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A STUDY ON INTERVAL-VALUED HESITANT FUZZY SET IN RESIDUATED LATTICES

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Abstract. *In order to exactly quantify the decision maker's opinions in the real decision making problems, interval-valued hesitant fuzzy set (IVHFS), to permit an element's membership degree to be a set of several possible interval values, was introduced by Chen, Xu and Xia. In this paper, we continue the study on interval-valued hesitant fuzzy set in a residuated lattice and apply the interval-valued hesitant fuzzy set to the filter theory of a general residuated lattice; the notions of interval-valued hesitant fuzzy (implicative, positive implicative, MV, regular) filters are firstly introduced. Consequently, their properties, and some equivalent characterizations of these interval-valued hesitant fuzzy filters are derived. Finally, the relations among these interval-valued hesitant fuzzy filters are investigated.*

Keywords: Residuated lattice, Interval-valued hesitant fuzzy set, Interval-valued hesitant fuzzy (implicative, positive implicative, MV, regular) filters

1. **Introduction.** The concept of fuzzy set was introduced by Zadeh [17]; consequently, many new methods and theories treating uncertainty have been proposed. Since then, some extensions of this concept have been developed. Recently, Torra [13, 14] introduced the concept of hesitant fuzzy sets (HFSs) for dealing with the situations in which the difficulty of establishing the membership degree is not due to the margin of error (as in intuitionistic fuzzy sets or interval-valued fuzzy sets) or some possibility distribution (as in type-2 fuzzy sets) of the possible values, but due to our hesitation between a few different values. Also, the hesitant fuzzy set (HFS) [13, 14, 15] is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinion of decision makers. However, in many real decision making problems, due to insufficiency in available information, it may be difficult for decision makers to exactly quantify their opinions with a crisp number, but can be represented by an interval number within $[0, 1]$. So, in order to exactly quantify the decision maker's opinions in the real decision making problems, Chen et al. [1, 2] defined the concept of an interval-valued hesitant fuzzy set (IVHFS) to permit an element's membership degree to be a set of several possible interval values, and studied the correlation coefficients of interval-valued hesitant fuzzy sets (IVHFSs) and their applications to clustering analysis. Consequently, many researchers have investigated this topic and have established many useful conclusions. Chen et al. [2] proposed some operational laws for IVHFSs based on the algebraic t-norms

and t-conorms, and developed some interval-valued hesitant fuzzy aggregation operators based on these operational laws. Moreover, Wei et al. [16] proposed some hesitant interval-valued fuzzy aggregation operators and their applications to multiple attribute decision making.

Among many studies on interval-valued hesitant fuzzy set, the vast majority of researches mainly focused on the decision making. However, for the studies of algebras, it is also significant work combining the interval-valued hesitant fuzzy set and corresponding algebraic systems. Especially, for a logic algebraic system, the filter theory and congruence theory are the two important research directions of logic algebraic systems. The filter theory plays a very important role in studying logical algebras [5, 6, 7, 8, 9, 10, 11, 12, 19] and the completeness of the corresponding non-classical logics [4]. From a logical point of view, all kinds of filters are corresponding to various sets of provable formulas. Still now, the filter theory of BL-algebras and residuated lattices has been widely investigated [12, 19], and some of their characterizations and relations were presented. In addition, some new types of filter of BL-algebras and lattice implication algebras are derived. As for lattice implication algebras, BL-algebras, R_0-algebras, MTL-algebras, MV-algebras, etc., all of them are particular types of residuated lattices. Therefore, it is meaningful to establish the fuzzy filter theory of a general residuated lattice for studying the common properties of the above-mentioned logical algebras. Applying the new tool, interval-valued hesitant fuzzy set, to the filter theory of a general residuated lattice, is very significant work in studying the residuated lattice and its filters. Furthermore, this work will play a vital role in corresponding logical system and further lay a solid foundation for (uncertain or automated) reasoning based on corresponding logic systems.

The current paper focuses on studying interval-valued hesitant fuzzy filter theory of a general residuated lattice by applying the interval-valued hesitant fuzzy sets to the filter theory of a general residuated lattice. In Section 2, we mainly list some basic concepts and some properties of residauted lattices, which will be used in other sections. In Section 3, we mainly investigate the interval-valued hesitant fuzzy filter in a residuated lattice, and the equivalent characterizations of interval-valued hesitant fuzzy filter are obtained. In Section 4, we investigate the interval-valued hesitant fuzzy implicative filter in a residuated lattice, the equivalent characterizations of interval-valued hesitant fuzzy implicative filter are got. In Section 5, the interval-valued hesitant fuzzy G-filters in a residuated lattice are investigated. In this section, some equivalent characterizations of interval-valued hesitant fuzzy G-filter are also studied; the relations among interval-valued hesitant fuzzy G-filter, interval-valued hesitant fuzzy positive implicative filter, interval-valued hesitant fuzzy Boolean filters are also got. In Section 6, the interval-valued hesitant fuzzy regular filters in a residuated lattice are investigated. In this section, we mainly study the relations among interval-valued hesitant fuzzy regular filers, interval-valued hesitant fuzzy positive implicative filter, interval-valued hesitant fuzzy G-filter. In Section 7, we conclude this paper.

2. Preliminaries.

Definition 2.1. [3] *A residuated lattice is an algebraic structure* $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ *of type* $(2, 2, 2, 2, 0, 0)$ *satisfying the following axioms.*

(C1) $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice.*

(C2) $(L, \otimes, 1)$ *is a commutative semigroup (with the unit element 1).*

(C3) $(\otimes, \to)$ *is an adjoint pair, i.e., for any* $x, y, z, w \in L$.

(R1) *if* $x \le y$ *and* $z \le w$*, then* $x \otimes z \le y \otimes w$.

(R2) *if* $x \le y$*, then* $y \to z \le x \to z$ *and* $z \to x \le z \to y$.

(R3) *(adjointness condition)* $x \otimes y \le z$ *if and only if* $x \le y \to z$.

In what follows, let $\mathcal{L}$ denote a residuated lattice unless otherwise specified.

Proposition 2.1. [3] *In each residuated lattice $\mathcal{L}$, the following properties hold for all $x, y, z \in L$.*
(P1) $(x \otimes y) \to z = x \to (y \to z)$.
(P2) $x \to (y \to z) = y \to (x \to z)$.
(P3) $0' = 1$, $1' = 0$, $x' = x'''$, $x \le x''$, *where* $x' = x \to 0$.
(P4) $x \vee y \to z = (x \to z) \wedge (y \to z)$.
(P5) $x \otimes x' = 0$.

Definition 2.2. [19]
(1) *A non-empty subset F of a residuated lattice $\mathcal{L}$ is called a* **filter** *of $\mathcal{L}$ if it satisfies*
(F1) $(\forall x, y \in L)$ $(x, y \in F \Rightarrow x \otimes y \in F)$ *and*
(F2) $(\forall x, y \in L)$ $(x \in F,\ x \le y \Rightarrow y \in F)$.
(2) *A non-empty subset F of a residuated lattice $\mathcal{L}$ is called an* **implicative filter** *of $\mathcal{L}$ if it satisfies*
(F3) $1 \in F$ *and*
(F5) $(\forall x, y, z \in L)$ $(z, z \to ((x \to y) \to x) \in F \Rightarrow x \in F)$.
(3) *A non-empty subset F of a residuated lattice $\mathcal{L}$ is called a* **positive implicative filter** *of $\mathcal{L}$ if it satisfies* (F3) *and*
(F6) $(\forall x, y, z \in L)$ $(z \to (x \to y),\ z \to x \in F \Rightarrow z \to y \in F)$.
(4) *A filter F of a residuated lattice $\mathcal{L}$ is called a* **Boolean filter** *if it satisfies the condition*
(F7) $(\forall x \in L)$ $(x \vee x' = 1)$.
(5) *A filter F of a residuated lattice $\mathcal{L}$ is called an* **MV-filter** *if it satisfies the condition*
(F8) $(\forall x, y \in L)$ $(y \to x \in F \Rightarrow ((x \to y) \to y) \to x \in F)$.
(6) *A filter F of a residuated lattice $\mathcal{L}$ is called an* **R-filter** *if it satisfies the condition*
(F9) $(\forall x \in L)$ $(x'' \to x \in F)$.

Proposition 2.2. [19] *A non-empty subset F of a residuated lattice $\mathcal{L}$ is a* **filter** *of $\mathcal{L}$ if it satisfies* (F3) *and* (F4) $(\forall x, y \in L)$ $(x \in F,\ x \to y \in F \Rightarrow y \in F)$.

By an interval $\tilde{a}$ we mean an interval $[a^-, a^+]$, where a^-, a^+ are real numbers and $0 \le a^- \le a^+ \le 1$. The set of all intervals is denoted by $D[0,1]$. The interval $[a, a]$ is identified with the real number a.

For intervals $\tilde{a}_i = \left[a_i^-, a_i^+\right]$, $\tilde{b}_i = \left[b_i^-, b_i^+\right]$, where $i \in I$, I is an index set, we define
$\mathrm{r}\max\left\{\tilde{a}_i, \tilde{b}_i\right\} = \left[\max\left\{a_i^-, b_i^-\right\}, \max\left\{a_i^+, b_i^+\right\}\right]$,
$\mathrm{r}\min\left\{\tilde{a}_i, \tilde{b}_i\right\} = \left[\min\left\{a_i^-, b_i^-\right\}, \min\left\{a_i^+, b_i^+\right\}\right]$,
$\wedge_{i\in I}\tilde{a}_i = \left[\wedge_{i\in I} a_i^-, \wedge_{i\in I} a_i^+\right]$,
$\vee_{i\in I}\tilde{a}_i = \left[\vee_{i\in I} a_i^-, \vee_{i\in I} a_i^+\right]$.
Furthermore, we have
(i) $\tilde{a}_i \le \tilde{b}_i$ if and only if $a_i^- \le b_i^-$ and $a_i^+ \le b_i^+$,
(ii) $\tilde{a}_i = \tilde{b}_i$ if and only if $a_i^- = b_i^-$ and $a_i^+ = b_i^+$,
(iii) $k\tilde{a} = \left[ka_i^-, ka_i^+\right]$, where $0 \le k \le 1$.

Then, it can be shown that $(D[0,1], \le, \wedge, \vee)$ is a complete lattice, $\tilde{0} = [0,0]$ as its least element and $\tilde{1} = [1,1]$ as its greatest element.

By an interval-valued fuzzy set F [18] on X, we mean that set $F = \left\{\left\langle x, \left[\mu_F^-(x), \mu_F^+(x)\right]\right\rangle \mid x \in X\right\}$, where μ_F^- and μ_F^+ are two fuzzy sets of X such that $\mu_F^-(x) \le \mu_F^+(x)$ for any $x \in X$. Putting $\tilde{\mu_F}(x) = [\mu_F^-(x), \mu_F^+(x)]$, we see that $F = \{(x, \tilde{\mu_F}(x)) | x \in X\}$, where $\tilde{\mu_F} : X \to D[0,1]$.

For any $\tilde{t} \in D[0,1]$, the set $U\left(F; \tilde{t}\right) = \left\{x \in X | \tilde{\mu_F}(x) \geq \tilde{t}\right\}$ is called the **interval-valued level subset** of F.

Definition 2.3. [13, 14, 15] *Let X be a reference set. A hesitant fuzzy set F on X is defined in terms of a function $h_F(x)$ that returns a subset of $[0,1]$ when it is applied to X, i.e., $F = \{\langle x, h_F(x) | x \in X \rangle\}$ where $h_F(x)$ is a set of some different values in $[0,1]$, representing the possible membership degrees of the element $x \in X$ to F. $h_F(x)$ is called a hesitant fuzzy element (HFE), a basic unit of HFS.*

Definition 2.4. [2] *Let X be a reference set. An interval-valued hesitant fuzzy set (IVHFS) on X is $\tilde{F} = \{\langle x, h_{\tilde{F}}(x)\rangle | x \in X\}$, where $h_{\tilde{F}}(x)$ denotes all possible interval-valued membership degrees of the element $x \in X$ to the set $\tilde{F}$. For convenience, we call $h_{\tilde{F}}(x)$ an interval-valued hesitant fuzzy element (IVHFE), which reads $h_{\tilde{F}}(x) = \{\tilde{r} | \tilde{r} \in h_{\tilde{F}}(x)\}$, where $\tilde{r}$ is an interval number. An IVHFE is the basic unit of an IVHFS, and it can be considered as a special case of the IVHFS. The relationship between IVHFE and IVHFS is similar to that between interval-valued fuzzy number and interval-valued fuzzy set.*

Remark 2.1. *From Definition 2.4, both hesitant fuzzy sets and interval-valued fuzzy sets are all particular interval-valued hesitant fuzzy sets.*

Definition 2.5. *Let X be a reference set and $\tilde{F}$ be an interval-valued hesitant fuzzy set on X. For any $x, y \in X$,*

(1) $h_{\tilde{F}}(x) \subseteq h_{\tilde{F}}(y)$, if and only if, for any interval $\tilde{r} \in h_{\tilde{F}}(x)$, there exists an interval $\tilde{s} \in h_{\tilde{F}}(y)$ such that $\tilde{r} \leq \tilde{s}$;

(2) $h_{\tilde{F}}(x) \cap h_{\tilde{F}}(y) = \{\mathrm{r}\min\{\tilde{r}, \tilde{s}\} | \tilde{r} \in h_{\tilde{F}}(x), \tilde{s} \in h_{\tilde{F}}(y)\}$.

Example 2.1 will explain the reasonability of Definition 2.5.

Example 2.1. *Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$ with the operations $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and $\otimes, \to$ as Table 1.*

TABLE 1. $\otimes, \to$ in L

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Then $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ is a residuated lattice. For any $x \in L$, we define $h_{\tilde{F}}(x)$ to a set $\tilde{F}$ as follows

$$h_{\tilde{F}}(0) = \left\{\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{2}{3}, \frac{3}{4}\right]\right\}, \quad h_{\tilde{F}}(a) = h_{\tilde{F}}(b) = h_{\tilde{F}}(1) = \left\{\left[\frac{4}{5}, \frac{9}{10}\right]\right\}$$

Then $\tilde{F}$ can be considered as an interval-valued hesitant fuzzy set, i.e.,

$$\tilde{F} = \left\{\left\langle 0, \left\{\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{2}{3}, \frac{3}{4}\right]\right\}\right\rangle, \left\langle a, \left\{\left[\frac{4}{5}, \frac{9}{10}\right]\right\}\right\rangle, \left\langle b, \left\{\left[\frac{4}{5}, \frac{9}{10}\right]\right\}\right\rangle, \left\langle 1, \left\{\left[\frac{4}{5}, \frac{9}{10}\right]\right\}\right\rangle\right\}$$

obviously, $h_{\tilde{F}}(0) \subseteq h_{\tilde{F}}(x)$, $x = a, b, 1$.

$$h_{\tilde{F}}(0) \cap h_{\tilde{F}}(x) = \left\{\mathrm{r}\min\left\{\left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{4}{5}, \frac{9}{10}\right]\right\}, \mathrm{r}\min\left\{\left[\frac{2}{3}, \frac{3}{4}\right], \left[\frac{4}{5}, \frac{9}{10}\right]\right\}\right\}$$

$$= \left\{ \left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{2}{3}, \frac{3}{4}\right] \right\},\ x = a, b, 1.$$

In the rest parts of this paper, we take a residuated lattice $\mathcal{L}$ as the reference set.

3. **Interval-Valued Hesitant Fuzzy Filters.** In this section, we will apply the interval-valued hesitant fuzzy set to the filter theory of a general residuated lattice. The notion of interval-valued hesitant fuzzy filter will be proposed and some equivalent characterizations will be given.

Definition 3.1. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued hesitant fuzzy set of $\mathcal{L}$. $\tilde{F}$ is called an interval-valued hesitant fuzzy filter if it satisfies the following conditions, for any $x, y \in L$,*
(3.1) $h_{\tilde{F}}(1) \supseteq h_{\tilde{F}}(x)$,
(3.2) $h_{\tilde{F}}(y) \supseteq h_{\tilde{F}}(x) \cap h_{\tilde{F}}(x \to y)$.

Example 3.1. *Let $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ be a residuated lattice as in Example 2.1. We define an interval-valued hesitant fuzzy set $\tilde{F}$ of $\mathcal{L}$ by*

$$\tilde{F} = \left\{ \left\langle 0, \left\{ \left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{2}{3}, \frac{3}{4}\right] \right\} \right\rangle, \left\langle a, \left\{ \left[\frac{4}{5}, \frac{9}{10}\right] \right\} \right\rangle, \left\langle b, \left\{ \left[\frac{4}{5}, \frac{9}{10}\right] \right\} \right\rangle, \left\langle 1, \left\{ \left[\frac{4}{5}, \frac{9}{10}\right] \right\} \right\rangle \right\}$$

It is routine to verify that $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$ by Definition 3.1.

Example 3.2. *Let $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ be a residuated lattice as in Example 2.1, and define an interval-valued hesitant fuzzy set $\tilde{F}$ of $\mathcal{L}$ by*

$$\tilde{F} = \left\{ \left\langle 0, \left\{ \left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{1}{2}, \frac{2}{3}\right] \right\} \right\rangle, \left\langle a, \left\{ \left[\frac{7}{10}, \frac{4}{5}\right] \right\} \right\rangle, \left\langle b, \left\{ \left[\frac{7}{10}, \frac{4}{5}\right] \right\} \right\rangle, \left\langle 1, \left\{ \left[\frac{5}{6}, \frac{6}{7}\right] \right\} \right\rangle \right\}$$

However, $\tilde{F}$ is not an interval-valued hesitant fuzzy filter. Because $h_{\tilde{F}}(0) = \left\{ \left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{1}{2}, \frac{2}{3}\right] \right\}$, $h_{\tilde{F}}(a) \cap h_{\tilde{F}}(a \to 0) = h_{\tilde{F}}(a) = \left\{ \left[\frac{7}{10}, \frac{4}{5}\right] \right\} \nsubseteq h_{\tilde{F}}(0)$.

Theorem 3.1. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued hesitant fuzzy set of $\mathcal{L}$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$ if and only if $\tilde{F}$ satisfies: for any $x, y \in L$,*
(3.3) $h_{\tilde{F}}(x \otimes y) \supseteq h_{\tilde{F}}(x) \cap h_{\tilde{F}}(y)$,
(3.4) *$x \leq y$ implies $h_{\tilde{F}}(y) \supseteq h_{\tilde{F}}(x)$.*

Proof: Let $\tilde{F}$ satisfy (3.3) and (3.4). For any $x \in L$, $x \leq 1$. It follows from (3.4) that $h_{\tilde{F}}(1) \supseteq h_{\tilde{F}}(x)$. As $x \leq (x \to y) \to y$, i.e., $x \otimes (x \to y) \leq y$, and so $h_{\tilde{F}}(y) \supseteq h_{\tilde{F}}(x \otimes (x \to y)) \supseteq h_{\tilde{F}}(x) \cap h_{\tilde{F}}(x \to y)$. Hence $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$ by Definition 3.1.

Conversely, let $\tilde{F}$ be an interval-valued hesitant fuzzy filter of $\mathcal{L}$. For any $x, y \in L$, $x \to (y \to (x \otimes y)) = 1$, i.e., $x \leq y \to (x \otimes y)$. Therefore, $h_{\tilde{F}}(x \otimes y) \supseteq h_{\tilde{F}}(y) \cap h_{\tilde{F}}(y \to (x \otimes y)) \supseteq h_{\tilde{F}}(y) \cap (h_{\tilde{F}}(x) \cap h_{\tilde{F}}(x \to (y \to (x \otimes y)))) \supseteq h_{\tilde{F}}(y) \cap (h_{\tilde{F}}(x) \cap h_{\tilde{F}}(1)) \supseteq h_{\tilde{F}}(x) \cap h_{\tilde{F}}(y)$, thus $\tilde{F}$ satisfies (3.2). Let $x, y \in L$ such that $x \leq y$, then $x \to y = 1$. And so $h_{\tilde{F}}(y) \supseteq h_{\tilde{F}}(x) \cap h_{\tilde{F}}(x \to y) = h_{\tilde{F}}(x) \cap h_{\tilde{F}}(1) = h_{\tilde{F}}(x)$. □

Theorem 3.2. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ an interval-valued hesitant fuzzy set of $\mathcal{L}$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy filter if and only if,*
(3.5) *$x \to (y \to z) = 1$ implies $h_{\tilde{F}}(z) \supseteq h_{\tilde{F}}(x) \cap h_{\tilde{F}}(y)$ for any $x, y, z \in L$.*

Proof: Assume $\tilde{F}$ satisfies (3.5), take $x = 1$, then $y \leq z$, and so $h_{\tilde{F}}(z) \supseteq h_{\tilde{F}}(y)$. That is, (3.4) holds. Since $x \to (y \to (x \otimes y)) = 1$, it follows from (3.5) that $h_{\tilde{F}}(x \otimes y) \supseteq h_{\tilde{F}}(x) \cap h_{\tilde{F}}(y)$. Therefore, $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$.

Conversely, let $\tilde{F}$ be an interval-valued hesitant fuzzy filter of $\mathcal{L}$ and $x \to (y \to z) = 1$ for any $x, y, z \in L$. And so $h_{\tilde{F}}(z) \supseteq h_{\tilde{F}}(y) \cap h_{\tilde{F}}(y \to z) \supseteq h_{\tilde{F}}(y) \cap (h_{\tilde{F}}(x) \cap h_{\tilde{F}}(x \to (y \to z))) \supseteq h_{\tilde{F}}(x) \cap h_{\tilde{F}}(y)$. □

Corollary 3.1. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ an interval-valued hesitant fuzzy set of $\mathcal{L}$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy filter if and only if, $x \otimes y \leq z$ implies $h_{\tilde{F}}(z) \supseteq h_{\tilde{F}}(x) \cap h_{\tilde{F}}(y)$ for any $x, y, z \in L$.*

Theorem 3.3. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ an interval-valued hesitant fuzzy set of $\mathcal{L}$. If $\tilde{F}$ is an interval-valued hesitant fuzzy filter if and only if $\tilde{F}$ satisfies following two conditions:*
(3.1) $h_{\tilde{F}}(1) \supseteq h_{\tilde{F}}(x)$ for any $x \in L$,
(3.6) $h_{\tilde{F}}(x \to z) \supseteq h_{\tilde{F}}(x \to y) \cap h_{\tilde{F}}(y \to z)$ for any $x, y, z \in L$.

Proof: Assume that $\tilde{F}$ is an interval-valued hesitant fuzzy filter. Since $(x \to y) \leq (y \to z) \to (x \to z)$, it follows from Theorem 3.1 that $h_{\tilde{F}}((y \to z) \to (x \to z)) \supseteq h_{\tilde{F}}(x \to y)$. We have $h_{\tilde{F}}(x \to z) \supseteq h_{\tilde{F}}(y \to z) \cap h_{\tilde{F}}((y \to z) \to (x \to z))$. It follows that $h_{\tilde{F}}(x \to z) \supseteq h_{\tilde{F}}(x \to y) \cap h_{\tilde{F}}(y \to z)$.

Conversely, if $h_{\tilde{F}}(x \to z) \supseteq h_{\tilde{F}}(x \to y) \cap h_{\tilde{F}}(y \to z)$ for any $x, y, z \in L$, then $h_{\tilde{F}}(1 \to z) \supseteq h_{\tilde{F}}(1 \to y) \cap h_{\tilde{F}}(y \to z)$, that is $h_{\tilde{F}}(z) \supseteq h_{\tilde{F}}(y) \cap h_{\tilde{F}}(y \to z)$. By Definition 3.1, $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$. □

Theorem 3.4. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ an interval-valued hesitant fuzzy set of $\mathcal{L}$. $\tilde{F}$ is an interval-valued hesitant fuzzy filter if and only if it satisfies the following conditions:*
(3.7) $h_{\tilde{F}}(x) \subseteq h_{\tilde{F}}(y \to x)$ for any $x, y \in L$;
(3.8) $h_{\tilde{F}}(a) \cap h_{\tilde{F}}(b) \subseteq h_{\tilde{F}}((a \to (b \to x)) \to x)$ for any $a, b, x \in L$.

Proof: Assume that $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$. As $x \leq y \to x$ for any $x, y \in L$. It follows from (3.4) that $h_{\tilde{F}}(y \to x) \supseteq h_{\tilde{F}}(x)$, i.e., (3.7) holds.

Since $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$, we have $h_{\tilde{F}}((a \to (b \to x)) \to x) \supseteq h_{\tilde{F}}(b \to ((a \to (b \to x)) \to x)) \cap h_{\tilde{F}}(b) \supseteq h_{\tilde{F}}(a) \cap h_{\tilde{F}}(b)$. So (3.8) holds.

Conversely, $\tilde{F}$ is an interval-valued hesitant fuzzy set of $\mathcal{L}$ and satisfies (3.7) and (3.8). It follows from (3.7) that $h_{\tilde{F}}(x) \subseteq h_{\tilde{F}}(1)$. By (3.8), we have $h_{\tilde{F}}(y) = h_{\tilde{F}}(1 \to y) = h_{\tilde{F}}(((x \to y) \to (x \to y)) \to y) \supseteq h_{\tilde{F}}(x \to y) \cap h_{\tilde{F}}(x)$. □

Theorem 3.5. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ an interval-valued hesitant fuzzy set of $\mathcal{L}$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$ if and only if the set*

$$\tilde{F}_{\tilde{\tau}} := \{x \in L | \tilde{\tau} \subseteq h_{\tilde{F}}(x)\}$$

is a filter of $\mathcal{L}$ for any $\tilde{\tau} \in \mathcal{P}(D[0,1])$ with $\tilde{F}_{\tilde{\tau}} \neq \emptyset$.

Proof: Let $\tilde{F}$ be an interval-valued hesitant fuzzy filter of $\mathcal{L}$ and $x, y \in L$, $\tilde{\tau} \in \mathcal{P}(D[0,1])$ be such that $x \in \tilde{F}_{\tilde{\tau}}$ and $x \to y \in \tilde{F}_{\tilde{\tau}}$. We have $\tilde{\tau} \subseteq h_{\tilde{F}}(x)$ and $\tilde{\tau} \subseteq h_{\tilde{F}}(x \to y)$. It follows from (3.1) that $h_{\tilde{F}}(1) \supseteq h_{\tilde{F}}(x) \supseteq \tilde{\tau}$, and so $1 \in \tilde{F}_{\tilde{\tau}}$. Meanwhile, we have $h_{\tilde{F}}(y) \supseteq h_{\tilde{F}}(x \to y) \cap h_{\tilde{F}}(x) \supseteq \tilde{\tau} \cap \tilde{\tau} = \tilde{\tau}$, and so $y \in \tilde{F}_{\tilde{\tau}}$. Therefore, $\tilde{F}_{\tilde{\tau}}$ is a filter of $\mathcal{L}$.

Conversely, assume that $\tilde{F}_{\tilde{\tau}}$ is a filter of $\mathcal{L}$ for any $\tilde{\tau} \in \mathcal{P}(D[0,1])$ with $\tilde{F}_{\tilde{\tau}} \neq \emptyset$. For any $x \in L$, let $h_{\tilde{F}}(x) = \tilde{\gamma}$. Then $x \in \tilde{F}_{\tilde{\gamma}}$ and $\tilde{F}_{\tilde{\gamma}}$ is a filter of $\mathcal{L}$. It follows that $1 \in \tilde{F}_{\tilde{\gamma}}$ and $h_{\tilde{F}}(x) = \tilde{\gamma} \subseteq h_{\tilde{F}}(1)$. For any $x, y \in L$, let $h_{\tilde{F}}(x) = \tilde{\tau}$ and $h_{\tilde{F}}(x \to y) = \tilde{s}$. Obviously,

$x \in \tilde{F}_{\tilde{\tau}\cap\tilde{s}}$ and $x \to y \in \tilde{F}_{\tilde{\tau}\cap\tilde{s}}$ which imply that $y \in \tilde{F}_{\tilde{\tau}\cap\tilde{s}}$. Therefore, $h_{\tilde{F}}(y) \supseteq \tilde{\tau} \cap \tilde{s} = h_{\tilde{F}}(x) \cap h_{\tilde{F}}(x \to y)$. Therefore, $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$. □

Corollary 3.2. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued hesitant fuzzy filter of $\mathcal{L}$. Then the set $\{x \in L | h_{\tilde{F}}(a) \subseteq h_{\tilde{F}}(x)\}$ is a filter of $\mathcal{L}$ for any $a \in L$, where $\emptyset \neq h_{\tilde{F}}(a)$.*

4. **Interval-Valued Hesitant Fuzzy Implicative Filters.**

Definition 4.1. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued hesitant fuzzy set of $\mathcal{L}$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter if $\tilde{F}$ satisfies the following conditions:*

(3.1) *$h_{\tilde{F}}(1) \supseteq h_{\tilde{F}}(x)$ for any $x \in L$,*

(4.1) *$h_{\tilde{F}}(x \to z) \supseteq h_{\tilde{F}}(x \to (z' \to y)) \cap h_{\tilde{F}}(y \to z)$ for any $x, y, z \in L$.*

Example 4.1. *Let $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ be a residuated lattice as in Example 2.1, and define an interval-valued hesitant fuzzy set of $\mathcal{L}$ by*

$$\tilde{F} = \left\{ \left\langle 0, \left\{ \left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{2}{3}, \frac{3}{4}\right] \right\} \right\rangle, \left\langle a, \left\{ \left[\frac{4}{5}, \frac{9}{10}\right] \right\} \right\rangle, \left\langle b, \left\{ \left[\frac{4}{5}, \frac{9}{10}\right] \right\} \right\rangle, \left\langle 1, \left\{ \left[\frac{4}{5}, \frac{9}{10}\right] \right\} \right\rangle \right\}$$

It is routine to verify that $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter of $\mathcal{L}$.

Example 4.2. *Let $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ be a residuated lattice as in Example 2.1, and define an interval-valued hesitant fuzzy set of $\mathcal{L}$ by*

$h_{\tilde{F}}(0) = \left\{\left[\frac{1}{2}, \frac{2}{3}\right]\right\}$, $h_{\tilde{F}}(a) = h_{\tilde{F}}(b) = \left\{\left[\frac{7}{10}, \frac{4}{5}\right]\right\}$,
$h_{\tilde{F}}(1) = \{[0.82, 0.9]\}$.

It is routine to verify that $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$. However, $\tilde{F}$ is not an interval-valued hesitant fuzzy implicative filter. In fact, taking $x = b$, $y = z = a$, $h_{\tilde{F}}(b \to (a' \to a)) \cap h_{\tilde{F}}(a \to a) = h_{\tilde{F}}(1) \cap h_{\tilde{F}}(1) = h_{\tilde{F}}(a) = \{[0.82, 0.9]\} \not\supseteq h_{\tilde{F}}(b \to a) = h_{\tilde{F}}(a) = \left\{\left[\frac{7}{10}, \frac{4}{5}\right]\right\}$.

Theorem 4.1. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ an interval-valued hesitant fuzzy filter of $\mathcal{L}$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter if and only it satisfies $h_{\tilde{F}}(x \to y) \supseteq h_{\tilde{F}}(z \to (x \to (y' \to y))) \cap h_{\tilde{F}}(z)$ for any $x, y, z \in L$.*

Proof: Assume that $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter. Then $h_{\tilde{F}}(x \to y) \supseteq h_{\tilde{F}}(x \to (y' \to y)) \cap h_{\tilde{F}}(y \to y) = h_{\tilde{F}}(x \to (y' \to y)) \cap h_{\tilde{F}}(1) \supseteq h_{\tilde{F}}(x \to (y' \to y))$. Since $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$, we have $h_{\tilde{F}}(x \to (y' \to y)) \supseteq h_{\tilde{F}}(z \to (x \to (y' \to y))) \cap h_{\tilde{F}}(z)$. And so, $h_{\tilde{F}}(x \to y) \supseteq h_{\tilde{F}}(x \to (y' \to y)) \supseteq h_{\tilde{F}}(z \to (x \to (y' \to y))) \cap h_{\tilde{F}}(z)$.

Conversely, assume that $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$ and $h_{\tilde{F}}(x \to y) \supseteq h_{\tilde{F}}(z \to (x \to (y' \to y))) \cap h_{\tilde{F}}(z)$. Since $x \to (z' \to y) = (x \otimes z') \to y$ and $(x \to y) \otimes (y \to z) \leq x \to z$, we have $(x \to (z' \to y)) \otimes (y \to z) = ((x \otimes z') \to y) \otimes (y \to z) \leq x \otimes z' \to z = x \to (z' \to z)$. It follows that $h_{\tilde{F}}(x \to (z' \to z)) \supseteq h_{\tilde{F}}(((x \otimes z') \to y) \otimes (y \to z))$ and $h_{\tilde{F}}(((x \otimes z') \to y) \otimes (y \to z)) \supseteq h_{\tilde{F}}(x \to (z' \to y)) \cap h_{\tilde{F}}(y \to z)$, hence $h_{\tilde{F}}(x \to (z' \to z)) \supseteq h_{\tilde{F}}(x \to (z' \to y)) \cap h_{\tilde{F}}(y \to z)$. By the hypothesis, we have $h_{\tilde{F}}(x \to z) \supseteq h_{\tilde{F}}(1 \to (x \to (z' \to z))) \cap h_{\tilde{F}}(1) \supseteq h_{\tilde{F}}(x \to (z' \to z)) \supseteq h_{\tilde{F}}(x \to (z' \to y)) \cap h_{\tilde{F}}(y \to z)$. Therefore, $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter of $\mathcal{L}$. □

Theorem 4.2. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued hesitant fuzzy set of $\mathcal{L}$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter if and only if $\tilde{F}$ satisfies $h_{\tilde{F}}(x \to y) \supseteq h_{\tilde{F}}(z \to (x \to (x \to y))) \cap h_{\tilde{F}}(z)$ for any $x, y, z \in L$.*

Proof: It is similar to Theorem 4.1. □

Theorem 4.3. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued hesitant fuzzy filter of $\mathcal{L}$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter if and only if $\tilde{F}$ satisfies $h_{\tilde{F}}(x \to z) \supseteq h_{\tilde{F}}(x \to (z' \to z))$ for any $x, y, z \in L$.*

Proof: Assume that $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter of $\mathcal{L}$, we have $h_{\tilde{F}}(x \to z) \supseteq h_{\tilde{F}}(x \to (z' \to y)) \cap h_{\tilde{F}}(y \to z)$. Taking $y = z$, we have $h_{\tilde{F}}(x \to z) \supseteq h_{\tilde{F}}(x \to (z' \to z)) \cap h_{\tilde{F}}(1) \supseteq h_{\tilde{F}}(x \to (z' \to z))$.

Conversely, assume that $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$. Then $h_{\tilde{F}}(x \to (z' \to z)) \supseteq h_{\tilde{F}}(y \to (x \to (z' \to z))) \cap h_{\tilde{F}}(y)$ and so $h_{\tilde{F}}(x \to z) \supseteq h_{\tilde{F}}(x \to (z' \to z)) \supseteq h_{\tilde{F}}(y \to (x \to (z' \to z))) \cap h_{\tilde{F}}(y)$. By Theorem 4.1, $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter of $\mathcal{L}$. □

Theorem 4.4. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued hesitant fuzzy filter of $\mathcal{L}$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter if and only if $h_{\tilde{F}}(x) \supseteq h_{\tilde{F}}(z \to ((x \to y) \to x)) \cap h_{\tilde{F}}(z)$ for any $x, y, z \in L$.*

Proof: This proof is similar to Theorem 4.12 in [19]. □

Definition 4.2. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued hesitant fuzzy filter of $\mathcal{L}$. $\tilde{F}$ is called an interval-valued hesitant fuzzy Boolean filter if $h_{\tilde{F}}(x \vee x') = h_{\tilde{F}}(1)$ for any $x \in L$.*

Theorem 4.5. *Let $\tilde{F}$ be an interval-valued hesitant fuzzy filter of $\mathcal{L}$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy Boolean filter of $\mathcal{L}$ if and only if $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter of $\mathcal{L}$.*

Proof: The proof is similar to proof of Theorem 3.11 in [8]. □

Theorem 4.6. *In a residuated lattice, every interval-valued hesitant fuzzy implicative (Boolean) filter is an interval-valued hesitant fuzzy filter.*

Proof: By Theorem 4.4 and Theorem 4.5, taking $y = 1$ in Theorem 4.4, it follows that $\tilde{F}$ is an interval-valued hesitant fuzzy filter. □

Theorem 4.7. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued hesitant fuzzy filter of $\mathcal{L}$. Then the following conditions are equivalent:*

(1) $\tilde{F}$ is an interval-valued hesitant fuzzy Boolean filter;
(2) $h_{\tilde{F}}(x) = h_{\tilde{F}}(x' \to x)$ for any $x \in L$;
(3) $h_{\tilde{F}}(x) = h_{\tilde{F}}((x \to y) \to x)$ for any $x, y \in L$.

Proof: This proof is similar to that of Theorem 3.8 in [10]. □

5. Interval-Valued Hesitant Fuzzy Positive Implicative (G-) Filter.

Definition 5.1. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued hesitant fuzzy filter of $\mathcal{L}$. Then $\tilde{F}$ is called an interval-valued hesitant fuzzy G-filter if it satisfies*

(5.1) $h_{\tilde{F}}(x \to (x \to y)) \subseteq h_{\tilde{F}}(x \to y)$ *for any* $x, y \in L$.

Remark 5.1. *Obviously, in Definition 5.1, the condition (5.1) could equivalently be replaced by the following condition: $h_{\tilde{F}}(x \to (x \to y)) = h_{\tilde{F}}(x \to y)$.*

Definition 5.2. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ an interval-valued hesitant fuzzy set of $\mathcal{L}$. Then $\tilde{F}$ is called an interval-valued hesitant fuzzy positive implicative filter if it satisfies (3.1) and*

(5.2) $h_{\tilde{F}}(x \to z) \supseteq h_{\tilde{F}}(x \to (y \to z)) \cap h_{\tilde{F}}(x \to y)$ *for any* $x, y, z \in L$.

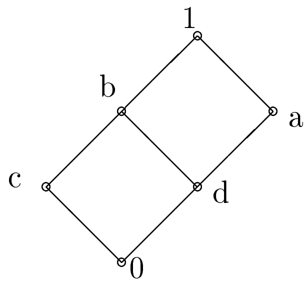

FIGURE 1. Hasse diagram of L

TABLE 2. $\to$ of $\mathcal{L}$

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

TABLE 3. $\otimes$ of $\mathcal{L}$

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	d	0	d	a
b	0	d	c	c	0	b
c	0	0	c	c	0	c
d	0	d	0	0	0	d
1	0	a	b	c	d	1

Example 5.1. *Let $L = \{0, a, b, c, d, 1\}$, the Hasse diagram of L be defined as Figure 1 and its implication operator $\to$ be defined as Table 2 and operator $\otimes$ be defined as Table 3. Then $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ is a residuated lattice. $\mathcal{L}$ is also a regular residuated lattice.*

We define an interval-valued hesitant fuzzy set $\tilde{F}$ of $\mathcal{L}$ by

$$\begin{aligned}\tilde{F} = \{&\langle 0, \{[0.1, 0.2], [0.21, 0.3]\}\rangle; \langle a, \{[0.5, 0.6], [0.7, 0.9]\}\rangle; \langle b, \{[0.5, 0.6], [0.7, 0.9]\}\rangle;\\ &\langle c, \{[0.31, 0.42]\}\rangle; \langle d, \{[0.3, 0.4]\}\rangle; \langle 1, \{[0.5, 0.6], [0.7, 0.9]\}\rangle\}\end{aligned}$$

It is easy to verify that $\tilde{F}$ is an interval-valued hesitant fuzzy positive implicative filter of $\mathcal{L}$, and also an interval-valued hesitant fuzzy G-filter of $\mathcal{L}$.

Taking $x = 1$ in (5.2), we have:

Theorem 5.1. *In a residuated lattice $\mathcal{L}$, each interval-valued hesitant fuzzy positive implicative filter is an interval-valued hesitant fuzzy filter.*

The converse of Theorem 5.1 does not hold in general. For example, in Example 2.1, we define an interval-valued hesitant fuzzy set as follows: $h_{\tilde{F}}(0) = \left\{\left[\frac{1}{2}, \frac{2}{3}\right]\right\}$, $h_{\tilde{F}}(a) = h_{\tilde{F}}(b) = \left\{\left[\frac{7}{10}, \frac{4}{5}\right]\right\}$, $h_{\tilde{F}}(1) = \{[0.85, 0.9]\}$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$. However, it is not an interval-valued hesitant fuzzy positive implicative filter. In fact, $h_{\tilde{F}}(a \to (a \to 0)) \cap h_{\tilde{F}}(a \to a) = h_{\tilde{F}}(1) \cap h_{\tilde{F}}(1) = \{[0.85, 0.9]\} \not\subseteq h_{\tilde{F}}(a \to 0) = h_{\tilde{F}}(a)$.

Theorem 5.2. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued hesitant fuzzy filter of $\mathcal{L}$. Then the followings are equivalent:*
(1) $\tilde{F}$ is an interval-valued hesitant fuzzy positive implicative filter;
(2) $h_{\tilde{F}}(x \to y) \supseteq h_{\tilde{F}}(x \to (x \to y))$ for any $x, y \in L$;
(3) $h_{\tilde{F}}((x \to y) \to (x \to z)) \supseteq h_{\tilde{F}}(x \to (y \to z))$ for any $x, y, z \in L$.

Proof: Assume that $\tilde{F}$ is an interval-valued hesitant fuzzy positive implicative filter, we have $h_{\tilde{F}}(x \to y) \supseteq h_{\tilde{F}}(x \to (x \to y)) \cap h_{\tilde{F}}(x \to x)) = h_{\tilde{F}}(x \to (x \to y)) \cap h_{\tilde{F}}(1) \supseteq h_{\tilde{F}}(x \to (x \to y))$. Thus (2) is valid.

Suppose that (2) holds. That is, assume that $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$ and $h_{\tilde{F}}(x \to y) \supseteq h_{\tilde{F}}(x \to (x \to y))$. Note that $x \to (y \to z) \leq x \to ((x \to y) \to (x \to z))$, it follows that $h_{\tilde{F}}((x \to y) \to (x \to z)) = h_{\tilde{F}}(x \to ((x \to y) \to z)) \supseteq h_{\tilde{F}}(x \to (x \to ((x \to y) \to z))) = h_{\tilde{F}}(x \to ((x \to y) \to (x \to z))) \supseteq h_{\tilde{F}}(x \to (y \to z))$.

Assume that (3) holds, since $\tilde{F}$ is an interval-valued hesitant fuzzy filter, we have $h_{\tilde{F}}(x \to z) \supseteq h_{\tilde{F}}(x \to y) \cap h_{\tilde{F}}((x \to y) \to (x \to z)) \supseteq h_{\tilde{F}}(x \to y) \cap h_{\tilde{F}}(x \to (y \to z))$. Therefore, from Definition 5.1, $\tilde{F}$ is an interval-valued hesitant fuzzy positive implicative filter of $\mathcal{L}$. □

Corollary 5.1. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued hesitant fuzzy filter of $\mathcal{L}$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy G-filter if and only if $\tilde{F}$ is an interval-valued hesitant fuzzy positive implicative filter of $\mathcal{L}$.*

Theorem 5.3. *Let $\tilde{F}$ be an interval-valued hesitant fuzzy filter of $\mathcal{L}$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy positive implicative filter if and only if $h_{\tilde{F}}(y \to x) \supseteq h_{\tilde{F}}(z \to (y \to (y \to x))) \cap h_{\tilde{F}}(z)$, for any $x, y, z \in L$.*

Proof: Since $\tilde{F}$ is an interval-valued hesitant fuzzy positive implicative filter of $\mathcal{L}$, we have $h_{\tilde{F}}(y \to (y \to x)) \supseteq h_{\tilde{F}}(z \to (y \to (y \to x))) \cap h_{\tilde{F}}(z)$. Since $y \to x = 1 \to (y \to x) = (y \to y) \to (y \to x)$, we have $h_{\tilde{F}}(y \to x) = h_{\tilde{F}}((y \to y) \to (y \to x)) \supseteq h_{\tilde{F}}(y \to (y \to x)) \supseteq h_{\tilde{F}}(z \to (y \to (y \to x))) \cap h_{\tilde{F}}(z)$.

Conversely, let $\tilde{F}$ be an interval-valued hesitant fuzzy filter and satisfy the condition: $h_{\tilde{F}}(y \to x) \supseteq h_{\tilde{F}}(z \to (y \to (y \to x))) \cap h_{\tilde{F}}(z)$ for any $x, y, z \in L$. And so $h_{\tilde{F}}(z \to x) \supseteq h_{\tilde{F}}((z \to y) \to (z \to (z \to x))) \cap h_{\tilde{F}}(z \to y)$. Since $z \to (y \to x) = y \to (z \to x) \leq (z \to y) \to (z \to (z \to x))$, we have $h_{\tilde{F}}((z \to y) \to (z \to (z \to x))) \supseteq h_{\tilde{F}}(z \to (y \to x))$. Therefore, we have $h_{\tilde{F}}(z \to x) \supseteq h_{\tilde{F}}((z \to y) \to (z \to (z \to x))) \cap h_{\tilde{F}}(z \to y) \supseteq h_{\tilde{F}}(z \to (y \to x)) \cap h_{\tilde{F}}(z \to y)$. Hence $\tilde{F}$ is an interval-valued hesitant fuzzy positive implicative filter. □

Theorem 5.4. *Let $\tilde{F}$ be an interval-valued hesitant fuzzy positive implicative filter of a residuated lattice $\mathcal{L}$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter of $\mathcal{L}$ if and only if $h_{\tilde{F}}((y \to x) \to x) = h_{\tilde{F}}((x \to y) \to y)$ for any $x, y \in L$.*

Proof: Assume that $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter of $\mathcal{L}$. We have, for any $x, y, z \in L$, $h_{\tilde{F}}((y \to x) \to x) \supseteq h_{\tilde{F}}(z \to (((y \to x) \to x) \to y) \to ((y \to x) \to x)) \cap h_{\tilde{F}}(z)$. Taking $z = 1$, we have $h_{\tilde{F}}((y \to x) \to x) \supseteq h_{\tilde{F}}((((y \to x) \to x) \to y) \to ((y \to x) \to x))$. Since $(x \to y) \to y \leq (y \to x) \to ((x \to y) \to x) = (x \to y) \to ((y \to x) \to x) \leq (((y \to x) \to x) \to y) \to ((y \to x) \to x)$. Since $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$, we have $h_{\tilde{F}}((((y \to x) \to x) \to y) \to (y \to x) \to x) \supseteq h_{\tilde{F}}((x \to y) \to y)$. Hence, $h_{\tilde{F}}((y \to x) \to x) \supseteq h_{\tilde{F}}((x \to y) \to y)$ for any $x, y \in L$. Similarly, we can prove $h_{\tilde{F}}((x \to y) \to y) \supseteq h_{\tilde{F}}((y \to x) \to x)$. Hence $h_{\tilde{F}}((y \to x) \to x) = h_{\tilde{F}}((x \to y) \to y)$.

Conversely, since $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$, we have, for any $x, y, z \in L$, $h_{\tilde{F}}((x \to y) \to x) \supseteq h_{\tilde{F}}(z \to ((x \to y) \to x)) \cap h_{\tilde{F}}(z)$. Since $(x \to y) \to x \leq$

$(x \to y) \to ((x \to y) \to y)$, we have $h_{\tilde{F}}((x \to y) \to ((x \to y) \to y)) \supseteq h_{\tilde{F}}((x \to y) \to x)$. By hypotheses, $h_{\tilde{F}}((y \to x) \to x) = h_{\tilde{F}}((x \to y) \to y)$. Since $\tilde{F}$ is an interval-valued hesitant fuzzy filter of $\mathcal{L}$, from Theorem 5.2, $h_{\tilde{F}}((x \to y) \to y) \supseteq h_{\tilde{F}}((x \to y) \to ((x \to y) \to y))$. It follows that $h_{\tilde{F}}((y \to x) \to x) \supseteq h_{\tilde{F}}((x \to y) \to x)$. Since $y \leq x \to y$ and $y \to x \leq z \to (y \to x)$, we get $(x \to y) \to x \leq y \to x \leq z \to (y \to x)$. It follows that $h_{\tilde{F}}(z \to (y \to x)) \supseteq h_{\tilde{F}}((x \to y) \to x)$. Hence $h_{\tilde{F}}(y \to x) \supseteq h_{\tilde{F}}(z \to (y \to x)) \cap h_{\tilde{F}}(z) \supseteq h_{\tilde{F}}((x \to y) \to x) \cap h_{\tilde{F}}(z)$. And so $h_{\tilde{F}}(x) \supseteq h_{\tilde{F}}((y \to x) \to x) \cap h_{\tilde{F}}(y \to x) \supseteq h_{\tilde{F}}((x \to y) \to x) \cap h_{\tilde{F}}(y \to x) \supseteq h_{\tilde{F}}((x \to y) \to x) \cap (h_{\tilde{F}}((x \to y) \to x) \cap h_{\tilde{F}}(z)) \supseteq (h_{\tilde{F}}(z) \cap h_{\tilde{F}}(z \to ((x \to y) \to x))) \cap h_{\tilde{F}}(z) = h_{\tilde{F}}(z) \cap h_{\tilde{F}}(z \to ((x \to y) \to x))$. Therefore, $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter of $\mathcal{L}$. □

Corollary 5.2. *Let $\tilde{F}$ be an interval-valued hesitant fuzzy implicative filter of a residuated lattice $\mathcal{L}$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy implicative positive filter of $\mathcal{L}$.*

Theorem 5.5. *In a residuated lattice, each interval-valued hesitant fuzzy Boolean filter of $\mathcal{L}$ is an interval-valued hesitant fuzzy positive implicative filter of $\mathcal{L}$.*

Proof: Let $\tilde{F}$ be an interval-valued hesitant fuzzy Boolean filter of $\mathcal{L}$, then $h_{\tilde{F}}(x \to z) \supseteq h_{\tilde{F}}((x \vee x') \to (x \to z)) \cap h_{\tilde{F}}(x \vee x') = h_{\tilde{F}}((x \vee x') \to (x \to z)) \cap h_{\tilde{F}}(1) \supseteq h_{\tilde{F}}((x \vee x') \to (x \to z))$. Since $(x \vee x') \to (x \to z) = (x \to (x \to z)) \wedge (x' \to (x \to z)) = (x \to (x \to z)) \wedge ((x' \otimes x) \to z) = x \to (x \to z)$, we have $h_{\tilde{F}}((x \vee x') \to (x \to z)) = h_{\tilde{F}}(x \to (x \to z))$. Hence $h_{\tilde{F}}(x \to z) \supseteq h_{\tilde{F}}(x \to (x \to z))$, it follows from Theorem 5.2 that $\tilde{F}$ is an interval-valued hesitant fuzzy positive implicative filter of $\mathcal{L}$. □

The converse of Theorem 5.5 does not hold in general; the following example exactly shows this judgement.

Example 5.2. *Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$ with the operations $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and $\otimes, \to$ as Table 4.*

TABLE 4. $\otimes, \to$ in L

$\to$	0	a	b	1	$\otimes$	0	a	b	1
0	1	1	1	1	0	0	0	0	0
a	0	1	1	1	a	0	a	a	a
b	0	a	1	1	b	0	a	b	b
1	0	a	b	1	1	0	a	b	1

Then $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ is a residuated lattice. For any $x \in L$, we define an interval-valued hesitant fuzzy set $\tilde{F}$ as follows

$$h_{\tilde{F}}(0) = \left\{ \left[\frac{1}{4}, \frac{1}{3}\right], \left[\frac{2}{3}, \frac{3}{4}\right] \right\} = h_{\tilde{F}}(a) = h_{\tilde{F}}(b),$$
$$h_{\tilde{F}}(1) = \left\{ \left[\frac{4}{5}, \frac{9}{10}\right] \right\}.$$

It is easy to verify the $\tilde{F}$ is an interval-valued hesitant fuzzy positive implicative filter of $\mathcal{L}$. However, it is not an interval-valued hesitant fuzzy Boolean filter of $\mathcal{L}$. In fact, we take $h_{\tilde{F}}(a \vee a') = h_{\tilde{F}}(a) \neq h_{\tilde{F}}(1)$.

Remark 5.2. *In Theorem 5.5, we investigated the relation between the interval-valued hesitant fuzzy Boolean filter and interval-valued hesitant fuzzy positive implicative filter. In next section, we will give an equivalent condition for them.*

Definition 5.3. *An interval-valued hesitant fuzzy set $\tilde{F}$ of a residuated lattice $\mathcal{L}$ is called an interval-valued hesitant fuzzy MV-filter of $\mathcal{L}$, if $\tilde{F}$ is an interval-valued hesitant fuzzy filter and satisfies*

(5.3) $h_{\tilde{F}}(((x \to y) \to y) \to x) \supseteq h_{\tilde{F}}(y \to x)$ *for any* $x, y \in L$.

Remark 5.3. *In lattice implication algebras, BL-algebras, R_0-algebras, the MV-filters are called fantastic filters.*

Theorem 5.6. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued hesitant fuzzy set of $\mathcal{L}$. Then $\tilde{F}$ be an interval-valued hesitant fuzzy MV-filter if and only if*

(3.1) $h_{\tilde{F}}(1) \supseteq h_{\tilde{F}}(x)$ *for any* $x \in L$;

(5.4) $h_{\tilde{F}}(((x \to y) \to y) \to x) \supseteq h_{\tilde{F}}(z) \cap h_{\tilde{F}}(z \to (y \to x))$ *for any* $x, y, z \in L$.

Proof: The proof is similar to that of Theorem 4.24 in [10]. □

In MV-algebras and lattice implication algebras, we have $(x \to y) \to y = (y \to x) \to x$ and $((x \to y) \to y) \to y = x \to y$. Therefore, we have the following corollary.

Corollary 5.3. *In MV-algebras and lattice implication algebras, interval-valued hesitant fuzzy filter and interval-valued hesitant fuzzy MV-filter are equivalent.*

Theorem 5.7. *In a residuated lattice, every interval-valued hesitant fuzzy implicative filter is an interval-valued hesitant fuzzy MV-filter.*

Proof: Assume that $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter of $\mathcal{L}$. Since $x \leq ((x \to y) \to y) \to x$, and so $(((x \to y) \to y) \to x) \to y \leq x \to y$. This implies that $((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x) \geq (x \to y) \to ((((x \to y) \to y) \to x)) = ((x \to y) \to y) \to ((x \to y) \to x) \geq y \to x$. It follows that $h_{\tilde{F}}(((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x)) \supseteq h_{\tilde{F}}(y \to x)$. Since $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter of $\mathcal{L}$, we have $h_{\tilde{F}}((((x \to y) \to y) \to x)) \supseteq h_{\tilde{F}}(z \to (((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x))) \cap h_{\tilde{F}}(z)$. Taking $z = 1$, we get $h_{\tilde{F}}((((x \to y) \to y) \to x)) \supseteq h_{\tilde{F}}(((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x)) \supseteq h_{\tilde{F}}(y \to x)$. Hence, $\tilde{F}$ is an interval-valued hesitant fuzzy MV-filter. □

The following example shows the converse of Theorem 5.7 may be not true in general.

Example 5.3. *Let $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ be a residuated lattice as in Example 2.1, and define an interval-valued hesitant fuzzy set $\tilde{F} = \{\langle x, h_{\tilde{F}}(x)\rangle | x \in L\}$ of $\mathcal{L}$ by $h_{\tilde{F}}(0) = \{[0.3, 0.4]\} = h_{\tilde{F}}(a)$, $h_{\tilde{F}}(b) = h_{\tilde{F}}(1) = \{[0.5, 0.6]\}$. It is easy to verify that $\tilde{F}$ is an interval-valued hesitant fuzzy MV-filter of $\mathcal{L}$. However, $\tilde{F}$ is not an interval-valued hesitant fuzzy Boolean filter, because $h_{\tilde{F}}(a \vee a') = h_{\tilde{F}}(a) = \{[0.3, 0.4]\} \neq h_{\tilde{F}}(1) = \{[0.5, 0.6]\}$.*

The following theorem gives the relation among the interval-valued hesitant fuzzy implicative filter, interval-valued hesitant fuzzy positive implicative filter and interval-valued hesitant fuzzy MV filter.

Theorem 5.8. *An interval-valued hesitant fuzzy set $\tilde{F}$ of a residuated lattice $\mathcal{L}$ is an interval-valued hesitant fuzzy implicative (Boolean) filter of $\mathcal{L}$ if and only if $\tilde{F}$ is both an interval-valued hesitant fuzzy positive implicative filter and an interval-valued hesitant fuzzy MV-filter of $\mathcal{L}$.*

Proof: Assume that $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter of $\mathcal{L}$. From Corollary 5.2 and Theorem 5.7, we know F is both an interval-valued hesitant fuzzy positive implicative filter and an interval-valued hesitant fuzzy MV filter of $\mathcal{L}$.

Conversely, suppose that $\tilde{F}$ is both an interval-valued hesitant fuzzy positive implicative filter and an interval-valued hesitant fuzzy MV filter of $\mathcal{L}$. It follows from Theorem 5.2(2) that $h_{\tilde{F}}((x \to y) \to y) \supseteq h_{\tilde{F}}((x \to y) \to ((x \to y) \to y))$. Since $(x \to y) \to x \leq (x \to y) \to ((x \to y) \to y)$, it follows that $h_{\tilde{F}}((x \to y) \to ((x \to y) \to y)) \supseteq h_{\tilde{F}}((x \to y) \to x)$. And so $h_{\tilde{F}}((x \to y) \to y) \supseteq h_{\tilde{F}}((x \to y) \to x)$. On the other hand, since $\tilde{F}$ is an interval-valued hesitant fuzzy MV-filter of $\mathcal{L}$, we have $h_{\tilde{F}}(((x \to y) \to y) \to x) \supseteq h_{\tilde{F}}(y \to x)$. Since $(x \to y) \to x \leq y \to x$, we have $h_{\tilde{F}}(y \to x) \supseteq h_{\tilde{F}}((x \to y) \to x)$. Thus, $h_{\tilde{F}}(((x \to y) \to y) \to x) \supseteq h_{\tilde{F}}((x \to y) \to x)$, and so $h_{\tilde{F}}(x) \supseteq h_{\tilde{F}}(((x \to y) \to y) \to x) \cap h_{\tilde{F}}((x \to y) \to y) \supseteq h_{\tilde{F}}((x \to y) \to x), h_{\tilde{F}}((x \to y) \to x) \supseteq h_{\tilde{F}}(z \to ((x \to y) \to x)) \cap h_{\tilde{F}}(z)$. Hence $h_{\tilde{F}}(x) \supseteq h_{\tilde{F}}(z \to ((x \to y) \to x)) \cap h_{\tilde{F}}(z)$. Therefore, $\tilde{F}$ is an interval-valued hesitant fuzzy implicative filter of $\mathcal{L}$. □

6. **Interval-Valued Hesitant Fuzzy Regular Filters.**

Definition 6.1. *An interval-valued hesitant fuzzy filter $\tilde{F}$ of a residuated lattice $\mathcal{L}$ is called an interval-valued hesitant fuzzy regular filter if it satisfies the condition: $h_{\tilde{F}}(x'' \to x) = h_{\tilde{F}}(1)$ for any $x \in L$.*

Example 6.1. *Let $L = [0, 1]$ (unit interval). For any $x, y \in L$, define $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$.*

$$x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ (1 - x) \vee y & \text{otherwise.} \end{cases}$$

$$x \otimes y = \begin{cases} 0 & \text{if } x + y \leq 1, \\ x \wedge y & \text{otherwise.} \end{cases}$$

Then $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ is a residuated lattice. Define an interval-valued hesitant fuzzy set $\tilde{F} = \{\langle x, h_{\tilde{F}}(x)\rangle | x \in L\}$ by

$$h_{\tilde{F}}(x) = \begin{cases} \{[0.5, 0.6], [0.7, 0.8]\} & \text{if } x \in (0.8, 1], \\ \{[0.1, 0.2], [0.3, 0.4]\} & \text{if } x \in [0, 0.8]. \end{cases}$$

It is easy to verify $\tilde{F}$ is an interval-valued hesitant fuzzy filter. Obviously, it is also an interval-valued hesitant fuzzy regular filter.

Theorem 6.1. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued fuzzy filter of $\mathcal{L}$. Then following assertions are equivalent, for any $x, y \in L$,*
(6.1) *$\tilde{F}$ is an interval-valued hesitant fuzzy regular filter;*
(6.2) *$h_{\tilde{F}}(y \to x) \supseteq h_{\tilde{F}}(x' \to y')$;*
(6.3) *$h_{\tilde{F}}(y' \to x) \supseteq h_{\tilde{F}}(x' \to y)$.*

Proof: The proof is similar to that of Theorem 5.14 in [19]. □

Theorem 6.2. *An interval-valued hesitant fuzzy set $\tilde{F}$ of $\mathcal{L}$ is an interval-valued hesitant fuzzy regular filter if and only if it satisfies*
(3.1) *$h_{\tilde{F}}(1) \supseteq h_{\tilde{F}}(x)$ for any $x \in L$;*
(6.4) *$h_{\tilde{F}}(y' \to x) \supseteq h_{\tilde{F}}(z \to (x' \to y)) \cap h_{\tilde{F}}(z)$ for any $x, y, z \in L$.*

Proof: The proof is similar to that of Theorem 5.17 in [19]. □

Theorem 6.3. *An interval-valued hesitant fuzzy set $\tilde{F}$ of a residuated lattice $\mathcal{L}$ is an interval-valued hesitant fuzzy regular filter if and only if it satisfies*
(3.1) *$h_{\tilde{F}}(1) \supseteq h_{\tilde{F}}(x)$ for any $x \in L$;*
(6.5) *$h_{\tilde{F}}(y \to x) \supseteq h_{\tilde{F}}(z \to (x' \to y')) \cap h_{\tilde{F}}(z)$ for any $x, y, z \in L$.*

Proof: It is similar to Theorem 3.26 in [19]. □

Theorem 6.4. *Let M be an R-filter of a residuated lattice $\mathcal{L}$. Then there exists an interval-valued hesitant fuzzy regular filter $\tilde{F}$ of $\mathcal{L}$ such that $\tilde{F}_{\tilde{\tau}} = M$ for some $\tilde{\tau} \in \mathcal{P}(D[0,1])$.*

Proof: Let $\tilde{F}$ be an interval-valued hesitant fuzzy set $\{\langle x, h_{\tilde{F}}(x)\rangle | x \in L\}$ on L defined by

$$h_{\tilde{F}}(x) = \begin{cases} \alpha, & x \in M, \\ \beta, & \text{otherwise.} \end{cases}$$

where $\alpha, \beta \in \mathcal{P}(D[0,1])$ and $\alpha \supseteq \beta$.

Let $x, y, z \in L$ be such that $z \to (x' \to y) \in M$ and $z \in M$, then $y' \to x \in M$, it follows that $h_{\tilde{F}}(z \to (x' \to y)) = h_{\tilde{F}}(z) = h_{\tilde{F}}(y' \to x) = \alpha$ and so $h_{\tilde{F}}(z \to (x' \to y)) \cap h_{\tilde{F}}(z) = h_{\tilde{F}}(y' \to x)$. If $z \to (x' \to y) \notin M$ or $z \notin M$, then $h_{\tilde{F}}(z \to (x' \to y)) = \beta$ or $h_{\tilde{F}}(z) = \beta$. We have $h_{\tilde{F}}(z \to (x' \to y)) \cap h_{\tilde{F}}(z) \subseteq h_{\tilde{F}}(y' \to x)$. Therefore, $h_{\tilde{F}}(z \to (x' \to y)) \cap h_{\tilde{F}}(z) \subseteq h_{\tilde{F}}(y' \to x)$ for any $x, y, z \in L$. On the other hand, we have $1 \in M$, thus $h_{\tilde{F}}(1) = \alpha \supseteq h_{\tilde{F}}(x)$ for any $x \in L$; this shows that $\tilde{F}$ is an interval-valued hesitant fuzzy regular filter. It is easy to verify that $\tilde{F}_{\tilde{\tau}} = M$. □

Theorem 6.5. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued hesitant fuzzy set of $\mathcal{L}$. Then the following conditions are equivalent:*

(1) $\tilde{F}$ is an interval-valued hesitant fuzzy (implicative, positive implicative, MV, Boolean, regular) filter;

(2) $\tilde{F}_{\tilde{\tau}}$ ($\neq \emptyset$) is an (implicative, positive implicative, MV, Boolean, R) filter of $\mathcal{L}$.

Proof: It is easily obtained by Theorem 3.5. □

Theorem 6.6. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued hesitant fuzzy set of $\mathcal{L}$. If $\tilde{F}$ is an interval-valued hesitant fuzzy MV-filter, then $\tilde{F}$ is an interval-valued hesitant fuzzy regular filter.*

Proof: Suppose that $\tilde{F}$ is an interval-valued hesitant fuzzy MV-filter of $\mathcal{L}$, then $h_{\tilde{F}}(((x \to y) \to y) \to x) \supseteq h_{\tilde{F}}(y \to x)$ for any $x, y \in L$. It follows that $h_{\tilde{F}}(x'' \to x) = h_{\tilde{F}}(((x \to 0) \to 0) \to x) \supseteq h_{\tilde{F}}(0 \to x) = h_{\tilde{F}}(1)$, which means $h_{\tilde{F}}(x'' \to x) = h_{\tilde{F}}(1)$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy regular filter of $\mathcal{L}$. □

The converse of Theorem 6.6 will not hold in general as the following example shows.

Example 6.2. *In Example 6.1, $\tilde{F}$ is an interval-valued hesitant fuzzy regular filter, but $\tilde{F}$ is not an interval-valued hesitant fuzzy MV-filter, because,*

$$\begin{aligned} h_{\tilde{F}}(((0.8 \to 0.3) \to 0.3) \to 0.8) &= h_{\tilde{F}}(0.8) = \{[0.1, 0.2], [0.3, 0.4]\} \not\supseteq h_{\tilde{F}}(0.3 \to 0.8) \\ &= \{[0.5, 0.6], [0.7, 0.8]\}. \end{aligned}$$

Theorem 6.7. *In a BL-algebra, an interval-valued hesitant fuzzy MV-filter and interval-valued hesitant fuzzy regular filter are equivalent.*

Proof: The proof is similar to Theorem 7.6 in [19]. □

Theorem 6.8. *Let $\mathcal{L}$ be a residuated lattice and $\tilde{F}$ be an interval-valued hesitant fuzzy set of $\mathcal{L}$. Then $\tilde{F}$ is an interval-valued hesitant fuzzy Boolean filter if and only if it is both an interval-valued hesitant fuzzy positive implicative filter and an interval-valued hesitant fuzzy regular filter of $\mathcal{L}$.*

Proof: From Theorem 5.8 and Theorem 6.6, we have the conclusion that an interval-valued hesitant fuzzy Boolean filter is both an interval-valued hesitant fuzzy positive implicative filter and an interval-valued hesitant fuzzy regular filter.

Conversely, assume that $\tilde{F}$ is an interval-valued hesitant fuzzy positive implicative filter and an interval-valued hesitant fuzzy regular filter. For any $x, y \in L$, since $x' \leq x \to y$, we have $(x \to y) \to x \leq x' \to x$; it follows that $h_{\tilde{F}}((x \to y) \to x) \subseteq h_{\tilde{F}}(x' \to x)$. Next, since $x' \to x = (x'')' \to x$, we have $h_{\tilde{F}}(x' \to x) \subseteq h_{\tilde{F}}(x' \to x'')$. And so, since $x' \to x'' = x' \to (x' \to 0)$, it follows from Remark 5.1 that $h_{\tilde{F}}(x' \to x'') = h_{\tilde{F}}(x' \to (x' \to 0)) = h_{\tilde{F}}(x' \to 0) = h_{\tilde{F}}(x'') = h_{\tilde{F}}(x)$. Therefore, $h_{\tilde{F}}((x \to y) \to x) \subseteq h_{\tilde{F}}(x' \to x) = h_{\tilde{F}}(x)$. Furthermore, we have $h_{\tilde{F}}((x \to y) \to x) = h_{\tilde{F}}(x)$. From Theorem 4.7, $\tilde{F}$ is an interval-valued hesitant fuzzy Boolean filter of $\mathcal{L}$. □

7. **Conclusions.** Filter theory plays a very important role in studying logical systems and the related algebraic structures. The filter theory of a residuated lattice, which is also studied in several algebraic structures, plays an important role in studying these algebras and the completeness of the corresponding non-classical logics. In this paper, we develop the interval-valued hesitant fuzzy filter theory of residuated lattices. Mainly, we give some new characterizations of interval-valued hesitant fuzzy (implicative, Boolean, positive implicative, MV, regular) filters in residuated lattices. The theory can be used in MV-algebras, lattice implication algebras, BL-algebras, MTL-algebras, etc. Meanwhile, we hope that it will be of great use to provide theoretical foundation to design intelligent information processing systems.

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