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SOME NEW CONSTRUCTIONS OF THE ALMOST DIFFERENCE SET PAIRS BASED ON GENERALIZED CYCLOTOMIC CLASSES

Yan-Guo Jia[1,2], Ying Shen[1,2,*], Xiu-Min Shen[1,2] and Jia-Qi Wang[1,2]

[1]College of Information Science and Engineering
[2]The Key Laboratory for Computer Virtual Technology and System Integration of Hebei Province
Yanshan University
No. 438, West Hebei Ave., Qinhuangdao 066004, P. R. China
jyg@ysu.edu.cn; *Corresponding author: 562413895@qq.com

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Abstract. *In this paper, the computer searches a large number of Almost Difference Set Pairs on two times prime v of residual class ring Z_{2v} by using generalized cyclotomic classes based on cyclotomic classes and Chinese Remainder Theorem. From the instance of the Almost Difference Set Pairs, seven construction methods of Almost Difference Set Pairs are presented. We can obtain more perfect discrete signal by utilizing the constructed Almost Difference Set Pairs.*

Keywords: Generalized cyclotomic classes, Chinese Remainder Theorem, Almost Difference Set pairs, Perfect discrete signal

1. Introduction.

In the fileds of spread spectrum communication, coding theory, applied mathematics and others, the ideal sequence with good autocorrelation properties [1] and high linear complexity [2] is scholars' research highlight.

The ideal optimum signal refers to sequence, which of out-phase cyclic autocorrelation function value is zero and in-phase cyclic autocorrelation function value is not zero [3,4]. Due to the limited number of such ideal sequence, scholars have begun to study multi-value sequence [5,6], sequence pairs [7,8] and other forms of sequence [9]. Difference Set [10] and Difference Set Pairs [11] are effective theory of combinatorial design of studying sequence and sequence pairs. In this paper, our study is about Almost Difference Set Pairs [12] that is an extended form of the Difference Set Pairs; this research can enrich theory of combinatorial designs, and provide more perfect discrete signal.

In the previous related works, some achievements are about making use of generalized cyclotomic classes [2] based on two times prime v of residual class ring to construct the kinds of sequence and sequence pairs and Difference Set Pairs, but no one uses it to construct Almost Difference Set Pairs.

This paper is on the basis of generalized cyclotomic classes based on two times prime v of residual class ring, drawn seven construction methods of Almost Difference Set Pairs. Section 2 gives some existing definitions and lemmas. In Section 3, we acquire three lemmas about generalized cyclotomic classes and construct seven kinds of Almost Difference Set Pairs of order $e \leq 6$ by using generalized cyclotomic classes. Not only do these construction methods enlarge the number of the Almost Difference Set Pairs, but also provide a greater range of options for practical engineering demand.

2. Preliminaries.

Definition 2.1. *Let $Z_v = \{0, 1, \cdots, v-1\}$ be the mod v residual class ring, U, W be two subsets of Z_v, $|U| = k_1$, $|W| = k_2$, $e = |U \cap W|$, if t nonzero element a in Z_v lets the* *equation:* $x - y \equiv a(\mathrm{mod}\, v)$ *have exactly* λ *ways, where* $\lambda < t$, $(x, y) \in (U, W)$, *and other* $v - 1 - t$ *nonzero element has exactly* $\lambda + 1$ *ways, then* (U, W) *is called a* $(v, k_1, k_2, e, \lambda, t)$ *Almost Difference Set Pairs.*

Definition 2.2. *Let* $v = ef + 1$ *be a prime power,* f *be a positive integer,* F_v *be a finite field of order* v*, set* ω *be a primitive element of* F_v, $\varepsilon = \omega^e$*, and let*

$$H_i^e = \left\{\omega^i, \omega^i\varepsilon, \omega^i\varepsilon^2, \cdots, \omega^i\varepsilon^{f-1}\right\}, \quad 0 \le i \le e-1$$

Then H_0^e, H_1^e, $\cdots$, H_{e-1}^e *is called e-order cyclotomic classes.*

Definition 2.3. *Let* $v = ef + 1$ *be a prime power,* H_i *is called cyclotomic classes of order* e, $Z_{2v} = \{0, 1, \cdots, 2v-1\}$ *be the mod* $2v$ *residual class ring, by the Chinese Remainder Theorem,* $Z_{2v} \cong Z_2 \otimes Z_v$*, relatively to isomorphism* $f(\omega) = (\omega_1, \omega_2)$*, where* $\omega_1 = \omega \; mod \; 2$, $\omega_2 = \omega \; mod \; v$*. Let denote* $f^{-1}(\{j\} \times H_i)$ *as* $H_{j,i}$, $j = 0, 1$; $i = 0, \cdots, e-1$*, and then* $H_{j,i}$ *is called e-order generalized cyclotomic classes.*

For example: Let $17 = 4f + 1$, then $H_0 = \{1, 13, 15, 4\}$, $H_1 = \{3, 5, 14, 12\}$, $H_2 = \{9, 15, 8, 2\}$, $H_3 = \{10, 11, 7, 6\}$, so generalized cyclotomic classes are $H_{0,0} = \{18, 30, 16, 4\}$, $H_{1,0} = \{1, 13, 33, 21\}$, $H_{0,1} = \{20, 22, 14, 12\}$, $H_{1,1} = \{3, 5, 31, 29\}$, $H_{0,2} = \{26, 32, 8, 2\}$, $H_{1,2} = \{9, 15, 25, 19\}$, $H_{0,3} = \{10, 28, 24, 6\}$, $H_{1,3} = \{27, 11, 7, 23\}$.

Lemma 2.1. *Let* $g \in H_k^e$*, then the equation* $x + g = y$, $x \in H_i^e$, $y \in H_j^e$*, has exactly* $(i-k, j-k)$ *solution.*

Lemma 2.2. *Some properties of cyclotomic number are as follows:*
(1) $(i', j')_e = (i, j)_e$*, when* $i' \equiv i(mod\ e)$, $j' \equiv j(mod\ e)$
(2) $(i, j)_e = (e - i, j - i)_e$
(3) $(i, j)_e = \begin{cases} (j, i)_e & if\ 2|f \\ (j + e/2, i + e/2)_e & otherwise \end{cases}$
(4) $\sum_{j=0}^{e-1}(i, j)_e = \begin{cases} f - 1 & if\ -1 \in H_i^e \\ f & otherwise \end{cases}$
(5) $\sum_{i=0}^{e-1}(i, j)_e = \begin{cases} f - 1 & if\ j = 0 \\ f & otherwise \end{cases}$

3. **New Construction of Almost Difference Set Pairs.** In this section, we will give three lemmas about generalized cyclotomic classes and prove them. Afterwards we will use the generalized cyclotomic classes to construct seven kinds of Almost Difference Set Pairs.

Lemma 3.1. *Let* $v = ef + 1$ *be a prime number,* $H_{j,i}$ $(j = 0, 1;\ i = 0, \cdots, e-1)$ *be generalized cyclotomic classes of order e in the* Z_{2v}*, then*
(1) *If* f *be an even number,*

$$|(H_{j,i} + g) \cap \{0\}| = \begin{cases} 1 & g \in H_{j,i} \\ 0 & g \notin H_{j,i} \end{cases}$$

(2) *If* f *be an odd number,*

$$|(H_{j,i} + g) \cap \{0\}| = \begin{cases} 1 & g \in H_{j,(i+e/2)mod\ e} \\ 0 & g \notin H_{j,(i+e/2)mod\ e} \end{cases}$$

Proof: When f be an even number, set H_i is cyclotomic classes of order e, $H_i = (h_0, h_1, \cdots, h_{f-1})$, $H_{j,i} = (x_0, x_1, \cdots, x_{f-1})$, $0 \le i \le f/2 - 1$, $s - t = f/2$, by the Chinese Remainder Theorem, we can get the following equation:

$$\begin{cases} x_t \; mod \; 2 = j \\ x_t \; mod \; v = h_t \end{cases}$$

and

$$\begin{cases} x_{f/2+t} \ mod\ 2 = j \\ x_{f/2+t} \ mod\ v = h_{f/2+t} \end{cases}$$

Since $(h_t + h_{f/2+t})\ mod\ v = 0$, then $(x_t + x_{f/2+t})\ mod\ v = 0$ $(*)$.

No matter when j is 0 or 1, we can get $(x_t + x_{f/2+t})\ mod\ 2 = 0$. With Formula $(*)$, we have

$$(x_t + x_{f/2+t})\ mod\ (2v) = 0$$

Therefore, Formula (1) is proved, and similarly we can prove Formula (2).

Lemma 3.2. *Let $v = ef + 1$ be a prime number, $H_{j,i}$ $(j = 0, 1; i = 0, \cdots, e-1)$ be generalized cyclotomic classes of order e in the Z_{2v}, then*
(1) *If f be an even number,*

$$|(H_{j,i} + g) \cap \{v\}| = \begin{cases} 1 & g \in H_{(j+1)mod\ 2,i} \\ 0 & g \notin H_{(j+1)mod\ 2,i} \end{cases}$$

(2) *If f be an odd number,*

$$|(H_{j,i} + g) \cap \{v\}| = \begin{cases} 1 & g \in H_{(j+1)mod\ 2,(i+e/2)mod\ e} \\ 0 & g \notin H_{(j+1)mod\ 2,(i+e/2)mod\ e} \end{cases}$$

Proof: When f be an even number, set H_i be cyclotomic classes of order e, $H_i = (h_0, h_1, \cdots, h_{f-1})$, $H_{j,i} = (x_0, x_1, \cdots, x_{f-1})$, $H_{(j+1)mod\ 2,i} = (y_0, y_1, \cdots, y_{f-1})$, $0 \le i \le f/2 - 1$, $s - t = f/2$, by the Chinese Remainder Theorem we can get the following equation:

$$\begin{cases} x_t \ mod\ 2 = j \\ x_t \ mod\ v = h_t \end{cases}$$

and

$$\begin{cases} y_{f/2+t} \ mod\ 2 = (j+1) mod\ 2 \\ y_{f/2+t} \ mod\ v = h_{f/2+t} \end{cases}$$

Since $(h_t + h_{f/2+t}) mod\ v = 0$, then $(x_t + x_{f/2+t}) mod\ v = 0$ $(**)$.

And as account of $(x_t + x_{f/2+t}) mod\ 2 = 1$, with Formula $(**)$, we have

$$(x_t + y_{f/2+t})\ mod\ (2v) = v$$

Therefore, Formula (1) is proved, and similarly we can prove Formula (2).

Lemma 3.3. *Let $v = ef + 1$ be a prime number, $H_{j,i}$ $(j = 0, 1;\ i = 0, \cdots, e-1)$ be generalized cyclotomic classes of order e in the Z_{2v}, then*

$$|(H_{j,i} + \{v\}) \cap H_{(j+1)mod\ 2,i}| = f$$

Proof: Set H_i to be cyclotomic classes of order e, $H_i = (h_0, h_1, \cdots, h_{f-1})$, $H_{j,i} = (x_0, x_1, \cdots, x_{f-1})$, $H_{(j+1)mod\ 2,i} = (y_0, y_1, \cdots, y_{f-1})$. By the Chinese Remainder Theorem we can get the following equation:

$$\begin{cases} 0(mod\ 2) = x_n \\ h_n(mod\ v) = x_n \end{cases}$$

and

$$\begin{cases} 1(mod\ 2) = y_n \\ h_n(mod\ v) = y_n \end{cases}$$

where $0 < n < f$, down from the above two equations can be obtained.

$$v + x_n = y_n(mod\ 2v)$$

so $|(H_{j,i}+\{v\})\cap H_{(j+1)mod\ 2,i}|=f$.

In the following section, by search through the computer, summarize seven kinds of the construction methods of Almost Difference Set Pairs by using the nature of generalized cyclotomic classes.

Theorem 3.1. *Let $v=ef+1$ be a prime number, $H_{j,i}$ $(j=0,1;\ i=0,\cdots,e-1)$ be generalized cyclotomic classes of order e in the Z_{2v}, the presence of set pair (U,W) constructs Almost Difference Set Pairs $(2ef+2,2ef,nf,nf,nf-1,2nf)$, $n=1,\cdots,e$.*

Proof: Let

$$U=\bigcup_{i=0}^{e-1}(H_{0,i}\cup H_{1,i}),\ W=\bigcup_{x=1}^{n}H_{j_x,i_x}$$

where $j_x=0,1$, $i_x=0,\cdots,e-1$, and $i_1\neq i_2\neq\cdots\neq i_n$.

Obviously, $|U|=2ef$, $|W|=nf$, $|U\cap W|=nf$. For every nonzero element $g\in H_{j,i}$, let

$$\Delta=|(U+g)\cap W|$$

In the following we discussed three cases of nonzero element g.

a) When $g\in H_{0,k}$ $(k=0,1,\cdots,e-1)$,

$$\Delta=\sum_{m=0}^{e-1}(m-k,i_1-k)+\cdots+\sum_{m=0}^{e-1}(m-k,i_n-k)=\begin{cases} nf-1 & k\in\{i_1,\cdots,i_n\}\\ nf & k\notin\{i_1,\cdots,i_n\}\end{cases}$$

b) When $g\in H_{1,k}$ $(k=0,1,\cdots,e-1)$,

$$\Delta=\sum_{m=0}^{e-1}(m-k,i_1-k)+\cdots+\sum_{m=0}^{e-1}(m-k,i_n-k)=\begin{cases} nf-1 & k\in\{i_1,\cdots,i_n\}\\ nf & k\notin\{i_1,\cdots,i_n\}\end{cases}$$

c) When $g\in\{v\}$, $\Delta=nf$

So $\lambda=nf-1$, $\lambda+1=nf$. Therefore, the theorem is proved.

Example 3.1. *When $11=2f+1$, $H_{0,0}=\{12,4,16,20,14\}$, $H_{1,0}=\{1,15,5,9,3\}$, $H_{0,1}=\{2,8,10,18,6\}$, $H_{1,1}=\{15,21,19,11,7\}$, then $(H_{0,0}\cup H_{1,0}\cup H_{0,1}\cup H_{1,1},H_{0,0}\cup H_{0,1})$ constructs an Almost Difference Set Pair $(22,20,10,10,9,20)$.*

Theorem 3.2. *Let $v=ef+1$ be a prime number, $H_{j,i}$ $(j=0,1;\ i=0,\cdots,e-1)$ is generalized cyclotomic classes of order e in the Z_{2v}, the presence of set pair (U,W) constructs Almost Difference Set Pairs $(2ef+2,ef,2nf,nf,nf-1,2nf)$, $n=1,\cdots,e$.*

Proof: (1) Let

$$U=\bigcup_{x=0}^{e-1}H_{j_x,x},\ W=\bigcup_{x=1}^{n}(H_{0,i_x}\cup H_{1,i_x})$$

where $i_x=0,\cdots,e-1$, $j_x=0,1$, and $j_0,j_1,\cdots,j_{e-1}$ cannot be equal at the same time.

Obviously, $|U|=ef$, $|W|=2nf$, $|U\cap W|=nf$. For every nonzero element $g\in H_{j,i}$, let

$$\Delta=|(U+g)\cap W|$$

In the following we discussed three cases of nonzero element g.

a) When $g\in H_{0,k}$ $(k=0,1,\cdots,e-1)$,

$$\Delta=\sum_{m=0}^{e-1}(m-k,i_1-k)+\cdots+\sum_{m=0}^{e-1}(m-k,i_n-k)=\begin{cases} nf-1 & k\in\{i_1,\cdots,i_n\}\\ nf & k\notin\{i_1,\cdots,i_n\}\end{cases}$$

b) When $g\in H_{1,k}$ $(k=0,1,\cdots,e-1)$,

$$\Delta=\sum_{m=0}^{e-1}(m-k,i_1-k)+\cdots+\sum_{m=0}^{e-1}(m-k,i_n-k)=\begin{cases} nf-1 & k\in\{i_1,\cdots,i_n\}\\ nf & k\notin\{i_1,\cdots,i_n\}\end{cases}$$

c) When $g \in \{v\}$, $\Delta = nf$
So $\lambda = nf - 1$, $\lambda + 1 = nf$. Therefore, the theorem is proved.
(2) Let

$$U = \bigcup_{i=0}^{e-1}(H_{j,i} \cup H_{1,i}),\ W = \bigcup_{x=1}^{n}(H_{0,i_x} \cup H_{1,i'_x})$$

where $i, i_x, i'_x = 0, \cdots, e-1$, $j = 0, 1$.

Similarly, this formula can be proved easily.

Example 3.2. *When* $19 = 3f+1$, $H_{0,0} = \{20, 8, 26, 18, 30, 12\}$, $H_{1,0} = \{1, 27, 7, 37, 11, 31\}$, $H_{1,1} = \{2, 16, 14, 36, 22, 24\}$, $H_{0,2} = \{4, 32, 28, 34, 6, 10\}$, *then* $(H_{0,0} \cup H_{1,1} \cup H_{0,2}, H_{0,0} \cup H_{0,1})$ *constructs an Almost Difference Set Pair* $(38, 18, 12, 6, 5, 12)$.

Theorem 3.3. *Let* $v = ef + 1$ *be a prime number,* $H_{j,i}$ $(j = 0, 1;\ i = 0, \cdots, e-1)$ *is generalized cyclotomic classes of order* e *in the* Z_{2v}*, the presence of set pair* (U, W) *constructs Almost Difference Set Pairs* $(2ef + 2, ef + 1, 2nf + 1, nf + 1, nf, 2nf + 1)$, $n = 1, \cdots, e$.

Proof: (1) Let

$$U = \bigcup_{x=0}^{e-1} H_{j_x,x} \cup \{v\},\quad W = \bigcup_{x=1}^{n}(H_{0,i_x} \cup H_{1,i_x}) \cup \{v\}$$

where $i_x = 0, \cdots, e-1$, $j_x = 0, 1$, and $j_0, j_1, \cdots, j_{e-1}$ cannot be equal at the same time.

Obviously, $|U| = ef + 1$, $|W| = 2nf + 1$, $|U \cap W| = nf + 1$. For every nonzero element $g \in H_{j,i}$, let

$$\Delta = |(U + g) \cap W|$$

In the following we discussed three cases of nonzero element g.
a) When $g \in H_{0,k}$ $(k = 0, 1, \cdots, e-1)$,

$$\Delta = \sum_{m=0}^{e-1}(m-k, i_1-k) + \cdots + \sum_{m=0}^{e-1}(m-k, i_n-k) + \theta = \begin{cases} nf - 1 + \theta & k \in \{i_1, \cdots, i_n\} \\ nf + \theta & k \notin \{i_1, \cdots, i_n\} \end{cases}$$

When $f\ mod\ 2 = 0$ and $H_{1,k} \in U$ or $f\ mod\ 2 = 1$ and $H_{1,k+e/2} \in U$,

$$\Delta = \begin{cases} nf + 1 & k \in \{i_1, \cdots, i_n\} \\ nf + 1 & k \notin \{i_1, \cdots, i_n\} \end{cases}$$

When $f\ mod\ 2 = 0$ and $H_{1,k} \notin U$ or $f\ mod\ 2 = 1$ and $H_{1,k+e/2} \notin U$,

$$\Delta = \begin{cases} nf & k \in \{i_1, \cdots, i_n\} \\ nf & k \notin \{i_1, \cdots, i_n\} \end{cases}$$

b) When $g \in H_{1,k}$ $(k = 0, 1, \cdots, e-1)$,
When $f\ mod\ 2 = 0$ and $H_{0,k} \in U$ or $f\ mod\ 2 = 1$ and $H_{0,k+e/2} \in U$

$$\Delta = \begin{cases} nf + 1 & k \in \{i_1, \cdots, i_n\} \\ nf + 1 & k \notin \{i_1, \cdots, i_n\} \end{cases}$$

When $f\ mod\ 2 = 0$ and $H_{0,k} \notin U$ or $f\ mod\ 2 = 1$ and $H_{0,k+e/2} \notin U$

$$\Delta = \begin{cases} nf & k \in \{i_1, \cdots, i_n\} \\ nf & k \notin \{i_1, \cdots, i_n\} \end{cases}$$

c) When $g \in \{v\}$, $\Delta = nf$
So $\lambda = nf - 1$, $\lambda + 1 = nf$. Therefore, the theorem is proved.
(2) Let

$$U = \bigcup_{i=0}^{e-1}(H_{j,i} \cup H_{1,i}),\ W = \bigcup_{x=1}^{n}(H_{0,i_x} \cup H_{1,i'_x})$$

where $i, i_x, i'_x = 0, \cdots, e-1$, $j = 0, 1$.

Similarly, this formula can be proved quickly.

Example 3.3. *When* $11 = 5f + 1$, $H_{0,0} = \{12, 10\}$, $H_{1,0} = \{1, 21\}$, $H_{0,1} = \{2, 20\}$, $H_{1,1} = \{13, 9\}$, $H_{0,2} = \{4, 18\}$, $H_{0,3} = \{8, 14\}$, $H_{0,4} = \{16, 6\}$, *then* $(H_{0,0} \cup H_{0,1} \cup H_{0,2} \cup H_{0,3} \cup H_{0,4} \cup \{v\}, H_{0,0} \cup H_{1,0} \cup H_{0,1} \cup H_{1,1} \cup \{v\})$ *constructs an Almost Difference Set Pair* $(22, 11, 9, 5, 4, 11)$.

Theorem 3.4. *Let* $v = 4f + 1 = 4y^2 + 1$ *be a prime number,* $H_{j,i}$ $(j = 0, 1; i = 0, \cdots, 3)$ *be generalized cyclotomic classes of order* 4 *in the* Z_{2v}, *the presence of set pair* (U, W) *constructs Almost Difference Set Pairs* $(2(t+1), t, t/2, t/4, (t-4)/4, t/2)$, $t = v - 1$.

Proof: (1) Let

$$U = \bigcup_{i=1}^{2} H_{j+1,i} \cup H_{j,0} \cup H_{j,3},\ W = H_{j',i'} \cup H_{j',i'+1}$$

where $i' = 0, 2$; $j, j' = 0, 1$.

Obviously, $|U| = 4f$, $|W| = 2f$, $|U \cap W| = f$. For every nonzero element $g \in H_{j,i}$, let

$$\Delta = |(U + g) \cap W|$$

In the case of $v = 4f + 1 = 4y^2 + x^2 = 4y^2 + 1$, then $x = 1$, and $x \equiv 1(mod\ 4)$, by Lemma 2.2 derived calculation shows, cyclotomic classes of order 4 have five basic cyclotomic number; if f be an even number, they are recorded as $(0, 0)$, $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 2)$; if f be an odd number, they are recorded as $(0, 0)$, $(0, 1)$, $(0, 2)$, $(0, 3)$, $(1, 0)$. When $x \equiv 1(mod\ 4)$, reference [3] provides the formula of these five basic cyclotomic number, as shown in Table 1.

TABLE 1. Formula of cyclotomic classes of order 4

	$f \equiv 0(mod\ 2)$		$f \equiv 1(mod\ 2)$
$16(0,0)$	$v - 11 - 6x$	$16(0,0)$	$v - 7 + 2x$
$16(0,1)$	$v - 3 + 2x + 8y$	$16(0,1)$	$v + 1 + 2x - 8y$
$16(0,2)$	$v - 3 + 2x$	$16(0,2)$	$v + 1 - 6x$
$16(0,3)$	$v - 3 + 2x - 8y$	$16(0,3)$	$v + 1 + 2x + 8y$
$16(1,2)$	$v + 1 - 2x$	$16(1,0)$	$v - 3 - 2x$

When $i' = 1, 2, 3$, corresponding $H_{j,i'}$ is a sample of $H_{j,0}$, and corresponding $H_{j,(i'+2)}$ is a sample of $H_{j,2}$, so we may only prove the case $i' = 0$, $j, j' = 0$.

In the following we discussed three cases of nonzero element g.

a) When $g \in H_{0,k}$ $(k = 0, 1, \cdots, e - 1)$,
$\Delta = (0 - k, 0 - k) + (0 - k, 1 - k) + (3 - k, 0 - k) + (3 - k, 1 - k)$
If f be an even number, then
$\Delta_{0,0} = (0, 0) + (0, 1) + (3, 0) + (3, 1) = (4v - 16 - 4x)/16 = f - 1$
$\Delta_{0,1} = (3, 3) + (3, 0) + (2, 3) + (2, 0) = (4v - 8 + 4x)/16 = f$
$\Delta_{0,2} = (2, 2) + (2, 3) + (1, 2) + (1, 3) = (4v - 4x)/16 = f$
$\Delta_{0,3} = (1, 1) + (1, 2) + (0, 1) + (0, 2) = (4v - 8 + 4x)/16 = f$
If f be an odd number, then
$\Delta_{0,0} = (0, 0) + (0, 1) + (3, 0) + (3, 1) = (4v - 8 + 4x)/16 = f$
$\Delta_{0,1} = (3, 3) + (3, 0) + (2, 3) + (2, 0) = (4v - 16 - 4x)/16 = f - 1$
$\Delta_{0,2} = (2, 2) + (2, 3) + (1, 2) + (1, 3) = (4v - 8 + 4x)/16 = f$
$\Delta_{0,3} = (1, 1) + (1, 2) + (0, 1) + (0, 2) = (4v - 4x)/16 = f$
b) When $g \in H_{1,k}$ $(k = 0, 1, \cdots, e - 1)$,
$\Delta = (1 - k, 0 - k) + (1 - k, 1 - k) + (2 - k, 0 - k) + (2 - k, 1 - k)$
If f be an even number, then
$\Delta_{1,0} = (1, 0) + (1, 1) + (2, 0) + (2, 1) = (4v - 8 + 4x)/16 = f$
$\Delta_{1,1} = (0, 3) + (0, 0) + (1, 3) + (1, 0) = (4v - 16 - 4x)/16 = f - 1$

$\Delta_{1,2} = (3,2) + (3,3) + (0,2) + (0,3) = (4v - 8 + 4x)/16 = f$
$\Delta_{1,3} = (2,1) + (2,2) + (3,1) + (3,2) = (4v - 4x)/16 = f$
If f be an odd number, then
$\Delta_{1,0} = (1,0) + (1,1) + (2,0) + (2,1) = (4v - 16 - 4x)/16 = f - 1$
$\Delta_{1,1} = (0,3) + (0,0) + (1,3) + (1,0) = (4v - 8 + 4x)/16 = f$
$\Delta_{1,2} = (3,2) + (3,3) + (0,2) + (0,3) = (4v - 4x)/16 = f$
$\Delta_{1,3} = (2,1) + (2,2) + (3,1) + (3,2) = (4v - 8 + 4x)/16 = f$
c) When $g \in \{v\}$, $\Delta = f$
So $\lambda = f - 1 = (t-4)/4$, $\lambda + 1 = f = t/4$. Therefore, the theorem is proved.
(2) Let

$$U = \bigcup_{x=i}^{i-1}(H_{0,x} \cup H_{1,x}), W = H_{j,i} \cup H_{j',i+1}$$
$$\text{or} \quad U = \bigcup_{x=i}^{i+1}(H_{0,x} \cup H_{1,x}), W = H_{j,i} \cup H_{j',i-1}$$
$$\text{or} \quad U = \bigcup_{x=0}^{1} H_{j,x} \cup \bigcup_{x=2}^{3} H_{j+1,x}, W = H_{j,i} \cup H_{j,i-1}$$

where $i, i_x, i_x^{'} = 0, \cdots, e-1$, $j = 0, 1$.
Similarly, this formula can be proved easily.

Example 3.4. *When* $37 = 4f + 1 = 4y^2 + 1$, $H_{0,0} = \{38, 16, 34, 26, 46, 70, 10, 12, 44\}$, $H_{0,1} = \{2, 32, 68, 52, 18, 66, 20, 24, 14\}$, $H_{1,2} = \{41, 27, 25, 67, 73, 21, 3, 11, 65\}$, $H_{0,3} = \{8, 54, 60, 72, 42, 6, 22, 56\}$, $H_{1,3} = \{45, 17, 13, 23, 35, 5, 43, 59, 19\}$, *then* $(H_{0,0} \cup H_{0,1} \cup H_{1,2} \cup H_{1,3}, H_{0,0} \cup H_{0,3})$ *constructs an Almost Difference Set Pair* $(74, 36, 18, 9, 8, 18)$.

Theorem 3.5. *Let* $v = 4f + 1 = 4y^2 + 1$ *be a prime number,* $H_{j,i}$ $(j = 0, 1;\ i = 0, \cdots, 3)$ *be generalized cyclotomic classes of order* 4 *in the* Z_{2v}, *the presence of set pair* (U', W') *constructs Almost Difference Set Pairs* $(2(t+1), t+1, (t+2)/2, t/4, t/4, t/2)$, $t = v - 1$.

Proof: According to the U and W of Theorem 3.4, we can let

$$U^{'} = U \cup \{0\},\ W^{'} = W \cup \{v\}$$

Combining Theorem 3.3 and Theorem 3.4, this formula can be proved quickly.

Example 3.5. *When* $37 = 4f + 1 = 4y^2 + 1$, $H_{1,0} = \{1, 53, 71, 63, 9, 33, 47, 49, 7\}$, $H_{1,1} = \{39, 69, 31, 15, 55, 29, 57, 61, 51\}$, $H_{0,2} = \{4, 64, 62, 30, 36, 58, 40, 48, 28\}$, $H_{0,3} = \{8, 54, 60, 72, 42, 6, 22, 56\}$, $H_{1,3} = \{45, 17, 13, 23, 35, 5, 43, 59, 19\}$, *then* $(H_{1,0} \cup H_{1,1} \cup H_{0,2} \cup H_{0,3} \cup \{0\}, H_{1,0} \cup H_{1,3} \cup \{v\})$ *constructs an Almost Difference Set Pair* $(74, 37, 19, 9, 9, 18)$.

Theorem 3.6. *Let* $v = 4f + 1 = 4y^2 + 1$ *be a prime number,* $H_{j,i}$ $(j = 0, 1;\ i = 0, \cdots, 3)$ *be generalized cyclotomic classes of order* 4 *in the* Z_{2v}, *the presence of set pair* (U, W) *constructs Almost Difference Set Pairs* $(2(t+1), t, t, t/2, (t-2)/2, t)$, $t = v - 1$.

Proof: Let

$$U = H_{j,0} \cup H_{j,3} \cup \bigcup_{i=1}^{2} H_{j+1,i},$$
$$W = \bigcup_{i=0}^{1} H_{j',i} \cup \bigcup_{i=2}^{3} H_{j'',i} \quad \text{or} \quad W = \bigcup_{j=0}^{1} H_{j,i} \cup \bigcup_{x=i'}^{i'+1} H_{j',x}$$
$$\text{or} \quad U = \bigcup_{i=0}^{1} H_{j,i} \cup \bigcup_{i=2}^{3} H_{j+1,i},\quad W = \bigcup_{j=0}^{1} H_{j,i} \cup \bigcup_{x=i'}^{3} H_{j',x}$$

where $i, i^{'} = 0, 2$; $j, j^{'}, j^{''} = 0, 1$.
This formula combining Theorem 3.4 can be proved easily.

Example 3.6. *When* $101 = 4f + 1 = 4y^2 + 1$, $H_{0,0} = \{102, 16, 54, 56, 88, 196, 106, 80, 68, 78, 36, 172, 126, 198, 138, 188, 180, 52, 24, 182, 84, 132, 92, 58, 120\}$, $H_{1,0} = \{1, 117, 155, 157, 189, 95, 5, 181, 169, 179, 137, 71, 25, 97, 37, 87, 79, 153, 125, 81, 185, 31, 193, 159, 19\}$, $H_{1,1} = \{7, 103, 133, 11, 75, 89, 111, 59, 35, 55, 173, 41, 151, 93, 175, 73, 57, 3, 149, 61, 67, 163, 83, 15, 139\}$, $H_{0,2} = \{4, 64, 14, 22, 150, 178, 20, 118, 70, 110, 144, 82, 100, 186, 148, 146, 114, 6, 96, 122, 134, 124, 166, 30, 76\}$, $H_{1,2} = \{105, 165, 115, 123, 49, 77, 121, 17, 171, 9, 43, 183, 201, 85, 47, 45, 13,$

$107, 197, 21, 33, 23, 65, 131, 177\}$, *then* $(H_{0,0} \cup H_{0,3} \cup H_{1,1} \cup H_{1,2}, H_{0,0} \cup H_{1,0} \cup H_{0,2} \cup H_{0,3})$ *constructs an Almost Difference Set Pair* $(202, 100, 100, 50, 49, 100)$.

Theorem 3.7. *Let* $v = 4f + 1 = 4y^2 + 1$ *be a prime number,* $H_{j,i}$ $(j = 0, 1;\ i = 0, \cdots, 3)$ *be generalized cyclotomic classes of order* 4 *in the* Z_{2v}*, the presence of set pair* (U, W) *constructs Almost Difference Set Pairs* $(2(t+1), t+1, 3t/2, 3t/4, (3t-4)/4, 3t/2)$, $t = v-1$.

Proof: Let

$$U = H_{j,0} \cup H_{j,3} \cup \bigcup_{i=1}^{2} H_{j+1,i},\ W = \bigcup_{j=0}^{1}(H_{j,i_1} \cup H_{j,i_2}) \cup \bigcup_{x=i_3}^{i_3+1} H_{j',x}$$
$$\text{or } U = \bigcup_{i=0}^{1} H_{j,i} \cup \bigcup_{i=2}^{3} H_{j+1,i},\ W = \bigcup_{x=i-1}^{i} H_{j',x} \cup \bigcup_{j=0}^{1}(H_{j,i+1} \cup H_{j,i+2})$$

where $i, i_1, i_2, i_3 = 0, \cdots, 3;\ j, j' = 0, 1$.

Combining Theorem 3.4, this formula can be proved easily.

Example 3.7. *When* $101 = 4f+1 = 4y^2+1$, $H_{0,0} = \{102, 16, 54, 56, 88, 196, 106, 80, 68, 78, 36, 172, 126, 198, 138, 188, 180, 52, 24, 182, 84, 132, 92, 58, 120\}$, $H_{0,1} = \{2, 32, 108, 112, 176, 190, 10, 160, 136, 156, 72, 142, 50, 194, 74, 174, 158, 104, 48, 162, 168, 62, 184, 116, 38\}$, $H_{1,1} = \{103, 133, 7, 11, 75, 89, 111, 59, 35, 55, 173, 41, 151, 93, 175, 73, 57, 3, 149, 61, 67, 163, 83, 176, 15, 139\}$, $H_{0,2} = \{4, 64, 14, 22, 150, 178, 20, 118, 70, 110, 144, 82, 100, 186, 148, 146, 114, 6, 96, 122, 134, 124, 166, 30, 76\}$, $H_{1,2} = \{105, 165, 115, 123, 49, 77, 121, 17, 171, 9, 43, 183, 201, 85, 47, 45, 13, 107, 197, 21, 33, 23, 65, 131, 177\}$, $H_{0,3} = \{8, 128, 28, 44, 98, 154, 40, 34, 140, 18, 86, 164, 200, 170, 94, 90, 26, 12, 192, 42, 66, 46, 130, 60, 152\}$, $H_{1,3} = \{109, 27, 129, 145, 199, 53, 141, 135, 39, 119, 187, 63, 99, 69, 195, 191, 127, 113, 91, 143, 167, 147, 29, 161, 51\}$, *then* $(H_{0,0} \cup H_{0,1} \cup H_{1,2} \cup H_{1,3}, H_{0,0} \cup H_{0,1} \cup H_{1,1} \cup H_{0,2} \cup H_{1,2} \cup H_{0,3})$ *constructs an Almost Difference Set Pair* $(202, 100, 150, 75, 74, 150)$.

4. Conclusions.

In this paper, seven construction methods of Almost Difference Set Pairs on two times prime v of residual classes ring $2v$ are proposed by using generalized cyclotomic classes based on cyclotomic classes and Chinese Remainder Theorem. The first three theorems of Almost Difference Set Pairs have high universality; the later four theorems only apply to the case of $v = 4f + 1 = 4y^2 + 1$. So through these construction methods we could construct a lot of Almost Difference Set Pairs with different parameters. Further more, according to equivalence relations between Almost Difference Set Pairs and all kinds of perfect discrete signal, we can also construct more and more perfect discrete signal of practical engineering demand.

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