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MODELING AND STATE FEEDBACK CONTROL OF MULTI-RATE NETWORKED CONTROL SYSTEMS WITH BOTH SHORT TIME DELAY AND PACKET DROPOUT

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Abstract. *The model of multi-rate networked control systems with both short time delay and packet dropout is set up. It is stated that multi-rate NCS with both short time delay and packet dropout can be modeled as a switched stochastic system. And multi-rate NCS with both short time delay and packet dropout can be modeled as a switched system when the actuator is time driven and the state noise is not considered. Moreover, corresponding state feedback controllers are proposed to guarantee the stability of multi-rate networked control systems with both short time delay and packet dropout. An example is given in the end to verify the theoretical results of this paper.*
Keywords: Multi-rate networked control systems, Modeling, Short time delay, Packet dropout, Robust control

1. **Introduction.** Feedback control systems wherein the control loops are closed through a real-time network are called networked control systems (NCSs). Networked control architecture has many advantages over a traditional point-to-point design including low cost of installation, ease of maintenance, lower cost and greater flexibility. For these reasons the networked control architecture is already used in many applications, particularly where weight and volume are of consideration. NCS received more and more attention and has been a very hot research topic, see [1-13,34], to name a few.

From the point of network control methodology, it can be divided into eight kinds approximately. The first one is the approach of augmented deterministic discrete-time model methodology [14]. The second one is queuing methodology [15,16]. The third one is optimal stochastic control methodology [2]. The fourth one is perturbation methodology [17,18]. The fifth one is switched system methodology [21,22,33]. The sixth one is robust control methodology [19,23,24]. The seventh one is fuzzy logic modulation methodology

[25]. The eighth one is event-based methodology [26]. Besides these approaches above, jumped system and adaptive control, etc., are often used to investigate NCS, see [5-9] for details.

The research achievements of NCS are almost all under the following assumption: the sampling rate of each node in NCS is the same. This brings more convenience for the theoretical research of NCS; however, the sampling rate of each node is not identical or different from each other in practical application. On the one hand, the nodes in NCS are in distribution, and it could not use a single sampling rate to control many different physical processes. On the other hand it is impossible to use the same sampling frequency to control all the actuators in a large area [5]. And then, the sensor has a lower sampling frequency that can effectively reduce the network flow in the network environment, and the controller or the actuator has a higher sampling frequency that will improve the system performance [29]. So multi-rate sampling is a natural selection and inevitable requirement in NCS; multi-rate sampling research of NCS has important theoretical significance and application value.

For multi-rate NCS, there are some works reported. Model of multi-rate NCS is set up when sensor, controller and actuator are all time driven, and the stability of the system and the interference suppression characteristics is analyzed in [27]. [28] is a simple expansion to [27]. Globe exponential stability of multi-rate NCS is analyzed when controller and actuator are all event driven in [29], and the exponential stability of multi-rate NCS is analyzed in three cases of perfect transmission, delayed transmission and time-varying transmission. The stability of multi-rate NCS is analyzed based on the model set up in [30], and a V-K iteration algorithm is used to get the stabilizing controller of multi-rate NCS. The modeling of multi-rate NCS with long time delay is presented in [20]. The analysis and modeling of multi-rate NCS with short time delay is presented in [32].

In multi-rate NCS, time delay and packet dropout are two most important issues and they often exist simultaneously. The complex fieldbus control systems applied in modern industry can be looked as multi-rate NCS with both time delay and packet dropout. Published literature shows that many questions about the multi-rate NCS should be investigated, and even the modeling of multi-rate NCS with both time delay and packet drop is still open. This motivates us to investigate this study. The main contributions of this paper include:

(1) The model of multi-rate networked control systems with both short time delay and packet dropout is proposed;

(2) Corresponding state feedback controller is proposed to guarantee the stability of multi-rate NCS with both short time delay and packet dropout.

The paper is organized into 6 sections including the introduction. Section 2 presents problem formulations and main assumptions. Section 3 presents the modeling of multi-rate NCS with both short time delay and packet dropout. State feedback controllers of multi-rate NCS with both short time delay and packet dropout are proposed in Section 4. There is an example to illustrate the results in Section 5. Section 6 summarized this paper.

2. **Problem Statement and Preliminaries.** It is assumed that the controlled process is linear time-invariant system, which can be expressed as

$$\begin{cases} \dot{x}(t) = A^c x(t) + B^c u(t) + E^c v(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $v(t) \in R^n$ and A^c, B^c, E^c are matrices of appropriate sizes, $v(t)$ is interference input. The sampling period of the sensor is noted as T_s, the sampling periods of controller and actuator are the same and noted as T_c. $T_c = T_s/N$ and N is a positive integer not less than 2. That is to say, the controller and actuator have a higher sampling period than the sensor.

For the convenience of the later formulation, four concepts can be introduced firstly.

Definition 2.1. [32] *The sampling periods of the sensor, the controller and the actuator in networked control systems are not the same, that is to say there are more than one sampling rate in networked control systems. Then such NCS is called* ***multi-rate networked control systems****.*

Definition 2.2. [32] *The sampling periods of the sensor, the controller and the actuator in networked control systems are the same, that is to say there is only one sampling rate in networked control systems. Then such NCS is called* ***single-rate networked control systems****.*

Definition 2.3. [4] *If network induced delay distributes in the interval* $[0 \ \ \alpha]$ *and* $\alpha \leq T_s$, *then such network-induced delay is called* ***short time delay****.*

Definition 2.4. [4] *If network induced delay distributes in the interval* $[0 \ \ \alpha]$ *and* $\alpha > T_s$, *then such network-induced delay is called* ***long time delay****.*

Remark 2.1. *By the definition of multi-rate networked control systems, the most complex case is that there are three sampling rates in multi-rate NCS, that is to say any two sampling periods of the sensor, the controller and the actuator are not the same each other. For the convenience of formulation, we consider the sampling rates of controller and actuator are the same in this paper, that is to say there are only two sampling rates in multi-rate NCS in this paper.*

In NCS, the sensor is almost always clock driven; the controller or the actuator is either clock driven or event driven. The clock of the sensor is the clock of NCS. There are four cases according to driving modes of nodes in NCS. These four cases are as Table 1 [4].

TABLE 1. The driving modes of nodes in NCS

	sensor	controller	actuator
Case 1	Clock driving	Clock driving	Clock driving
Case 2	Clock driving	Event driving	Clock driving
Case 3	Clock driving	Clock driving	Event driving
Case 4	Clock driving	Event driving	Event driving

For convenience of investigation, we make the following rational assumptions.

A1: The network-induced delay is short time delay. The network-induced delay at time step k is denoted by τ^k, $\tau^k = \tau_{sc}^k + \tau_{ca}^k$ is bounded by $\tau^k < T_s$.

A2: The clock difference of different nodes in NCS is not considered.

A3: The disorder of the data packets is not considered, which means the most recent data can be used in the node.

A4: The maximum number of consecutive dropped data packets is bounded, by positive integer D.

3. Modeling of Multi-rate NCS with Both Short Time Delay and Packet Dropout.

We first consider the simplified case when the packets are dropped periodically, with the period T_m. Note that T_m is integer multiple of the sampling period T_s, i.e., $T_m = mT_s$. In the case of $m = T_m/T_s \geq 2$, the first $(m-1)$ packets are drops. We present the model of multi-rate NCS in 4 cases.

(1) Case 1. For these first $(m-1)$ steps, the previous control signal is used; it is assumed that the data packet drop begins from $u(kT_m)$, and that is to say $u(kT_m - T_s)$ will be used in the following $(m-1)$ steps. Therefore, one can obtain

$$\begin{aligned}
&x[kT_m + (m-1)T_s] \\
&= e^{A^c(m-1)T_s}x(kT_m) + \int_{kT_m}^{kT_m+(m-1)T_s} e^{A^c[kT_m+(m-1)T_s-s]}B^c ds u(kT_m - T_s) \\
&\quad + \int_{kT_m}^{kT_m+(m-1)T_s} e^{A^c[kT_m+(m-1)T_s-s]}E^c v(s)ds \\
&= e^{A^c(m-1)T_s}x(kT_m) + \int_0^{(m-1)T_s} e^{A^c s}B^c ds u(kT_m - T_s) + \int_0^{(m-1)T_s} e^{A^c s}E^c v(s)ds \\
&= e^{A^c(m-1)T_s}x(kT_m) + \int_0^{(m-1)T_s} e^{A^c(T_s-s)}B^c ds u(kT_m - T_s) \\
&\quad + \int_0^{(m-1)T_s} e^{A^c(T_s-s)}E^c v(s)ds \\
&= A^{m-1}x(kT_m) + \sum_{i=0}^{m-2} A^i B_1 u(kT_m - T_s) + \sum_{i=0}^{m-2} A^i E v(kT_m + iT_s) \\
&= A^{m-1}x(kT_m) + \sum_{i=1}^{m-1} B_i u(kT_m - T_s) + \sum_{i=0}^{m-2} A^i E v(kT_m + iT_s)
\end{aligned} \tag{2}$$

where $A = e^{A^c T_s}$, $B_1 = \int_0^{T_s} e^{A^c\eta}B^c d\eta$, $B_k = A_0^{k-1}B_1$, $1 \le k \le m-1$.

Remark 3.1. *Iteration method is used to get the value of* $x[kT_m + (m-1)T_s]$ *in [28], and the result of (2) is similar to that of [28]. Equality (2) is an accurate result of* $x[kT_m+(m-1)T_s]$*, while result in [28] is just an approximated result of* $x[kT_m+(m-1)T_s]$*.*

During the interval $t \in [kT_m + (m-1)T_s, (k+1)T_m)$, the new packet is transmitted successfully with some delay, say $\tau = h(T_s/N)$, where $h = 0, 1, 2, \ldots, N$. It is assumed that $v(kT_m) = v(kT_m + T_s) = \cdots = v(kT_m + (m-1)T_s)$, then

$$\begin{aligned}
x[(k+1)T_m] &= Ax[kT_m + (m-1)T_s] + \sum_{i=N-h+1}^{N} B_i u(kT_m - T_s) \\
&\quad + \sum_{i=1}^{N-h} B_i u(kT_m + (m-1)T_s) + Ev(kT_m + (m-1)T_s) \\
&= A^m x(kT_m) + \left(\sum_{i=1}^{m-1} A^i B_1 + \sum_{i=N-h+1}^{N} B_i\right) u(kT_m - T_s) \\
&\quad + \sum_{i=1}^{N-h} B_i u(kT_m + (m-1)T_s) + \sum_{i=0}^{m-1} A^i E v(kT_m)
\end{aligned} \tag{3}$$

Note that $x(kT_m + mT_s)$ equals $x((k+1)T_m)$, and $x(kT_m + (m-1)T_s) = x((k+1)T_m - T_s)$. It is assumed that the controller uses just the time-invariant linear feedback control law, $u(t) = Kx(t)$. Then, we may substitute the $u(\cdot)$ to obtain

$$x((k+1)T_m - T_s) = A^{m-1}x(kT_m) + \sum_{i=0}^{m-2} A^i B_1 K x(kT_m - T_s) + \sum_{i=0}^{m-2} A^i E V(kT_m) \tag{4}$$

and

$$\begin{aligned} x[(k+1)T_m] = & A^m x(kT_m) + \left(\sum_{i=1}^{m-1} A^i B_1 + \sum_{i=N-h+1}^{N} B_i \right) K x(kT_m - T_s) \\ & + \sum_{i=1}^{N-h} B_i K x((k+1)T_m - T_s) + \sum_{i=0}^{m-1} A^i E v(kT_m) \end{aligned} \tag{5}$$

Substitute (4) into (5), then

$$\begin{aligned} x[(k+1)T_m] = & \left(A^m + \sum_{i=1}^{N-h} B_i K A^{m-1} \right) x(kT_m) \\ & + \left(\sum_{i=1}^{m-1} A^i B_1 + \sum_{i=N-h+1}^{N} B_i + \sum_{i=1}^{N-h} B_i K \sum_{i=0}^{m-2} A^i B_1 \right) K x(kT_m - T_s) \\ & + \left(\sum_{i=0}^{m-1} A^i + \sum_{i=1}^{N-h} B_i K \sum_{i=0}^{m-2} A^i \right) E v(kT_m) \end{aligned} \tag{6}$$

If we let $\hat{x}[k] = \begin{bmatrix} x(kT_m - T_s) \\ x(kT_m) \end{bmatrix}$ and $v(k) = v(kT_m)$, then the above equations can be written as

$$\hat{x}[k+1] = \Phi_{(m,h)} \hat{x}[k] + E_m v(k) \tag{7}$$

where

$$\Phi_{(m,h)} = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}, \quad \Pi_{11} = \sum_{i=0}^{m-2} A^i B_1 K, \quad \Pi_{12} = A^{m-1},$$

$$\Pi_{21} = \left(\sum_{i=1}^{m-1} A^i B_1 + \sum_{i=N-h+1}^{N} B_i + \sum_{i=1}^{N-h} B_i K \sum_{i=0}^{m-2} A^i B_1 \right) K,$$

$$\Pi_{22} = A^m + \sum_{i=1}^{N-h} B_i K A^{m-1}, \quad E_m = \begin{bmatrix} \sum_{i=0}^{m-2} A^i E \\ \sum_{i=0}^{m-1} A^i E + \sum_{i=1}^{N-h} B_i K \sum_{i=0}^{m-2} A^i E \end{bmatrix}.$$

In this case, $m = T_m/T_s \geq 2$, and $h = 0, 1, 2, \ldots, N$.

For the case of $m = 1$, namely no packet dropout, the following dynamic equation is derived

$$\hat{x}[k+1] = \Phi_{(1,h)} \hat{x}[k] + E_1 v(k) \tag{8}$$

where

$$\Phi_{(1,h)} = \begin{bmatrix} 0 & I \\ \sum_{i=N-h+1}^{N} B_i & A + \sum_{i=1}^{N-h} B_i K \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 \\ E \end{bmatrix}.$$

For the case of aperiodic packet dropouts, from [28] one knows the NCS along with a typical aperiodic delay and packet dropout can be seen as a dynamic system switching among the dynamics with different periodic delay and packet dropout pattern $\Phi_{(m,h)}$ for $m = 1, 2, \ldots, D$ and $h = 0, 1, 2, \ldots, N$. This observation leads to modeling the multi-rate NCS with both short time delay and packet dropout in this case as a switching system namely as

$$\hat{x}[k+1] = \Phi_{(m,h)} \hat{x}[k] + E_m v[k] \tag{9}$$

where

$$\Phi_{(m,h)} = \begin{cases} \begin{bmatrix} 0 & I \\ \sum_{i=N-h+1}^{N} B_i K & A + \sum_{i=1}^{N-h} B_i K \end{bmatrix}, & m = 1 \\ \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}, & m \geq 2 \end{cases},$$

$$E_m = \begin{cases} \begin{bmatrix} 0 \\ E \end{bmatrix}, & m = 1 \\ \begin{bmatrix} \sum_{i=0}^{m-2} A^i E \\ \sum_{i=0}^{m-1} A^i + \sum_{i=1}^{N-h} B_i K \sum_{i=0}^{m-2} A^i \end{bmatrix}, & m \geq 2 \end{cases}$$

where $\Phi_{(m,h)} \in \{\Phi_{(1,0)}, \Phi_{(1,1)}, \ldots, \Phi_{(1,N)}, \Phi_{(2,0)}, \ldots, \Phi_{(D,0)}, \ldots, \Phi_{(D,N)}\}$.

(2) Case 2. As the actuator is still time driven; the character of the network-induced delay is the same as that of Case 1. The model of multi-rate NCS with packet drop and short time delay is the same as Case 1.

(3) Case 3. For these first $(m-1)$ steps, the previous control signal is used; it is assumed that the data packet drop begins from $u(kT_m)$, and that is to say $u(kT_m - T_s)$ will be used in the following $(m-1)$ steps. From the case 1 part in this section we know

$$x[kT_m + (m-1)T_s] = A^{m-1}x(kT_m) + \sum_{i=0}^{m-2} A^i B_1 u(kT_m - T_s) + \sum_{i=0}^{m-2} A^i E v(kT_m + iT_s) \quad (10)$$

During the period $t \in [kT_m + (m-1)T_s, (k+1)T_m)$, the new packet is transmitted successfully with some delay, say k, where $h = 0, 1, 2, \ldots, N$. Then

$$\begin{aligned} x[(k+1)T_m] &= Ax(kT_m + (m-1)T_s) + \int_{T_s-\tau^k}^{T_s} e^{A^c s} B^c ds u(kT_m - T_s) \\ &\quad + \int_0^{T_s-\tau^k} e^{A^c s} B^c ds u(kT_m + (m-1)T_s) + Ev(kT_m + (m-1)T_s) \\ &= A^m x(kT_m) + \left(\sum_{i=1}^{m-1} A^i B_1 + \Gamma_1(\tau^k) \right) u(kT_m - T_s) \\ &\quad + \Gamma_0(\tau^k) u(kT_m + (m-1)T_s) + \sum_{i=0}^{m-1} A^i E v(kT_m + iT_s) \end{aligned} \quad (11)$$

where $\Gamma_0(\tau) = \int_0^{T_s-\tau^k} e^{A^c s} B^c ds$, $\Gamma_1(\tau) = \int_{T_s-\tau^k}^{T_s} e^{A^c s} B^c ds$.

Note that $x(kT_m + mT_s)$ equals $x((k+1)T_m)$, and $x(kT_m + (m-1)T_s) = x((k+1)T_m - T_s)$. Let assume that $v(kT_m) = v(kT_m + T_s) = \cdots = v(kT_m + (m-1)T_s)$, and that the controller uses just the time-invariant linear feedback control law, $u(t) = Kx(t)$. Then, we may substitute the $u(\cdot)$ to obtain

$$x((k+1)T_m - T_s) = A^{m-1}x(kT_m) + \sum_{i=0}^{m-2} A^i B_1 K x(kT_m - T_s) + \sum_{i=0}^{m-2} A^i E v(kT_m) \quad (12)$$

and

$$x[(k+1)T_m] = A^m x(kT_m) + \left(\sum_{i=1}^{m-1} A^i B_1 + \Gamma_1(\tau^k)\right) Kx(kT_m - T_s) \\ + \Gamma_0(\tau^k)Kx((k+1)T_m - T_s) + \sum_{i=0}^{m-1} A^i Ev(kT_m) \tag{13}$$

Substitute $x((k+1)T_m - T_s)$ into (13), then

$$\begin{aligned} x[(k+1)T_m] = & \left(A^m + \Gamma_0(\tau^k)KA^{m-1}\right) x(kT_m) \\ & + \left(\sum_{i=1}^{m-1} A^i B_1 + \Gamma_1(\tau^k) + \Gamma_0(\tau^k)K \sum_{i=0}^{m-2} A^i B_1\right) Kx(kT_m - T_s) \\ & + \left(\sum_{i=0}^{m-1} A^i + \Gamma_0(\tau^k)K \sum_{i=0}^{m-2} A^i\right) Ev(kT_m) \end{aligned} \tag{14}$$

Further, $\Gamma_1(\tau^k)$ and $\Gamma_0(\tau^k)$ can be distributed as

$$\Gamma_1(\tau^k) = \Gamma_1 + \Delta\Gamma_1(\tau^k), \quad \Gamma_0(\tau^k) = \Gamma_0 + \Delta\Gamma_0(\tau^k) \tag{15}$$

where Γ_1 and Γ_0 are the deterministic parts of the matrices $\Gamma_1(\tau^k)$ and $\Gamma_0(\tau^k)$ separately, while $\Delta\Gamma_1(\tau^k)$ and $\Delta\Gamma_0(\tau^k)$ are the uncertain parts.

Moreover, $\Delta\Gamma_1(\tau^k)$ and $\Delta\Gamma_0(\tau^k)$ always can be written as

$$\left[\Delta\Gamma_1(\tau^k) \quad \Delta\Gamma_0(\tau^k)\right] = DF(\tau^k)[\Omega_1 \quad \Omega_2] \tag{16}$$

$F(\tau^k)$ satisfies $F(\tau^k)F(\tau^k) \leq I$, while D, Ω_1 and Ω_2 are constant matrices with appropriate dimensions. The calculation method and expressions of these matrices can refer to [32].

If we let $\hat{x}[k] = \begin{bmatrix} x(kT_m) \\ x(kT_m - T_s) \\ Kx(kT_m - T_s) \end{bmatrix}$ and $v(k) = v(kT_m)$, then the above equations can be written as

$$\begin{cases} \hat{x}[k+1] = \left(\Phi_{(m)} + \Delta\Phi_{(m,\tau^k)}\right)\hat{x}[k] + \left(E_{(m)} + \Delta E_{(m,\tau^k)}\right) v(k) \\ y(k) = \hat{C}\hat{x}(k) \end{cases} \tag{17}$$

where

$$\Phi_{(m)} = \begin{bmatrix} A^m + \Gamma_0 K A^{m-1} & \sum\limits_{i=1}^{m-1} A^i B_1 K + \Gamma_1 K & \Gamma_0 K \sum\limits_{i=0}^{m-2} A^i B_1 \\ A^{m-1} & \sum\limits_{i=0}^{m-2} A^i B_1 K & 0 \\ KA^{m-1} & 0 & K\sum\limits_{i=0}^{m-2} A^i B_1 \end{bmatrix},$$

$$\Delta\Phi_{(m,\tau^k)} = \begin{bmatrix} \Delta\Gamma_0(\tau^k)KA^{m-1} & \Delta\Gamma_1(\tau^k)K & \Delta\Gamma_0(\tau^k)K\sum\limits_{i=0}^{m-2} A^i B_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{(m)} = \begin{bmatrix} \sum_{i=0}^{m-1} A^i E + \Gamma_0 K \sum_{i=0}^{m-2} A^i E \\ \sum_{i=0}^{m-2} A^i E \\ K \sum_{i=0}^{m-2} A^i E \end{bmatrix}, \quad \Delta E_{(m,\tau^k)} = \begin{bmatrix} \Delta\Gamma_0(\tau^k) K \sum_{i=0}^{m-2} A^i E \\ 0 \\ 0 \end{bmatrix},$$

$$\hat{C} = [C \ \ 0 \ \ 0].$$

In this case, $m = T_m/T_s \geq 2$, and $h = 0, 1, 2, \ldots, N$.

For the case of $m = 1$, namely no packet dropout, the following dynamic equation is derived

$$\begin{cases} \hat{x}[k+1] = \left(\Phi_{(1)} + \Delta\Phi_{(1,\tau^k)}\right)\hat{x}[k] + \left(E_{(1)} + \Delta E_{(1,\tau^k)}\right) v(k) \\ y(k) = \hat{C}\hat{x}(k) \end{cases} \tag{18}$$

where

$$\Phi_{(1)} = \begin{bmatrix} A + \Gamma_0 K & \Gamma_1 K & 0 \\ I & 0 & 0 \\ K & 0 & 0 \end{bmatrix}, \quad E_{(1,\tau^k)} = \begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix},$$

$$\Delta\Phi_{(1,\tau^k)} = \begin{bmatrix} \Delta\Gamma_0(\tau^k) K & \Delta\Gamma_1(\tau^k) K & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Delta E_{1,\tau^k} = 0.$$

For the case of aperiodic packet dropouts, from [27] one knows the NCS along with a typical aperiodic delay and packet dropout can be seen as a dynamic system switching among the dynamics with different periodic delay and packet dropout pattern $\Phi_{(m,\tau^k)}$. This observation leads to modeling the multi-rate NCS with packet drop and short time delay in this case as a switching system namely as

$$\begin{cases} \hat{x}[k+1] = \left(\Phi_{(m)} + \Delta\Phi_{(m,\tau^k)}\right)\hat{x}[k] + \left(E_{(m)} + \Delta E_{(m,\tau^k)}\right) v(k) \\ y(k) = \hat{C}\hat{x}(k) \end{cases} \tag{19}$$

where

$$\Phi_{(m)} = \begin{cases} \begin{bmatrix} A + \Gamma_0 K & \Gamma_1 K & 0 \\ I & 0 & 0 \\ K & 0 & 0 \end{bmatrix}, & m = 1 \\ \begin{bmatrix} A^m + \Gamma_0 K A^{m-1} & \sum_{i=1}^{m-1} A^i B_1 K + \Gamma_1 K & \Gamma_0 K \sum_{i=0}^{m-2} A^i B_1 \\ A^{m-1} & \sum_{i=0}^{m-2} A^i B_1 K & 0 \\ K A^{m-1} & 0 & K \sum_{i=0}^{m-2} A^i B_1 \end{bmatrix}, & m \geq 2 \end{cases},$$

$$E_{(m)} = \begin{cases} \begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix}, & m = 1 \\ \begin{bmatrix} \sum_{i=0}^{m-1} A^i E + \Gamma_0 K \sum_{i=0}^{m-2} A^i E \\ \sum_{i=0}^{m-2} A^i E \\ K \sum_{i=0}^{m-2} A^i E \end{bmatrix}, & m \geq 2 \end{cases},$$

$$\Delta\Phi_{(m,\tau^k)} = \begin{cases} \begin{bmatrix} \Delta\Gamma_0(\tau^k)K & \Delta\Gamma_1(\tau^k)K & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & m = 1 \\ \begin{bmatrix} \Delta\Gamma_0(\tau^k)KA^{m-1} & \Delta\Gamma_1(\tau^k)K & \Delta\Gamma_0(\tau^k)K\sum\limits_{i=0}^{m-2} A^i B_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & m \geq 2 \end{cases},$$

$$\begin{bmatrix}\Delta\Gamma_1(\tau^k) & \Delta\Gamma_0(\tau^k)\end{bmatrix} = DF(\tau^k)[\Omega_1 \quad \Omega_2] \text{ and } F(\tau^k)F(\tau^k) \leq I,$$

$$\Delta E_{(m,\tau^k)} = \begin{cases} 0, & m = 1 \\ \begin{bmatrix} \Delta\Gamma_0(\tau^k)K\sum\limits_{i=0}^{m-2} A^i E \\ 0 \\ 0 \end{bmatrix}, & m \geq 2 \end{cases}, \quad \hat{C} = [0 \quad C].$$

(4) Case 4. As the actuator is still event driving, the character of network-induced delay is the same as that in Case 3. Therefore, the model of multi-rate NCS is the same as that in Case 3.

Remark 3.2. *From the models of multi-rate NCS with both short time delay and packet dropout at the 4 cases, we know multi-rate NCS with both short time delay and packet dropout can be modeled as a switched stochastic system. And multi-rate NCS with both short time delay and packet dropout can be modeled as a switched system when the actuator is time driven and the state noise is not considered.*

4. State Feedback Control of Multi-rate NCS.

Assume the states of multi-rate NCS all above are completely measurable. Moreover, following state feedback is introduced:

$$u(k) = Kx(k) \tag{20}$$

To analyze the stability of the system expediently, following lemma is introduced.

Lemma 4.1. *W, M, N, F are matrices with suitable dimensions, satisfying $F^TF \leq I$, and W is symmetric matrix, then $W + MFN + N^TF^TM^T < 0$ is equivalent to $W + \varepsilon MM^T + \varepsilon^{-1}N^TN < 0$, where the scalar $\varepsilon > 0$.*

Theorem 4.1. *For all allowable uncertainties in system (19), given symmetric positive definite matrices S_1, S_2 and S_3, if there exists gain matrix K, as well as constants $\varepsilon, \mu > 0$, letting the LMIs satisfy*

$$\begin{bmatrix} \varepsilon I & 0 & -\hat{\Omega}_1 & 0 \\ * & S^{-1} - \varepsilon\hat{D}\hat{D}^T & -\Phi_{(1)} & -E_{(1)} \\ * & * & S - \hat{C}^T\hat{C} & 0 \\ * & * & * & \mu I \end{bmatrix} > 0, \quad m = 1 \tag{21}$$

$$\begin{bmatrix} \varepsilon I & 0 & -\hat{\Omega}_2 & -\hat{\Omega}_3 \\ * & S^{-1} - \varepsilon\hat{D}\hat{D}^T & -\Phi_{(m)} & -E_{(m)} \\ * & * & S - \hat{C}^T\hat{C} & 0 \\ * & * & * & \mu I \end{bmatrix} > 0, \quad m = 2, 3, \ldots, D \tag{22}$$

then it is called that system (19) is asymptotically stable with H_∞ norm bound γ, where $\hat{D} = \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix}$, $\hat{\Omega}_1 = [\Omega_2 K \quad \Omega_1 K \quad 0]$, $S = diag(S_1, S_2, S_3)$, $\hat{\Omega}_3 = \Omega_2 K \sum_{i=0}^{m-2} A^i E$, $\hat{\Omega}_2 = \left[\Omega_2 K A^{m-1} \quad \Omega_1 K \quad \Omega_2 K \sum_{i=0}^{m-2} A^i B\right]$*; others refer to Equation (19).*

Proof: Consider the following Lyapunov function,

$$V(k) = z^T(k) S z(k)$$

S is a symmetric positive definite matrix because S_1, S_2 and S_3 are symmetric positive definite matrices.

(1) If $m = 1$, along arbitrary trajectory of system (19), one can have

$$\Delta V(k) = V(k+1) - V(k) = \begin{bmatrix} z^T(k) \\ v^T(k) \end{bmatrix} \Pi [z(k) \quad v(k)] \tag{23}$$

where $\Pi = \begin{bmatrix} \left(\Phi_{(1)} + \Delta\Phi_{(1,\tau^k)}\right)^T S \left(\Phi_{(1)} + \Delta\Phi_{(1,\tau^k)}\right) - S & \left(\Phi_{(1)} + \Delta\Phi_{(1,\tau^k)}\right)^T S E_{(1)} \\ * & \left(E_{(1)} + \Delta E_{(1,\tau^k)}\right)^T S E_{(1)} \end{bmatrix}$.

Plusing $y^T(k)y(k) - \gamma^2 v^T(k)v(k)$ at two ends of equality (23), we have

$$\Delta V(k) + y^T(k)y(k) - \gamma^2 v^T(k)v(k) = \begin{bmatrix} z^T(k) \\ v^T(k) \end{bmatrix} \Pi' [z(k) \quad v(k)] \tag{24}$$

where $\Pi' = \begin{bmatrix} \Pi'_1 & \Pi'_2 \\ * & E_{(1)}^T S E_{(1)} - \gamma^2 I \end{bmatrix}$, $\Pi'_1 = \left(\Phi_{(1)} + \Delta\Phi_{(1,\tau^k)}\right)^T S \left(\Phi_{(1)} + \Delta\Phi_{(1,\tau^k)}\right) - S + \hat{C}^T\hat{C}$. $\Pi'_2 = \Phi_{(1)}^T S E_{(1)}$.

Consider following inequality

$$\Pi' < 0 \tag{25}$$

1) If $v(k) \equiv 0$, based on equality (24), we have $\Delta V(k) < 0$;

2) In zero initial case, we know $V(0) = 0$. And it can be obtained that $V(\infty) \geq 0$. Therefore,

$$\begin{aligned} &\sum_{k=0}^{\infty} \left[\Delta V(k) + y^T(k)y(k) - \gamma^2 v^T(k)v(k)\right] \\ &= V(\infty) - V(0) + \sum_{k=0}^{\infty} \left[y^T(k)y(k) - \gamma^2 v^T(k)v(k)\right] < 0. \end{aligned}$$

Moreover,

$$\sum_{k=0}^{\infty} \left[y^T(k)y(k) - \gamma^2 v^T(k)v(k)\right] < -V(\infty) \leq 0 \tag{26}$$

Therefore, $\sum_{k=0}^{\infty} y^T(k)y(k) < \sum_{k=0}^{\infty} \gamma^2 v^T(k)v(k)$. Namely, $\|y(k)\|_2 < \gamma \|v(k)\|_2$.

From 1) and 2), it knows that system (19) is asymptotically stable with H_∞ norm bound γ.

Using Schur complement theory, inequality (25) is equivalent to

$$\begin{bmatrix} S^{-1} & -\left(\Phi_{(1)} + \Delta\Phi_{(1,\tau^k)}\right) & -E_{(1)} \\ * & S - \hat{C}^T\hat{C} & 0 \\ * & * & \gamma^2 I \end{bmatrix} > 0 \tag{27}$$

Inequality (27) can be written as

$$\begin{bmatrix} S^{-1} & -\Phi_{(1)} & -E_{(1)} \\ * & S-\hat{C}^T\hat{C} & 0 \\ * & * & \gamma^2 I \end{bmatrix} - \begin{bmatrix} \hat{D} \\ 0 \\ 0 \end{bmatrix} F(t) \begin{bmatrix} 0 \\ \hat{\Omega}_1^T \\ 0 \end{bmatrix}^T - \begin{bmatrix} 0 \\ \hat{\Omega}_1^T \\ 0 \end{bmatrix} F^T(t) \begin{bmatrix} \hat{D} \\ 0 \\ 0 \end{bmatrix}^T > 0 \quad (28)$$

Based on Lemma 4.1, inequality (28) is equivalent to

$$\begin{bmatrix} S^{-1} & -\Phi_{(1)} & -E_{(1)} \\ * & S-\hat{C}^T\hat{C} & 0 \\ * & * & \gamma^2 I \end{bmatrix} - \varepsilon \begin{bmatrix} \hat{D} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \hat{D} \\ 0 \\ 0 \end{bmatrix}^T - \varepsilon^{-1} \begin{bmatrix} 0 \\ \hat{\Omega}_1^T \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \hat{\Omega}_1^T \\ 0 \end{bmatrix}^T > 0 \quad (29)$$

By applying the Schur complement to LMI (29) again and assuming $\mu = \gamma^2$, inequality above is equivalent to (21).

(2) Homoplastically, if $m \geq 2$, inequality (22) can also be obtained. Therefore, this completes the proof.

Remark 4.1. *The state-feedback control of multi-rate NCS with no network-induced delay with both short time delay and packet dropout have been discussed. Moreover, inequalities (21) and (22) belong to linear matrix inequalities, so LMI-Toolbox can be used to help solve the problems.*

5. **Example.** Now, we give an example to verify the theory results of this paper. This example is the case of multi-rate NCS with both short time delay and packet dropout. Consider the continuous-time integrator with disturbances as follows

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0.2 & -1.4 \end{bmatrix} x(t) + \begin{bmatrix} 20 \\ 10 \end{bmatrix} u(t) + \begin{bmatrix} 0.01 \\ 0.1 \end{bmatrix} v(t) \\ y(t) = \begin{bmatrix} 2 & 1 \\ 2.001 & -0.0103 \end{bmatrix} x(t) \end{cases}$$

Assume that the sampling period of sensor is 0.1ms. The controller is time driving, and the actuator is event driving. The controller reads the receiving buffer every 0.05ms, that is to say $T_c = 0.05$ms, $T_s = 0.1$ms, $N = T_s/T_c = 2$, the maximum number of successively dropped packets $D = 3$ and random short time delay $\tau^k \in [0, T_s]$.

Then we have

$$A = e^{A^c T_s} = \begin{bmatrix} 1 & 0.1 \\ 0.02 & 0.86 \end{bmatrix}, \; B_1 = \int_0^{T_c} e^{A^c t} B^c d_t = \begin{bmatrix} 1.0125 \\ 0.4875 \end{bmatrix},$$

$$E = \int_0^{T_s} e^{A^c (T_s - t)} E^c d_t = \begin{bmatrix} 0.0011 \\ 0.0104 \end{bmatrix}, \quad \Gamma_0(\tau) = \begin{bmatrix} 2 - 20\tau^k + 5(0.1-\tau^k)^2 \\ 1 - 10\tau^k - 5(0.1-\tau^k)^2 \end{bmatrix},$$

$$\Gamma_1(\tau) = \begin{bmatrix} 0.05 + 20\tau^k - 5(0.1-\tau^k)^2 \\ -0.05 + 10\tau^k + 5(0.1-\tau^k)^2 \end{bmatrix}.$$

Here select,

$$\Gamma_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0.05 \\ -0.05 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$F(\tau) = \begin{bmatrix} -\tau^k & 0.5(0.1-\tau^k)^2 \\ 0.1(0.1-\tau^k)^2 & -\tau^k - 0.7(0.1-\tau^k)^2 \end{bmatrix}.$$

Obviously, $F^T(t)F(t) < I$.

Therefore,

$$\Omega_1 = \begin{bmatrix} -20 \\ -10 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} 20 \\ 10 \end{bmatrix}, \quad \Delta\Gamma_0(\tau) = \begin{bmatrix} -20\tau^k + 5(0.1 - \tau^k)^2 \\ -10\tau^k - 5(0.1 - \tau^k)^2 \end{bmatrix},$$

$$\Delta\Gamma_1(\tau) = \begin{bmatrix} 20\tau^k - 5(0.1 - \tau^k)^2 \\ 10\tau^k + 5(0.1 - \tau^k)^2 \end{bmatrix}.$$

Substitute these parameters into (24) and let

$$S_1 = \begin{bmatrix} 3 \times 10^{18} & 0 \\ 0 & 0.02 \times 10^{18} \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 \times 10^{14} & 0 \\ 0 & 1 \times 10^{19} \end{bmatrix}, \quad S_3 = 5 \times 10^{18}.$$

By researching the feasibility of the solution to LMIs (23) and (24), then we have $\gamma = \sqrt{7.7477 \times 10^8} = 2.7835 \times 10^4$, $K = [-0.0104 \quad -7.0295 \times 10^{-7}]$.

Consider the initial state of system: $x(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, and noise:

$$v(t) = \begin{cases} 0.5 & 55 \le k \le 66 \\ 0 & \text{others} \end{cases}$$

The system is stable based on the state response curve shown as Figure 1 and control signal curve shown as Figure 2.

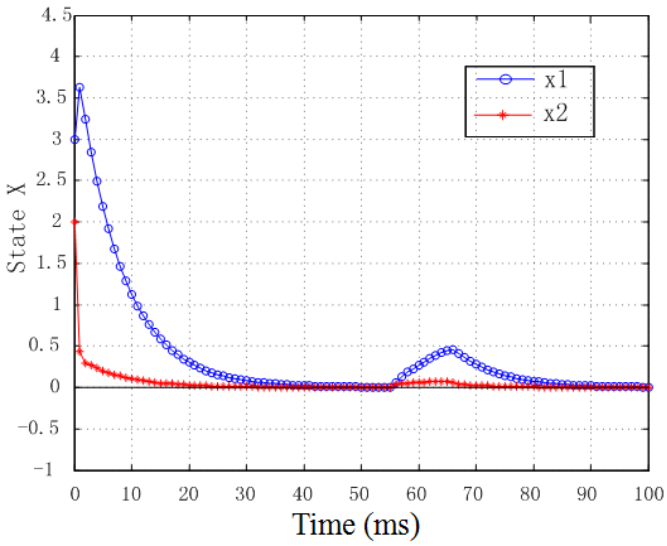

FIGURE 1. The state response curve of multi-rate NCS with packet drop and short time delay

Obviously, state feedback control strategies introduced in this paper can guarantee the stabilities of multi-rate NCS with both short time delay and packet dropout. When the approaches proposed by [29] are used, the state response curves of closed-loop multi-rate NCS with both short time delay and packet dropout are shown in Figure 3 and the closed loop system is unstable. The effectiveness and advantage of the method in this work are sufficiently illustrated.

6. **Conclusions.** The model of multi-rate networked control systems with both short time delay and packet dropout is set up. It is stated that the multi-rate networked control systems can be formulated as a discrete-time switched stochastic system. Especially the multi-rate networked control systems can be formulated as a discrete-time switched system when the actuator is clock driven and the state noise is not considered. Moreover,

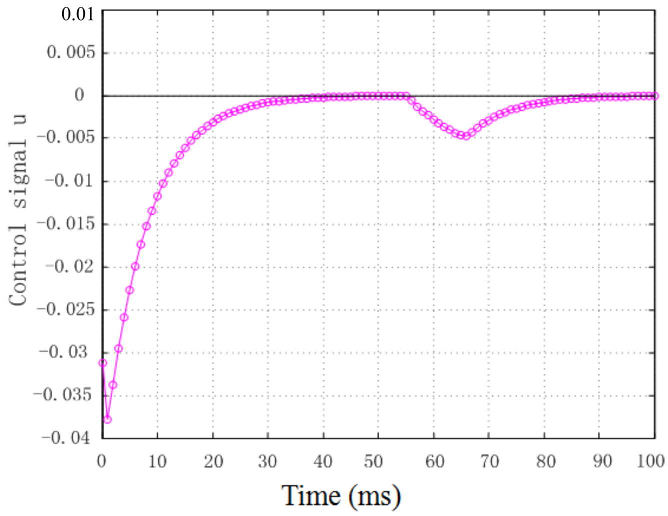

FIGURE 2. The control signal curve of multi-rate NCS with packet drop and short time delay

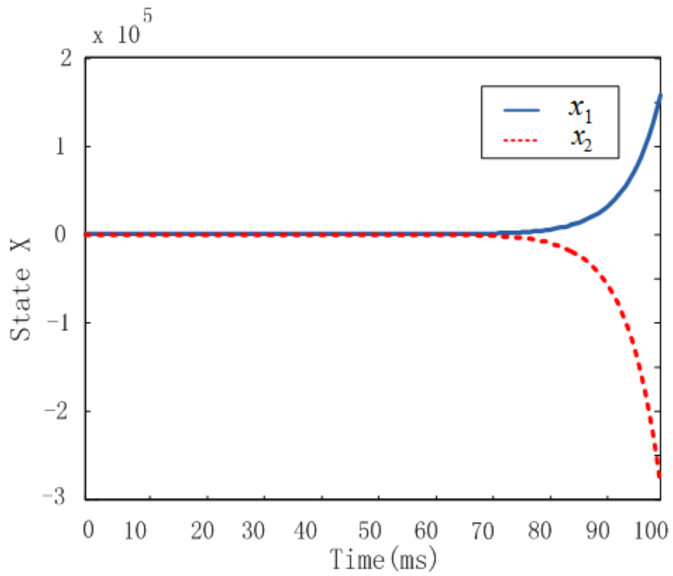

FIGURE 3. The state response curves of multi-rate NCS by the method in [29]

corresponding state feedback controllers are proposed to guarantee the stabilities of multi-rate NCS with both short time delay and packet dropout, while the modeling and stability analysis of multi-rate NCS with both packet drop and long time delay is very challenging work and will be studied in detail in the future.

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REFERENCES

[1] L. Zou, Z. D. Wang and H. J. Gao, Observer-based H-infinity control of networked systems with stochastic communication protocol: The finite-horizon case, *Automatica*, vol.63, no.1, pp.366-373, 2016.

[2] S. S. Hu and Q. X. Zhu, Stochastic optimal control and analysis of stability of networked control systems with long delay, *Automatica*, vol.39, no.11, pp.1877-1884, 2003.

[3] B. Tang, J. Wang and Y. Zhang, A delay-distribution approach to stabilization of networked control systems, *IEEE Trans. Control of Network Systems*, vol.2, no.4, pp.382-392, 2015.

[4] Q. X. Zhu, H. L. Liu and S. S. Hu, Uniformed model of networked control systems with long time delay, *Journal of Systems Engineering and Electronics*, vol.19, no.2, pp.385-390, 2008.

[5] Y. Tipsuwan and M.-Y. Chow, Control methodologies in networked control systems, *Control Eng. Practice*, vol.11, no.10, pp.1099-1111, 2003.

[6] R. A. Gupta and M. Y. Chow, Networked control systems: Overview and research trends, *IEEE Trans. Industrial Electronics*, vol.57, no.10, pp.2527-2535, 2010.

[7] P. Antsaklis and J. Baillieul, Guest editorial special issue on networked control systems, *IEEE Trans. Automatic Control*, vol.49, no.9, pp.1421-1423, 2004.

[8] J. P. Hespanha, P. Naghshtabrizi and Y. Xu, A survey of recent results in networked control systems, *Proc. of IEEE*, vol.95, no.1, pp.138-162, 2007.

[9] T. C. Yang, Networked control system: A brief survey, *IEE Proceedings – Control Theory & Applications*, vol.153, no.4, pp.403-412, 2006.

[10] R. A. Gupta, A. A. Masoud and M.-Y. Chow, A delay-tolerant potential field-based network implementation of an integrated navigation system, *IEEE Trans. Ind. Electron.*, vol.57, no.2, pp.769-783, 2010.

[11] A. M. Memon and M. S. Mahmoud, Two-level design for aperiodic networked control systems, *Signal Processing*, vol.120, no.8, pp.43-55, 2015.

[12] Y.-L. Wang and Q.-L. Han, Modelling and controller design for discrete-time networked control systems with limited channels and data drift, *Information Sciences*, vol.269, no.4, pp.332-348, 2014.

[13] H. Lin, H. Y. Su, P. Shi et al., LQG control for networked control systems over packet drop links without packet acknowledgment, *Journal of the Franklin Institute*, vol.352, no.11, pp.5042-5060, 2015.

[14] Y. Halevi and A. Ray, Integrated communication and control systems: Part 1 – Analysis, *Trans. of the ASME, Journal of Dynamic Systems, Measurement, and Control*, vol.110, no.4, pp.367-373, 1988.

[15] R. Luck and A. Ray, An observer-based compensator for distributed delays, *Automatica*, vol.26, no.5, pp.903-908, 1990.

[16] R. Luck and A. Ray, Experimental verification of a delay compensation algorithm for integrated communication and control systems, *International Journal of Control*, vol.59, no.6, pp.1357-1372, 1994.

[17] G. C. Walsh, O. Beldiman and L. Bushnell, Asymptotic behavior of networked control systems, *Proc. of the 1999 IEEE International Conference on Control Applications*, Kohala Coast, Hawaii, pp.1448-1453, 1999.

[18] G. C. Walsh, O. Beldiman and L. Bushnell, Error encoding algorithms for networked control systems, *Automatica*, vol.38, no.2, pp.261-267, 2002.

[19] K. K. Ahn and Q. T. Dinh, Robust variable sampling period control for networked control systems, *IEEE Trans. Industrial Electronics*, vol.62, no.9, pp.5630-5643, 2015.

[20] Q. Zhu and G. Xie, Analysis and modelling of multi-rate networked control systems with long time delay, *Proc. of Chinese Control and Decision Conference*, Taiyuan, China, pp.2990-2995, 2012.

[21] K. Lee and R. Bhattacharya, Stability analysis of large-scale distributed networked control systems with random communication delays: A switched system approach, *Systems & Control Letters*, vol.85, no.3, pp.77-83, 2015.

[22] D. M. Tilbury and D. Georgiev, Packet-based control: The h2-optimal solution, *Automatica*, vol.42, no.1, pp.137-144, 2006.

[23] H. J. Gao and T. W. Chen, Network-based H_∞ output tracking control, *IEEE Trans. Automatic Control*, vol.53, no.3, pp.655-667, 2008.

[24] X. F. Jiang, Q. L. Han, S. R. Liu and A. K. Xue, A new H_∞ stabilization criterion for networked control systems, *IEEE Trans. Automatic Control*, vol.53, no.4, pp.1025-1032, 2008.

[25] S. M. Zhang, D. Z. Zhao, C. W. Li et al., Fuzzy speed control of networked motion control systems, *Journal of Computational and Nonlinear Dynamics*, vol.10, no.6, pp.1555-1563, 2015.

[26] D. Goerges, S. AI-Areqi and S. Liu, Event-based control and scheduling codesign: Stochastic and robust approaches, *IEEE Trans. Automatic Control*, vol.60, no.5, pp.1291-1303, 2015.

[27] H. Lin, G. S. Zhai and P. J. Antsaklis, Robust stability and disturbance attenuation analysis of a class of networked control systems, *Proc. of the 42nd IEEE Conference on Decision and Control*, Lahaina, Hawaii, pp.1182-1187, 2003.

[28] H. Lin and P. J. Antsaklis, Stability and persistent disturbance attenuation properties for a class of networked control systems: Switched system approach, *International Journal of Control*, vol.78, no.18, pp.1447-1458, 2006.

[29] L. Y. Xiang and G. Guo, Model-based networked control systems with multi-rate sampling, *Information and Control*, vol.34, no.3, pp.373-378,380, 2005 (in Chinese).

[30] Z. H. Guan, C. X. Yang and J. Huang, Stabilization of networked control systems with random delays: A new multirate method, *Proc. of the 17th World Congress of IFAC*, South Korea, pp.4204-4209, 2008.

[31] C. Xie, W. Fan and W. Hu, Modeling and control method of a class of networked control systems with short time delay, *Journal of Nanjing University of Science and Technology*, vol.33, no.2, pp.156-160, 2009 (in Chinese).

[32] H. L. Liu and Q. X. Zhu, Analysis and modeling of multi-rate networked control systems with short time delay, *Advanced Science Letters*, vol.11, no.1, pp.745-750, 2012.

[33] W. A. Zhang and L. Yu, Modelling and control of networked control systems with both network-induced delay and packet-dropout, *Automatica*, vol.44, no.12, pp.3181-3188, 2008.

[34] C. Tan, L. Li and H. S. Zhang, Stabilization of networked control systems with both network-induced delay and packet dropout, *Automatica*, vol.59, no.9, pp.194-199, 2015.