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UNRELATED-PARALLEL MACHINE SCHEDULING WITH CONTROLLABLE AND GENERAL POSITION-DEPENDENT PROCESSING TIMES FOR MINIMIZING TOTAL ABSOLUTE DEVIATION OF TIMES PROBLEMS

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Abstract. *In a service oriented environment, it is important to reduce the variability in the completion times of jobs and the waiting times in a system. This paper investigates unrelated parallel-machine scheduling problems with simultaneous considerations of controllable and general position-dependent processing times. We examine two models of resource allocation, namely the linear resource allocation model and the convex resource allocation model. We aim to find the optimal resource allocations and the optimal job sequence to minimize the cost function containing total absolute deviation of completion times and the resource allocation and the cost function containing the total absolute deviation of waiting times and the resource allocation, respectively. If the number of machines is fixed, we show that all the considered problems can be solved in polynomial time.*
Keywords: Scheduling, Unrelated parallel-machine, Controllable processing times, General position-dependent processing times

1. Introduction.

Scheduling problems with controllable processing times have become a popular topic among researchers in the past years. In many cases schedulers may control the actual processing time of a job by varying the allocation of additional resource. Variants of such problems have found many applications in areas such as controlling the ingot preheating process in metal production [12], part manufacturing [18], machine tooling environment [24], assembly environment [5,6] and VLSI circuit design [25]. There are two models of scheduling with resource allocations considered in the literature, namely the linear resource allocation model and the convex resource allocation model [23]. Vickson [26] is among the pioneers to provide such a problem with a linear resource allocation model of processing times in terms of assigned amounts of resources. On the other hand, for many resource allocation problems in physical or economic systems, they do not use a linear resource consumption function, since it fails to reflect the law of diminishing marginal returns. This law states that productivity increases at a decreasing rate with the amount of resource allocated [23]. In order to model this, some studies on scheduling problems with resource allocation assumed that the job processing time is a convex decreasing function of the amount of resource allocated to the processing of the job, such as [3], [8], and [28]. Survey on this area of scheduling research is provided by Shabtay and Steiner [23]. For new trends in scheduling with controllable processing time, we refer the reader to Karimi-Nasab and Fatemi Ghomi [15], Li et al. [17], Niu et al. [19], Oron [20], Rudek and Rudek [21,22], Yin et al. [30] and Yin et al. [31].

In the literature of scheduling with controllable processing times, the single-machine problems have received most of the attention. Nevertheless, the parallel-machine problems are interesting and closer to the real problems industrials in practice. Comprehensive survey of different scheduling problems concerning parallel-machine scheduling problems with additional resources is presented by Edis et al. [9]. Recently, Yang et al. [27] consider unrelated parallel-machine scheduling involving controllable processing times and rate-modifying activities simultaneously. They assume that the actual processing time of a job can be compressed by allocating a greater amount of a common resource to process the job. The objective is to determine the optimal job compressions, the optimal positions of the rate-modifying activities and the optimal schedule to minimize a total cost function that depends on the total completion time and total job compressions. They proposed an efficient polynomial time algorithm to solve the problem under study. Chang et al. [7] further investigate unrelated parallel-machine scheduling problems with simultaneous considerations of resource allocation and rate-modifying activities. They examine two types of resource allocation. They aim to find the optimal resource allocations, the optimal rate-modifying activity positions, and the optimal job sequence to minimize the cost function containing the total completion time plus the resource allocation and the cost function containing the total machine load plus the resource allocation, respectively. They show that the problem under study can be formulated as an assignment problem and thus can be solved in a polynomial time algorithm.

On the other hand, we often encounter environments in which the processing times of jobs may be subject to change due to various possible changes of the positions of jobs in a sequence. Two different models to position-dependent processing times in scheduling settings have been introduced. If the job processing times increase with the number of jobs already processed that results in decreasing of the production efficiency, this phenomenon is called the position-dependent deteriorating effect; while if the job processing times decrease with the number of jobs already processed that results in increasing of the production efficiency, this phenomenon is called the learning effect. Scheduling problems with position-dependent processing times have received increasing attention in the last decade. For details on this area of research, we refer the reader to [1,2,13,14], among others.

It is natural to study scheduling problems combining controllable and position-dependent processing times. To the best of our knowledge, however, the scheduling problem with simultaneous considerations of controllable and general position-dependent processing times has never been investigated on an unrelated-parallel setting. The general position-dependent processing times mean that the processing time of the job is not restricted to any specific function dependent on its position in a sequence. Motivated in a service oriented environment, it is important to reduce the variability in the completion times of jobs and the waiting times in a system. In this paper, we consider unrelated-parallel machine scheduling problems involving controllable and general position-dependent processing times. The objective is to find the optimal resource allocations and the optimal job sequence to minimize the cost function containing total absolute deviation of completion times ($TADC$) plus the resource allocation and the cost function containing the total absolute deviation of waiting times ($TADW$) plus the resource allocation, respectively. The $TADC$ is a function for completion time variance of jobs. It is deemed desirable that each job spends approximately the same time in the system as every other job. The $TADW$ is a measure of waiting time variation in scheduling. It is important in reducing the variability in the waiting times. Obviously, the first objective function under consideration is related to the variability in the completion times of jobs. This type of problems has applications in many manufacturing or service environments whenever it is deemed desirable to provide jobs the same treatment. The second objective function under study is related to the variability in the waiting times in a system. The supplier might be interested in providing as much uniform quality of production or service as possible based on the jobs' waiting times in system [16].

The remainder of this paper is organized as follows. We formulate the problem under study in Section 2. In Sections 3 and 4, we provide polynomial time solutions for solving the proposed problems. We conclude the paper and suggest some topics for future research in the last section.

2. **Problem Description.** In this section we first introduce the notations to be used throughout the paper, followed by formulation of the problem.

n: the number of jobs;
m: the number of machines;
J_j: job j, $j = 1, 2, \dots, n$;
M_i: machine i, $i = 1, 2, \dots, m$;
n_i: the number of jobs assigned to process on machine M_i, $n = \sum_{i=1}^{m} n_i$, $i = 1, 2, \dots, m$;
S_i: the set of jobs assigned to process on machine M_i, $i = 1, 2, \dots, m$;
$\bar{p}_{ij}$: the normal processing time of job J_j on machine M_i, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$;
v_{ij}: the compression rate of job J_j on machine M_i, $v_{ij} > 0$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$;
$\bar{u}_{ij}$: the upper bound on the amount of resource allocated to job J_j on machine M_i, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$;
u_{ij}: the amount of resource allocated to job J_j on machine M_i, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$;
$f(r)$: a general position-dependent function, $f(r) > 0$, $r = 1, 2, \dots, n$;
p_{ijr}: the actual processing time of job J_j scheduled in the rth position on machine M_i, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$ and $r = 1, 2, \dots, n_i$;
C_{ij}: the completion time of job J_j on machine M_i, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$;
W_{ij}: the waiting time of job J_j on machine M_i, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$;
G_{ij}: the per unit time cost associated with the resource allocation to job J_j on machine M_i, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$;
$O(\cdot)$: big O notation for the time complexity;

$TADC$: the total absolute differences in completion times;
$TADW$: the total absolute differences in waiting times.

Subscript

$[ir]$: the job scheduled in the rth position on machine M_i, $i = 1, 2, \dots, m$ and $r = 1, 2, \dots, n_i$.

We are giving a set of n independent jobs J_j ($j = 1, 2, \dots, n$) to be processed on m ($m < n$) unrelated parallel machines M_i ($i = 1, 2, \dots, m$). All the jobs are simultaneously available at time zero and job preemption is not allowed. A machine can process at most one job at a time and cannot stand idle until the last job assigned to it is finished. Let $S = (S_1, S_2, \dots, S_m)$ be a schedule for the machines, where S_i denotes the set of jobs assigned to process on machine M_i. Then $S_i \cap S_j = \emptyset$, $\forall i \neq j$, and $\bigcup_{i=1}^{m} S_i = \{J_1, \dots, J_n\}$.

In this paper we study the unrelated parallel-machine scheduling with simultaneous considerations of controllable and general position-dependent processing times. We consider two models of resource allocation. In the first one that describes the linear resource allocation, the actual processing time p_{ijr} of job J_j scheduled in position r on machine M_i is given by the following function:

$$p_{ijr} = \bar{p}_{ij} f(r) - v_{ij} u_{ij}, \quad i = 1, 2, \dots, m, \ j = 1, 2, \dots, n \text{ and } r = 1, 2, \dots, n_i, \tag{1}$$

where $0 \leq u_{ij} \leq \bar{u}_{ij} < \frac{\bar{p}_{ij} f(r)}{v_{ij}}$. The second model concerns a convex resource allocation and the actual processing time p_{ijr} of job J_j scheduled in position r on machine M_i is given as follows:

$$p_{ijr} = \left(\frac{\bar{p}_{ij} f(r)}{u_{ij}}\right)^k, \quad i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n \text{ and } r = 1, 2, \ldots, n_i, \tag{2}$$

where $u_{ij} > 0$ and k is a positive constant. It should be noted that in models (1) and (2), $f(r) > 0$ is a general position-dependent function. We do not restrict $f(r)$ to any specific function. Clearly, the actual processing time of a job is not only a function of the amount of resource allocated to the processing of the job, but also a function of the job's scheduled position.

In this study we aim to determine the optimal resource allocation and the optimal schedule such that the corresponding value of the following objective functions is minimized, respectively:

$$f(u_{ij}, S_i) = TADC + \sum_{i=1}^{m} \sum_{r=1}^{n_i} G_{ij} u_{ij} \tag{3}$$

and

$$g(u_{ij}, S_i) = TADW + \sum_{i=1}^{m} \sum_{r=1}^{n_i} G_{ij} u_{ij}, \tag{4}$$

where

$$TADC = \sum_{i=1}^{m} \sum_{l=1}^{n_i} \sum_{k=l}^{n_i} \left| C_{[il]} - C_{[ik]} \right| = \sum_{i=1}^{m} \sum_{r=1}^{n_i} (r-1)(n_i - r + 1) p_{[ir]} \tag{5}$$

and

$$TADW = \sum_{i=1}^{m} \sum_{l=1}^{n_i} \sum_{k=l}^{n_i} \left| W_{[il]} - W_{[ik]} \right| = \sum_{i=1}^{m} \sum_{r=1}^{n_i} r(n_i - r) p_{[ir]}, \tag{6}$$

respectively.

The considered problem, according to the three-field notation scheme $\alpha/\beta/\gamma$ [10], with model (1) will be denoted as $Rm/PR, lin/TADC$ and $Rm/PR, lin/TADW$, respectively, and with model (2) as $Rm/PR, con/TADC$ and $Rm/PR, con/TADW$, respectively, where PR denotes "position-dependent and resource-dependent processing times" and lin and con denote the linear resource allocation model and the convex resource allocation model, respectively.

3. The Linear Resource Allocation Model.

In this section we consider the linear resource allocation model. We will show that both $Rm/PR, lin/TADC$ and $Rm/PR, lin/TADW$ problems can be solved in polynomial time.

3.1. Optimal solution for $Rm/PR, lin/TADC$.

In this subsection we investigate the $Rm/PR, lin/TADC$ problem. If we substitute $C_{[ih]} = \sum_{r=1}^{h} p_{[ir]}$ into (5), then we obtain that

$$\begin{aligned} f(u_{ij}, S_i) &= TADC + \sum_{i=1}^{m} \sum_{r=1}^{n_i} G_{ij} u_{ij} \\ &= \sum_{i=1}^{m} \sum_{r=1}^{n_i} \left[(r-1)(n_i - r + 1) p_{[ir]} + G_{[ir]} u_{[ir]} \right] \\ &= \sum_{i=1}^{m} \sum_{r=1}^{n_i} \left\{ [(r-1)(n_i - r + 1)] \left(\bar{p}_{[ir]} f(r) - v_{[ir]} u_{[ir]} \right) + G_{[ir]} u_{[ir]} \right\} \end{aligned}$$

$$= \sum_{i=1}^{m}\sum_{r=1}^{n_i}\alpha_{ir}\bar{p}_{[ir]}f(r) + \sum_{i=1}^{m}\sum_{r=1}^{n_i}\left(G_{[ir]} - \alpha_{ir}v_{[ir]}\right)u_{[ir]}, \quad (7)$$

where $\alpha_{ir} = (r-1)(n_i - r + 1)$, $i = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, n_i$.

Lemma 3.1. *Given a sequence, the optimal resource allocation for the $Rm/PR, lin/TA$-DC problem can be determined by*

$$u^*_{[ir]} = \begin{cases} \bar{u}_{[ir]}, & \text{if } G_{[ir]} - \alpha_{ir}v_{[ir]} < 0, \\ 0, & \text{if } G_{[ir]} - \alpha_{ir}v_{[ir]} \geq 0, \end{cases} \quad (8)$$

where $u^*_{[ir]}$ denotes the optimal resource allocation of a job scheduled in the rth position on machine M_i.

Proof: From (7), we know that

$$f(u_{ij}, S_i) = TADC + \sum_{i=1}^{m}\sum_{r=1}^{n_i}G_{ij}u_{ij} = \sum_{i=1}^{m}\sum_{r=1}^{n_i}\alpha_{ir}\bar{p}_{[ir]}f(r) + \sum_{i=1}^{m}\sum_{r=1}^{n_i}\left(G_{[ir]} - \alpha_{ir}v_{[ir]}\right)u_{[ir]}.$$

Thus, for any job sequence, for minimizing the objective $f(u_{ij}, S_i)$, the optimal resource allocation of a job in a position with $G_{[ir]} - \alpha_{ir}v_{[ir]} < 0$ should be its upper bound on the amount of resource $\bar{u}_{[ir]}$, and the optimal resource allocation of a job in a position with $G_{[ir]} - \alpha_{ir}v_{[ir]} \geq 0$ should be 0. Therefore, the optimal resource allocation of a job scheduled in the rth position on machine M_i is

$$u^*_{[ir]} = \begin{cases} \bar{u}_{[ir]}, & \text{if } G_{[ir]} - \alpha_{ir}v_{[ir]} < 0, \\ 0, & \text{if } G_{[ir]} - \alpha_{ir}v_{[ir]} \geq 0. \end{cases}$$

This completes the proof of Lemma 3.1.

In what follows we prove that the $Rm/PR, lin/TADC$ problem can be optimally solved in $O(n^{m+2})$ time.

Theorem 3.1. *The $Rm/PR, lin/TADC$ problem can be solved in $O(n^{m+2})$ time.*

Proof: First, we denote by $P(n, m) = (n_1, n_2, \ldots, n_m)$ the job-to-machine allocation vector [29], where n_i denotes the number of jobs assigned to process on machine M_i and $n = \sum_{i=1}^{m} n_i$. Next, let $x_{ijr} \in \{0, 1\}$ such that $x_{ijr} = 1$ if job J_j is scheduled in the rth position on machine M_i and $x_{ijr} = 0$ otherwise. Then, we can minimize the objective (7) via solving the following problem:

$$\text{Minimize} \quad \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{r=1}^{n_i}\lambda_{ijr}x_{ijr} \quad (9)$$

$$\text{subject to} \quad \sum_{i=1}^{m}\sum_{r=1}^{n_i}x_{ijr} = 1, \ j = 1, 2, \ldots, n, \quad (10)$$

$$\sum_{j=1}^{n}x_{ijr} = 1, \ i = 1, 2, \ldots, m, \ r = 1, 2, \ldots, n_i, \quad (11)$$

$$x_{ijr} \in \{0, 1\}, \ i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n, \ r = 1, 2, \ldots, n_i, \quad (12)$$

where

$$\lambda_{ijr} = \begin{cases} \alpha_{ir}\bar{p}_{ij}f(r), & \text{if } G_{ij} - \alpha_{ir}v_{ij} \geq 0, \\ \alpha_{ir}\bar{p}_{ij}f(r) + (G_{ij} - \alpha_{ir}v_{ij})\bar{u}_{ij}, & \text{if } G_{ij} - \alpha_{ir}v_{ij} < 0. \end{cases} \quad (13)$$

The constraints ensure that each job is scheduled exactly once and each position on each machine is occupied by one job. Thus, the problem (9)-(12) can be viewed as an assignment problem and, therefore, can be solved in $O(n^3)$ time [4].

Moreover, for a given vector $P(n,m) = (n_1, n_2, \ldots, n_m)$, the number of jobs n_i on machine M_i may be $0, 1, 2, \ldots, n$, for $i = 1, 2, \ldots, m$. Thus, if we get the numbers of jobs on the first $(m-1)$ machines, the number of jobs assigned to the last machine is then determined uniquely because $n = \sum_{i=1}^{m} n_i$. As a result, the upper bound on the number of allocation vectors $P(n,m)$ is $(n+1)^{m-1}$. Therefore, we conclude that the time complexity for the $Rm/PR, lin/TADC$ problem is $O(n^{m+2})$. This completes the proof of Theorem 3.1.

Algorithm 3.1 gives a method to find the local optimal solution for the objective (7) when vector $P(n,m) = (n_1, n_2, \ldots, n_m)$ is known.

Algorithm 3.1.

Step 1. Calculate α_{ir}, for $i = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, n_i$.
Step 2. Calculate λ_{ijr} by using (13), for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and $r = 1, 2, \ldots, n_i$.
Step 3. Solve the assignment problem (9)-(12) to determine the local optimal job sequence.
Step 4. Calculate the local optimal resources by using (8).
Step 5. Calculate the local optimal actual processing times by using (1).
Step 6. Calculate the corresponding value of the objective by using (7).

The global optimal solution is the one with the minimum value of the objective for all possible allocation vectors $P(n,m) = (n_1, n_2, \ldots, n_m)$.

3.2. **Optimal solution for *Rm/PR, lin/TADW*.** Similar to the above analysis, if we substitute $C_{[ih]} = \sum_{r=1}^{h} p_{[ir]}$ into (6), then we obtain that

$$\begin{aligned} g(u_{ij}, S_i) &= TADW + \sum_{i=1}^{m}\sum_{r=1}^{n_i} G_{ij}u_{ij} \\ &= \sum_{i=1}^{m}\sum_{r=1}^{n_i} \left[r(n_i - r)p_{[ir]} + G_{[ir]}u_{[ir]}\right] \\ &= \sum_{i=1}^{m}\sum_{r=1}^{n_i} \left\{[r(n_i - r)]\left(\bar{p}_{[ir]}f(r) - v_{[ir]}u_{[ir]}\right) + G_{[ir]}u_{[ir]}\right\} \\ &= \sum_{i=1}^{m}\sum_{r=1}^{n_i} \omega_{ir}\bar{p}_{[ir]}f(r) + \sum_{i=1}^{m}\sum_{r=1}^{n_i}\left(G_{[ir]} - \omega_{ir}v_{[ir]}\right)u_{[ir]}, \end{aligned} \tag{14}$$

where $\omega_{ir} = (n_i - r)r$, $i = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, n_i$. Following the analysis in the previous subsection, we have the following results.

Lemma 3.2. *Given a sequence, the optimal resource allocation for the $Rm/PR, lin/TADW$ problem can be determined by*

$$u^*_{[ir]} = \begin{cases} \bar{u}_{[ir]}, & \text{if } G_{[ir]} - \omega_{ir}v_{[ir]} < 0, \\ 0, & \text{if } G_{[ir]} - \omega_{ir}v_{[ir]} \geq 0. \end{cases} \tag{15}$$

Proof: The proof is similar to that of Lemma 3.1.

Theorem 3.2. *The $Rm/PR, lin/TADW$ problem can be solved in $O(n^{m+2})$ time.*

Proof: The proof is similar to that of Theorem 3.1.

Obviously, for a given vector $P(n,m) = (n_1, n_2, \ldots, n_m)$, we can obtain the optimal value of the objective (14) in a similar manner of Algorithm 3.1.

4. **The Convex Resource Allocation Model.** In this section, we consider the convex resource allocation model. First, Lemma 4.1 is a useful lemma which will be applied to solving the considered problem.

Lemma 4.1 ([11])**.** *Let there be two sequences of non-negative numbers x_i and y_i. The sum of the products of the corresponding elements $\sum_{i=1}^{n} x_i y_i$ is the least if the sequences are monotonic in the opposite sense.*

4.1. **Optimal solution for $Rm/PR, con/TADC$.** In this subsection we consider the $Rm/PR, con/TADC$ problem. If we substitute $C_{[ih]} = \sum_{r=1}^{h} p_{[ir]}$ into (5), then we obtain that

$$f(u_{ij}, S_i) = TADC + \sum_{i=1}^{m}\sum_{r=1}^{n_i} G_{ij}u_{ij} = \sum_{i=1}^{m}\sum_{r=1}^{n_i} \alpha_{ir}\left(\frac{\bar{p}_{[ir]}f(r)}{u_{[ir]}}\right)^k + \sum_{i=1}^{m}\sum_{r=1}^{n_i} G_{[ir]}u_{[ir]}, \quad (16)$$

where $\alpha_{ir} = (r-1)(n_i - r + 1)$, $i = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, n_i$.

Lemma 4.2. *Given a sequence, the optimal resource allocation for the $Rm/PR, con/TA$-DC problem can be determined by*

$$u^*_{[ir]} = \left(\frac{\alpha_{ir}k}{G_{[ir]}}\right)^{\frac{1}{k+1}} \left(\bar{p}_{[ir]}f(r)\right)^{\frac{k}{k+1}}, \quad (17)$$

where $i = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, n_i$.

Proof: We take the first derivative of $f(u_{ij}, S_i)$ with respect to $u_{[ir]}$ and let it be equal to 0. We obtain that

$$u^*_{[ir]} = \left(\frac{\alpha_{ir}k}{G_{[ir]}}\right)^{\frac{1}{k+1}} \left(\bar{p}_{[ir]}f(r)\right)^{\frac{k}{k+1}},$$

where $i = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, n_i$. Since the objective (16) is a convex function, (17) provides necessary and sufficient conditions for optimality. This completes the proof of Lemma 4.2.

By substituting (17) into (16), we obtain

$$f(u_{ij}, S_i) = \left(k^{\frac{-k}{k+1}} + k^{\frac{1}{k+1}}\right)\sum_{i=1}^{m}\sum_{r=1}^{n_i} \theta_{[ir]}\phi_{ir}, \quad (18)$$

where $\theta_{[ir]} = \left(G_{[ir]}\bar{p}_{[ir]}\right)^{\frac{k}{k+1}}$ and $\phi_{ir} = \left(\alpha_{ir}(f(r))^k\right)^{\frac{1}{k+1}}$, $i = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, n_i$.

In what follows we show that the $Rm/PR, con/TADC$ problem can be solved in $O(n^m \log n)$ time.

Theorem 4.1. *The Rm/PR, $con/TADC$ problem can be solved in $O(n^m \log n)$ time.*

Proof: Clearly, (18) can be viewed as the scalar product of the $\theta_{[ir]}$ and ϕ_{ir} vectors, for $i = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, n_i$. By Lemma 4.1, the optimal job sequence is obtained by matching the smallest ϕ_{ir} value to the job with the largest $\theta_{[ir]}$ value, the second smallest ϕ_{ir} value to the job with the second largest $\theta_{[ir]}$ value, and so on. The time complexity of a sorting algorithm is $O(n \log n)$. In addition, the upper bound on the number of allocation vectors $P(n, m)$ is $(n+1)^{m-1}$. Therefore, we conclude that the time complexity for the $Rm/PR, con/TADC$ problem is $O(n^m \log n)$. This completes the proof of Theorem 4.1.

For a given vector $P(n, m) = (n_1, n_2, \ldots, n_m)$, we can obtain the local optimal solution by Algorithm 4.1

Algorithm 4.1.

Step 1. Calculate α_{ir}, for $i = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, n_i$.
Step 2. Calculate the θ_{ir} and ϕ_{ir}, for $i = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, n_i$.
Step 3. By Lemma 4.1, assign the job with the largest θ_{ir} value to the position with the smallest value of ϕ_{ir}, the job with the second largest θ_{ir} with the second smallest value of ϕ_{ir} and so on. Then, obtain the local optimal job sequence.
Step 4. Calculate the local optimal resources by using (17).
Step 5. Calculate the local optimal actual processing times by using (2).
Step 6. Calculate the corresponding value of the objective by using (18).

Clearly, the global optimal solution is the one with the minimum value of the objective for all possible allocation vectors $P(n, m) = (n_1, n_2, \ldots, n_m)$.

4.2. **Optimal solution for *Rm/PR, con/TADW*.** Performing a similar analysis of the $Rm/PR, con/TADC$ problem, we obtain that

$$g(u_{ij}, S_i) = TADW + \sum_{i=1}^{m}\sum_{r=1}^{n_i} G_{ij}u_{ij} = \sum_{i=1}^{m}\sum_{r=1}^{n_i} \omega_{ir}\left(\frac{\bar{p}_{[ir]}f(r)}{u_{[ir]}}\right)^k + \sum_{i=1}^{m}\sum_{r=1}^{n_i} G_{[ir]}u_{[ir]} \quad (19)$$

where $\omega_{ir} = (n_i - r)r$, $i = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, n_i$. Based on the analysis in the previous subsection, we have the following results.

Lemma 4.3. *Given a sequence, the optimal resource allocation for the $Rm/PR, con/TADW$ problem can be determined by*

$$u^*_{[ir]} = \left(\frac{\omega_{ir}k}{G_{[ir]}}\right)^{\frac{1}{k+1}} \left(\bar{p}_{[ir]}f(r)\right)^{\frac{k}{k+1}}, \quad (20)$$

where $i = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, n_i$.

Proof: The proof is similar to that of Lemma 4.2.

Theorem 4.2. *The $Rm/PR, con/TADW$ problem can be solved in $O(n^m \log n)$ time.*

Proof: The proof is similar to that of Theorem 4.1.

Again, for a given vector $P(n, m) = (n_1, n_2, \ldots, n_m)$, we can find the optimal value of the objective (19) in a similar manner of Algorithm 4.1.

5. **Conclusions.** In this paper we investigated scheduling problems with simultaneous considerations of controllable and general position-dependent processing times. The goal of this study was to find the optimal resource allocation and the optimal job sequence to minimize the objective functions. The linear and convex resource allocation models are examined, respectively. Two objective functions are examined, namely the total cost function consisted of *TADC* and the resource allocation and the total cost function consisted of *TADW* and the resource allocation. If the number of machines is given, we showed that all the studied problems are polynomially solvable. It is worthy of future research to consider the problem with variable rate-modifying activity durations or multiple rate-modifying activities, or in more complicated machine setting, or optimizing other performance measures.

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REFERENCES

[1] A. Bachman and A. Janiak, Scheduling jobs with position-dependent processing times, *Journal of the Operational Research Society*, vol.55, pp.257-264, 2004.

[2] D. Biskup, A state-of-the-art review on scheduling with learning effects, *European Journal of Operational Research*, vol.188, pp.315-329, 2008.

[3] D. Biskup and H. Jahnke, Common due date assignment for scheduling on a single machine with jointly reducible processing times, *International Journal of Production Economics*, vol.69, pp.317-322, 2001.

[4] P. Brucker, *Scheduling Algorithms*, Springer-Verlag Inc., New York, 2007.

[5] R. L. Burdett and E. Kozan, Evolutionary algorithms for resource constrained non-serial mixed flow shops, *International Journal of Computational Intelligence and Application*, vol.3, pp.411-435, 2003.

[6] R. L. Burdett and E. Kozan, Resource aggregation issues and effects in mixed model assembly, *Proc. of the ASOR Qld Branch 5th OR Conference: OR into the 21st Century*, Sunshine Coast, Australia, pp.35-53, 2013.

[7] T.-R. Chang, H.-T. Lee, D.-L. Yang and S.-J. Yang, Unrelated-parallel machine scheduling with simultaneous considerations of resource-dependent processing times and rate-modifying activities, *International Journal of Innovative Computing, Information and Control*, vol.10, no.4, pp.1587-1600, 2014.

[8] T. C. E. Cheng, C. Oguz and X. D. Qi, Due-date assignment and single machine scheduling with compressible processing times, *International Journal of Production Economics*, vol.43, pp.29-35, 1996.

[9] E. B. Edis, C. Oguz and I. Ozkarahan, Parallel machine scheduling with additional resources: Notation, classification, models and solution methods, *European Journal of Operational Research*, vol.230, pp.449-463, 2013.

[10] R. L. Graham, E. L. Lawler, J. K. Lenstra and A. H. G. Rinnooy Kan, Optimization and approximation in deterministic sequencing and scheduling: A survey, *Annals of Discrete Mathematics*, vol.5, pp.287-326, 1979.

[11] G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, London, 1967.

[12] A. Janiak, Minimization of the blooming mill stand stills mathematical model, suboptimal algorithms, *Mechanika*, vol.8, no.2, pp.37-49, 1989.

[13] A. Janiak and R. Rudek, Scheduling problems with position dependent job processing times, in *Scheduling in Computer and Manufacturing Systems*, A. Janiak (ed.), Warszawa, WKL, Poland, 2006.

[14] A. Janiak and R. Rudek, Experience based approach to scheduling problems with the learning effect, *IEEE Trans. Systems, Man, and Cybernetics – Part A*, vol.39, pp.344-357, 2009.

[15] M. Karimi-Nasab and S. M. T. Fatemi Ghomi, Multi-objective production scheduling with controllable processing times and sequence-dependent setups for deteriorating items, *International Journal of Production Research*, vol.50, pp.7378-7400, 2012.

[16] H.-T. Lee and S.-J. Yang, Parallel machines scheduling with deterioration effects and resource allocations, *Journal of the Chinese Institute of Industrial Engineers*, vol.29, pp.534-543, 2012.

[17] K. Li, Y. Shi, S.-L. Yang and B.-Y. Cheng, Parallel machine scheduling problem to minimize the makespan with resource dependent processing times, *Applied Soft Computing*, vol.11, pp.5551-5557, 2011.

[18] C. T. D. Ng, T. C. E. Cheng, A. Janiak and M. Y. Kovalyov, Group scheduling with controllable setup and processing times: Minimizing total weighted completion time, *Annals of Operations Research*, vol.133, pp.163-174, 2005.

[19] G. Niu, S. Sun, P. Lafon, Y. Zhang and J. Wang, Two decompositions for the bicriteria job-shop scheduling problem with discretely controllable processing times, *International Journal of Production Research*, vol.50, pp.7415-7427, 2012.

[20] D. Oron, Scheduling controllable processing time jobs in a deteriorating environment, *Journal of the Operational Research Society*, vol.65, pp.49-56, 2014.

[21] A. Rudek and R. Rudek, A note on optimization in deteriorating systems using scheduling problems with the aging effect and resource allocation models, *Computers & Mathematics with Applications*, vol.62, pp.1870-1878, 2011.

[22] A. Rudek and R. Rudek, On flowshop scheduling problems with the aging effect and resource allocation, *International Journal of Advanced Manufacturing Technology*, vol.62, pp.135-145, 2012.

[23] D. Shabtay and G. Steiner, A survey of scheduling with controllable processing times, *Discrete Applied Mathematics*, vol.155, pp.1643-1666, 2007.

[24] D. Shabtay and G. Steiner, Optimal due date assignment and resource allocation to minimize the weighted number of tardy jobs on a single machine, *Manufacturing & Service Operations Management*, vol.9, pp.332-350, 2007.

[25] D. Shabtay and G. Steiner, Single machine batch scheduling to minimize total completion time and resource consumption costs, *Journal of Scheduling*, vol.10, pp.255-261, 2007.

[26] R. G. Vickson, Two single-machine sequencing problems involving controllable processing times, *AIIE Transactions*, vol.12, pp.258-262, 1980.

[27] D.-L. Yang, T. C. E. Cheng and S.-J. Yang, Parallel-machine scheduling with controllable processing times and rate-modifying activities to minimize total cost involving total completion time and job compressions, *International Journal of Production Research*, vol.52, pp.1133-1141, 2014.

[28] D.-L. Yang, C.-J. Lai and S.-J. Yang, Scheduling problems with multiple due windows assignment and controllable processing times on a single machine, *International Journal of Production Economics*, vol.150, pp.96-103, 2014.

[29] S.-J. Yang and D.-L. Yang, Minimizing the total completion time in single-machine scheduling with aging/deteriorating effects and deteriorating maintenance activities, *Computers & Mathematics with Applications*, vol.60, pp.2161-2169, 2010.

[30] Y. Yin, T. C. E. Cheng, C.-C. Wu and S.-R. Cheng, Single-machine common due-date scheduling with batch delivery costs and resource-dependent processing times, *International Journal of Production Research*, vol.51, pp.5083-5099, 2013.

[31] Y. Yin, T. C. E. Cheng, C.-C. Wu and S.-R. Cheng, Single-machine due window assignment and scheduling with a common flow allowance and controllable job processing times, *Journal of the Operational Research Society*, vol.65, pp.1-13, 2014.