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MODELLING FOR MECHANICAL ELEMENTS OF RECTANGULAR MEMBERS WITH STRAIGHT HAUNCHES USING SOFTWARE: PART 1

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Abstract. *This paper presents a mathematical model for mechanical elements of rectangular members with straight haunches for the general case (symmetrical and/or nonsymmetrical) subjected to a uniformly distributed load considering the bending and shear deformations to obtain the fixed-end moments, carry-over factors and stiffness factors, which is novelty of this research. The properties of rectangular cross section the member are: the width "b" is constant and the height "h" varies along beam, this variation is linear type. The superposition method is used to obtain the solution of such problems, and the deformations anywhere of beam are found by the conjugate beam method through exact integrations using the software "Derive" to obtain some results. The traditional model takes into account only bending deformations. Also a comparison is made between proposed model (bending and shear deformations are considered), and traditional model (bending deformations are taken into account) to show the differences. Besides the effectiveness and accuracy of the developed model, a significant advantage is that fixed-end moments, carry-over factors and stiffness factors are obtained for any rectangular cross section of beam using the mathematical equations.*
Keywords: Straight haunches for the general case (symmetrical and/or nonsymmetrical), Bending and shear deformations, Uniformly distributed load, Fixed-end moments, Carry-over factors, Stiffness factors, Superposition method, Conjugate beam method

1. **Introduction.** The members with haunches of reinforced concrete are distinguished from prismatic ones because the beam height has a gradual variation in all or part of its length, its application in buildings of moderate elevation, as well as on bridges and viaducts of various functions. In buildings, the beams with haunches of reinforced concrete offer the following advantages over prismatic beams: 1) The lateral stiffness of buildings is increased substantially; 2) These beams types lead to a more efficient use of concrete and steel reinforcement; 3) The weight of the structure is reduced to optimize the strength and stability or to meet architectural requirements and specific functions of service; 4) The use of beams with haunches eases the placement of the electrical installation, air conditioning, water and sewage equipment, etc.

During the last century, between 1950 and 1960 were developed several design aids, as those presented by Guldan [1], and the most popular tables published by the Portland Cement Association (PCA) in 1958 "Handbook" [2].

Traditional methods used for the variable cross section members, the deflections by Simpson's rule are obtained or some other techniques to perform numerical integration and the tables presenting some books are limited to certain relationships [3-5].

The most relevant papers addressing the issue of structural members with haunches are: *Stiffness Formulation for Nonprismatic Beam Elements*, which paper presents a method to define two-dimensional (2D) and three-dimensional (3D) elastic-stiffness matrices for nonprismatic elements (tapered or haunched), based on traditional beam theory and the flexibility method [6]; *Plane Frameworks of Tapering Box and I-Section*, which is based on the classical theory of beams of Bernoulli-Euler for two-dimensional member without including axial deformations [7]; *Elementary Theory for Linearly Tapered Beams*, which proposed a more rigorous theory of beams for members varying linearly, in which the hypothesis generalized by Kirchhoff were introduced to take account the shear deformations [8]; *Nonprismatic Shear Beams*, which shows a solution of the problem of bending of nonprismatic beams, including the effect of shear deformations [9]; *Approximate Stiffness Matrix for Tapered Beams*, which proposed a method for finding a modified bending stiffness matrix for a member of varying section [10]; *An Efficient Procedure to Find Shape Functions and Stiffness Matrices of Nonprismatic Euler-Bernoulli and Timoshenko Beam Elements* [11]; *A Mathematical Model for Rectangular Beams of Variable Cross Section of Symmetrical Parabolic Shape for Uniformly Distributed Load* [12], *Mathematical Model for Rectangular Beams of Variable Cross Section of Symmetrical Linear Shape for Uniformly Distributed Load* [13], *Mathematical Model for Rectangular Beams of Variable Cross Section of Symmetrical Linear Shape for Concentrated Load* [14], *A Mathematical Model for Fixed-End Moments for Two Types of Loads for Variable Rectangular Cross Section* of parabolic shape [15], which are shown for the cases of symmetrical haunches and shear deformations are neglected; *Modeling for Beams of Cross Section "I" Subjected to a Uniformly Distributed Load with Straight Haunches* [16].

This paper presents a mathematical model for mechanical elements of rectangular members with straight haunches for the general case (symmetrical and/or nonsymmetrical) subjected to a uniformly distributed load considering the bending and shear deformations to obtain the fixed-end moments, carry-over factors and stiffness factors, and the deformations anywhere of beam are found by the conjugate beam method through exact integrations using the software "Derive" to obtain some results, which is novelty of this research. The properties of the rectangular cross section of beam vary along its axis "x", i.e., the width "b" is constant and the height "h" varies along the beam, and this variation is straight type. Also a comparison is made between proposed model (bending and shear deformations are considered), and traditional model (bending deformations are taken into account) to show the differences.

The paper is organized as follows. Section 2 shows the formulation of the mathematical model. Section 2.1 shows the derivation of the equations for fixed-end moments. Section 2.2 presents the derivation of the equations for carry-over factors and stiffness factors. Section 3 is dedicated to the results through the comparison of the two models, the proposed model (PM), and the traditional model (TM). Section 4 presents the conclusions.

2. Formulation of the Mathematical Model.

Figure 1 shows a beam in elevation and also presents its rectangular cross-section taking into account that the width "b" is constant and height "h_x" varying of straight shape in three different parts.

Table 1 shows the equations of the heights "h_x", shear areas "A_{sx}" to a distance "x", and the moment of inertia "I_z" around the axis "z" for each interval.

2.1. Derivation of the equations for fixed-end moments.

Figure 2(a) presents the beam AB subjected to a uniformly distributed load and fixed-ends. The fixed-end moments are found by the sum of the effects. The moments are considered positive in counterclockwise and the moments are considered negative in clockwise. Figure 2(b) shows the

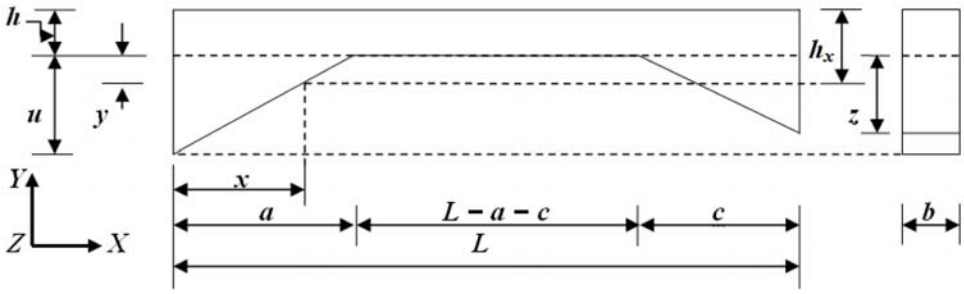

FIGURE 1. Rectangular section with straight haunches

TABLE 1. Properties of the rectangular section

Concept	Equations		
Interval	$0 \le x \le a$	$a \le x \le L-c$	$L-c \le x \le L$
h_x	$\frac{ah+u(a-x)}{a}$	h	$\frac{ch+z(x-L+c)}{c}$
A_{sx}	$\frac{5b[ah+u(a-x)]}{6a}$	$\frac{5bh}{6}$	$\frac{5b[ch+z(x-L+c)]}{6c}$
I_z	$\frac{b[ah+u(a-x)]^3}{12a^3}$	$\frac{bh^3}{12}$	$\frac{b[ch+z(x-L+c)]^3}{12c^3}$

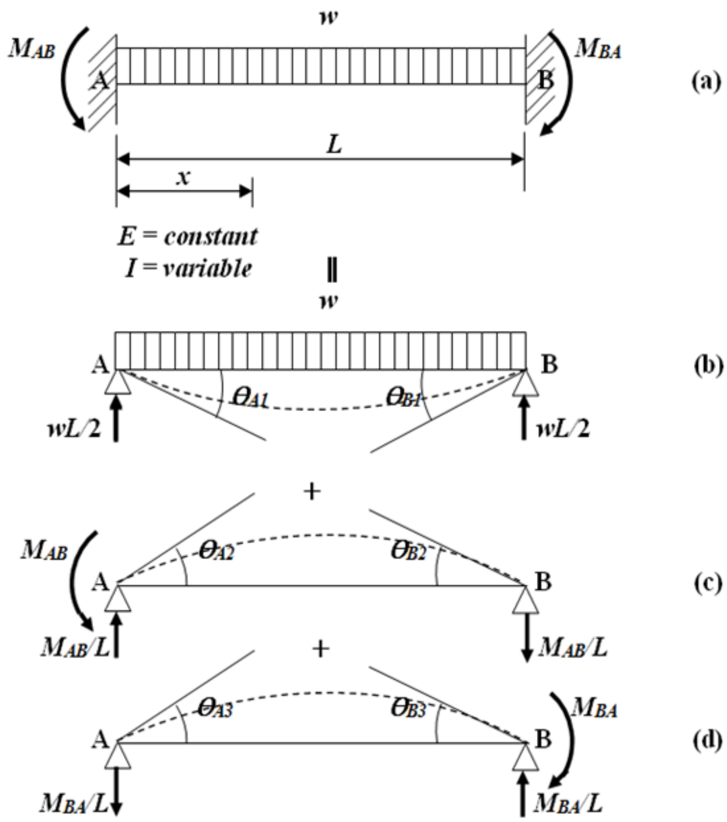

FIGURE 2. Beam fixed at its ends

same simply supported beam at their ends and its load applied to find the rotations "θ_{A1}" and "θ_{B1}". Now, the rotations "θ_{A2}" and "θ_{B2}"are caused by the moment "M_{AB}"applied in the support A, according to Figure 2(c), and in terms of "θ_{A3}" and "θ_{B3}" are caused by the moment "M_{BA}"applied in the support B, seen in Figure 2(d) [12-22].

By superposition method the conditions of geometry are obtained [12-22]:

$$\theta_{A1} - \theta_{A2} - \theta_{A3} = 0 \tag{1}$$

$$\theta_{B1} - \theta_{B2} - \theta_{B3} = 0 \tag{2}$$

By the conjugate beam method the rotations of "θ_{Ai}" and "θ_{Bi}" for non-prismatic members are obtained [23]:

$$\theta_{Ai} = \int_0^L \frac{M_x}{EI_z}dx - \frac{1}{L}\int_0^L \frac{V_x}{GA_{sx}}dx - \frac{1}{L}\int_0^L \frac{M_x x}{EI_z}dx \tag{3}$$

$$\theta_{Bi} = \frac{1}{L}\int_0^L \frac{V_x}{GA_{sx}}dx + \frac{1}{L}\int_0^L \frac{M_x x}{EI_z}dx \tag{4}$$

where i takes the values of 1, 2 and 3, E is the modulus of elasticity, G is shear modulus, V_x and M_x are shear forces and moments to a distance "x".

Table 2 shows the equations of the shear forces and moments anywhere of the beam to a distance "x" [24].

Table 2. Shear forces and moments

Concept	Equations		
	Figure 2(b)	Figure 2(c)	Figure 2(d)
Shear forces	$V_x = \frac{wx(L-2x)}{2}$	$V_x = -\frac{M_{AB}}{L}$	$V_x = \frac{M_{BA}}{L}$
Moments	$M_x = \frac{wx(L-x)}{2}$	$M_x = \frac{M_{AB}(L-x)}{L}$	$M_x = \frac{M_{BA}x}{L}$

The values of "θ_{A1}", "θ_{B1}", "θ_{A2}", "θ_{B2}", "θ_{A3}" and "θ_{B3}" by Equations (3) and (4) are obtained:

$$\begin{aligned}\theta_{A1} = &-\frac{6w}{EbL}\left\{\frac{1+\nu}{5}\left[\int_0^a \frac{a(L-2x)}{ah+u(a-x)}dx + \int_a^{L-c}\frac{(L-2x)}{h}dx\right.\right.\\ &\left.+\int_{L-c}^L \frac{c(L-2x)}{ch+z(x-L+c)}dx\right] - \int_0^a \frac{a^3(L-x)^2x}{[ah+u(a-x)]^3}dx\\ &\left.-\int_a^{L-c}\frac{(L-x)^2x}{h^3}dx - \int_{L-c}^L \frac{c^3(L-x)^2x}{[ch+z(x-L+c)]^3}dx\right\}\end{aligned} \tag{5}$$

$$\begin{aligned}\theta_{B1} = &\frac{6w}{EbL}\left\{\frac{1+\nu}{5}\left[\int_0^a \frac{a(L-2x)}{ah+u(a-x)}dx + \int_a^{L-c}\frac{(L-2x)}{h}dx\right.\right.\\ &\left.+\int_{L-c}^L \frac{c(L-2x)}{ch+z(x-L+c)}dx\right] + \int_0^a \frac{a^3(L-x)x^2}{[ah+u(a-x)]^3}dx\\ &\left.+\int_a^{L-c}\frac{(L-x)x^2}{h^3}dx + c^3\int_{L-c}^L \frac{(L-x)x^2}{[ch+z(x-L+c)]^3}dx\right\}\end{aligned} \tag{6}$$

$$\begin{aligned}\theta_{A2} = &\frac{12M_{AB}}{EbL^2}\left\{\frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx + \int_a^{L-c}\frac{dx}{h}\right.\right.\\ &\left.+\int_{L-c}^L \frac{c}{ch+z(x-L+c)}dx\right] + \int_0^a \frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx\\ &\left.+\int_a^{L-c}\frac{(L-x)^2}{h^3}dx + \int_{L-c}^L \frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx\right\}\end{aligned} \tag{7}$$

$$\theta_{B2} = -\frac{12M_{AB}}{EbL^2}\left\{\frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx + \int_a^{L-c}\frac{dx}{h}\right.\right.$$
$$\left.+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right] - \int_0^a \frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx \tag{8}$$
$$\left.-\int_a^{L-c}\frac{(L-x)x}{h^3}dx - \int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx\right\}$$

$$\theta_{A3} = -\frac{12M_{BA}}{EbL^2}\left\{\frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx + \int_a^{L-c}\frac{dx}{h}\right.\right.$$
$$\left.+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right] - \int_0^a \frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx \tag{9}$$
$$\left.-\int_a^{L-c}\frac{(L-x)x}{h^3}dx - \int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx\right\}$$

$$\theta_{B3} = \frac{12M_{BA}}{EbL^2}\left\{\frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx + \int_a^{L-c}\frac{dx}{h}\right.\right.$$
$$\left.+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right] + \int_0^a \frac{a^3x^2}{[ah+u(a-x)]^3}dx \tag{10}$$
$$\left.+\int_a^{L-c}\frac{x^2}{h^3}dx + \int_{L-c}^{L}\frac{c^3x^2}{[ch+z(x-L+c)]^3}dx\right\}$$

Equations (5), (7) and (9) corresponding to the support A are substituted into Equation (1), and Equations (6), (8) and (10) corresponding to the support B are substituted into Equation (2). Subsequently, generated equations are solved and the values of "M_{AB}" and "M_{BA}" are obtained:

$$M_{AB} = m_{AB}wL^2 \tag{11}$$

$$M_{BA} = m_{BA}wL^2 \tag{12}$$

Equations for fixed-end moments factors of "m_{AB}" and "m_{BA}" are:

$$m_{AB} = \frac{1}{2L}\left[\left\{\int_0^a \frac{a^3(L-x)^2x}{[ah+u(a-x)]^3}dx + \int_a^{L-c}\frac{(L-x)^2x}{h^3}dx\right.\right.$$
$$+\int_{L-c}^{L}\frac{c^3(L-x)^2x}{[ch+z(x-L+c)]^3}dx - \frac{1+\nu}{5}\left[\int_0^a\frac{a(L-2x)}{ah+u(a-x)}dx\right.$$
$$\left.\left.+\int_a^{L-c}\frac{L-2x}{h}dx + \int_{L-c}^{L}\frac{c(L-2x)}{ch+z(x-L+c)}dx\right]\right\}\left\{\int_0^a\frac{a^3x^2}{[ah+u(a-x)]^3}dx\right.$$
$$+\int_a^{L-c}\frac{x^2}{h^3}dx + \int_{L-c}^{L}\frac{c^3x^2}{[ch+z(x-L+c)]^3}dx + \frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx\right.$$
$$\left.\left.+\int_a^{L-c}\frac{dx}{h} + \int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\right\} - \left\{\int_0^a\frac{a^3(L-x)x^2}{[ah+u(a-x)]^3}dx\right.$$
$$+\int_a^{L-c}\frac{(L-x)x^2}{h^3}dx + \int_{L-c}^{L}\frac{c^3(L-x)x^2}{[ch+z(x-L+c)]^3}dx$$
$$+\frac{1+\nu}{5}\left[\int_0^a\frac{a(L-2x)}{ah+u(a-x)}dx + \int_a^{L-c}\frac{L-2x}{h}dx\right.$$
$$\left.\left.+\int_{L-c}^{L}\frac{c(L-2x)}{ch+z(x-L+c)}dx\right]\right\}\left\{\int_0^a\frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx\right.$$

$$
\begin{aligned}
&+\int_a^{L-c}\frac{(L-x)x}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx\\
&-\frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\bigg\}\bigg]\\
&\bigg/\bigg[\bigg\{\int_0^a\frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-x)^2}{h^3}dx\\
&+\int_{L-c}^{L}\frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\right.\\
&\left.+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\bigg\}\bigg\{\int_0^a\frac{a^3x^2}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{x^2}{h^3}dx \qquad (13)\\
&+\int_{L-c}^{L}\frac{c^3x^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx\right.\\
&\left.+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\bigg\}-\bigg\{\int_0^a\frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{(L-x)x}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx\\
&-\frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\bigg\}^2\bigg]
\end{aligned}
$$

$$
\begin{aligned}
m_{BA}=\frac{1}{2L}\bigg[\bigg\{&\int_0^a\frac{a^3(L-x)x^2}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-x)x^2}{h^3}dx\\
&+\int_{L-c}^{L}\frac{c^3(L-x)x^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\left[\int_0^a\frac{a(L-2x)}{ah+u(a-x)}dx\right.\\
&\left.+\int_a^{L-c}\frac{L-2x}{h}dx+\int_{L-c}^{L}\frac{c(L-2x)}{ch+z(x-L+c)}dx\right]\bigg\}\bigg\{\int_0^a\frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{(L-x)^2}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx\\
&+\frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\bigg\}\\
&-\bigg\{\int_0^a\frac{a^3(L-x)^2x}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-x)^2x}{h^3}dx\\
&+\int_{L-c}^{L}\frac{c^3(L-x)^2x}{[ch+z(x-L+c)]^3}dx-\frac{1+\nu}{5}\left[\int_0^a\frac{a(L-2x)}{ah+u(a-x)}dx\right.\\
&\left.+\int_a^{L-c}\frac{L-2x}{h}dx+\int_{L-c}^{L}\frac{c(L-2x)}{ch+z(x-L+c)}dx\right]\bigg\}\bigg\{\int_0^a\frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{(L-x)x}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx\\
&-\frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\right.\\
&\left.+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\bigg\}\bigg]\bigg/\bigg[\bigg\{\int_0^a\frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{a}^{L-c}\frac{(L-x)^2}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx\\
&+\frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\right.\\
&\left.+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\Bigg\}\Bigg\{\int_0^a\frac{a^3x^2}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{x^2}{h^3}dx+\int_{L-c}^{L}\frac{c^3x^2}{[ch+z(x-L+c)]^3}dx \qquad (14)\\
&+\frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\Bigg\}\\
&-\Bigg\{\int_0^a\frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-x)x}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx\\
&-\frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\Bigg\}^2\Bigg]
\end{aligned}
$$

2.2. **Derivation of the equations for carry-over factors and stiffness factors.** In order to develop the method to obtain the carry-over factors and stiffness factors, it will be helpful to consider the following problem: If a clockwise moment of "M_{AB}" is applied at the simple support of a straight member of variable cross section simply supported at one end and fixed at the other, find the rotation "θ_A" at the simple support and the moment "M_{BA}" at the fixed end, as shown in Figure 3.

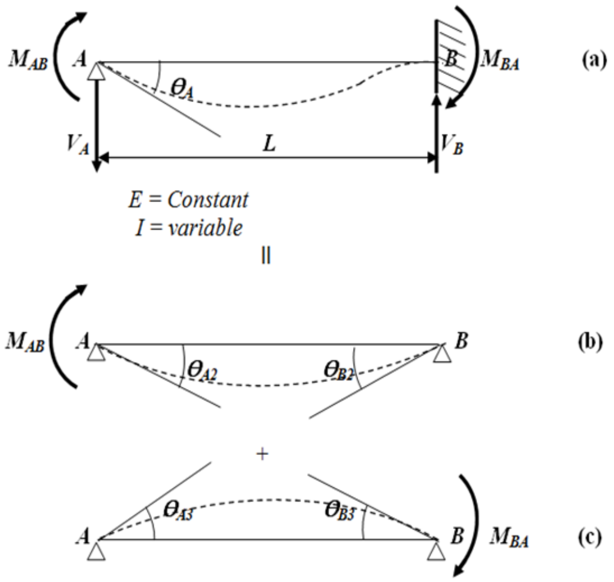

Figure 3. A simply supported beam at one end and fixed at the other

The additional end moments, "M_{AB}" and "M_{BA}" should be such as to cause rotations of "θ_A" and "θ_B", respectively. If "θ_{A2}" and "θ_{B2}" are the end rotations caused by "M_{AB}", according to Figure 3(b), and "θ_{A3}" and "θ_{B3}" by "M_{BA}", these are observed in Figure 3(c).

The conditions of required geometry are [25]:

$$\theta_A = \theta_{A2} - \theta_{A3} \qquad (15)$$

$$0 = \theta_{B2} - \theta_{B3} \qquad (16)$$

Substitute Equations (8) and (10) into Equation (16), and "M_{BA}" is presented in function of "M_{AB}". The carry-over factor of "A" to "B" is the ratio of moment induced at support "B" due to a moment applied at support "A"; the same above procedure is used to find the carry-over factor of "B" to "A". Equations are presented as follows:

$$M_{BA} = C_{AB} M_{AB} \tag{17}$$

$$M_{AB} = C_{BA} M_{BA} \tag{18}$$

Equations for the carry-over factors "C_{AB}" and "C_{BA}" are shown:

$$\begin{aligned} C_{AB} = &\left\{ \int_0^a \frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx + \int_a^{L-c} \frac{(L-x)x}{h^3}dx + \int_{L-c}^L \frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx \right. \\ &\left. - \frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx + \int_a^{L-c}\frac{dx}{h} + \int_{L-c}^L \frac{c}{ch+z(x-L+c)}dx\right]\right\} \\ &\Big/ \left\{ \int_0^a \frac{a^3x^2}{[ah+u(a-x)]^3}dx + \int_a^{L-c} \frac{x^2}{h^3}dx + \int_{L-c}^L \frac{c^3x^2}{[ch+z(x-L+c)]^3}dx \right. \\ &\left. + \frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx + \int_a^{L-c}\frac{dx}{h} + \int_{L-c}^L \frac{c}{ch+z(x-L+c)}dx\right]\right\} \end{aligned} \tag{19}$$

$$\begin{aligned} C_{BA} = &\left\{ \int_0^a \frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx + \int_a^{L-c} \frac{(L-x)x}{h^3}dx + \int_{L-c}^L \frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx \right. \\ &\left. - \frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx + \int_a^{L-c}\frac{dx}{h} + \int_{L-c}^L \frac{c}{ch+z(x-L+c)}dx\right]\right\} \\ &\Big/ \left\{ \int_0^a \frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx + \int_a^{L-c} \frac{(L-x)^2}{h^3}dx + \int_{L-c}^L \frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx \right. \\ &\left. + \frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx + \int_a^{L-c}\frac{dx}{h} + \int_{L-c}^L \frac{c}{ch+z(x-L+c)}dx\right]\right\} \end{aligned} \tag{20}$$

Substitute Equations (7) and (9) into Equation (15), and subsequently Equation (17) is substituted in this same equation to find "M_{AB}" in function of "θ_A". The stiffness "K_{AB}" is the moment applied at support "A" to cause a rotation of 1 radian at support "A"; the same above procedure is used to find the stiffness "K_{BA}" at support "B". Equations are presented:

$$M_{AB} = K_{ABA} = \frac{k_{AB}EI}{L}A \tag{21}$$

$$M_{BA} = K_{BAB} = \frac{k_{BA}EI}{L}B \tag{22}$$

where: I is the moment of inertia of the minimum section.

Equations for the stiffness factors "k_{AB}" and "k_{BA}" are:

$$\begin{aligned} k_{AB} = &\left\{ \int_0^a \frac{a^3x^2}{[ah+u(a-x)]^3}dx + \int_a^{L-c} \frac{x^2}{h^3}dx + \int_{L-c}^L \frac{c^3x^2}{[ch+z(x-L+c)]^3}dx \right. \\ &\left. + \frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx + \int_a^{L-c}\frac{dx}{h} + \int_{L-c}^L \frac{c}{ch+z(x-L+c)}dx\right]\right\}\frac{L^4}{h^3} \\ &\Big/ \left[\left\{ \int_0^a \frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx + \int_a^{L-c} \frac{(L-x)^2}{h^3}dx \right.\right. \end{aligned}$$

$$
\begin{aligned}
&+\int_{L-c}^{L}\frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\right.\\
&\left.+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\Bigg\}\Bigg\{\int_0^a\frac{a^3x^2}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{x^2}{h^3}dx+\int_{L-c}^{L}\frac{c^3x^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx\right.\\
&\left.+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\Bigg\}-\Bigg\{\int_0^a\frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{(L-x)x}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx\\
&\left.-\frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\Bigg\}^2\right]
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
k_{BA}=&\Bigg\{\int_0^a\frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-x)^2}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx\\
&+\frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\Bigg\}\frac{L^4}{h^3}\\
&\Bigg/\left[\Bigg\{\int_0^a\frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-x)^2}{h^3}dx\right.\\
&+\int_{L-c}^{L}\frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\right.\\
&\left.+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\Bigg\}\Bigg\{\int_0^a\frac{a^3x^2}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{x^2}{h^3}dx+\int_{L-c}^{L}\frac{c^3x^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx\right.\\
&\left.+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\Bigg\}-\Bigg\{\int_0^a\frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{(L-x)x}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx\\
&\left.-\frac{1+\nu}{5}\left[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\Bigg\}^2\right]
\end{aligned}
\tag{24}
$$

3. **Results.** Tables 3, 4 and 5 show the comparison of the two models; the proposed model (PM) is the mathematical model presented in this paper taking into account the bending and shear deformations, and the traditional model (TM) considering the bending deformations only. Table 3 shows the fixed-end moments factors (m_{AB} and m_{BA}) subjected to a uniformly distributed load, and Table 4 presents the carry-over factors (C_{AB} and C_{BA}), and Table 5 exhibits the stiffness factors (k_{AB} and k_{BA}). Such comparisons were realized for $\nu = 0.20$ of concrete; $c = 0.1L$, $0.2L$, $0.3L$, $0.4L$, $0.5L$; $z = 0.4h$, $0.6h$, h, $1.5h$, $2h$; $a = 0.3L$ and $u = h$; $a = 0.2L$ and $u = 1.5h$; and $h = 0.1L$, because these values are presented in the tables on page 619 [3]. Results shown in the tables mentioned above are identical to the traditional model.

Also an other way to validate the proposed model is as follows: substituting "$a = 0L$ and $c = 0L$" or "$u = 0$ and $z = 0$" into Equations (11) and (12) to obtain the fixed-end

Table 3. Fixed-end moments factors

c	z/h	m_{AB}		m_{BA}		m_{AB}		m_{BA}	
		$a = 0.3L$; $u/h = 1$; $h = 0.1L$				$a = 0.2L$; $u/h = 1.5$; $h = 0.1L$			
		PM	**TM**	**PM**	**TM**	**PM**	**TM**	**PM**	**TM**
$0.1L$	0.4	0.1232	0.1237	0.0732	0.0728	0.1201	0.1205	0.0740	0.0736
	0.6	0.1209	0.1213	0.0763	0.0759	0.1180	0.1184	0.0770	0.0767
	1.0	0.1177	0.1181	0.0806	0.0803	0.1151	0.1154	0.0813	0.0810
	1.5	0.1153	0.1157	0.0839	0.0836	0.1129	0.1132	0.0845	0.0843
	2.0	0.1138	0.1141	0.0860	0.0857	0.1115	0.1118	0.0866	0.0863
$0.2L$	0.4	0.1190	0.1194	0.0795	0.0791	0.1162	0.1166	0.0802	0.0799
	0.6	0.1149	0.1152	0.0854	0.0851	0.1124	0.1127	0.0860	0.0858
	1.0	0.1088	0.1089	0.0943	0.0942	0.1068	0.1069	0.0948	0.0947
	1.5	0.1037	0.1037	0.1018	0.1018	0.1021	0.1021	0.1021	0.1021
	2.0	0.1004	0.1002	0.1068	0.1069	0.0991	0.0990	0.1070	0.1071
$0.3L$	0.4	0.1170	0.1175	0.0825	0.0822	0.1144	0.1148	0.0832	0.0829
	0.6	0.1118	0.1120	0.0904	0.0902	0.1095	0.1098	0.0910	0.0907
	1.0	0.1034	0.1034	0.1034	0.1034	0.1018	0.1018	0.1037	0.1037
	1.5	0.0959	0.0956	0.1154	0.1157	0.0949	0.0947	0.1154	0.1156
	2.0	0.0906	0.0901	0.1240	0.1246	0.0900	0.0897	0.1238	0.1242
$0.4L$	0.4	0.1162	0.1167	0.0832	0.0826	0.1136	0.1141	0.0838	0.0833
	0.6	0.1104	0.1109	0.0919	0.0915	0.1083	0.1087	0.0924	0.0920
	1.0	0.1008	0.1010	0.1074	0.1071	0.0995	0.0996	0.1075	0.1073
	1.5	0.0916	0.0914	0.1230	0.1232	0.0910	0.0908	0.1227	0.1229
	2.0	0.0847	0.0841	0.1353	0.1360	0.0846	0.0842	0.1347	0.1353
$0.5L$	0.4	0.1155	0.1161	0.0824	0.0818	0.1130	0.1137	0.0829	0.0823
	0.6	0.1095	0.1101	0.0911	0.0905	0.1076	0.1081	0.0914	0.0908
	1.0	0.0995	0.0999	0.1071	0.1066	0.0984	0.0989	0.1070	0.1065
	1.5	0.0897	0.0896	0.1245	0.1246	0.0894	0.0895	0.1239	0.1237
	2.0	0.0819	0.0814	0.1393	0.1401	0.0822	0.0820	0.1382	0.1385

moments "$M_{AB} = wL^2/12$" and "$M_{BA} = wL^2/12$"; substituting "$a = 0L$ and $c = 0L$" or "$u = 0$ and $z = 0$" into Equations (19) and (20), and the shear deformations are neglected to find the carry-over factor "$C_{AB} = C_{BA} = 0.5$"; substituting "$a = 0L$ and $c = 0L$" or "$u = 0$ and $z = 0$" into Equations (21) and (22), and the shear deformations are neglected to obtain the stiffness "$K_{AB} = K_{BA} = 4EI/L$". The values presented above correspond to a constant cross section.

Therefore, the proposed model in this paper is valid and is not limited to certain dimensions or proportions as shown in some books, and the bending and shear deformations are considered.

According to the results, if the fixed-end moment of a member in a support is greater than the other support, in a support the traditional model is greater, and in the other support, the proposed model is greater, where the biggest difference is of 0.62%. The traditional model is greater in all cases for the carry-over factors and the stiffness factors, where the biggest difference is of 3.89% for the carry-over factors, and for the stiffness factors is of 11.45%.

Table 3 shows that when the loads and the haunches are symmetrical the fixed-end moments for the two models are not affected, and also these are the same for the traditional model and the proposed model.

TABLE 4. Carry-over factors

c	z/h	C_{AB}		C_{BA}		C_{AB}		C_{BA}	
		$a = 0.3L$; $u/h = 1$; $h = 0.1L$				$a = 0.2L$; $u/h = 1.5$; $h = 0.1L$			
		PM	TM	PM	TM	PM	TM	PM	TM
$0.1L$	0.4	0.4885	0.4994	0.7639	0.7841	0.5116	0.5227	0.7126	0.7301
	0.6	0.5035	0.5148	0.7619	0.7819	0.5272	0.5387	0.7108	0.7282
	1.0	0.5236	0.5354	0.7591	0.7789	0.5482	0.5603	0.7084	0.7256
	1.5	0.5388	0.5510	0.7570	0.7766	0.5640	0.5765	0.7064	0.7235
	2.0	0.5484	0.5608	0.7556	0.7750	0.5740	0.5867	0.7052	0.7221
$0.2L$	0.4	0.5313	0.5435	0.7465	0.7662	0.5566	0.5691	0.6971	0.7142
	0.6	0.5627	0.5759	0.7392	0.7585	0.5895	0.6030	0.6905	0.7074
	1.0	0.6078	0.6225	0.7291	0.7480	0.6367	0.6517	0.6816	0.6981
	1.5	0.6440	0.6597	0.7212	0.7397	0.6746	0.6907	0.6746	0.6907
	2.0	0.6676	0.6840	0.7161	0.7342	0.6992	0.7161	0.6700	0.6858
$0.3L$	0.4	0.5653	0.5789	0.7221	0.7413	0.5930	0.6070	0.6752	0.6920
	0.6	0.6133	0.6286	0.7074	0.7261	0.6435	0.6592	0.6621	0.6784
	1.0	0.6872	0.7052	0.6872	0.7052	0.7212	0.7397	0.6440	0.6597
	1.5	0.7508	0.7713	0.6714	0.6888	0.7881	0.8091	0.6299	0.6451
	2.0	0.7946	0.8166	0.6611	0.6781	0.8340	0.8566	0.6207	0.6356
$0.4L$	0.4	0.5887	0.6036	0.6945	0.7133	0.6188	0.6341	0.6502	0.6666
	0.6	0.6511	0.6686	0.6712	0.6895	0.6851	0.7031	0.6294	0.6453
	1.0	0.7545	0.7765	0.6392	0.6566	0.7947	0.8173	0.6008	0.6159
	1.5	0.8513	0.8778	0.6143	0.6309	0.8972	0.9243	0.5785	0.5930
	2.0	0.9222	0.9521	0.5981	0.6141	0.9721	1.0026	0.5641	0.5780
$0.5L$	0.4	0.5999	0.6160	0.6669	0.6856	0.6326	0.6491	0.6250	0.6413
	0.6	0.6724	0.6919	0.6349	0.6529	0.7105	0.7305	0.5963	0.6119
	1.0	0.8008	0.8269	0.5904	0.6074	0.8482	0.8751	0.5565	0.5713
	1.5	0.9319	0.9657	0.5556	0.5717	0.9887	1.0233	0.5254	0.5394
	2.0	1.0357	1.0760	0.5328	0.5483	1.0995	1.1407	0.5052	0.5186

4. **Conclusions.** Traditional methods used for the variable cross section members are by the Simpson's rule to obtain the rotations or some other technique to perform numerical integration [3-5], and other authors present some tables considering the bending deformations and shear, but are limited to certain relationships and also the heights of the haunches are the same at both ends [23].

This paper presents a mathematical model for mechanical elements of rectangular members with straight haunches for the general case (symmetrical and/or nonsymmetrical) subjected to a uniformly distributed load considering the bending and shear deformations to obtain the fixed-end moments, carry-over factors and stiffness factors, which is novelty of this research. The traditional model takes into account bending deformations.

The mathematical technique presented in this research is very adequate for the fixed-end moments, rotations, carry-over factors and stiffness for beams of variable rectangular cross section subjected to a uniformly distributed load, because it presents the mathematical expression, and with the support of some software, we obtain the values exactly.

The significant application of fixed-end moments is in the matrix methods of structural analysis to obtain the moments acting and the stiffness of a member. The carry-over factor is used in the moment distribution method or Hardy Cross method.

Table 5. Stiffness factors

c	z/h	k_{AB}		k_{BA}		k_{AB}		k_{BA}	
		$a = 0.3L$; $u/h = 1$; $h = 0.1L$				$a = 0.2L$; $u/h = 1.5$; $h = 0.1L$			
		PM	TM	PM	TM	PM	TM	PM	TM
$0.1L$	0.4	8.4360	8.7903	5.3953	5.5988	7.3519	7.6329	5.2785	5.4644
	0.6	8.5804	8.9525	5.6704	5.8936	7.4730	7.7678	5.5427	5.7462
	1.0	8.7814	9.1784	6.0574	6.3092	7.6413	7.9555	5.9137	6.1429
	1.5	8.9376	9.3537	6.3620	6.6369	7.7719	8.1010	6.2053	6.4550
	2.0	9.0376	9.4657	6.5591	6.8490	7.8554	8.1938	6.3938	6.6569
$0.2L$	0.4	8.7895	9.1910	6.2554	6.5200	7.6537	7.9722	6.1113	6.3527
	0.6	9.0919	9.5349	6.9211	7.2391	7.9089	8.2600	6.7513	7.0407
	1.0	9.5547	10.0644	7.9655	8.3752	8.2986	8.7020	7.7524	8.1240
	1.5	9.9508	10.5198	8.8854	9.3828	8.6315	9.0812	8.6315	9.0812
	2.0	10.2213	10.8315	9.5293	10.0910	8.8586	9.3404	9.2455	9.7524
$0.3L$	0.4	9.0334	9.4700	7.0718	7.3957	7.8672	8.2144	6.9094	7.2055
	0.6	9.4796	9.9826	8.2184	8.6418	8.2473	8.6474	8.0161	8.4027
	1.0	10.2292	10.8547	10.2292	10.8547	8.8854	9.3828	9.9508	10.5198
	1.5	10.9432	11.6970	12.2382	13.0973	9.4925	10.0914	11.8767	12.6558
	2.0	11.4731	12.3282	13.7885	14.8468	9.9427	10.6219	13.3591	14.3167
$0.4L$	0.4	9.1820	9.6390	7.7830	8.1565	8.0004	8.3653	7.6141	7.9575
	0.6	9.7335	10.2767	9.4425	9.9662	8.4758	8.9104	9.2263	9.7078
	1.0	10.7344	11.4568	12.6706	13.5492	9.3405	9.9204	12.3556	13.1628
	1.5	11.7918	12.7343	16.3417	17.7180	10.2564	11.0151	15.9056	17.1688
	2.0	12.6531	13.7971	19.5101	21.3900	11.0042	11.9271	18.9640	20.6882
$0.5L$	0.4	9.2847	9.7501	8.3520	8.7608	8.0890	8.4618	8.1871	8.5654
	0.6	9.9007	10.4633	10.4864	11.0895	8.6244	9.0770	10.2767	10.8363
	1.0	11.0707	11.8525	15.0141	16.1356	9.6507	10.2853	14.7094	15.7551
	1.5	12.4063	13.4948	20.8106	22.7964	10.8348	11.7276	20.3877	22.2476
	2.0	13.5908	15.0024	26.4186	29.4427	11.8950	13.0629	25.8874	28.7307

Also, a significant advantage is with the support of some software can be generated a large number of tables for different values or relationships of "a", "c", "u" and "z".

The proposed model passes to be the more appropriate model for structural analysis, and also is adjusted to the real conditions, since the shear forces and bending moments act in any structures, and therefore the bending and shear deformations are presented.

The mathematical model developed in this paper applies only for rectangular beams subjected to a uniformly distributed load of variable cross section with straight haunches for the general case (symmetrical and/or nonsymmetrical) considering the bending and shear deformations. The suggestions for future research are: 1) when the member presented another type of cross section; 2) when the member is subjected to another type of load.

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