International Journal of Innovative Computing, Information and Control
Volume **12**, Number **2**, April **2016**

ICIC International ©2016 ISSN 1349-4198
pp. **503–518**

PROBABILITY OF FUZZY SET THEORY AND PROBABILITY AMPLITUDE OF QUANTUM NEURONS (SIMILARITIES AND PHYSICAL QUANTITIES OF QUANTUM NEURAL NETWORKS)

Hiroyuki Matsuura

Department of Medical Educational Center
Ryotokuji University
1-1-1, Meikai, Urayasu, Chiba 279-8567, Japan
nobelpensuke@gakushikai.jp; matsuura@ryotokuji.ac.jp

Received August 2015; revised December 2015

Abstract. *We have proposed the method of the neural network based on quantum theory (wave equation and path integrals) of polaritons, and made some relation's tools and descriptions for calculations for arbitrary neural circuits developed. The most important difference between the common (classical) neural network and quantum one existed in whether there were interferences between both systems. The quantum system had essentially many interferences' relationships in its system, and so its probability was related to the probability amplitude, wave functions and propagators, which were commonly complex functions. On the other hand, the classical probability never contained any interferences since it had in the real number field. And concretely we showed how those quantum methods, whose system contained much interference, were applied to the Bayes' theory, entropy of information theory, and the two-step neural network of multi channels. And we found that our quantum neural network and polariton's model were connected with the common quantum information theory, classical neural system and information theory, and quantum network contained many branches of soft science. Moreover, when we attempt to practice that calculation on classical fuzzy probability and quantum amplitude, we immediately find that fuzzy probability is equivalent to Choquet integral. However, we recognize the difference between Choquet integral and path integral. As Choquet integral is always real number, but quantum integral means complex number. Thus, Choquet integral has sometimes divergence of integral values in spite of finite integral value of quantum computation. Thus, we showed that our methods were related to various areas as applications of fuzzy controls, classical neural systems, the classical information theory and so on.*
Keywords: Neural network, Quantum computation, Interference, Path integral, Wave function, Propagator, Bayes' theory, Entropy, Quantum neuron, Fuzzy probability, Choquet integral, Quantum mechanical expectation value

1. **Introduction.** The most famous research on muscular systems is a model of actin-myosin proposed by Hodgkin & Huxley, whose theory is based on mathematical cable theory, ionic current (Na+, K+), local currents and conductions of action potentials [1]. This model won the great success in branch of biological and physiological neuron's model and neural networks. Modern various neuro-physiology is constructed on their theory, whose theory mentions each of neural impulse never interferes with each other, for each neural axon is insulated by its myelin sheath. Arvanitaki, however, proved an existence of ephapse, which is an electrical interference between two axons of neuron [2,3]. His discovery and experiment are thought to be made up a kind of artificial neuron [4,5]. So, we have been studying a model of electromagnetic interferences between each of neural

axon and the network, and finally we proposed new engineering neuron's model based on the polaritons.

They are quantum quasi particles, massive photons and quantized polarization waves, arising on the axon through the physiological process. In the other word, its process is a series of neuron's activities, polarization-depolarization-re-polarization [2]. Then, we have begun to study the theory and tools of description for neural circuits and logical circuits (AND, NOT, OR) by Feynman path integrals [5]. Thus, we could perform to give an expression of path integral for various neural networks and Beyes' system [1,5].

In this paper, differences and similarities between classical system and quantum one have been shown. For example, classical neural network and quantum network, fuzzy probability, Choquet integral and Feynman path integral, common Bayes' theory and quantum style, classical entropy and quantum expression.

2. **Porariton's Equation and Rules of Quantum Neural Conduction.** In previous paper, we showed the equation of polaritons on neural axons, and the polarities are exactly governed by Proca equation, Equation (1), which was relativistic one [18]

$$\begin{aligned} &(\partial_\mu \partial^\mu + m^2) A^\mu = J^\mu \\ &J^\nu(x) \equiv (\rho(\mathbf{x}, t), i(\mathbf{x}, t)) \approx j^\nu_{Na} + j^\nu_K \end{aligned} \tag{1}$$

The symbol m is polariton's mass, and the J^ν means the quaternary vector currents. According to classical neural theory as Hodgkin & Huxley model, the polariton means a quantized polarization wave, which is an impulse from neurons and an action potential. So, the total current j^ν_a, ($a = Na$ or K) is generated by major two ionic currents, which correspond to the sodium current j^ν_{Na} and to the potassium current j^ν_K through neural axon. To derive non-relativistic polariton's equation from relativistic Equation (1), we need return from the wave function A^μ of natural unite to that of MKS unite:

$$A^\mu(\mathbf{x}, t) = \varphi^\mu(\mathbf{x}, t) \cdot \exp\left(-\frac{i}{\hbar} mc^2 t\right) \tag{2}$$

Then, we split the time dependent of A^μ into two terms, then the one containing the rest polariton's mass, m. In the non-relativistic limit, the kinetic energy E_k is so small that we can define it as

$$E_k = E - mc^2, \quad E' << mc^2 \tag{3}$$

non-relativistic kinetic energy E_k means

$$\left| i\frac{\partial \varphi^\mu}{\partial t} \right| \approx E_k \varphi^\mu << mc^2 \varphi^\mu \tag{4-1}$$

$$\frac{\partial A^\mu}{\partial t} \approx -i\frac{mc^2}{\hbar}\varphi^\mu \cdot \exp\left(-\frac{i}{\hbar} mc^2 t\right) \tag{4-2}$$

$$\frac{\partial^2 A^\mu}{\partial t^2} \approx \left[-i\frac{2mc^2}{\hbar}\frac{\partial \varphi^\mu}{\partial t} - i\frac{m^2c^4\varphi^\mu}{\hbar^2}\right] \cdot \exp\left(-\frac{i}{\hbar} mc^2 t\right) \tag{4-3}$$

Inserting this results into following relativistic relation:

$$p^\mu p_\mu A^\nu + m^2c^2 A^\nu = j^\nu / c \tag{5}$$

We finally obtain the 4-components' non-relativistic expressions as Schrödinger equation. The result is non-relativistic polariton's relationship,

$$\begin{aligned} &i\hbar\frac{\partial A^\mu}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\right] A^\mu \\ &A^\mu = (\phi, \mathbf{A}), \quad j^\nu \hbar^2/(2mc) \Leftrightarrow \hat{V} A^\nu \end{aligned} \tag{6}$$

We reach the final polariton's equation with 4-components. The motion of polaritons is described by above 4-components' equations: they are scalar potential $A_0 = \phi$ and vector potential is $\mathbf{A}$. If the quaternary vector potential of electromagnetic field of polaritons are having $\mathbf{A}$ = constant or $\mathbf{A}$ changing much slowly (i.e., stationary magnetic field), then Equation (6) becomes common Schrödinger equation for polariton with only having the scalar potential ϕ,

$$\begin{aligned} i\hbar\frac{\partial\phi(\mathbf{x},t)}{\partial t} &= \left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\right]\phi(\mathbf{x},t) = \hat{H}\phi(\mathbf{x},t) \\ \because \mathbf{B}(\mathbf{x},t) &= rot\, A(\mathbf{x},t) \approx 0, \quad \mathbf{E}(\mathbf{x},t) = -grad\, \phi(\mathbf{x},t) \end{aligned} \tag{7}$$

To simply our problem we discuss the near static magnetic field being accompanied with scalar potential case, whose quaternary solution nearly equals $A^\mu = (\phi, \text{Constant } \mathbf{A})$.

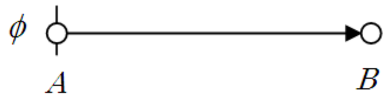
3. **Description of Quantum Neural Network Using Feynman Path Integral.** According to our previous papers, we have developed quantum description tools of polariton's models [17,18]. We would like to write down the useful relationships between kernels and wave functions, whose expressions are based on quantum mechanics and path integral method [13,14,19-22].

(1) The solution ϕ of Equation (7) is written down by using kernel $K(B,A)$ of ϕ for free propagation of polariton ($\phi(\mathbf{A})$ is an initial condition):

$$\boxed{\text{Quaternary potential}}$$

$$i\hbar\frac{\partial\phi}{\partial t} = \hat{H}\phi \Rightarrow \phi(\mathbf{x},t) = \int K(B,A)\phi(A)dA \Rightarrow A^\mu = (\phi(\mathbf{x},t), \text{constant Ac}) \tag{8}$$

$$\because B \equiv (\mathbf{x},t),\ A \equiv (\mathbf{x}_0, t_0)$$



The $K(B,A)$ of free polatriton is represented as

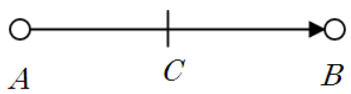
$$K(B,A) = \left[\frac{2\pi i\hbar(t-t_0)}{m}\right]^{-1/2} \exp\left[\frac{im(x-x_0)^2}{2\hbar(t-t_0)}\right] = {}_t\langle B/A\rangle_0 \tag{9}$$

And the position B becomes

$$x(t) = x_0 + \frac{x-x_0}{t-t_0}(t-t_0) \tag{10}$$

(2) If the kernel $K_C(B,A)$ is divided into two parts by a relay's point C, then its kernel,

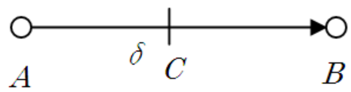
$$K_C(B,A) \equiv {}_t\left\langle B\right|_C A\rangle_0 = \int K(B,C)K(C,A)dC \tag{11-1}$$



is given by Feynman path integral.

If the polariton is diffracted by potentials at point D, then we have a similar relation with using slit width δ:

$$K_C(B,A) \equiv {}_t\left\langle B\right|_C A\rangle_0 = \int_0^\delta K(B,D)K(D,A)dD \tag{11-2}$$



The kernel $K(B,A)$ should be governed with Schrödinger equation:

$$i\hbar\frac{\partial K(B,A)}{\partial t} = \hat{H}K(B,A) \tag{12}$$

(3) When a state vector $|\phi(t)\rangle$ is projected into x-axis of Cartesian coordinate, the wave function $\phi(\mathbf{x}, t)$ has an expression,

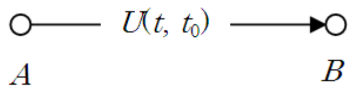
$$\phi(\mathbf{x}, t) \equiv \langle x|\phi(t)\rangle\,, \quad \because |\phi(t)\rangle = U(t, t_0)|\phi(t_0)\rangle \tag{13}$$

(4) When we substitute Equation (13) into Equation (8), an explicit description of unitary operator $U(t, t_0)$ obeys the same Schrödinger equation. The unitary operator,

$$U(t, t_0) = \exp\left(-i\hat{H}(t - t_0)/\hbar\right) \tag{14}$$

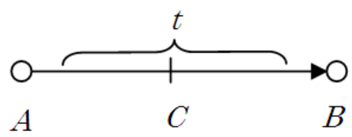
is finally applied for the kernel $K(B, A)$, so the time-development's form of kernel becomes

$$K(B, A) = {}_t\langle B|A\rangle_0 = \left\langle B\left|\hat{U}(t, t_0)\right|A\right\rangle \tag{15}$$



(5) The special case of kernel,

$$K_C(B, A) = {}_t\langle B|_C A\rangle_t = \left\langle B\left|\hat{U}(t, t_C)\hat{U}(t_C, t)\right|A\right\rangle \tag{16}$$

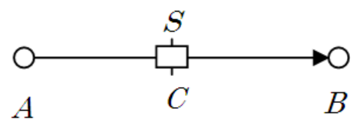


equals this delta function at fixed time t, and we have

$$\int K(B, C)K(C, A)dC = \int {}_t\langle B|C\rangle_{tc}\langle C|A\rangle_t dC = \langle B|A\rangle_t = \delta(\mathbf{x} - \mathbf{x}_0) \tag{17}$$
$$\because \int dX|X\rangle\langle X| = 1$$

(6) If the free polariton is scattered by general potentials V as being observed in atomic structures or by switch function S of electronic circuit at point C, we have a similar scattering representation to the diffraction's Equation (11-2) by using Equation (15):

$$\begin{aligned} &K_C(B, A) \\ &\equiv {}_t\left\langle B\left|\hat{S}_C\right|A\right\rangle_0 = \textstyle\int K(B, C)S(C)K(C, A)dC \\ &= \int \left\langle B\left|\hat{U}(t, t_C)\right|C\right\rangle S(C)\left\langle C\left|\hat{U}(t_C, t_0)\right|A\right\rangle dC \end{aligned} \tag{18}$$



(7) When the scalar potential of polariton is governed by ϕ of that quaternary Schrödinger Equation (8), then a time-development state $|\phi(t)\rangle$ of the formal expression for Equation (8) is

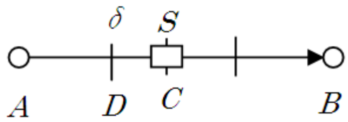
$$|\phi(t)\rangle = e^{-i\hat{H}t/\hbar}|\phi(0)\rangle \tag{19}$$

And completeness of the eigen-state vector $|\Psi_i(t)\rangle$, which is applied for Equation (16), leads us to the kernel expression of proper wave function $\Psi_i(x, t)$.

$$K(B, A) = {}_t\langle B|A\rangle_0 = \sum_j \psi_j(\mathbf{x})\psi_j^*(\mathbf{x}_0)\exp(-iE_j(t - t_0)) \tag{20}$$
$$\because |\phi(t)\rangle = e^{-i\hat{H}t/\hbar}|\phi(0)\rangle, \quad \sum_j |\psi_j(\mathbf{x})\rangle\left\langle\psi_j^*(\mathbf{x}_0)\right| = 1, \quad \langle x|\phi(t)\rangle = \phi(x, t)$$

(8) Both rules of the diffraction at point D and the potential scattering at point C are described by the form of path integral, and then we have the kernel $K_{DC}(B, A)$:

$$\begin{aligned}
&K_{DC}(B, A)\\
&= {}_t\left\langle B \middle| \hat{S}_C \middle|_D A \right\rangle_0\\
&= \int dDdCdEK(B, E)K(E, C)S(C)K(C, D)K(D, A)\\
&= \int_0^\delta \int dCK(B, C)S(C)K(C, A)
\end{aligned} \tag{21}$$



If we use those kernels descriptions, we can transform many classical neural networks into quantum neural ones. For example, we would like to obtain a quantum expression of the network by applying above relations for following classical neural network.

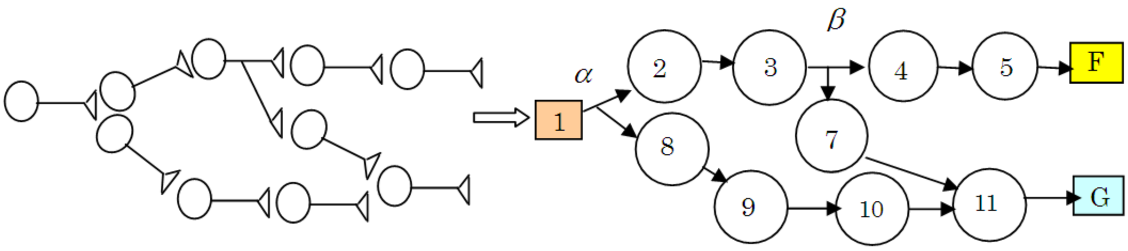


Figure 1. Quantum calculation of neural network

When an action potential, which is quantized polarization vector (polariton in our models), conducts from neuron-1 to neuron-5 (point F) or to neuron-11 (point G), we are able to calculate the state of wave function at the point F or the point G. In the other word, an initial wave function $\Psi(1)$ propagates from the point-1 to the point F or point G, and our methods enable to know the final wave function $\Psi(F)$ or $\Psi(G)$. The $\Psi(F)$ is given as

$$\Psi(F) = \int K(F, 1)\Psi(1)dx_1, \quad K(F, 1) \equiv K(F, x_1) \tag{22}$$

from using Equation (8). And if we can write down the expression of the kernel $K(F, 1)$, the final result of wave function at the point F:

$$\begin{aligned}
&K(F, 1)\\
&= \int dx_5 \cdots dx_1 d\beta d\alpha K(F, 5)S(5)K(5, 4)S(4)K(4, \beta)K(\beta, 3)S(3)K(3, 2)S(2)K(2, \alpha)K(\alpha, 1)
\end{aligned}$$

We apply the same method for the point G, and the wave function $\Psi(G)$ at point G becomes the sum of two different paths, which are both $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 7 \Rightarrow 11 \Rightarrow G$ & $1 \Rightarrow 8 \Rightarrow 9 \Rightarrow 10 \Rightarrow 11 \Rightarrow G$. The one path is shown as

$$\begin{aligned}
&K_A(G, 1)\\
&= \int dx_{11} \cdots dx_1 d\beta d\alpha K(G, 11)S(11)K(11, 7)S(7)K(7, \beta)K(\beta, 3)S(3)K(3, 2)S(2)K(2, \alpha)K(\alpha, 1)
\end{aligned}$$

and another is

$$\begin{aligned}
&K_B(G, 1)\\
&= \int dx_{11} \cdots dx_1 d\alpha K(G, 11)S(11)K(11, 10)S(10)K(10, 9)S(9)K(9, 8)S(8)K(8, \alpha)K(\alpha, 1)
\end{aligned}$$

So, notice that the final wave function $\Psi(G)$ is given as the sum of two paths,

$$K(G,1) = K_A(G,1) + K_B(G,1) \quad \because \Psi(G,1) = \int K(G,1)\Psi(1) \tag{23}$$

Thus, we can rewrite various classical neural networks into the quantum ones by using above formulas, and those expressions are not the static expressions of quantum state but they are dynamic descriptions of the propagations and the time developments of systems, which correspond to polariton conductions and their motions. We would like to mention the state vectors $|\phi(t)\rangle$ or $\langle x|\phi(t)\rangle$ in the following section.

4. Similarities and Differences between Classical and Quantum Neural System.

We would like to show two examples of simple application of the quantum neural systems. One is an example of quantum neural network, which looks like classical neural network's model; the other is probability of fuzzy set theory called the fuzzy probability.

4.1. Quantum expression of neural network.

The classical neural networks are described as famous following relations: if inputs signal X_j $(j = 1, N)$, weighted by W_{Kj}, are added to the K-th neuron, then the changes of activity of membrane potential U_k are commonly expressed as

$$U_K = \sum_j^N W_{Kj} X_j - h_j \tag{24}$$

A classical output, which is controlled signal, Y_K is determined by propagator function $f(\,)$ and the potential U_K. Thus, the Y_K becomes output of the classical networks:

$$Y_K = f(U_K) = \frac{1}{1 + \exp(-aU_K)} \tag{25}$$

On the other hand, if we pay attention to a quantum neural network, its networks can be written by the same manner to classical network, and then the state $\left|A_K^B(t)\right\rangle$ is

$$\left|A_K^B(t)\right\rangle = \sum_j^N C_{Kj}(t)\left|A_j(t)\right\rangle = \sum_j^N \left\{ C_{Kj}(t) \exp\left(\frac{-i\varepsilon_j t}{\hbar}\right)\left|A_j\right\rangle - h_j\left|A_j(0)\right\rangle \right\}, \tag{26}$$
$$\because \left|A_j(0)\right\rangle = const$$

The weight W_{Kj} and signal X_j correspond to the weight C_{Kj} (coefficient) of superposition of the quantum state vector $|A_j(t)\rangle$, and the final state $\left|A_K^B(t)\right\rangle$ is regarded as the classical potential term U_K. The classical output Y_K is determined by propagator function $f(\,)$ and potential U_K. By the same reason, the quantum outputs are given by the following relation

$$\begin{aligned}
\Phi_K = \left\langle x\middle|A_K^B(t)\right\rangle &= \sum_j^N C_{Kj}(t)\left\langle x|A_j(t)\right\rangle - h_j C_{j0} \\
&= \sum_j^N C_{Kj}(t)\exp\left(\frac{-i\varepsilon_j t}{\hbar}\right)\left\langle x|A_j\right\rangle - h_j C_{j0} \\
&= \sum_j^N \left\{ C_{Kj}(t)\exp\left(\frac{-i(\mathbf{px} + \varepsilon_j t)}{\hbar}\right) - h_j C_{j0} \right\}, \quad \because C_{j0} = \langle x|A_j(0)\rangle
\end{aligned} \tag{27}$$

in the projection of the coordinate space. So, we easily find, the classical output Y_k can be replaced by the quantum expression Ψ_K. Thus, we have an equation of

$$\Psi_K = f(\Phi_K) = \frac{1}{1+\exp(-a\Phi_K)} \tag{28}$$

Two expressions of output functions are much similar to each other; however, the quantum outputs truly contain various quantum effects which are essentially different from the classical networks, because the quantum output function Ψ_K allows complex number's functions, and it does not mean the probability but corresponds to the probability amplitude. The other hand, the parameters of classical networks Y_k, U_K and X_j, are quite real numbers since they do not have interferences among others.

4.2. **Formal similarity between fuzzy probability and quantum expectation value.** We would like to refer to an example of a fuzzy probability by taking up a dice. The A is defined as the set of numbers of the dice

$$\text{Set } A\text{: } A(X) = \{1,2,3,4,5,6\} \tag{29}$$

We consider a fuzzy event as an element of set A taking nearly equal to the value 6, which means the fuzzy probability $P_E(\approx 6)$. To calculate the fuzzy probability $P(\approx 6)$, it is necessary to introduce a membership function of the set A. For example, each element of the membership function is given as $A(X)$, $(X = 1, 6)$,

$$A(1) = 0,\ A(2) = 0.1,\ A(3) = 0.3,\ A(4) = 0.6,\ A(5) = 0.9,\ A(6) = 1 \tag{30}$$

Then we can calculate the fuzzy probability by using probability $P(X)$, since we are having the membership function. Thus, the fuzzy probability $P_E(\approx 6)$ is obtained by procedure,

$$P_E(\approx 6) = A(1)P(1) + A(2)P(2) + A(3)P(3) + A(4)P(4) + A(5)P(5) + A(6)P(6) \tag{31}$$

We assume that the dice has an equivalent probability for each value:

$$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$$

So we have final result $P_E(\approx 6) = 0.483$. According to common probability method, the probability, that we can obtain the value 5 or 6 of the dice, has the same expression,

$$A(1) = 0,\ A(2) = 0,\ A(3) = 0,\ A(4) = 0,\ A(5) = 1,\ A(6) = 1 \tag{32}$$

Thus, we have

$$\begin{aligned} P_E(5 \vee 6) &= A(1)P(1) + A(2)P(2) + A(3)P(3) + A(4)P(4) + A(5)P(5) + A(6)P(6) \\ &= 1 \times 1/6 + 1 \times 1/6 = 1/3 \end{aligned} \tag{33}$$

Hitherto based on the above discussion, both probabilities, $P_E(X_J)$ can be written down by using the probability density $P_\rho(X)$ and membership function $F_J(X)$ for $X = X_J$,

$$P_E(\approx X_J) = \int_{all\,X} P_\rho(X) F_J(X) dX \tag{34}$$

In order to expand Equation (34) by regarding sub index J, we consider, a set of membership function F, that of probability density P_ρ. We make an inner product of the elements:

$$F = \{F_1(X), F_2(X), \cdots, F_M(X)\}, \quad P_\rho = \{P_{\rho 1}(X), P_{\rho 2}(X), \cdots, P_{\rho M}(X)\} \tag{35}$$

$$\xrightarrow[product]{} I = \left\{ \int_{all\,X} P_{\rho 1}(X)F_1(X)dX, \int_{all\,X} P_{\rho 2}(X)F_2(X)dX, \cdots, \int_{all\,X} P_{\rho M}(X)F_M(X)dX \right\}$$

So we have an expression of fuzzy probability of two variables, when we regard the indexes of $P_\rho(X)$ and F_J as variable y:

$$P_E(\approx y) = \int P_\rho(X,y)F(X,y)dX \tag{36}$$

That is the fuzzy probability when it takes the value to be about y. Thus, we find that those equations from Equation (34) to Equation (36) show the fuzzy probability, and we notice that the quantum description of the expectation value has mathematically a kind of similarity between each other. As the quantum mechanics mentions, its probability density $P_E(\approx X)$ is defined as $|\Psi|^2$, and it is possible to translate the fuzzy probability into quantum language. Then an expectation is to be, according to quantum mechanics,

$$\langle F_J(\approx X_J)\rangle = \int P_\rho(X)F_J(X)dX = \int \Psi^*(X)F_J(X)\Psi(X)dX \tag{37}$$

Notice that the fuzzy probability Equation (34), by the membership function, has similarity to the expectation value of quantum mechanics. Thus, we can estimate the various physical quantities and the controls of quantum neural networks, since the fuzzy probability is contained in a kind of quantum probability. The fuzzy probability $P_E(\approx X)$ can directly be translated into the expectation value of membership function $\langle F_J(X)\rangle$. And we find the membership function $F_J(X)$ to be equal to a physical observable, which means the operator of physical quantity $F_J(X)$-hat. If the polariton, conducting on axon, has an eigen value E_J and eigen function Ψ_J belonging to Schrödinger Equation (7), then the quantum mechanical expectation of the membership function (strictly speaking, that is a membership operator) is given by

$$\left\langle \hat{F}_J(\approx X_J, P)\right\rangle = \int \psi_J^*(X)\hat{F}_J(X,P)\psi_J(X)dX, \quad \because \hat{F}_J(X,P) \equiv F_J\left(X, -i\hbar\nabla\right) \tag{38}$$

After all, those equations, from Equation (34) to Equation (38), show the similarity of the fuzzy probability and the quantum description of the expectation process.

4.3. **Equivalence of fuzzy probability and Choquet integral [22,23].** We showed that the fuzzy probability $P_E(\approx 6)$ was obtained by Equation (31),

$$P_E(\approx 6) = A(1)P(1) + A(2)P(2) + A(3)P(3) + A(4)P(4) + A(5)P(5) + A(6)P(6) \tag{39}$$

If we consider an independent variable X to have continuity, then Equation (34) is described by an integral form,

$$P_E(\approx 6) = \sum_X P(X)\cdot A(X) \cong \int P(X)\cdot A(X)dX \tag{40}$$

as we wrote in Equation (32). The Choquet integral of this case, the value $A(X)$ means to be fuzzy measure and the probability $P(X)$ corresponds to its counter grade. So we are able to have an expression for the Choquet integral (Figure 2) [42,43].

$$\begin{aligned} P_{Choquet}(\approx 6) \equiv (C)\int_X f d\mu &= (P(1)-P(0))\cdot\sum_{X=1}^{X=6}A(X) + (P(2)-P(1))\cdot\sum_{X=2}^{X=6}A(X) \\ &+ (P(3)-P(2))\cdot\sum_{X=3}^{X=6}A(X) + (P(4)-P(3))\cdot\sum_{X=4}^{X=6}A(X) \\ &+ (P(5)-P(4))\cdot\sum_{X=5}^{X=6}A(X) + (P(6)-P(5))\cdot\sum_{X=6}^{X=6}A(X) \end{aligned} \tag{41}$$

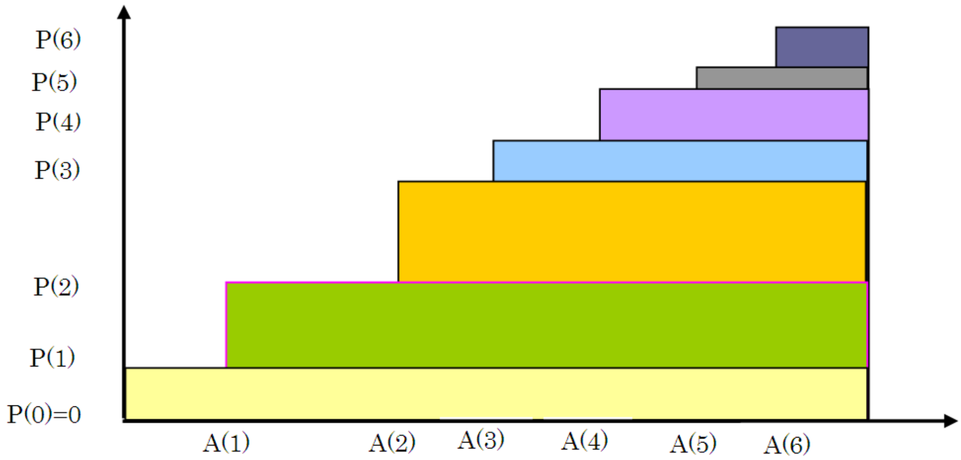


FIGURE 2. Choquet integral of fuzzy set theory

And we define as $P(0) = 0$. Simplifying Equation (41), we immediately notice that the fuzzy probability $P_E(\approx 6)$ of Equation (39) or Equation (40) is equal to the results of Equation (41), which is the definition of Choquet integral in real number's area.

4.4. **Difference between Choquet integral and quantum integral.** Calculating fuzzy probability (Choquet integral or Sugano integral), all functions, $A(X)$, $P(X)$ and its variable X are always real numbers. And we never encounter complex numbers under its calculation process. However, the quantum mechanical expectation is essentially different from those fuzzy integrals except the similarity of formal style. The wave function (probability amplitude) Ψ of Equation (37) generally means complex function. However, its expectation and variable X have to take real values, because the expectation should be observable and X is coordinate of our space. In Equation (37), we assume that the Ψ takes a plane wave $\exp(-ikX)$, and we adopt its complex conjugate wave function $\Psi^* = \exp\{i(k + \Delta k)X\}$ with slight difference of momentum. And if $F_J(X)$ (i.e., $A(X)$) is momentum operator, then the quantum mechanical expectation becomes

$$\begin{aligned} \langle F_J(\approx X_J)\rangle &= \int P_\rho(X)F_J(X)dX \\ &= \int \Psi^*(X)F_J(X)\Psi(X)dX \\ &= \int e^{ikX}\left(-i\hbar\frac{d}{dX}\right)e^{-i(k+\Delta k)X}dX \\ &= -\int_{-\infty}^{\infty} e^{-i\Delta kX}\cdot\hbar(k+\Delta k)dX \\ &= -2\pi\delta(\Delta k)\cdot\hbar(k+\Delta k) \end{aligned} \tag{42}$$

So, $\delta(\Delta k)$ means Dirac delta function. The Δk is nearly to zero, and then $\delta(\Delta k)$ becomes a very sharp function, and we perform an integral for Equation (42) at near to zero. We have the result:

$$\int \langle F_J(\approx X_J)\rangle\, d(\Delta k) = -\int_{-\varepsilon}^{\varepsilon} 2\pi\delta(\Delta k)\cdot\hbar(k+\Delta k)\cdot d(\Delta k) = -2\pi\hbar k \tag{43}$$

It is much important to notice that the result of calculation is not infinite, but it becomes a finite value. In the case of Choquet integral, we can adopt $\Psi = \cos(kX)$, and Ψ^* is $\cos\{(k + \Delta k)X\}$, and moreover, $F_J(X)$ means momentum operator. And we obtain the calculating result of Equation (34):

$$
\begin{aligned}
P_E(\approx X_J) &= \int\limits_{all\,X} P_\rho(X)F_J(X)dX \\
&= \int\limits_{-\infty}^{\infty} \cos(kX) \cdot \left(-i\hbar\frac{d}{dX}\right)\cos(k + \Delta k)XdX \\
&= i\hbar \int\limits_{-\infty}^{\infty} \cos(kX) \cdot (k + \Delta k)\sin\{(k + \Delta k)X\}dX = 0
\end{aligned} \tag{44}
$$

The result of Equation (44) always takes zero value because of orthogonality of trigonometric function. If the above $F_J(X)$ takes real number A, its result becomes divergence and infinite,

$$
P_E(\approx X_J) = \int\limits_{\text{all }X} P_\rho(X)F_J(X)dX = \int\limits_{-\infty}^{\infty} \cos(kX) \cdot (A)\cos\{(k + \Delta k)X\}dX = \infty \tag{45}
$$

if the Δk is much near to zero. And if the Δk is not equal to zero, we always obtain zero momentum, and those results are not significant. Thus, if we adopt probability amplitude Ψ which is complex number, we should naturally be led to quantum mechanical expectation so as to prevent from giving a nonsensical result, instead of Choquet integral or fuzzy probability.

5. **Bayes' Theory and Its Quantum Expression.** We would like to mention the famous classical Bayes' theory and to our style of quantum Bayes' form. As you know, Bayes' statistics, which is often used in an inferential of causality, is said to be subjective probability when the Bayes' method is compared with normal probability theory [6-12]. The classical system is essentially an apparent pathway independently, and it is deterministic method. On the other hand, quantum mechanics is essentially world of interference and superposition, and is described by complex numbers [13,14]. We showed polariton was governed by massive relativistic equation, Proca equation, or non-relativistic quaternary Schrödinger equations [17,18].

5.1. **Concept of quantum Bayes' system.** When we know a final result for an event B, the Bayes' probability is defined as the ratio that an event A_k (where $k = 1$ to N) arises. Then we have the common formula of Bayes':

$$
P_{Cl}(A_K|B) = \frac{P(B|A_K) \cdot P(A_K)}{P(B)}, \quad \because P(B) = \sum_K^n P(B|A_K) \cdot P(A_K) \tag{46}
$$

We are able to regard $P(A_K)$ as a probability of occurrence of event A, and $P(B|A_K)$ means to be a correspondence probability when initial probability is $P(A_K)$. The probability $P(B|A_K)$ represents a condition that an event A_K is propagated to the state B, when the event A_K took place at an occurrence probability $P(A_K)$. So, the symbol $P(B|A_K)$ is regarded as a kind of classical propagator of probability $P(A_K)$, or transitional probability. We are ordinary able to regard Equation (46) as the theory of classical Bayes' theory. And we attempted to expand the propagator's concepts from the classical standpoint into the quantum mechanical one. To expand from the above classical Bayes' theory to the quantum versions, we need a rule that the classical Bayes' theory should be reproduced if expectation values of quantum operators are expressed by eigen values and pure states. The expectation values of quantum Maxwell equations (quantum electrodynamics) should obey the rule of the classical Maxwell equations. Thus, $P(A_K)$ and $P(B|A_K)$ should be regarded as operators of quantum expression, and those eigen functions of both operators should be regarded as complex probability amplitudes. Performing to reinterpret classical relation into quantum one, we would like to show one of the simplest cases of quantum expressions. Notice that the simplest quantum form is given as following forms:

$$\left\langle A_K \left| \hat{P}(A|B) \right| B \right\rangle \equiv \frac{\left\langle B \left| \hat{P}(B|A) \cdot \hat{P}(A) \right| A_K \right\rangle}{\sum\limits_{j}^{n} \left\langle B \left| \hat{P}(B|A) \cdot \hat{P}(A) \right| A_j \right\rangle} \tag{47}$$

The quantum form is similar to classical Bays' theory; however, all probabilities' relations are not c-numbers but q-numbers of operators in quantum Bayes'.

One of initial state vectors is $|A_K\rangle$, and the final state vector is represented as $|B\rangle$. Equation (47) should be more simplified by a relationship between the initial vectors and the final vector (Figure 3). We know, Figure 3 mentions that quantum neural network Figure 3A is similar to natural neural one, Figure 3B. And some quantum neural networks are composed of many axons and many synapses, which cause the quantum interferences.

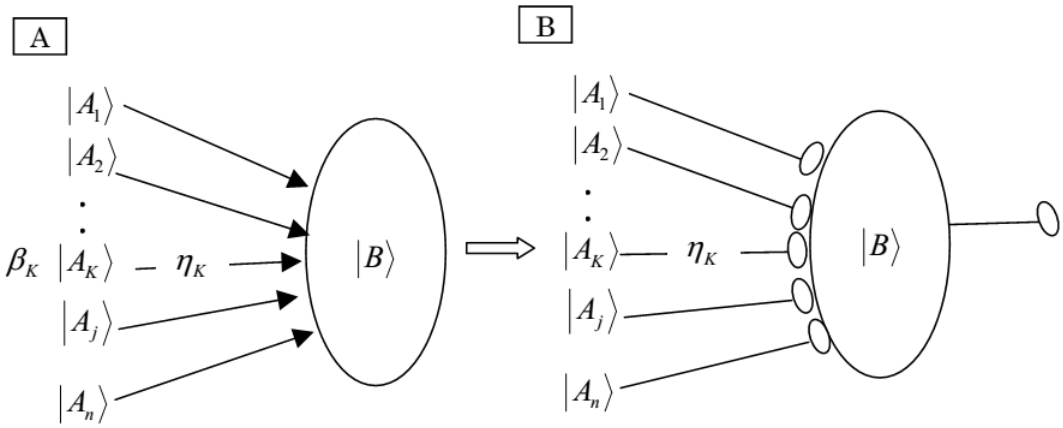


FIGURE 3. Connection type of state vectors and Bayes' form in quantum system (propagators and convergence of neural network)

In order to calculate Equation (47), we would like to introduce some rules that define eigen state vectors having the completeness and orthonormality.

5.2. **Multi classical and quantum channels with errors.** We are discussing quantum channel without noise and its Bayes' form, and herefrom would like to study the channels with multi-dimensional channels with errors in this subsection. Now we have two channels, whose one is classical case and another means quantum system as shown in Figures 4(a) and 4(b).

According to explanation of previous section, the $P(A_s)$ and the $P(B_j|A_s)$ correspond to the occurrence of probability of an event A_s and the propagating probability from the event A_s to the final result B_j. Thus, we know the classical channels of Bayes' form:

$$P_{Cl}(A_K|B_j) = \frac{P(B_j|A_K) \cdot P(A_K)}{P(B)}, \quad \because P(B_j) = \sum_K^n P(B_j|A_K) \cdot P(A_K) \tag{48}$$

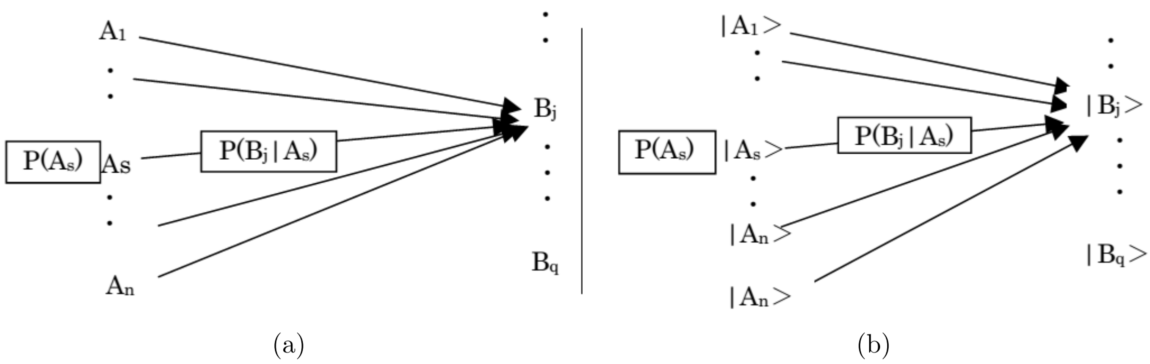


FIGURE 4. (a) Classical multi channels; (b) Quantum multi channels

That representation is a Bayes' probability of multi channels as same as Equation (46). On the other hand, quantum case is acquired by practicing to change those probabilities into the corresponding quantum operators, $P(A_s)$-hat and $P(B_j|A_s)$-hat. On the other hand, the classical event A_s is translated into a state vector $|A_s\rangle$. The simplest multi quantum channels are drawn as following forms of above Figure 4(b), and then we have following relation Equation (49).

$$\left\langle A_K \middle| \hat{P}(A|B) \middle| B_j \right\rangle \equiv \frac{\left\langle B_j \middle| \hat{P}(B|A)\cdot\hat{P}(A) \middle| A_K \right\rangle}{\sum_j^n \left\langle B_j \middle| \hat{P}(B|A)\cdot\hat{P}(A) \middle| A_j \right\rangle} \tag{49}$$

We would like to introduce both classical and quantum expressions of error's propagating probability, $1-P(B_j|A_s)$ and $1-P(B_j|A_s)$-hat, into our Equation (48) or Equation (49). Thus, we define the similar rules to simplify quantum calculations and observations as previous subsection.

1. Base set: the state vectors $|A_s\rangle$, $(s = 1$ to $n)$ make a complete set, and they are in pure state. States vectors $|B_j\rangle$, $(s = 1$ to $q)$ are in not pure states but they belong to the mixed states of all pure $|A_s\rangle$.
2. Orthonormality of base set: the pure state vectors hold on orthonormality.

$$\langle A_j(t_j)|A_K(t_K)\rangle = \delta_{jK}\delta(t_j - t_s) \tag{50}$$

   We introduce the following relation being used in ordinary quantum mechanics: we have completeness between the following bra vectors & ket vectors by Dirac's expression.

$$\sum_j^n |A_j\rangle\langle A_j| = 1 \tag{51}$$

3. An eigen function and eigen state, and propagating operators. The probability of occurrence of state A_s becomes

$$\hat{P}(A)|A_j\rangle = \beta_j|A_j\rangle \tag{52}$$

4. Propagating operators with errors and correct propagation in quantum channels: If the correct probability $P(B_j|A_s)$-hat is in state $|A_s\rangle$, then the error probability's operator is expressed as $1-P(B_j|A_s)$-hat. We have the p-numbers correct channels, and so the rests $(n-p)$ numbers are in wrong. Then the correct and wrong propagating cases are Correct case:

$$\hat{P}(B|A)|A_j\rangle \equiv \hat{\eta}(A)|A_j\rangle = \xi_j\eta_j|A_j\rangle, \quad 1\le j\le p \tag{53-1}$$

Wrong case:

$$\left\{1-\hat{P}(B|A)\right\}|A_j\rangle \equiv \hat{\eta}^w(A)\,|A_j\rangle = \xi_j(1-\eta_j)\,|A_j\rangle \tag{53-2}$$

The propagating operator η-hat commonly conveys probability amplitude of a correct information and ξ means a conduction's rate of propagating processes; however, sometimes it fails to transmit the correct information from $|A_s\rangle$ to $|B_j\rangle$. If we assume that the p channels are in correct states and the other $(n-p)$ channels propagate the signals to be wrong, we can define two cases, which one to be correct and another to correspond to wrong case. Our propagating operator of neuron's model is to have four effects, which mainly contain neural conductions, ephapse among axons, thermal noise, and interferences nearby synaptic junction. And errors are induced by various interference and noise. The correct propagating operators $\eta(A)$-hat is composed of those factors:

$$\begin{aligned}\eta(A) = {}&(\text{neural conduction}) + (\text{ephapse}) + (\text{noise \& attenuation})\\ &+ (\text{synaptic interferences})\end{aligned}$$

5. Each final state $|B_j\rangle$, $(j = 1 \sim q)$ is written down as summing up pure initial states. Thus, the $|B_j\rangle$, $(j = 1 \sim q)$, is mixed and superposed by a lot of pure states $|A_s\rangle$. So, final mixed states enable to be expanded by n-numbers bases of orthonormal pure states.

So we have some final states written down as

$$\text{A final state of } B\text{: } |B_j\rangle = \sum_s^q C_s^j\,|A_s\rangle\,, \quad 1 \le j \le q \tag{54}$$

As we assume that the p channels are in correct and the others $(n-p)$ are in wrong conditions, the numerator of Equation (58) becomes by applying Equations (50)-(57),

$$\begin{aligned}\left\langle B_j\left|\hat{P}(B|A)\cdot\hat{P}(A)\right|A_s\right\rangle &= \left\langle B_j\left|\hat{P}(B|A)\cdot\beta_s\right|A_s\right\rangle\\ &= \beta_s\left\langle B_j\left|\hat{P}(B|A)\right|A_s\right\rangle\\ &= C_s^{*j}\beta_s\xi_s\eta_s\end{aligned} \tag{55}$$

The denominator's Equation (49) is given by the similar way as Equation (55), except an existence of both channels being correct and wrong. We can decide the expression of denominator,

$$\begin{aligned}\sum_s^n\left\langle B^j\left|\hat{P}(B|A)\hat{P}(A)\right|A_s\right\rangle &= \sum_s^n\left\langle B\left|\hat{P}(B|A)\right|A_s\right\rangle\beta_s\\ &= \sum_{s=p+1}^n C_s^{*j}\beta_s\xi_s(1-\eta_s) + \sum_{s=1}^p C_s^{*j}\beta_s\xi_s\eta_s\\ &= \sum_{s=1}^p C_s^{*j}\beta_s\xi_s\eta_s + \sum_{s=1}^{n-p} C_{p+s}^{*j}\beta_{p+s}\xi_{p+s}(1-\eta_{p+s})\end{aligned} \tag{56}$$

and final quantum Bayes's form for state $|B_j\rangle$ becomes

$$\left\langle A_s\left|\hat{P}(A|B)\right|B_j\right\rangle \equiv \frac{C_s^{*j}\beta_s\xi_s\eta_s}{\sum\limits_{s=1}^p C_s^{*j}\beta_s\xi_s\eta_s + \sum\limits_{s=1}^{n-p} C_{p+s}^{*j}\beta_{p+s}\xi_{p+s}(1-\eta_{p+s})} \tag{57}$$

The denominator of Equation (57) contains two kinds of term. So, we notice that the first term represents the correct propagating amplitude, and that the second term is the case of the wrong (an error) propagation or communication. We can find, the result has complex interferences between correct channels (i.e., axons of neurons) and wrong ones, because of taking absolute value of Equation (57). They are two types of interferences: one type belongs to each of correct channel, and the other is in wrong channels. Moreover, we know that a new interference Z_q in Equation (58), emerges in the term of P_Q as shown in Equation (58):

$$P_Q(A_s|B_j) = \frac{\left|C^{*j}_{s=1}\beta_s\xi_s\eta_s\right|}{\left(\left|\sum_{s=1}^{p} C^{*j}_{s=1}\beta_s\xi_s\eta_s\right|^2 + \left|\sum_{s=1}^{n-p} C^{*j}_{p+s}\beta_{p+s}\xi_{p+s}(1-\eta_{p+s})\right|^2 + Z_q\right)^{1/2}} \tag{58}$$

$$Z_q \equiv \left(\sum_{s=1}^{p} C^{*j}_{s=1}\beta_s\xi_s\eta_s\right)\cdot\left(\sum_{s=1}^{n-p} C^{*j}_{p+s}\beta_{p+s}\xi_{p+s}(1-\eta_{p+s})\right)$$

That Z_q says an existence of interferences in between correct channels and wrong channels. So we calculate both an amplitude of entropy for all paths $\sigma_{tA}(B_j|A)$, from A_s $(s = 1, n)$ to B_j, and finally we obtain the total amplitude of entropy for the mixed state for all B_j, $(j = 1, q)$. That is described by the symbol $\sigma(B|A)$. We know the result $\sigma(B_j|A_s)$:

$$\begin{aligned}\sigma_{tA}(B_j|A) &\equiv \sum_{s}^{n}\sigma(B_j|A_s) = \left\langle B_j \left|\hat{P}(B|A)\hat{P}(A)\cdot\log\left(\hat{P}(B|A)\hat{P}(A)\right)\right| A_s\right\rangle \\ &= -\sum_{s=1}^{p} C^{*j}_{s}\beta_s\xi_s\eta_s\cdot\log_2(\beta_s\eta_s) - \sum_{s=1}^{n-p} C^{*j}_{p+s}\beta_{p+s}\xi_{p+s}\bar{\eta}_{p+s}\cdot\log_2\left(\beta_{p+s}\bar{\eta}_{p+s}\right)\end{aligned} \tag{59}$$

$\because\ \bar{\eta}_{p+q} \equiv 1 - \eta_{p+s}$

And then $\sigma(B|A)$ is expressed as

$$\begin{aligned}\sigma(B|A) &\equiv \sum_{j}^{q}\sigma_{tA}(B_j|A) \\ &= -\sum_{j}^{q}\left(\sum_{s=1}^{p} C^{*j}_{s}\beta_s\xi_s\eta_s\cdot\log_2(\beta_s\eta_s)\right) \\ &\quad -\sum_{j}^{q}\left(\sum_{s=1}^{n-p} C^{*j}_{p+s}\beta_{p+s}\xi_{p+s}\bar{\eta}_{p+s}\cdot\log_2\left(\beta_{p+s}\bar{\eta}_{p+s}\right)\right)\end{aligned} \tag{60}$$

(ξ means a conduction's rate of propagating processes). The total entropy $H(B|A)$ from state A to state B is calculated as

$$\begin{aligned}H(B|A) &\equiv -\left[\sigma^*(B|A)\cdot\sigma(B|A)\right]^{1/2} \\ &= -\left|\sum_{j}^{q}\left(\sum_{s=1}^{p} C^{*j}_{s}\beta_s\xi_s\eta_s\cdot\log_2(\beta_s\eta_s)\right)\right. \\ &\quad \left. +\sum_{j}^{q}\left(\sum_{s=1}^{n-p} C^{*j}_{p+s}\beta_{p+s}\xi_{p+s}\bar{\eta}_{p+s}\cdot\log_2\left(\beta_{p+s}\bar{\eta}_{p+s}\right)\right)\right|\end{aligned} \tag{61}$$

From Equation (61), we find not only interferences of correct channels and those of wrong ones, but also a lot of interferences between correct and wrong channels, which are truly quantum effects without being in classical systems. In following section, we would like to discuss an approximate solution's method, being called perturbation.

6. **Summary and Conclusions.** We, at first, showed an expression of motion of polariton based on Proca equation, which we can reduce into Schrödinger equation with only scalar potential ϕ by ignoring vector potential $\mathbf{A}$, if polariton's mass is so large that it cannot move fast on axons. The interference among many neurons was expressed by description of path integral. And its method of path integral is closely related to Feynman kernel, whose expression corresponds to the states of motion and propagation of polariton.

We attempted to compare classical Bayes' theorem with quantum Bayes' form. The quantum Bayes has a kind of operator (q-number) style, and whose expression is related to operator and complex value. However, those counter observable and eigen values are real numbers. On the other hand, the classical Bayes' form takes only real observable and c-number. The essential difference between the quantum neural system and the classical one is whether those concerned systems have some kind of interference or not. Thus, quantum Bayes' form contains much interference between each quantum states vector. However, there is not interference among neural systems based on the classical Bayes' theory. And, we knew that result of the quantum Bayes' form was equivalent to the classical Bayes' theorem if it were not for the interferences and superposition between each quantum state vector; for example, both types of entropy, probabilities, and neural networks.

We knew that there were formal similarities between in the fuzzy probability (equivalent to Choquet integral) and the expectation values of quantum theory. Both fuzzy probability's calculation and the above quantum integral take same formal style except either those numbers are complex or not. That difference is much important, because quantum integral for momentum is convergence (some definite value) and fuzzy probability's calculation runs to infinite (divergence).

The fuzzy set theory and soft computing are thought to be much similar to human sense compared with ordinary physics, mathematics and engineering. And quantum calculation method has a formal commonality with fuzzy probability and Choquet integral, except either complex or real numbers. Human has some kind of fuzziness, and fuzziness exists in human thing and actions. Thus, it is safe to say that our thinking of brain mimics a kind of quantum calculation, if we introduce complex number and variable into ordinary real fuzzy number. So, we have some problems of complex fuzzy theory, which is possible to make up some similar theory to quantum mechanics. What kind of inclusive relations between quantum theory and complex fuzzy theory? The quantum expectation's expressions had the same descriptions with the fuzzy probability or Choquet integral, and so the membership function is regarded as the corresponding potentials of the wave function. Wave function corresponds to complex fuzzy probability function, and we shall have conditions or some equation governing phenomena based on principle of least action (action S) like physics.

Moreover, if our brain has quantum computation process, various characters of human fuzziness can be created by quantum interference between many neural networks.

We will notice that our brain should be a kind of quantum computer, and quantum process be related to generate a part of our feeling.

Acknowledgment. The authors also gratefully acknowledge the helpful comments and useful suggestions of the reviewers and Prof. K. Wasaki, who have improved the presentation of this paper.

REFERENCES

[1] H. Matsuura, Quantum theory of polariton on neuro-conduction and ephapse (quantum current and role of polariton on axon), *ICIC Express Letters*, vol.6, no.6, pp.1671-1675, 2012.

[2] H. Gary and K. Christof, Electrical interactions via the extracelluar potential near cell bodies, *Journal of Computational Neuroscience*, vol.6, pp.169-184, 1999.

[3] N. Garret and R. John, *A Coupled Compartmental Models of Ephaptic Coupling in Avian Nucleus Laminaris and Its Effect on Auditory Coincidence Defection*, http:// files.nyu.edu/gtn206/public/ NeskeRinzel_?2010.pdf.

[4] A. Arvanitaki, Effects evoked in an axon by the activity of a contiguous one, *Journal of Neurophysiol*, vol.5, no.2, pp.89-108, 1942.

[5] H. Matsuura, M. Nakano and L. Wasaki, Quantum circuit, lots, interference and basis of neuro-computations, *ICIC Express Letters*, vol.3, no.1, pp.7-14, 2009.

[6] M. Howard and J. Gerard, *Quantum Measurement and Control*, Cambridge University Press, 2010.

[7] Z. Hradil, R. Myska, J. Perian, M. Zawisky, Y. Hasegawa and H. Rauch, Quantum phase interferometry, *Physical Review Letters*, vol.76, pp.4295-4299, 1996.

[8] S. Benjamin and W. Michael, *Quantum Processes Systems & Information*, Cambridge University Press, 2010.

[9] D. W. Berry and H. M. Wiseman, Optical states and almost optical adaptive measurement for quantum interferometry, *Physical Review Letters*, vol.85, pp.5098-5102, 2000.

[10] D. W. Berry, H. M. Wiseman and J. K. Breslin, Optimal input states and feedback for interferometric phase estimation, *Physical Review A*, vol.63, pp.053804-053810, 2001.

[11] H. Matsuura and K. Wasaki, Expression of quantum Bayes' form and its calculations, *ICIC Express Letters*, vol.6, no.6, pp.1653-1657, 2012.

[12] H. Matsuura, Quantum neural net, polariton, and classical neural networks, *ICIC Express Letters*, vol.6, no.6, pp.1665-1669, 2012.

[13] R. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill Inc., 1965.

[14] R. Feynman, Space-time approach to non-relativistic quantum mechanics, *Reviews of Modern Physics*, vol.20, pp.367-387, 1948.

[15] W. Pedrycz, Fuzzy neural networks and neurocomputations, *Fuzzy Sets and Systems*, vol.56, no.1, pp.1-28, 1993.

[16] M. Gupta and D. Rao, On the principles of fuzzy neural networks, *Fuzzy Sets and Systems*, vol.61, no.1, pp.1-18, 1995.

[17] H. Matsuura and K. Wasaki, Quantum theory of fundamental network (path integral expression circuits and network's quantization), *International Journal of Innovative Computing, Information and Control*, vol.10, no.5, pp.1601-1623, 2014.

[18] H. Matsuura and K. Wasaki, Quantization of artificial neuron (quantum current, model of polariton on axon), *International Journal of Innovative Computing, Information and Control*, vol.10, no.3, pp.1121-1133, 2014.

[19] R. Shankar, *Principles of Quantum Mechanics*, 2nd Edition, Plenum, New York & London, 1994.

[20] P. M. Dirac, The Lagrangian in quantum mechanics, *Physikalische zeitschrift der Sowjetunion*, 1933.

[21] P. A. Dirac, *Lecture on Qunatum Mechanics*, Dover Publications, New York, 2010.

[22] L. M. Brown, *Feynman's Thesis, A New Approch to Quantum Mechanics*, World Scientific Publishing, London, 2010.

[23] G. Choquet, Theory of capacities, *Ann. Inst. Fourier*, vol.5, pp.131-295, 1953.

[24] D. R. Adams., *Choquet Integrals in Potential Theory*, Publications Matematiques, 1998.