a_3	a_4	a_5
x_1	$s_2^{(1)}$	$s_3^{(1)}$	$s_2^{(1)}$	$s_3^{(1)}$	$s_4^{(1)}$
x_2	$s_3^{(1)}$	$s_3^{(1)}$	$s_3^{(1)}$	$s_4^{(1)}$	$s_3^{(1)}$
x_3	$s_2^{(1)}$	$s_3^{(1)}$	$s_3^{(1)}$	$s_2^{(1)}$	$s_4^{(1)}$
x_4	$s_4^{(1)}$	$s_3^{(1)}$	$s_3^{(1)}$	$s_3^{(1)}$	$s_2^{(1)}$

TABLE 2. The decision matrix given by expert d_2 based on $S^{(2)}$

	a_1	a_2	a_3	a_4	a_5
x_1	$s_3^{(2)}$	$s_5^{(2)}$	$s_4^{(2)}$	$s_6^{(2)}$	$s_5^{(2)}$
x_2	$s_4^{(2)}$	$s_3^{(2)}$	$s_6^{(2)}$	$s_5^{(2)}$	$s_5^{(2)}$
x_3	$s_4^{(2)}$	$s_5^{(2)}$	$s_3^{(2)}$	$s_3^{(2)}$	$s_6^{(2)}$
x_4	$s_3^{(2)}$	$s_4^{(2)}$	$s_4^{(2)}$	$s_5^{(2)}$	$s_5^{(2)}$

TABLE 3. The decision matrix given by expert d_3 based on $S^{(3)}$

	a_1	a_2	a_3	a_4	a_5
x_1	$s_5^{(3)}$	$s_6^{(3)}$	$s_6^{(3)}$	$s_7^{(3)}$	$s_6^{(3)}$
x_2	$s_4^{(3)}$	$s_5^{(3)}$	$s_6^{(3)}$	$s_4^{(3)}$	$s_7^{(3)}$
x_3	$s_7^{(3)}$	$s_5^{(3)}$	$s_7^{(3)}$	$s_6^{(3)}$	$s_5^{(3)}$
x_4	$s_4^{(3)}$	$s_5^{(3)}$	$s_4^{(3)}$	$s_4^{(3)}$	$s_5^{(3)}$

Obviously, in this practical case, each decision maker uses the same linguistic term set to provide his/her assessments. That is, the linguistic values in a 2-tuple linguistic decision matrix come from the same linguistic term set. Therefore, **Procedure 2** in Section 3 is used and applied to solving this group decision making problem.

As per Equation (3), one can easily convert the above three linguistic matrices into 2-tuple linguistic decision matrices below.

TABLE 4. The 2-tuple linguistic decision matrix transformed from Table 1

	a_1	a_2	a_3	a_4	a_5
x_1	$(s_2^{(1)},0)$	$(s_3^{(1)},0)$	$(s_2^{(1)},0)$	$(s_3^{(1)},0)$	$(s_4^{(1)},0)$
x_2	$(s_3^{(1)},0)$	$(s_3^{(1)},0)$	$(s_3^{(1)},0)$	$(s_4^{(1)},0)$	$(s_3^{(1)},0)$
x_3	$(s_2^{(1)},0)$	$(s_3^{(1)},0)$	$(s_3^{(1)},0)$	$(s_2^{(1)},0)$	$(s_4^{(1)},0)$
x_4	$(s_4^{(1)},0)$	$(s_3^{(1)},0)$	$(s_3^{(1)},0)$	$(s_3^{(1)},0)$	$(s_2^{(1)},0)$

TABLE 5. The 2-tuple linguistic decision matrix transformed from Table 2

	a_1	a_2	a_3	a_4	a_5
x_1	$(s_3^{(2)},0)$	$(s_5^{(2)},0)$	$(s_4^{(2)},0)$	$(s_6^{(2)},0)$	$(s_5^{(2)},0)$
x_2	$(s_4^{(2)},0)$	$(s_3^{(2)},0)$	$(s_6^{(2)},0)$	$(s_5^{(2)},0)$	$(s_5^{(2)},0)$
x_3	$(s_4^{(2)},0)$	$(s_5^{(2)},0)$	$(s_3^{(2)},0)$	$(s_3^{(2)},0)$	$(s_6^{(2)},0)$
x_4	$(s_3^{(2)},0)$	$(s_4^{(2)},0)$	$(s_4^{(2)},0)$	$(s_5^{(2)},0)$	$(s_5^{(2)},0)$

TABLE 6. The 2-tuple linguistic decision matrix transformed from Table 3

	a_1	a_2	a_3	a_4	a_5
x_1	$(s_5^{(3)},0)$	$(s_6^{(3)},0)$	$(s_6^{(3)},0)$	$(s_7^{(3)},0)$	$(s_6^{(3)},0)$
x_2	$(s_4^{(3)},0)$	$(s_5^{(3)},0)$	$(s_6^{(3)},0)$	$(s_4^{(3)},0)$	$(s_7^{(3)},0)$
x_3	$(s_7^{(3)},0)$	$(s_5^{(3)},0)$	$(s_7^{(3)},0)$	$(s_6^{(3)},0)$	$(s_5^{(3)},0)$
x_4	$(s_4^{(3)},0)$	$(s_5^{(3)},0)$	$(s_4^{(3)},0)$	$(s_4^{(3)},0)$	$(s_5^{(3)},0)$

According to Equation (19), apply the MG2TLWA operator to aggregating the 2-tuple linguistic decision matrices, and obtain a collective 2-tuple linguistic decision matrix in Table 7.

It can be seen from Table 7 that the aggregated linguistic 2-tuples are normalized, and $S^{(*)} = S^{(3)}$.

Therefore, employ Equation (20) to synthesize linguistic 2-tuples in each row of Table 7, and we have the comprehensive score of each alternative as follows.

$$C_{x_1} = \left(s_6^{(*)}, 0.0079\right), C_{x_2} = \left(s_6^{(*)}, 0.0241\right), C_{x_3} = \left(s_6^{(*)}, -0.0402\right), C_{x_4} = \left(s_5^{(*)}, 0.0316\right).$$

TABLE 7. The collective 2-tuple linguistic decision matrix

	a_1	a_2	a_3	a_4	a_5
x_1	$(s_4^{(*)}, 0.0433)$	$(s_6^{(*)}, 0.0258)$	$(s_5^{(*)}, 0.0132)$	$(s_7^{(*)}, -0.0042)$	$(s_7^{(*)}, -0.0132)$
x_2	$(s_5^{(*)}, 0.0126)$	$(s_5^{(*)}, 0.0042)$	$(s_7^{(*)}, -0.0475)$	$(s_6^{(*)}, 0.0252)$	$(s_7^{(*)}, -0.0559)$
x_3	$(s_5^{(*)}, 0.0565)$	$(s_6^{(*)}, -0.0174)$	$(s_6^{(*)}, -0.0342)$	$(s_5^{(*)}, -0.0384)$	$(s_7^{(*)}, -0.0048)$
x_4	$(s_5^{(*)}, 0.0469)$	$(s_5^{(*)}, 0.0559)$	$(s_5^{(*)}, 0.0126)$	$(s_6^{(*)}, -0.0607)$	$(s_5^{(*)}, 0.0216)$

Thus, the alternatives is ranked as $x_2 > x_1 > x_3 > x_4$, which means that x_2 should be selected.

It is clear that the linguistic values in Tables 1-3 represent identical information with those in [10]. By utilizing the method in [10], the final ranking is determined as $x_2 > x_3 > x_1 > x_4$, indicating that x_2 is the best alternative, and the rank results obtained by the method in [10] and our proposed approach are the overall consistent. It is noted that the ranking result $x_4 > x_2 > x_3 > x_1$ given in [10] is incorrect because there is a computational error in obtaining the comprehensive value of the alternative x_4. Besides, in some group decision making problems, decision makers provide their assessments of an attribute under the same linguistic term set, and different attributes are needed to be assessed in various multi-granularity linguistic term sets, such as the case illustrated in Section 4.2. In this case, the method in [10] is unable to solve such group decision making problems, but our proposed approach is suitable to address this type of cases.

4.2. **A practical example.** To illustrate how to apply **Procedure 1** in detail, a practical example about the recommendation of undergraduate students for graduate admission is put forward in this section.

By adapting the numerical case presented in [22], in which the linguistic assessments were obtained from one linguistic term set and transformed into interval intuitionistic values, we furnish a justified example where the judgements provided by decision makers are converted into linguistic values according to multi-granularity linguistic term sets.

In China, large quantities of undergraduate students are looking forward to being recommended for a further study, so that they do not have to participate in the post-graduate entrance examination and save their time on the exam preparation and thus decompressing themselves. In this framework, a fraction of these students, who possess excellent quality and potential, would be selected. However, there are not a few outstanding students competing for such a precious opportunity. Thus, the selection of a student who performs better than others both in character and learning becomes a group decision making problem.

For the sake of tractability, assume a committee of four decision makers $D = \{d^1, d^2, d^3, d^4\}$ participate in the proceeding of evaluating three students $X = \{x_1, x_2, x_3\}$ with respect to five attributes: teamwork spirits and skills (a_1), English level (a_2), research potentials (a_3), academic records (a_4) and the ability of pressure resistance (a_5). The weighting vector of the four decision makers is $\lambda = (0.3, 0.25, 0.25, 0.2)^T$. Suppose all decision makers use the same linguistic term set for evaluating students under attributes a_1, a_2, a_4, a_5, i.e., $S^{(1)} = S^{(2)} = S^{(4)} = S^{(5)}$, where $S^{(j)} = \Big\{s_0^{(j)} = \textit{very poor}, s_1^{(j)} = \textit{poor}, s_2^{(j)} = \textit{medium}, s_3^{(j)} = \textit{good}, s_4^{(j)} = \textit{very good}\Big\}$ for $j = 1, 2, 4, 5$, and assess students regarding to a_3 with a linguistic label set $S^{(3)} = \Big\{s_0^{(3)} = \textit{extremely low}, s_1^{(3)} = \textit{very low}, s_2^{(3)} = \textit{low}, s_3^{(3)} = \textit{medium},$

$s_4^{(3)} = high, s_5^{(3)} = very\ high, s_6^{(3)} = extremely\ high\}$. Each decision maker gives his/her judgements on each student and the original assessment results are listed in Tables 8-11.

TABLE 8. The linguistic decision matrix given by d^1

	a_1	a_2	a_3	a_4	a_5
x_1	$s_4^{(1)}$	$s_2^{(2)}$	$s_4^{(3)}$	$s_1^{(4)}$	$s_3^{(5)}$
x_2	$s_3^{(1)}$	$s_3^{(2)}$	$s_5^{(3)}$	$s_3^{(4)}$	$s_2^{(5)}$
x_3	$s_3^{(1)}$	$s_4^{(2)}$	$s_6^{(3)}$	$s_4^{(4)}$	$s_3^{(5)}$

TABLE 9. The linguistic decision matrix given by d^2

	a_1	a_2	a_3	a_4	a_5
x_1	$s_4^{(1)}$	$s_3^{(2)}$	$s_5^{(3)}$	$s_2^{(4)}$	$s_4^{(5)}$
x_2	$s_4^{(1)}$	$s_4^{(2)}$	$s_3^{(3)}$	$s_4^{(4)}$	$s_3^{(5)}$
x_3	$s_2^{(1)}$	$s_2^{(2)}$	$s_4^{(3)}$	$s_3^{(4)}$	$s_1^{(5)}$

TABLE 10. The linguistic decision matrix given by d^3

	a_1	a_2	a_3	a_4	a_5
x_1	$s_4^{(1)}$	$s_1^{(2)}$	$s_5^{(3)}$	$s_0^{(4)}$	$s_2^{(5)}$
x_2	$s_3^{(1)}$	$s_4^{(2)}$	$s_6^{(3)}$	$s_2^{(4)}$	$s_4^{(5)}$
x_3	$s_4^{(1)}$	$s_2^{(2)}$	$s_5^{(3)}$	$s_4^{(4)}$	$s_3^{(5)}$

TABLE 11. The linguistic decision matrix given by d^4

	a_1	a_2	a_3	a_4	a_5
x_1	$s_4^{(1)}$	$s_2^{(2)}$	$s_4^{(3)}$	$s_1^{(4)}$	$s_4^{(5)}$
x_2	$s_4^{(1)}$	$s_3^{(2)}$	$s_4^{(3)}$	$s_3^{(4)}$	$s_3^{(5)}$
x_3	$s_3^{(1)}$	$s_4^{(2)}$	$s_6^{(3)}$	$s_3^{(4)}$	$s_4^{(5)}$

It is noticed that the original judgements with respect to different attributes are in different linguistic term sets. Then by utilizing Equation (3), convert the cells in Tables 8-11 into 2-tuple linguistic decision matrices as shown in Tables 12-15.

As per **Step 2** in **Procedure 1**, employ Equation (17) to synthesize individual linguistic 2-tuples in the above four matrices, and obtain a group linguistic 2-tuple matrix in Table 16.

Let the weighting vector of the attributes be $W = (0.232, 0.191, 0.209, 0.168, 0.200)^T$. Utilize Equation (18) to synthesize the linguistic 2-tuples in each row of the above matrix. Then, we have the comprehensive scores of all alternatives, which are expressed by linguistic 2-tuples $C_{x_i} = \left(s_{\varepsilon_i}^{(*)}, \alpha_i\right)$ $(i = 1, 2, 3)$ as:

$$C_{x_1} = \left(s_4^{(*)}, 0.0292\right), C_{x_2} = \left(s_5^{(*)}, 0.1273\right), C_{x_3} = \left(s_5^{(*)}, 0.1269\right)$$

where $S^{(*)} = S^{(3)}$.

By applying the MG2TLWA operator in this step, it is clear that the linguistic 2-tuples for one alternative are integrated into a collective value expressed by a linguistic

TABLE 12. The 2-tuple linguistic decision matrix for d^1

	a_1	a_2	a_3	a_4	a_5
x_1	$(s_4^{(1)}, 0)$	$(s_2^{(2)}, 0)$	$(s_4^{(3)}, 0)$	$(s_1^{(4)}, 0)$	$(s_3^{(5)}, 0)$
x_2	$(s_3^{(1)}, 0)$	$(s_3^{(2)}, 0)$	$(s_5^{(3)}, 0)$	$(s_3^{(4)}, 0)$	$(s_2^{(5)}, 0)$
x_3	$(s_3^{(1)}, 0)$	$(s_4^{(2)}, 0)$	$(s_6^{(3)}, 0)$	$(s_4^{(4)}, 0)$	$(s_3^{(5)}, 0)$

TABLE 13. The 2-tuple linguistic decision matrix for d^2

	a_1	a_2	a_3	a_4	a_5
x_1	$(s_4^{(1)}, 0)$	$(s_3^{(2)}, 0)$	$(s_5^{(3)}, 0)$	$(s_2^{(4)}, 0)$	$(s_4^{(5)}, 0)$
x_2	$(s_4^{(1)}, 0)$	$(s_4^{(2)}, 0)$	$(s_3^{(3)}, 0)$	$(s_4^{(4)}, 0)$	$(s_3^{(5)}, 0)$
x_3	$(s_2^{(1)}, 0)$	$(s_2^{(2)}, 0)$	$(s_4^{(3)}, 0)$	$(s_3^{(4)}, 0)$	$(s_1^{(5)}, 0)$

TABLE 14. The 2-tuple linguistic decision matrix for d^3

	a_1	a_2	a_3	a_4	a_5
x_1	$(s_4^{(1)}, 0)$	$(s_1^{(2)}, 0)$	$(s_5^{(3)}, 0)$	$(s_0^{(4)}, 0)$	$(s_2^{(5)}, 0)$
x_2	$(s_3^{(1)}, 0)$	$(s_4^{(2)}, 0)$	$(s_6^{(3)}, 0)$	$(s_2^{(4)}, 0)$	$(s_4^{(5)}, 0)$
x_3	$(s_4^{(1)}, 0)$	$(s_2^{(2)}, 0)$	$(s_5^{(3)}, 0)$	$(s_4^{(4)}, 0)$	$(s_3^{(5)}, 0)$

TABLE 15. The 2-tuple linguistic decision matrix for d^4

	a_1	a_2	a_3	a_4	a_5
x_1	$(s_4^{(1)}, 0)$	$(s_2^{(2)}, 0)$	$(s_4^{(3)}, 0)$	$(s_1^{(4)}, 0)$	$(s_4^{(5)}, 0)$
x_2	$(s_4^{(1)}, 0)$	$(s_3^{(2)}, 0)$	$(s_4^{(3)}, 0)$	$(s_3^{(4)}, 0)$	$(s_3^{(5)}, 0)$
x_3	$(s_3^{(1)}, 0)$	$(s_4^{(2)}, 0)$	$(s_6^{(3)}, 0)$	$(s_3^{(4)}, 0)$	$(s_4^{(5)}, 0)$

TABLE 16. The group linguistic 2-tuple matrix

	a_1	a_2	a_3	a_4	a_5
x_1	$(s_4^{(1)}, 0)$	$(s_2^{(2)}, 0.015)$	$(s_4^{(3)}, 0.07)$	$(s_1^{(4)}, 0.015)$	$(s_3^{(5)}, 0.085)$
x_2	$(s_4^{(1)}, -0.12)$	$(s_3^{(2)}, 0.105)$	$(s_4^{(3)}, 0.07)$	$(s_3^{(4)}, 0.015)$	$(s_3^{(5)}, -0.03)$
x_3	$(s_3^{(1)}, -0.015)$	$(s_3^{(2)}, 0.04)$	$(s_5^{(3)}, 0.057)$	$(s_3^{(4)}, 0.12)$	$(s_3^{(5)}, -0.05)$

2-tuple whose linguistic variable is evaluated in the linguistic term set that has the largest granularity. That is to say, during the aggregation process, the linguistic 2-tuples are normalized as well. If one employs the weighted aggregation operator presented in [2] or [15], since they cannot deal with group decision-making problems with multiple linguistic term sets, the aggregation process will be in a mess and a good result would not be obtained.

Evidently, according to the comparison laws of linguistic 2-tuples, the ranking of the comprehensive values is $\left(s_4^{(*)}, 0.0292\right) < \left(s_5^{(*)}, 0.1269\right) < \left(s_5^{(*)}, 0.1273\right)$, which directly indicates that the best choice is x_2.

5. **Conclusions.** In this work, based on some transformation functions between a linguistic 2-tuple and a crisp value, we intuitively provide a 2-tuple linguistic weighted average

operator and a generalized 2-tuple linguistic weighted average operator, which are intrinsically identical to the weighted average aggregation operators in [2] and [15]. In order to handle group decision making problems with multi-granularity linguistic label sets, in which linguistic term sets with different granularities are utilized to evaluate attribute values, we make an extension of the presented generalized 2-tuple linguistic weighted average operator, to simultaneously accomplish the aggregation and normalization process of linguistic 2-tuples assessed in different linguistic term sets. By reasonably combining the presented aggregation operators, two systematical approaches are developed to deal with multiple attribute group decision making problems with 2-tuple linguistic information denoted by different multi-granularity linguistic term sets. While the first procedure is put forward to deal with multi-attribute multi-granularity linguistic group decision-making problems where different linguistic term sets are used under different attributes, the second procedure is presented to handle problems in which different decision makers choose linguistic term sets with different granularities. A numerical example including a comparative study and a practical example are examined to illustrate the validity and advantage of the proposed methods.

In the future, we will concentrate on the aggregation of interval-valued linguistic 2-tuples and consensus measures of group decision making with multi-granularity linguistic term sets.

Acknowledgments. This research is supported by the Zhejiang Provincial Natural Science Foundation of China under Grant LY15G010004 and the National Natural Science Foundation of China under Grant 71271188.

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