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MODELLING FOR MECHANICAL ELEMENTS OF RECTANGULAR MEMBERS WITH STRAIGHT HAUNCHES USING SOFTWARE: PART 2

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Abstract. *The part 1 of this paper presents a mathematical model for mechanical elements of rectangular members with straight haunches for the general case (symmetrical and/or nonsymmetrical) subjected to a uniformly distributed load considering the bending and shear deformations to obtain the fixed-end moments, carry-over factors and stiffness factors. Now, this paper presents a mathematical model for members of rectangular cross section subjected to a concentrated load localized anywhere the member considering the bending and shear deformations with straight haunches (general case) to obtain the fixed-end moments, which is novelty of this research. The properties of rectangular cross section member are: the width "b" is constant and the height "h" varies along beam, and this variation is linear type. The compatibility equations and equilibrium are used to solve such problems, and the deformations anywhere of beam are found by the virtual work principle through exact integrations using the software "Derive" to obtain some results. The traditional model considers the bending deformations. Also a comparison is made between the proposed model and traditional model to show the differences. Besides the effectiveness and accuracy of the developed model, a significant advantage is that the fixed-end moments are calculated for any cross section of beam using the mathematical equations.*

Keywords: Straight haunches for the general case (symmetrical and/or nonsymmetrical), Bending and shear deformations, Concentrated load, Fixed-end moments, Compatibility equations and equilibrium, Virtual work principle

1. **Introduction.** The members with haunches of reinforced concrete are distinguished from prismatic because the beam height has a gradual variation in all or part of its length, its application in buildings of moderate elevation, as well as on bridges and viaducts of various functions. In buildings, the beams with haunches of reinforced concrete offer the following advantages over prismatic beams: 1) the lateral stiffness of buildings is increased substantially; 2) these beams types lead to a more efficient use of concrete and steel reinforcement; 3) the weight of the structure is reduced to optimize the strength and stability or to meet architectural requirements and specific functions of service; 4) the use of beams with haunches eases the placement of the electrical installation, air conditioning, water and sewage equipment, etc.

Members of rectangular cross section are subjected to a concentrated load where its main application is found in the live loads (mobile loads) of bridges corresponding to concentrated loads transmitted by the vehicles through their wheels to the road surface on the board.

During the last century, between 1950 and 1960 there developed several design aids, such as those presented by Guldan [1], and the most popular tables published by the Portland Cement Association (PCA) in 1958 "Handbook" [2].

Traditional methods used for the variable cross section members, the deflections by Simpson's rule are obtained or some other techniques to perform numerical integration and the tables presenting some books are limited to certain relationships [3-5].

The most relevant papers addressing the issue of structural members with haunches are shown in part 1 [6-16].

This paper presents a mathematical model for members of rectangular cross section subjected to a concentrated load localized anywhere the member considering the bending and shear deformations with straight haunches (general case) to obtain the fixed-end moments, which is novelty of this research. The properties of rectangular cross section member are: the width "b" is constant and the height "h" varies along beam, and this variation is linear type. The compatibility equations and equilibrium are used to solve such problems, and the deformations anywhere of beam are found by the virtual work principle through exact integrations using the software "Derive" to obtain some results. The traditional model considers the bending deformations. Also a comparison is made between the proposed model and traditional model to show the differences. Besides the effectiveness and accuracy of the developed model, a significant advantage is that the fixed-end moments are calculated for any cross section of beam using the mathematical equations.

The paper is organized as follows. Section 2 shows the formulation of the mathematical model for fixed-end moments subjected to a concentrated load localized anywhere of the member. Section 3 is dedicated to the results through the comparison of the two models, the Proposed Model (PM), and the Traditional Model (TM). Section 4 presents the conclusions.

2. **Formulation of the Mathematical Model.** Figure 1 of the part 1 presents a beam in elevation and also presents its rectangular cross section taking into account that the width "b" is constant and height "h_x" varies of straight shape in three different parts.

Table 1 of the part 1 shows the properties of the rectangular section.

Figure 1(a) presents the beam "AB" under a concentrated load localized anywhere on beam and fixed ends. The fixed-end moments are found by the sum of the effects. The moments are considered positive in counterclockwise, and negative in clockwise. Figure 1(b) shows the same simplily supported beam at their ends and its load applied to finding the rotations "α_{Ai}" and "α_{Bi}", where "i" takes the values of 1, 2 and 3. The rotations "α_{A1}" and "α_{B1}" are when the concentrated load is placed on $0 \leq x \leq a$. The rotations "α_{A2}" and "α_{B2}" are when the concentrated load is located in $a \leq x \leq L - c$. The rotations "α_{A3}" and "α_{B3}" are when the concentrated load is found of $L - c \leq x \leq L$. Now, the rotations "f_{11}" and "f_{21}" are caused by the unitary moment applied in the support "A", according to Figure 1(c), and in terms of "f_{12}" and "f_{22}" caused by the unitary moment applied in the support "B", see Figure 1(d) [12-22].

The compatibility equations and equilibrium of the beam are [12-22]:

To $0 \leq x \leq a$:

$$-f_{11}M_{AB} + f_{12}M_{BA} = \alpha_{A1} \tag{1}$$

$$-f_{21}M_{AB} + f_{22}M_{BA} = \alpha_{B1} \tag{2}$$

To $a \leq x \leq L - c$:

$$-f_{11}M_{AB} + f_{12}M_{BA} = \alpha_{A2} \tag{3}$$

$$-f_{21}M_{AB} + f_{22}M_{BA} = \alpha_{B2} \tag{4}$$

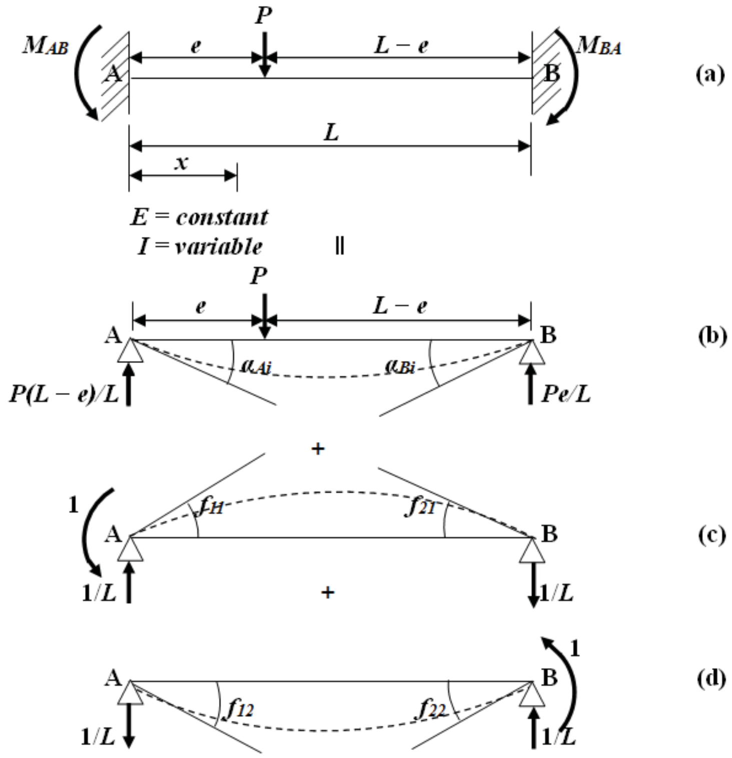

FIGURE 1. Beam fixed at its ends

To $L-c \leq x \leq L$:

$$-f_{11}M_{AB} + f_{12}M_{BA} = \alpha_{A3} \tag{5}$$

$$-f_{21}M_{AB} + f_{22}M_{BA} = \alpha_{B3} \tag{6}$$

Figure 1(b) is analyzed to find "α_{Ai}" and "α_{Bi}", the virtual work principle and take account of the bending and shear deformations used to obtain the rotations.

The rotations of "α_{Ai}" and "α_{Bi}" for non-prismatic members are [19]:

$$\alpha_{Ai} = \int_0^L \frac{V_x V_1}{GA_{sx}} dx + \int_0^L \frac{M_x M_1}{EI_z} dx \tag{7}$$

$$\alpha_{Bi} = \int_0^L \frac{V_x V_2}{GA_{sx}} dx + \int_0^L \frac{M_x M_2}{EI_z} dx \tag{8}$$

where: E is the modulus of elasticity, G is shear modulus, V_x and M_x are shear forces and moments of the real concentrated load, V_1 and M_1 are shear forces and moments due the unitary moment applied in the support "A", and V_2 and M_2 are shear forces and moments due the unitary moment applied in the support "B" to a distance "x".

The shear modulus is:

$$G = \frac{E}{2(1+\nu)} \tag{9}$$

where: ν is Poisson's ratio.

Table 1 shows the equations of the shear forces and moments anywhere of the beam to a distance "x" [23].

Table 1. Shear forces and moments

Concept	Equations			
Shear forces	To the left of P	$V_x = \frac{P(L-e)}{L}$	$V_1 = \frac{1}{L}$	$V_2 = \frac{1}{L}$
	To the right of P	$V_x = -\frac{Pe}{L}$		
Moments	To the left of P	$M_x = \frac{P(L-e)x}{L}$	$M_1 = -\frac{(L-x)}{L}$	$M_2 = \frac{x}{L}$
	To the right of P	$M_x = \frac{Pe(L-x)}{L}$		

The rotations of "α_{A1}", "α_{B1}", "α_{A2}", "α_{B2}", "α_{A3}" and "α_{B3}" by Equations (7) and (8) are obtained:

$$\begin{aligned}\alpha_{A1} = \frac{12P}{EbL^2}\Bigg\{ & \int_0^e \frac{a^3(L-e)(L-x)x}{[ah+u(a-x)]^3}dx + \int_e^a \frac{a^3e(L-x)^2}{[ah+u(a-x)]^3}dx \\ & + \int_a^{L-c} \frac{e(L-x)^2}{h^3}dx + \int_{L-c}^{L} \frac{c^3e(L-x)^2}{[ch+z(x-L+c)]^3}dx \\ & - \frac{1+\nu}{5}\Bigg[\int_0^e \frac{a(L-e)}{ah+u(a-x)}dx - \int_e^a \frac{ae}{ah+u(a-x)}dx - \int_a^{L-c} \frac{e}{h}dx \\ & - \int_{L-c}^{L} \frac{ce}{ch+z(x-L+c)}dx\Bigg]\Bigg\}\end{aligned} \tag{10}$$

$$\begin{aligned}\alpha_{B1} = \frac{12P}{EbL^2}\Bigg\{ & \int_0^e \frac{a^3(L-e)x^2}{[ah+u(a-x)]^3}dx + \int_e^a \frac{a^3e(L-x)x}{[ah+u(a-x)]^3}dx \\ & + \int_a^{L-c} \frac{e(L-x)x}{h^3}dx + \int_{L-c}^{L} \frac{c^3e(L-x)x}{[ch+z(x-L+c)]^3}dx \\ & + \frac{1+\nu}{5}\Bigg[\int_0^e \frac{a(L-e)}{ah+u(a-x)}dx - \int_e^a \frac{ae}{ah+u(a-x)}dx - \int_a^{L-c} \frac{e}{h}dx \\ & - \int_{L-c}^{L} \frac{ce}{ch+z(x-L+c)}dx\Bigg]\Bigg\}\end{aligned} \tag{11}$$

$$\begin{aligned}\alpha_{A2} = \frac{12P}{EbL^2}\Bigg\{ & \int_0^a \frac{a^3(L-e)(L-x)x}{[ah+u(a-x)]^3}dx + \int_a^e \frac{(L-e)(L-x)x}{h^3}dx \\ & + \int_e^{L-c} \frac{e(L-x)^2}{h^3}dx + \int_{L-c}^{L} \frac{c^3e(L-x)^2}{[ch+z(x-L+c)]^3}dx \\ & - \frac{1+\nu}{5}\Bigg[\int_0^a \frac{a(L-e)}{ah+u(a-x)}dx + \int_a^e \frac{L-e}{h}dx - \int_e^{L-c} \frac{e}{h}dx \\ & - \int_{L-c}^{L} \frac{ce}{ch+z(x-L+c)}dx\Bigg]\Bigg\}\end{aligned} \tag{12}$$

$$\begin{aligned}\alpha_{B2} = \frac{12P}{EbL^2}\Bigg\{ & \int_0^a \frac{a^3(L-e)x^2}{[ah+u(a-x)]^3}dx + \int_a^e \frac{(L-e)x^2}{h^3}dx \\ & + \int_e^{L-c} \frac{e(L-x)x}{h^3}dx + \int_{L-c}^{L} \frac{c^3e(L-x)x}{[ch+z(x-L+c)]^3}dx\end{aligned}$$

$$+\frac{1+\nu}{5}\left[\int_0^a \frac{a(L-e)}{ah+u(a-x)}dx+\int_a^e \frac{L-e}{h}dx-\int_e^{L-c}\frac{e}{h}dx\right.$$
$$\left.\left.-\int_{L-c}^{L}\frac{ce}{ch+z(x-L+c)}dx\right]\right\} \tag{13}$$

$$\alpha_{A3}=\frac{12P}{EbL^2}\left\{\int_0^a \frac{a^3(L-e)(L-x)x}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-e)(L-x)x}{h^3}dx\right.$$
$$+\int_{L-c}^{e}\frac{c^3(L-e)(L-x)x}{[ch+z(x-L+c)]^3}dx+\int_e^L \frac{c^3e(L-x)^2}{[ch+z(x-L+c)]^3}dx$$
$$-\frac{1+\nu}{5}\left[\int_0^a \frac{a(L-e)}{ah+u(a-x)}dx+\int_a^{L-c}\frac{L-e}{h}dx\right.$$
$$\left.\left.+\int_{L-c}^{e}\frac{c(L-e)}{ch+z(x-L+c)}dx-\int_e^L \frac{ce}{ch+z(x-L+c)}dx\right]\right\} \tag{14}$$

$$\alpha_{B3}=\frac{12P}{EbL^2}\left\{\int_0^a \frac{a^3(L-e)x^2}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-e)x^2}{h^3}dx\right.$$
$$+\int_{L-c}^{e}\frac{c^3(L-e)x^2}{[ch+z(x-L+c)]^3}dx+\int_e^L \frac{c^3e(L-x)x}{[ch+z(x-L+c)]^3}dx$$
$$+\frac{1+\nu}{5}\left[\int_0^a \frac{a(L-e)}{ah+u(a-x)}dx+\int_a^{L-c}\frac{L-e}{h}dx\right.$$
$$\left.\left.+\int_{L-c}^{e}\frac{c(L-e)}{ch+z(x-L+c)}dx-\int_e^L \frac{ce}{ch+z(x-L+c)}dx\right]\right\} \tag{15}$$

The coefficients of flexibilities through the virtual work principle are obtained [24]:

$$f_{11}=\int_0^L \frac{V_1V_1}{GA_{sx}}dx+\int_0^L \frac{M_1M_1}{EI_z}dx \tag{16}$$

$$f_{22}=\int_0^L \frac{V_2V_2}{GA_{sx}}dx+\int_0^L \frac{M_2M_2}{EI_z}dx \tag{17}$$

$$f_{12}=f_{21}=\int_0^L \frac{V_1V_2}{GA_{sx}}dx+\int_0^L \frac{M_1M_2}{EI_z}dx \tag{18}$$

Substituting the values of Table 1 of the part 1 and Table 1 of the part 2 into Equations (16), (17) and (18), the coefficients of flexibilities are found:

$$f_{11}=\frac{12}{EbL^2}\left\{\int_0^a \frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-x)^2}{h^3}dx\right.$$
$$+\int_{L-c}^{L}\frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx\right.$$
$$\left.\left.+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\right\} \tag{19}$$

$$f_{22}=\frac{12}{EbL^2}\left\{\int_0^a \frac{a^3x^2}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{x^2}{h^3}dx\right.$$
$$+\int_{L-c}^{L}\frac{c^3x^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx\right.$$
$$\left.\left.+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\right\} \tag{20}$$

$$
\begin{aligned}
f_{12} = f_{21} = &-\frac{12}{EbL^2}\left\{\int_0^a \frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx + \int_a^{L-c}\frac{(L-x)x}{h^3}dx\right. \\
&+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx - \frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx\right. \\
&\left.\left.+\int_a^{L-c}\frac{dx}{h} + \int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\right\}
\end{aligned} \tag{21}
$$

First condition: The concentrated load "P" is located in $0 \leq x \leq a$. Equations (10), (19) and (21) corresponding to the support "A" are substituted into Equation (1) and Equations (11), (20) and (21) corresponding to the support "B" are substituted into Equation (2). Subsequently, generated equations are solved and the values of "M_{AB}" and "M_{BA}" are obtained:

$$M_{AB} = \frac{f_{22}\alpha_{A1} - f_{12}\alpha_{B1}}{f_{12}^2 - f_{11}f_{22}} = m_{AB}PL \tag{22}$$

$$M_{BA} = \frac{f_{12}\alpha_{A1} - f_{11}\alpha_{B1}}{f_{12}^2 - f_{11}f_{22}} = m_{BA}PL \tag{23}$$

Second condition: The concentrated load "P" is situated in $a \leq x \leq L-c$. Equations (12), (19) and (21) corresponding to the support "A" are substituted into Equation (3) and Equations (13), (20) and (21) corresponding to the support "B" are substituted into Equation (4). Subsequently, generated equations are solved and the values of "M_{AB}" and "M_{BA}" are found:

$$M_{AB} = \frac{f_{22}\alpha_{A2} - f_{12}\alpha_{B2}}{f_{12}^2 - f_{11}f_{22}} = m_{AB}PL \tag{24}$$

$$M_{BA} = \frac{f_{12}\alpha_{A2} - f_{11}\alpha_{B2}}{f_{12}^2 - f_{11}f_{22}} = m_{BA}PL \tag{25}$$

Third condition: The concentrated load "P" is placed on $L-c \leq x \leq L$. Equations (14), (19) and (21) corresponding to the support "A" are substituted into Equation (1) and Equations (15), (20) and (21) corresponding to the support "B" are substituted into Equation (2). Subsequently, generated equations are solved and the values of "M_{AB}" and "M_{BA}" are:

$$M_{AB} = \frac{f_{22}\alpha_{A3} - f_{12}\alpha_{B3}}{f_{12}^2 - f_{11}f_{22}} = m_{AB}PL \tag{26}$$

$$M_{BA} = \frac{f_{12}\alpha_{A3} - f_{11}\alpha_{B3}}{f_{12}^2 - f_{11}f_{22}} = m_{BA}PL \tag{27}$$

Equations (22) to (27) are shown in Appendix.

3. **Results.**

Table 2 shows the comparison of the two models, the Proposed Model (PM) is the mathematical model presented in this paper taking account of the bending and shear deformations, and the Traditional Model (TM) takes account of only the bending deformations. This table presents the fixed-end moments factors (m_{AB} and m_{BA}) subjected to a concentrated load localized anywhere of the member. Such comparisons were realized for $\nu = 0.20$ of concrete; $e = 0.1L$, $0.3L$, $0.5L$, $0.7L$, $0.9L$; $c = 0.2L$, $0.3L$; $z = 0.4h$, $0.6h$, $1h$, $1.5h$, $2h$; $a = 0.3L$ and $u = 1h$; $a = 0.2L$ and $u = 1.5h$; $h = 0.1L$, because these values are presented in the tables on page 619 [3]. The results shown in the tables mentioned above are identical to the traditional model.

Another way to validate the proposed model is as follows (shear deformations are neglected). To $0 \leq x \leq a$ substitute "$u = 0h$ and $z = 0h$" or "$a = 0L$ and $c = 0L$" into Equations (22) and (23). To $a \leq x \leq L-c$ substitute "$u = 0h$ and $z = 0h$" or "$a = 0L$ and $c = 0L$" into Equations (24) and (25). To $L-c \leq x \leq L$ substitute "$u = 0h$ and

TABLE 2. Fixed-end moments factors

c	z/h	m_{AB}		m_{BA}		m_{AB}		m_{BA}		m_{AB}		m_{BA}		m_{AB}		m_{BA}		m_{AB}		m_{BA}	
		$e = 0.1L$				$e = 0.3L$				$e = 0.5L$				$e = 0.7L$				$e = 0.9L$			
		PM	TM	PM	TM	PM	TM	PM	TM	PM	TM	PM	TM	PM	TM	PM	TM	PM	TM	PM	TM
$a = 0.3L$; $u/h = 1$; $h = 0.1L$																					
0.2L	0.4	0.0920	0.0935	0.0046	0.0034	0.2151	0.2184	0.0412	0.0385	0.1943	0.1956	0.1156	0.1146	0.0912	0.0890	0.1583	0.1601	0.0114	0.0095	0.0855	0.0870
	0.6	0.0917	0.0933	0.0051	0.0038	0.2124	0.2158	0.0451	0.0422	0.1873	0.1884	0.1258	0.1250	0.0827	0.0801	0.1705	0.1727	0.0093	0.0074	0.0883	0.0898
	1.0	0.0912	0.0929	0.0058	0.0043	0.2084	0.2118	0.0511	0.0480	0.1768	0.1773	0.1415	0.1410	0.0702	0.0668	0.1888	0.1918	0.0065	0.0047	0.0919	0.0935
	1.5	0.0908	0.0926	0.0064	0.0048	0.2050	0.2085	0.0562	0.0530	0.1678	0.1680	0.1550	0.1548	0.0598	0.0559	0.2042	0.2078	0.0045	0.0029	0.0945	0.0960
	2.0	0.0906	0.0924	0.0068	0.0051	0.2027	0.2061	0.0598	0.0565	0.1618	0.1617	0.1642	0.1643	0.0530	0.0487	0.2144	0.2185	0.0034	0.0019	0.0959	0.0974
0.3L	0.4	0.0918	0.0933	0.0051	0.0038	0.2130	0.2164	0.0448	0.0419	0.1896	0.1909	0.1236	0.1225	0.0877	0.0856	0.1631	0.1649	0.0118	0.0100	0.0844	0.0860
	0.6	0.0913	0.0930	0.0058	0.0043	0.2092	0.2127	0.0509	0.0477	0.1799	0.1808	0.1386	0.1379	0.0774	0.0747	0.1783	0.1807	0.0098	0.0079	0.0870	0.0887
	1.0	0.0905	0.0924	0.0071	0.0052	0.2028	0.2063	0.0612	0.0577	0.1640	0.1640	0.1640	0.1640	0.0612	0.0577	0.2028	0.2063	0.0071	0.0052	0.0905	0.0924
	1.5	0.0898	0.0918	0.0083	0.0062	0.1967	0.2002	0.0712	0.0675	0.1492	0.1483	0.1882	0.1892	0.0471	0.0428	0.2248	0.2294	0.0051	0.0033	0.0932	0.0951
	2.0	0.0893	0.0914	0.0092	0.0069	0.1923	0.1957	0.0788	0.0750	0.1384	0.1367	0.2062	0.2081	0.0374	0.0326	0.2401	0.2455	0.0038	0.0022	0.0948	0.0967
$a = 0.2L$; $u/h = 1.5$; $h = 0.1L$																					
0.2L	0.4	0.0954	0.0966	0.0029	0.0019	0.2152	0.2186	0.0406	0.0377	0.1836	0.1847	0.1192	0.1183	0.0842	0.0821	0.1608	0.1626	0.0104	0.0087	0.0859	0.0873
	0.6	0.0952	0.0965	0.0032	0.0021	0.2129	0.2162	0.0444	0.0413	0.1770	0.1778	0.1295	0.1288	0.0763	0.0738	0.1729	0.1751	0.0084	0.0068	0.0886	0.0901
	1.0	0.0949	0.0963	0.0037	0.0023	0.2092	0.2127	0.0501	0.0468	0.1671	0.1675	0.1453	0.1449	0.0646	0.0616	0.1910	0.1940	0.0058	0.0043	0.0922	0.0937
	1.5	0.0947	0.0961	0.0040	0.0026	0.2062	0.2097	0.0551	0.0515	0.1587	0.1587	0.1587	0.1587	0.0551	0.0515	0.2062	0.2097	0.0040	0.0026	0.0947	0.0961
	2.0	0.0945	0.0960	0.0043	0.0028	0.2041	0.2077	0.0585	0.0548	0.1530	0.1528	0.1679	0.1681	0.0487	0.0449	0.2163	0.2202	0.0030	0.0017	0.0961	0.0975
0.3L	0.4	0.0952	0.0965	0.0032	0.0021	0.2134	0.2168	0.0441	0.0410	0.1792	0.1801	0.1273	0.1264	0.0809	0.0789	0.1655	0.1674	0.0108	0.0091	0.0848	0.0863
	0.6	0.0949	0.0963	0.0037	0.0024	0.2099	0.2134	0.0499	0.0465	0.1700	0.1706	0.1424	0.1418	0.0713	0.0688	0.1807	0.1831	0.0089	0.0072	0.0873	0.0890
	1.0	0.0945	0.0960	0.0045	0.0029	0.2042	0.2078	0.0598	0.0559	0.1550	0.1548	0.1678	0.1680	0.0562	0.0530	0.2050	0.2085	0.0064	0.0048	0.0908	0.0926
	1.5	0.0940	0.0957	0.0053	0.0034	0.1987	0.2024	0.0694	0.0651	0.1411	0.1401	0.1920	0.1931	0.0432	0.0393	0.2267	0.2311	0.0046	0.0030	0.0935	0.0953
	2.0	0.0937	0.0955	0.0059	0.0037	0.1947	0.1985	0.0765	0.0721	0.1310	0.1294	0.2098	0.2117	0.0342	0.0299	0.2418	0.2470	0.0035	0.0020	0.0951	0.0968

$z = 0h$" or "$a = 0L$ and $c = 0L$" into Equations (26) and (27). The fixed-end moments obtained for the three cases are: "$M_{AB} = Pe(L - e)^2/L^2$" and "$M_{BA} = Pe^2(L - e)/L^2$". The values presented above correspond to a constant cross section.

A way to validate the continuity of the cross section is as follows: in which the beam changes of positive slope to horizontal slope of the straight line in "a" and also when the beam changes of horizontal slope to negative slope in "$L - c$", load is placed on these points and these are the same results in the fixed-end moments. By example in Equations (22) and (24) substitute the value of "$e = a$" to obtain "M_{AB}", and Equations (23) and (25) to find "M_{BA}", and in Equations (24) and (26) substitute "$e = L - c$" to find "M_{AB}" and also in Equations (25) and (27) to obtain "M_{BA}".

Therefore, the proposed model in this paper is valid and is not limited to certain dimensions or proportions as shown in some books, and the bending and shear deformations are considered.

4. **Conclusions.** This paper developed a mathematical model for rectangular beams with straight haunches for the general case (symmetrical and/or nonsymmetrical) subjected to a concentrated load localized anywhere on beam considering the bending and shear deformations to obtain the fixed-end moments, which is novelty of this research. The compatibility equations and equilibrium are used to solve such problems, and the deformations anywhere of beam are found by the virtual work principle through exact integrations using the software "Derive" to obtain some results. The traditional model takes account of bending deformations.

The mathematical technique presented in this research is very adequate for obtaining the fixed-end moments for beams of variable rectangular cross section subjected to a concentrated load localized anywhere on beam, because it presents the mathematical expression, and with the support of some software, we obtain the values exactly.

Also, a significant advantage is that with the support of some software there can generate a large number of tables for different values or relationships of "a", "c", "e", "u" and "z".

The proposed model passes to be a more appropriate model for structural analysis, and also is adjusted to the real conditions; since the shear forces and moments act in any structures, the bending and shear deformations are presented.

The suggestions for future research are: 1) when the member presented another type of cross section; 2) when the member is subjected to another type of load.

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Appendix. Equations (22) and (23) present the fixed-end moments, when the concentrated load "P" is located of $0 \leq x \leq a$:

$$\begin{aligned}
M_{AB} = P\Bigg[\Bigg\{ & \int_0^e \frac{a^3(L-e)(L-x)x}{[ah+u(a-x)]^3}dx + \int_e^a \frac{a^3 e(L-x)^2}{[ah+u(a-x)]^3}dx + \int_a^{L-c} \frac{e(L-x)^2}{h^3}dx \\
& + \int_{L-c}^{L} \frac{c^3 e(L-x)^2}{[ch+z(x-L+c)]^3}dx - \frac{1+\nu}{5}\Bigg[\int_0^e \frac{a(L-e)}{ah+u(a-x)}dx \\
& - \int_e^a \frac{ae}{ah+u(a-x)}dx - \int_a^{L-c} \frac{e}{h}dx \\
& - \int_{L-c}^{L} \frac{ce}{ch+z(x-L+c)}dx\Bigg]\Bigg\}\Bigg\{\int_0^a \frac{a^3x^2}{[ah+u(a-x)]^3}dx + \int_a^{L-c} \frac{x^2}{h^3}dx \\
& + \int_{L-c}^{L} \frac{c^3x^2}{[ch+z(x-L+c)]^3}dx + \frac{1+\nu}{5}\Bigg[\int_0^a \frac{a}{ah+u(a-x)}dx + \int_a^{L-c} \frac{dx}{h} \\
& + \int_{L-c}^{L} \frac{c}{ch+z(x-L+c)}dx\Bigg]\Bigg\} - \Bigg\{\int_0^e \frac{a^3(L-e)x^2}{[ah+u(a-x)]^3}dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_e^a \frac{a^3e(L-x)x}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{e(L-x)x}{h^3}dx+\int_{L-c}^{L}\frac{c^3e(L-x)x}{[ch+z(x-L+c)]^3}dx\\
&+\frac{1+\nu}{5}\bigg[\int_0^e \frac{a(L-e)}{ah+u(a-x)}dx-\int_e^a\frac{ae}{ah+u(a-x)}dx-\int_a^{L-c}\frac{e}{h}dx\\
&-\int_{L-c}^{L}\frac{ce}{ch+z(x-L+c)}dx\bigg]\bigg\}\bigg\{\int_0^a\frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-x)x}{h^3}dx\\
&+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx-\frac{1+\nu}{5}\bigg[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\\
&+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\bigg]\bigg\}\bigg]\bigg/\bigg[\bigg\{\int_0^a\frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{(L-x)^2}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx\\
&+\frac{1+\nu}{5}\bigg[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\\
&+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\bigg]\bigg\}\bigg\{\int_0^a\frac{a^3x^2}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{x^2}{h^3}dx+\int_{L-c}^{L}\frac{c^3x^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\bigg[\int_0^a\frac{a}{ah+u(a-x)}dx\\
&+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\bigg]\bigg\}-\bigg\{\int_0^a\frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{(L-x)x}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx\\
&-\frac{1+\nu}{5}\bigg[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\bigg]\bigg\}\bigg]
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
M_{BA}=P\bigg[\bigg\{&\int_0^e\frac{a^3(L-e)x^2}{[ah+u(a-x)]^3}dx+\int_e^a\frac{a^3e(L-x)x}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{e(L-x)x}{h^3}dx\\
&+\int_{L-c}^{L}\frac{c^3e(L-x)x}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\bigg[\int_0^e\frac{a(L-e)}{ah+u(a-x)}dx\\
&-\int_e^a\frac{ae}{ah+u(a-x)}dx-\int_a^{L-c}\frac{e}{h}dx\\
&-\int_{L-c}^{L}\frac{ce}{ch+z(x-L+c)}dx\bigg]\bigg\}\bigg\{\int_0^a\frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-x)^2}{h^3}dx\\
&+\int_{L-c}^{L}\frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\bigg[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\\
&+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\bigg]\bigg\}-\bigg\{\int_0^e\frac{a^3(L-e)(L-x)x}{[ah+u(a-x)]^3}dx\\
&+\int_e^a\frac{a^3e(L-x)^2}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{e(L-x)^2}{h^3}dx\\
&+\int_{L-c}^{L}\frac{c^3e(L-x)^2}{[ch+z(x-L+c)]^3}dx-\frac{1+\nu}{5}\bigg[\int_0^e\frac{a(L-e)}{ah+u(a-x)}dx
\end{aligned}
$$

$$
\begin{aligned}
&- \int_e^a \frac{ae}{ah+u(a-x)}dx - \int_a^{L-c} \frac{e}{h}dx \\
&- \int_{L-c}^{L} \frac{ce}{ch+z(x-L+c)}dx \Bigg] \Bigg\} \Bigg\{ \int_0^a \frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx + \int_a^{L-c} \frac{(L-x)x}{h^3}dx \\
&+ \int_{L-c}^{L} \frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx - \frac{1+\nu}{5}\Bigg[\int_0^a \frac{a}{ah+u(a-x)}dx + \int_a^{L-c} \frac{dx}{h} \\
&+ \int_{L-c}^{L} \frac{c}{ch+z(x-L+c)}dx \Bigg] \Bigg\} \Bigg] \Bigg/ \Bigg[\Bigg\{ \int_0^a \frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx \\
&+ \int_a^{L-c} \frac{(L-x)^2}{h^3}dx + \int_{L-c}^{L} \frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx \\
&+ \frac{1+\nu}{5}\Bigg[\int_0^a \frac{a}{ah+u(a-x)}dx + \int_a^{L-c} \frac{dx}{h} \\
&+ \int_{L-c}^{L} \frac{c}{ch+z(x-L+c)}dx \Bigg] \Bigg\} \Bigg\{ \int_0^a \frac{a^3x^2}{[ah+u(a-x)]^3}dx + \int_a^{L-c} \frac{x^2}{h^3}dx \\
&+ \int_{L-c}^{L} \frac{c^3x^2}{[ch+z(x-L+c)]^3}dx + \frac{1+\nu}{5}\Bigg[\int_0^a \frac{a}{ah+u(a-x)}dx \\
&+ \int_a^{L-c} \frac{dx}{h} + \int_{L-c}^{L} \frac{c}{ch+z(x-L+c)}dx \Bigg] \Bigg\} - \Bigg\{ \int_0^a \frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx \\
&+ \int_a^{L-c} \frac{(L-x)x}{h^3}dx + \int_{L-c}^{L} \frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx \\
&- \frac{1+\nu}{5}\Bigg[\int_0^a \frac{a}{ah+u(a-x)}dx + \int_a^{L-c} \frac{dx}{h} + \int_{L-c}^{L} \frac{c}{ch+z(x-L+c)}dx \Bigg] \Bigg\} \Bigg]
\end{aligned} \tag{23}
$$

Equations (24) and (25) present the fixed-end moments, when the concentrated load "P" is located of $a \le x \le L-c$:

$$
\begin{aligned}
M_{AB} = P\Bigg[\Bigg\{ &\int_0^a \frac{a^3(L-e)(L-x)x}{[ah+u(a-x)]^3}dx + \int_a^e \frac{(L-e)(L-x)x}{h^3}dx + \int_e^{L-c} \frac{e(L-x)^2}{h^3}dx \\
&+ \int_{L-c}^{L} \frac{c^3e(L-x)^2}{[ch+z(x-L+c)]^3}dx - \frac{1+\nu}{5}\Bigg[\int_0^a \frac{a(L-e)}{ah+u(a-x)}dx + \int_a^e \frac{L-e}{h}dx \\
&- \int_e^{L-c} \frac{e}{h}dx - \int_{L-c}^{L} \frac{ce}{ch+z(x-L+c)}dx \Bigg] \Bigg\} \Bigg\{ \int_0^a \frac{a^3x^2}{[ah+u(a-x)]^3}dx \\
&+ \int_a^{L-c} \frac{x^2}{h^3}dx + \int_{L-c}^{L} \frac{c^3x^2}{[ch+z(x-L+c)]^3}dx + \frac{1+\nu}{5}\Bigg[\int_0^a \frac{a}{ah+u(a-x)}dx \\
&+ \int_a^{L-c} \frac{dx}{h} + \int_{L-c}^{L} \frac{c}{ch+z(x-L+c)}dx \Bigg] \Bigg\} - \Bigg\{ \int_0^a \frac{a^3(L-e)x^2}{[ah+u(a-x)]^3}dx \\
&+ \int_a^e \frac{(L-e)x^2}{h^3}dx + \int_e^{L-c} \frac{e(L-x)x}{h^3}dx + \int_{L-c}^{L} \frac{c^3e(L-x)x}{[ch+z(x-L+c)]^3}dx \\
&+ \frac{1+\nu}{5}\Bigg[\int_0^a \frac{a(L-e)}{ah+u(a-x)}dx + \int_a^e \frac{L-e}{h}dx - \int_e^{L-c} \frac{e}{h}dx \\
&- \int_{L-c}^{L} \frac{ce}{ch+z(x-L+c)}dx \Bigg] \Bigg\} \Bigg\{ \int_0^a \frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_a^{L-c}\frac{(L-x)x}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx\\
&-\frac{1+\nu}{5}\bigg[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\\
&+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\bigg]\bigg\}\bigg]\bigg/\bigg[\bigg\{\int_0^a\frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{(L-x)^2}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx\\
&+\frac{1+\nu}{5}\bigg[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\\
&+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\bigg]\bigg\}\bigg\{\int_0^a\frac{a^3x^2}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{x^2}{h^3}dx\\
&+\int_{L-c}^{L}\frac{c^3x^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\bigg[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\\
&+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\bigg]\bigg\}-\bigg\{\int_0^a\frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{(L-x)x}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx\\
&-\frac{1+\nu}{5}\bigg[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\bigg]\bigg\}\bigg]
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
M_{BA}=P\bigg[\bigg\{&\int_0^a\frac{a^3(L-e)x^2}{[ah+u(a-x)]^3}dx+\int_a^e\frac{(L-e)x^2}{h^3}dx+\int_e^{L-c}\frac{e(L-x)x}{h^3}dx\\
&+\int_{L-c}^{L}\frac{c^3e(L-x)x}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\bigg[\int_0^a\frac{a(L-e)}{ah+u(a-x)}dx+\int_a^e\frac{L-e}{h}dx\\
&-\int_e^{L-c}\frac{e}{h}dx-\int_{L-c}^{L}\frac{ce}{ch+z(x-L+c)}dx\bigg]\bigg\}\bigg\{\int_0^a\frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{(L-x)^2}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx\\
&+\frac{1+\nu}{5}\bigg[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\bigg]\bigg\}\\
&-\bigg\{\int_0^a\frac{a^3(L-e)(L-x)x}{[ah+u(a-x)]^3}dx+\int_a^e\frac{(L-e)(L-x)x}{h^3}dx+\int_e^{L-c}\frac{e(L-x)^2}{h^3}dx\\
&+\int_{L-c}^{L}\frac{c^3e(L-x)^2}{[ch+z(x-L+c)]^3}dx-\frac{1+\nu}{5}\bigg[\int_0^a\frac{a(L-e)}{ah+u(a-x)}dx+\int_a^e\frac{L-e}{h}dx\\
&-\int_e^{L-c}\frac{e}{h}dx-\int_{L-c}^{L}\frac{ce}{ch+z(x-L+c)}dx\bigg]\bigg\}\bigg\{\int_0^a\frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{(L-x)x}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx\\
&-\frac{1+\nu}{5}\bigg[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\Big]\Big\}\Big]\Big/\Big[\Big\{\int_0^a\frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{(L-x)^2}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\Big[\int_0^a\frac{a}{ah+u(a-x)}dx\\
&+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\Big]\Big\}\Big\{\int_0^a\frac{a^3x^2}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{x^2}{h^3}dx+\int_{L-c}^{L}\frac{c^3x^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\Big[\int_0^a\frac{a}{ah+u(a-x)}dx\\
&+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\Big]\Big\}-\Big\{\int_0^a\frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{(L-x)x}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx\\
&-\frac{1+\nu}{5}\Big[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\Big]\Big\}\Big]
\end{aligned}
\tag{25}
$$

Equations (26) and (27) present the fixed-end moments, when the concentrated load "P" is located of $L-c\le x\le L$:

$$
\begin{aligned}
M_{AB}=P\Big[\Big\{&\int_0^a\frac{a^3(L-e)(L-x)x}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-e)(L-x)x}{h^3}dx\\
&+\int_{L-c}^{e}\frac{c^3(L-e)(L-x)x}{[ch+z(x-L+c)]^3}dx+\int_e^{L}\frac{c^3e(L-x)^2}{[ch+z(x-L+c)]^3}dx\\
&-\frac{1+\nu}{5}\Big[\int_0^a\frac{a(L-e)}{ah+u(a-x)}dx+\int_a^{L-c}\frac{L-e}{h}dx+\int_{L-c}^{e}\frac{c(L-e)}{ch+z(x-L+c)}dx\\
&-\int_e^{L}\frac{ce}{ch+z(x-L+c)}dx\Big]\Big\}\Big\{\int_0^a\frac{a^3x^2}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{x^2}{h^3}dx\\
&+\int_{L-c}^{L}\frac{c^3x^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\Big[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\\
&+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\Big]\Big\}-\Big\{\int_0^a\frac{a^3(L-e)x^2}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-e)x^2}{h^3}dx\\
&+\int_{L-c}^{e}\frac{c^3(L-e)x^2}{[ch+z(x-L+c)]^3}dx+\int_e^{L}\frac{c^3e(L-x)x}{[ch+z(x-L+c)]^3}dx\\
&+\frac{1+\nu}{5}\Big[\int_0^a\frac{a(L-e)}{ah+u(a-x)}dx+\int_a^{L-c}\frac{L-e}{h}dx+\int_{L-c}^{e}\frac{c(L-e)}{ch+z(x-L+c)}dx\\
&-\int_e^{L}\frac{ce}{ch+z(x-L+c)}dx\Big]\Big\}\Big\{\int_0^a\frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-x)x}{h^3}dx\\
&+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx-\frac{1+\nu}{5}\Big[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\\
&+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\Big]\Big\}\Big]\Big/\Big[\Big\{\int_0^a\frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx\\
&+\int_a^{L-c}\frac{(L-x)^2}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\right.\\
&\left.\left.+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\right\}\left\{\int_0^a \frac{a^3x^2}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{x^2}{h^3}dx\right.\\
&+\int_{L-c}^{L}\frac{c^3x^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\right.\\
&\left.\left.+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\right\}-\left\{\int_0^a \frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx\right.\\
&+\int_a^{L-c}\frac{(L-x)x}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx\\
&\left.\left.-\frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\right\}\right]
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
M_{BA}=P&\left[\left\{\int_0^a \frac{a^3(L-e)x^2}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-e)x^2}{h^3}dx+\int_{L-c}^{e}\frac{c^3(L-e)x^2}{[ch+z(x-L+c)]^3}dx\right.\right.\\
&+\int_e^{L}\frac{c^3e(L-x)x}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\left[\int_0^a \frac{a(L-e)}{ah+u(a-x)}dx\right.\\
&+\int_a^{L-c}\frac{L-e}{h}dx+\int_{L-c}^{e}\frac{c(L-e)}{ch+z(x-L+c)}dx\\
&\left.\left.-\int_e^{L}\frac{ce}{ch+z(x-L+c)}dx\right]\right\}\left\{\int_0^a \frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx\right.\\
&+\int_a^{L-c}\frac{(L-x)^2}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx\\
&\left.+\frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\right\}\\
&-\left\{\int_0^a \frac{a^3(L-e)(L-x)x}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-e)(L-x)x}{h^3}dx\right.\\
&+\int_{L-c}^{e}\frac{c^3(L-e)(L-x)x}{[ch+z(x-L+c)]^3}dx+\int_e^{L}\frac{c^3e(L-x)^2}{[ch+z(x-L+c)]^3}dx\\
&-\frac{1+\nu}{5}\left[\int_0^a \frac{a(L-e)}{ah+u(a-x)}dx+\int_a^{L-c}\frac{L-e}{h}dx+\int_{L-c}^{e}\frac{c(L-e)}{ch+z(x-L+c)}dx\right.\\
&\left.\left.-\int_e^{L}\frac{ce}{ch+z(x-L+c)}dx\right]\right\}\left\{\int_0^a \frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-x)x}{h^3}dx\right.\\
&+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx-\frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\right.\\
&\left.\left.\left.+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\right]\right\}\right]\Big/\left[\left\{\int_0^a \frac{a^3(L-x)^2}{[ah+u(a-x)]^3}dx\right.\right.\\
&+\int_a^{L-c}\frac{(L-x)^2}{h^3}dx+\int_{L-c}^{L}\frac{c^3(L-x)^2}{[ch+z(x-L+c)]^3}dx\\
&+\frac{1+\nu}{5}\left[\int_0^a \frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\Big]\Big\}\Big\{\int_0^a\frac{a^3x^2}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{x^2}{h^3}dx\\
&+\int_{L-c}^{L}\frac{c^3x^2}{[ch+z(x-L+c)]^3}dx+\frac{1+\nu}{5}\Big[\int_0^a\frac{a}{ah+u(a-x)}dx+\int_a^{L-c}\frac{dx}{h}\\
&+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\Big]\Big\}-\Big\{\int_0^a\frac{a^3(L-x)x}{[ah+u(a-x)]^3}dx+\int_a^{L-c}\frac{(L-x)x}{h^3}dx \qquad (27)\\
&+\int_{L-c}^{L}\frac{c^3(L-x)x}{[ch+z(x-L+c)]^3}dx-\frac{1+\nu}{5}\Big[\int_0^a\frac{a}{ah+u(a-x)}dx\\
&+\int_a^{L-c}\frac{dx}{h}+\int_{L-c}^{L}\frac{c}{ch+z(x-L+c)}dx\Big]\Big\}\Big]
\end{aligned}
$$