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MATHEMATICAL MODELING AND POTENTIAL FUNCTION OF A PRODUCTION SYSTEM CONSIDERING THE STOCHASTIC RESONANCE

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Abstract. *Why does stochastic resonance occur in production system? There is the motivation of this paper. The noises regard as the probability element that the worker affects the process progress, or the supply chain has an impact in the process. The probability element represents a working ability to have a probability distribution. The stochastic resonance represents the relationship between the volatility of the working ability as the noise intensity and the throughput. We construct a mathematical model utilizing a Langevin-type equation for propagation of throughput under a stochastic resonance. We also deeply analyze the fluctuation of production processes. The model includes the supply chain to be produced in collaboration with external companies. A flow-shop-type production method, which generally constitutes a line, is utilized in this paper. The mathematical model ultimately becomes a diffusion-type equation. Moreover, with respect to fluctuation, we report that a diffusion coefficient results in a synchronous status. The validation of evaluation based on the data throughput of the production process is presented. The synchronous process is shown to be a much better method. For further verification, we confirm the benefit of using the synchronous process for performing dynamic simulations.*

Keywords: Stochastic resonance, Langevin-type equation, Potential function, Diffusion coefficient, Throughput propagation

1. **Introduction.** Based on mathematical and physical understandings of production engineering, we are conducting research aimed at establishing an academic area called mathematical production engineering. As our business size is a small-to-medium-sized enterprise, human intervention constitutes a significant part of the production process, and revenue can sometimes be greatly affected by human behavior. Therefore, when considering human intervention from outside companies, a deep analysis of the production process and human collaboration is necessary to understand the potential negative effects of such intervention.

With respect to mathematical modeling of deterministic systems in our studies, a physical model of the production process was constructed using a one-dimensional diffusion equation in 2012 [1, 22]. However, many concerns that occur in the supply chain are major problems facing production efficiency and business profitability. A stochastic partial bilinear differential equation with time delay was derived for outlet processes. The supply chain was modeled by considering as time delay [3]. With respect to the analysis of production processes in stochastic systems based on financial engineering, we have proposed that a production throughput rate can be estimated utilizing a Kalman filter based on a stochastic differential equation [2]. We have also proposed a stochastic differential equation (SDE) for the mathematical model describing production processes from the input of materials to the end. We utilized a risk-neutral principal in stochastic calculus based on the SDE [4].

With respect to the analysis of production processes based on physics in our studies, we have clarified that phenomena such as power-law distributions, self-similarity, phase transitions, and on-off intermittency can occur in production processes [5, 6, 7, 8, 9]. On the other hand, there is the famous theory of constraints (TOC) that describes the importance of avoiding bottlenecks in production processes [10]. We proposed that small fluctuations in an upstream subsystem appear as large fluctuations in the downstream (the so-called bullwhip effect) [13]. The bullwhip effect generates a large gap between the demand forecasts of the market and suppliers. Large fluctuations can be suppressed by the following mechanisms.

(1) Reducing the lead time, improving the throughput, and synchronizing the production process by the TOC.
(2) Sharing the demand information and performing mathematical evaluations.
(3) Analyzing the reduction and fluctuating demands of the subsystem (using nonlinear vibration theory).
(4) Basing the inventory management approach on stochastic demand.

In our studies, when using manufacturing equipment, delays in one production step are propagated to the next. Hence, the use of manufacturing equipment itself may lead to delays. The improvement of production processes was presented that the "Synchronization with preprocess" method was the most desirable in practice using the actual data in production flow process based on the cash flow model by using the SDE of log-normal type [11]. In essence, we have proposed the best way, which is a synchronous method using the Vasicek model for mathematical finance [12]. Then, the supply chain theme, which was a time delay in the production processes, was proposed for the throughput improvement based on a stochastic differential equation of log-normal type [13].

Moreover, with respect to the analysis of the synchronized state in our studies, we indicated that this state was a much better method from the viewpoint of potential energy [13, 14]. We have also shown that the phase difference between stages in a process corresponded to the standard deviation of the working time [15]. When the phase difference was constant, the total throughput could be minimized. We showed that a synchronous process could be realized by the gradient system. The above problem is not limited to small- and medium-sized companies; in all cases, human interventions that directly affect the production process present a major challenge.

In general, we may reasonably consider that human interventions within and outside of the production system (internal and external forces, respectively) introduce uncertainties into the system's progress [4, 12]. The production system is formed by connecting both elements. When human intervention from outside companies involves an uncertainty, the noise element is frequently overlooked; instead, researchers have focus on efficient production or manufacturing the best system. Moreover, by including the noise element, we can recognize the unique advantage of the system. We consider internal and external forces as two parameters in the production system. Rather than selecting the ratio between lead time and throughput that optimizes an individual's productivity, we select the parameters that achieve overall synchronization [4, 12]. In our previous study of a production system involving worker intervention, the specific abilities of workers required empirical analysis. To optimize typical modern production systems, we must recognize the importance of biological fluctuations. For example, the following aims typify technical innovation in the engineering industries.

(1) Detecting a small signal using the noise in the force.
(2) Synchronizing the circuit groups using the noise power.

With respect to stochastic resonance (SR) in our studies, we utilized in physical systems such electronic circuits, and even in biological systems such as neurotransmission; as a result, the same phenomenon has been confirmed [17, 18]. However, there have been no reports on application of SR in production processes for the improvement of throughput. Accordingly, we present the improvement of throughput in production processes using SR in the present study.

With respect to SR in our previous study, worker productivity in a high-mix, low-volume production process is optimized for the market demand, rather than the mass production process. To demonstrate the effectiveness of the throughput when the worker productivity is analyzed in this manner, we extract the probability distribution of the productivities of workers in a real production firm. Analyzing the actual results, we ascertain the probabilities of human factors in a production process.

Fujisaka and colleagues modeled the production process as a circuit system with an annular structure and coupled synchronization loops [19]. A production flow process used in our actual processes is regarded as the coupled synchronization loops reported in Fujisaka's reference [19]. Here, we apply their model to a relatively simple cascaded system, and model the dynamics using their derived Fokker-Planck equation (FPE). The FPE applies the modulation content of the equilibrium solution to the operator as the stochastic variation, and seeks the response and correlation functions. In their numerical calculations, Fujisaka and colleagues obtained the output signal-to-noise ratio, but did not calculate the eigenvalues and eigenfunctions of the operators in the fluctuating solution.

As described above, we consider that the noise (stochastic component) in workers' capability follows a probability distribution. We study the relationship between the intensity of SR (volatility in workers' ability) and the throughput (lead time) by capturing the process as a type of threshold reaction element. The proposed concept can potentially lead to innovative productivity by companies implementing a production system. Although the test system is small, it contains useful data for analyzing an innovative production system.

This study is a continuation of our previous research manuscript on SR [20]. We utilize a Langevin-type equation because we need to describe the mathematical model of production flow and also the relation between potential energy and fluctuations [6]. In a previous research, we have reported that the phase difference between stages in a process corresponds to the volatility of the working time and also that the parameters of the potential energy function can affect the stabilization of the process [15]. Assumptions of this paper are as follows.

- One of the processes flows sequentially next process. Moreover, there is always a worker in the production process.
- There is a correlation between the process θ_i and the upstream processes θ_{i+1} in close proximity to it in Figure 1.
- An outside company (Supply chain) is regarded as delay of supply.
- Self-similarity exists in production system.

Here, we further develop this previous study to obtain the relation between the parameters of the potential energy function and diffusion coefficient; that is, when the set of parameters is larger, the synchronous system becomes unstable due to changes in the diffusion coefficient. Furthermore, the value of the transition probability density function when processing flow a line due to changes in the diffusion coefficient changes. When the diffusion coefficient is relatively small, the value of the transition-probability-density function uniformly attenuates with change in the value of phase-difference variable. To the best of our knowledge, the analysis of production processes under noise and the determination of the relation between the parameters of the potential energy function and diffusion coefficient have not been previously undertaken.

2. Mathematical Modeling by Using Fokker-Plank Equation under Noise.

2.1. Topological concept of production process. In Figure 1, process $(i-1)$ and process $(i+1)$ are uncorrelated and θ_i denotes the phase at process i. Let the deviation of phase between processes $h_{i-1} = \theta_{i-1} - \theta_i$ and $h_i = \theta_i - \theta_{i+1}$. In Figure 2, there exists correlation between the processes proximate to one another in the production. In other words, the autocorrelation of $h_i(t)$ only is enabled.

As mentioned previously, the rate-of-return-deviation model in the production business can be described as a Langevin-type equation [6]. Figure 3 shows an equivalent model of flow-shop type production processes. Let $h_i \equiv d_i - d_{i-1}$ and $d\theta_i/dt = d_i$.

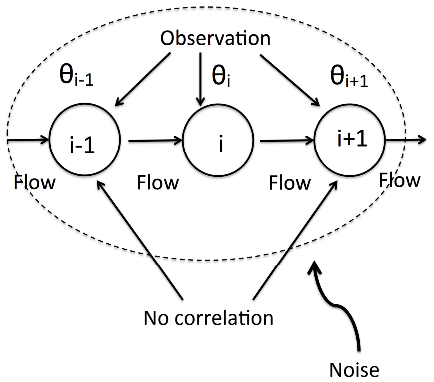

FIGURE 1. Throughput propagation under noise

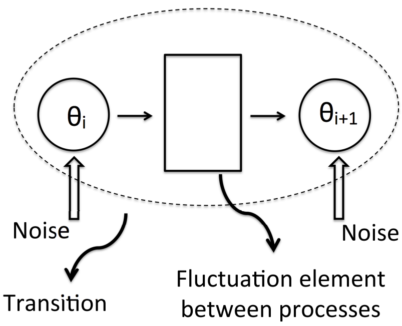

FIGURE 2. Fluctuation between processes

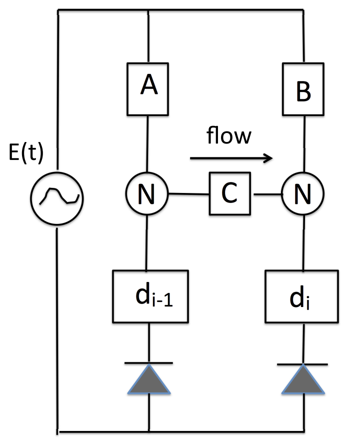

FIGURE 3. Equivalent model of throughput propagation

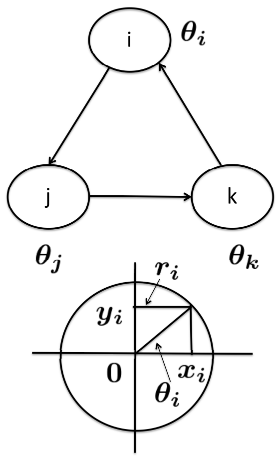

FIGURE 4. Production flow process by polar coordinate

We call θ_i the phase parameter of the processes. A, B, C are coupling coefficients. d_i, d_{i-1} denote equivalents to the potential energies of processes. N denotes a node in the circuit. $E(t) = E_0 + e_m \sin(wt - \varphi)$ has an alternating current.

With respect to $d_i - d_{i-1} = \Delta d_i$, $\Delta d_i = 0$ basically impossible by means of the coupling coefficients A, B, C with no harmony. Therefore, the deviation signal, h_i, undergoes fluctuations. In Figure 3, asynchronous phenomena are realistically evoked in the processes due to fluctuations affected by the variable parameter C. A detailed analysis is omitted here.

h_i is represented by Langevin type equation as follows:

$$\frac{dh_i}{dt} = f_i(h_i; t) + \sqrt{H} r_i(t) \tag{1}$$

where $f_i(h_i; t)$ denotes a probability throughput. $h \in [h_1, h_2, \cdots, h_{N'}]$, and $\sqrt{H} r_i(t)$ denotes the noise term.

We derive the Fokker-Planck equation (FPE) to be satisfied for $W(h, t)$. We can put the transition probability density function $W(h, t + \Delta t)$ at $t' = t + \Delta t$.

Assumption 2.1. *Transition probability $P(h, t + \Delta t | h', t)$ is a transition probability.*

$$W(h, t + \Delta t | h', t) = \int P(h, t + \Delta t | h', t) W(h', t) dh' \tag{2}$$

where let $\Delta_h = h - h'$ and we execute Taylor expansion of Equation (2).

$$\begin{aligned} &P(h + \Delta_h, t + \tau | h - \Delta_h, t) \times W(h - \Delta_h, t) \\ &= P(h + \Delta_h, t + \tau | h, t) W(h, t) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(- \sum_{i=1}^{N} \Delta_{h_i} \frac{\partial}{\partial h_i} \right)^n \\ &\times P(h + \Delta_h, t + \tau | h, t) W(h, t) \end{aligned} \tag{3}$$

Substitute Equation (3) to Equation (2) and execute the substitution of Δ_h, and then integrate Equation (2). As a result, the first term of Equation (3) is

$$\int P(h + \Delta_h, t + \tau | h, t) W(h, t) d\Delta_h = W(h, t) \tag{4}$$

Further, integrate to get the sum of second term of Equation (3),

$$-\tau \sum_{i=1} \frac{\partial}{\partial h_i} D_i(h, t, \tau) W(h, t) + \sum_{i,j} \frac{1}{2} \frac{\partial^2}{\partial h_i \partial h_j} D_{ij}(h, t, \tau) W(h, t) + \cdots \tag{5}$$

Then, let $h' = h(t)$,

$$D_i(h, t, \tau) = \frac{1}{\tau} \int \Delta_{h_i} P(h + \Delta_h, t + \tau | h, t) d\Delta_h = \frac{1}{\tau} < h_i(t + \tau) - h_i(t) > \tag{6}$$

$$\begin{aligned} D_{i,j}(h, t, \tau) &= \frac{1}{2\tau} \int \Delta_{h_i} \Delta_{h_j} P(h + \Delta_h, t + \tau | h, t) d\Delta_h \\ &= \frac{1}{2\tau} < h_i(t + \tau) - h_i(t) >< h_j(t + \tau) - h_j(t) > \end{aligned} \tag{7}$$

A correlation with only adjacent upstream process exists in production process. Therefore,

$$D_{i,j}(h, t, \tau) \equiv 0, \quad i \neq j \tag{8}$$

When $h(t)$ is described as Equation (1), we obtain as follows:

$$h_i(t + \tau) - h_i(t) = \int_t^{t+\tau} \left\{ f_i(h_i(t'), t') + \sqrt{H} r_i(t') \right\} dt' \tag{9}$$

Further, when we execute Taylor expansion of $f_i(h_i(t'), t')$,

$$f_i(h_i(t'), t') = f_i(h_i(t), t') + \sum_i \frac{\partial}{\partial h_i} f_i(h_i(t), t')\{h_i(t') - h_i(t)\} + \cdots \tag{10}$$

Equation (9) is

$$\begin{aligned} h_i(t+\tau) - h_i(t) = \int_t^{t+\tau} &\left\{ f_i(h_i(t), t') + \sum_i \frac{\partial}{\partial h_i} f_i(h_i(t), t')\{h_i(t') - h_i(t)\} dt' \right\} \\ &+ \cdots + \int_t^{t+\tau} \sqrt{H} r_i(t') dt' \end{aligned} \tag{11}$$

In Equation (11), as $h_i(t') - h_i(t)$ with respect to $t \to 0$, a high-order terms than τ can be neglected in the multiple integration of partial derivative.

$$\lim_{\tau \to 0} \frac{1}{\tau} < h_i(t+\tau) - h_i(t) >= f_i(h_i(t), t) \tag{12}$$

According to Equation (2),

$$W(h, t+\tau) = \int P(h, t+\tau | h', t) W(h', t) dh' \tag{13}$$

A deviation between Equation (13) and Equation (4) becomes Equation (5).

$$\begin{aligned} \frac{\partial W(h,t)}{\partial t} &= \lim_{\tau \to 0} \frac{W(h, t+\tau) - W(h,t)}{\tau} \\ &= -\sum_{i=1} \frac{\partial}{\partial h_i} f_i(h,t) W(h,t) + H \sum_i \frac{\partial^2}{\partial h_i^2} W(h,t) \end{aligned} \tag{14}$$

Equation (14) was derived as FPE with respect to $W(h, t)$. If an exact form of Equation (1) can be determined, we can present the FPE of a target system. However, this will be reported in a future study. Now, according to Equation (1), the mathematical model with respect to the phase parameter θ_i is

$$\frac{d\theta_i}{dt} = -\frac{\partial U_i(\theta_i)}{\partial \theta_i} + \sqrt{H} r_i(t) \tag{15}$$

where $U_i(\theta_i)$ represents a potential energy.

$$U_i(\theta_i) = \int \left[f_0(\theta_i) + k \sum_i f_i(\theta_i, \theta_{i-1}, \theta_{i+1}) + f_s\{g(\theta_i, t)\} \right] d\theta_i \tag{16}$$

According to Equation (16), the potential energy $U(\theta_i)$ is

$$\begin{aligned} U(\theta_i) = A_0 \sin\theta_i + k \sum_{j=1}^{L} \{\sin(\theta_i - \theta_{i-j}) + \sin(\theta_i - \theta_{i+j})\} \\ + A_m \{g_1(t) \sin\theta_i + g_2(t) \cos\theta_i\} \end{aligned} \tag{17}$$

where i is a number of process and j is a process connection.

If it is written in another equation,

$$\begin{aligned} U(\theta_i) = \sum \int f_i(\theta_i) d\theta_i + \sum \int \Bigg[A_0 \sin\theta_i + \sum_{j=1}^{L} k\{\sin(\theta_i - \theta_{i-j}) + \sin(\theta_i - \theta_{i+j})\} \\ + A_m\{g_1(t) \sin\theta_i + g_2(t) \cos\theta_i\} \Bigg] d\theta_i + \sqrt{H} r_i(t) \end{aligned} \tag{18}$$

From Equation (18), the mathematical model for θ_i is

$$\begin{aligned}\frac{d\theta_i}{dt} &= -\frac{\partial U(\theta_i)}{\partial \theta_i} + \sqrt{H} r_i(t) \\ &= -\left[A_0 \cos\theta_i + k\sum_{j=1}^{L}\{\cos(\theta_i + \theta_{i-j}) + \cos(\theta_i - \theta_{i+j})\}\right. \\ &\quad \left. + A_m\{g_1(t)\cos\theta_i - g_2(t)\sin\theta_i\}\right] + \sqrt{H} r_i(t)\end{aligned} \tag{19}$$

Therefore, FPE is as follows:

$$\frac{\partial W(\theta,t)}{\partial t} = \left[-\sum_i \frac{\partial}{\partial\theta_i} f_i(\theta_i) + H\sum_i \frac{\partial^2}{\partial\theta_i^2}\right] W(\theta,t) \tag{20}$$

where,

$$\begin{aligned}f_i(\theta_i) = A_0\cos\theta_i + k\sum_{i=1}^{L}\{\cos(\theta_i - \cos\theta_{i-j}) + \cos(\theta_i - \cos\theta_{i+j})\} \\ + A_m\{g_1(t)\cos\theta_i - g_2\sin\theta_i\}\end{aligned} \tag{21}$$

Thus, FPE has a following operator:

$$\mathcal{L}_{FP} \equiv \sum\left[-\frac{\partial}{\partial\theta_i}\left(\frac{\partial U}{\partial\theta_i}\right) + H\frac{\partial^2}{\partial\theta_i^2}\right] \tag{22}$$

From Fujisaka et al. [19], let $D^{(i)} \equiv \partial U/\partial\theta_i$,

$$\mathcal{L}_{FP} \equiv \sum\left[-\frac{\partial}{\partial\theta_i}D^{(i)} + H\frac{\partial^2}{\partial\theta_i^2}\right] \tag{23}$$

A fluctuation in the equilibrium state is as follows [6].

$$< |h_n(t)|^2 >= e^{-2w_0^n \cdot t} < |T_n(0)|^2 > \tag{24}$$

$$\phi_{h_n(t)} = D_f < |h_n|^2 > e^{-\frac{t}{\tau}} \tag{25}$$

where $\phi_{h_n(t)}$ represents an autocorrelation function of fluctuation, and τ is a time constant.

The velocity of fluctuations has a time constant of autocorrelation function.

3. **Potential Function between Processes.** Let $V(D)$ be the potential energy between processes. The potential energy undergoes fluctuation if an external force is added [15].

$$V(\varphi_{ij}) = F\varphi_{ij} + B(-4C\cos\varphi_{ij} + \cos 2\varphi_{ij}) \tag{26}$$

where F is a real number, B is a system parameter and C is a synchronizing parameter.

$$\frac{d\varphi_{ij}}{dt} = -\frac{\partial V(\varphi_{ij})}{\partial\varphi_{ij}} + \sqrt{H} r_{ij}(t) = -F - B\{4C\sin\varphi_{ij} - 2\sin 2\varphi_{ij}\} + \sqrt{H} r_{ij}(t) \tag{27}$$

With respect to Equation (27), let $F = A_0\sin(2\pi f t)$. Then, the potential energy $V(\varphi_{ij})$, which represents an system effectiveness, is

$$V(\varphi_{ij}) = A_0\sin(2\pi f t)\cdot\varphi_{ij} + B(-4C\cos\varphi_{ij} + \cos 2\varphi_{ij}) \tag{28}$$

where a phase difference $\varphi_{ij} = \theta_i - \theta_j$ is shown in Figure 5.

First- and second-order derivatives of $V(\varphi_{ij})$ are

$$\frac{\partial V(\varphi_{ij})}{\partial\varphi_{ij}} = F + B\{-4C\sin\varphi_{ij} - 2\sin 2\varphi_{ij}\} \tag{29}$$

$$\frac{\partial^2 V(\varphi_{ij})}{\partial \varphi_{ij}^2} = B\{4C\cos\varphi_{ij} - 4\cos 2\varphi_{ij}\} \tag{30}$$

Approximation equation with $\varphi_{ij} = 0$ near derived by Taylor expansion is

$$\begin{aligned} V(\varphi_{ij})\big|_{\varphi_{ij}=0} &= V(0) + \frac{\partial V(\varphi_{ij})}{\partial \varphi_{ij}}\bigg|_{\varphi_{ij}=0} \cdot \varphi_{ij} + \frac{1}{2}\frac{\partial^2 V(\varphi_{ij})}{\partial \varphi_{ij}^2}\bigg|_{\varphi_{ij}=0} \cdot \varphi_{ij}^2 + \cdots \\ &= B\{-4C+1\} + \{F\}\varphi_{ij} + \frac{1}{2}4B(C-1)\varphi_{ij}^2 + \cdots \end{aligned} \tag{31}$$

where hereafter, let $\varphi \equiv \varphi_{ij}$.

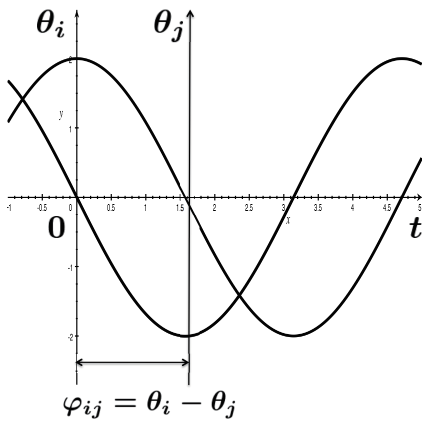

Figure 5. Production flow process by polar coordinate

Therefore, the first-order approximation equation of $V(\varphi)$ is

$$V(\varphi_{ij})|_{\varphi=0} = B(-4C+1) + F \cdot \varphi \tag{32}$$

the second-order approximation equation of $V(\varphi)$ is

$$V(\varphi)|_{\varphi=0} = B(-4C+1) + F \cdot \varphi + 2B(C-1)\varphi^2 \tag{33}$$

Using the first-order approximation equation of $V(\varphi)$ and according to Equation (20), the transition probability density function (pdf) is

$$\frac{\partial W(\varphi,t)}{\partial t} + F\frac{\partial W(\varphi,t)}{\partial \varphi} = H\frac{\partial^2 W(\varphi,t)}{\partial \varphi^2} \tag{34}$$

Moreover, using the second-order approximation,

$$\frac{\partial W(\varphi,t)}{\partial t} + (F + 2B(C-1)\varphi)\frac{\partial W(\varphi,t)}{\partial \varphi} = H\frac{\partial^2 W(\varphi,t)}{\partial \varphi^2} \tag{35}$$

Above analysis considers around $\varphi \approx 0$ shown as Figure 6. We reported that the potential exact function was determined by the set of parameters (F, B, C) [15].

Assumption 3.1.

$$\frac{d\varphi}{dt} = -\Gamma\varphi + \sqrt{H}r(t) \tag{36}$$

where, Γ *and* H *are cnnstants.* $(\partial V(\varphi))/(\partial\varphi) \approx \Gamma\varphi$.

Equation (36) indicates that $(\partial V(\varphi))/(\partial\varphi)$ denotes a first-order approximation with respect to φ. To establish Equation (36), it is limited to $\varphi \approx 0$. When assessed by actual data, it corresponds to Test Runs 2 and 3 as a synchronous status.

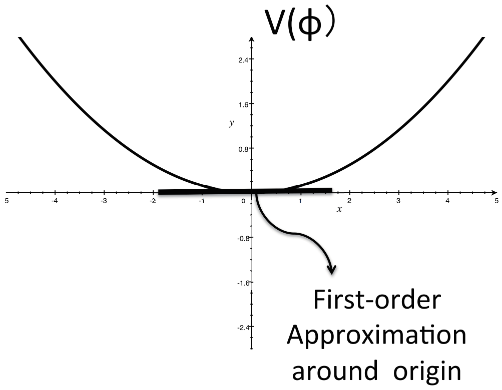

FIGURE 6. FPE in near origin point

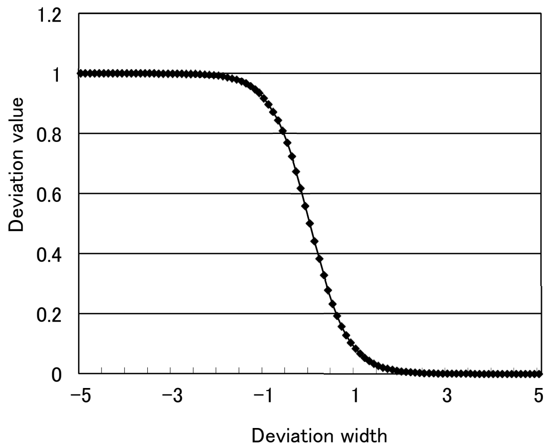

FIGURE 7. A particular solution of Burgers' equation

3.1. **A particular solution of Burgers' equation.** The second-order approximation is

$$\frac{\partial W}{\partial t} + (F + 4B(C-1)\varphi)\frac{\partial W}{\partial \varphi} = H\frac{\partial^2 W}{\partial \varphi^2} \tag{37}$$

In Equation (37), let $F \equiv 0$,

$$\frac{\partial W}{\partial t} + b\varphi\frac{\partial W}{\partial \varphi} = H\frac{\partial^2 W}{\partial \varphi^2} \tag{38}$$

where $b = 4B(C-1)$.

Dividing both sides by b in Equation (38),

$$b^{-1}\frac{\partial W}{\partial t} + \varphi\frac{\partial W}{\partial \varphi} = \frac{H}{b}\frac{\partial^2 W}{\partial \varphi^2} \tag{39}$$

Let $a = b^{-1}$, then Burgers' equation is

$$a\frac{\partial W}{\partial t} + \varphi\frac{\partial W}{\partial \varphi} = \kappa\frac{\partial^2 W}{\partial \varphi^2} \tag{40}$$

where $\kappa = aH$.

Assuming that when $\varphi \to +\infty$, $W = A$ and when $\varphi \to -\infty$, $W = 0$, a steady-state solution is

$$W = \frac{A}{a}\left\{1 - \tanh\frac{A}{2a\kappa}\left(\varphi - \frac{A}{2a}\tau\right)\right\} \tag{41}$$

3.2. **Analysis of a perturbation term added.** From Equation (31), we can derive as follows by neglecting the term of subsequent secondary approximation:

$$V(\varphi) \approx \hat{\Gamma}\varphi + A_0 f(\varphi, t) \tag{42}$$

where $\hat{\Gamma}$ is a constant and $A_0 f(\varphi, t)$ denotes a perturbation term added for the first-order approximation equation.

Equation (42) can be solved by Dr. Fujisaka's method [19]. However, we have assumed that an external force acting on the process is independent of the potential in our paper. Therefore, according to Equation (19), Equation (42) is

$$\frac{d\varphi}{dt} = -\frac{\partial}{\partial\varphi}\left(\hat{\Gamma}\varphi + A_0 f(t)\right) + \sqrt{H}r(t) \tag{43}$$

Assuming that $f(t) \in C^\infty$ class of function, and then, the operator $\mathcal{L}_{FP}$ of FPE is

$$\mathcal{L}_{FP} \approx \left(-\frac{\partial}{\partial\varphi}\hat{\Gamma} + H\frac{\partial^2}{\partial\varphi^2}\right) \tag{44}$$

Then FPE is

$$\frac{\partial W(\varphi, t)}{\partial t} = \left(-\hat{\Gamma}\frac{\partial W(\varphi, t)}{\partial\varphi} + H\frac{\partial^2 W(\varphi, t)}{\partial t^2}\right) \tag{45}$$

A particular solution of Equation (45) is

$$W(\varphi, t) = \frac{H}{\hat{\Gamma}}\frac{1}{\sqrt{4\pi H t}}\exp\left\{-\frac{\left(\varphi - \hat{\Gamma}t\right)^2}{4Ht}\right\} \tag{46}$$

Equation (46) is a diffusion equation that constrains the fluctuation in the neighborhood of a synchronization point. Moreover, the following is established in the neighborhood of the synchronization point.

$$\frac{\partial W(\varphi, t)}{\partial t} + \hat{\Gamma}\frac{\partial W(\varphi, t)}{\partial\varphi} = H\frac{\partial^2 W(\varphi, t)}{\partial t^2} \tag{47}$$

where $-\varphi_L \le \varphi \le \varphi_L$ and $0 \le T$, φ_L is a critical point of fluctuation.

Assumption 3.2. *The boundary condition of Equation (47)*

$$\left.\frac{\partial W(\varphi, t)}{\partial\varphi}\right|_{\varphi=-\varphi_L} = \left.\frac{\partial W(\varphi, t)}{\partial\varphi}\right|_{\varphi=\varphi_L} = 0 \tag{48}$$

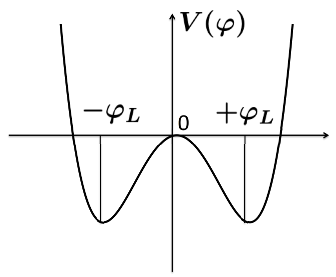

Figure 8. FPE in near origin point

Assumption 3.3. *The initial condition of Equation (47)*

$$W(\varphi,0)=\delta(\varphi),\ \delta(\varphi)=\lim_{\epsilon\to 0}\delta_\epsilon(\varphi)=\frac{1}{\epsilon}\ \left(|\varphi|<\frac{\epsilon}{2}\right)\ \textit{or}\ 0\ \left(|\varphi|>\frac{\epsilon}{2}\right) \tag{49}$$

where let $\hat{\Gamma}=F$.

4. Numerical Simulation.

4.1. **Numerical results of the potential energy function.** We represent the variation status of workers in the stages using the potential function. We use Equation (26) for the numerical calculation. Equation (26) is a potential function that includes the constant term F. Figures 9-16 show the potential function with constant terms F, B, and C. If we choose a significantly large value, the process deviates from synchronization, i.e., if $|F|\geq 3\sqrt{3}B$, the process cannot be synchronized [21]. We refer to Yuasa's report for the numerical simulations. From Figures 9-16, it is evident that $F=0.01$ does not affect

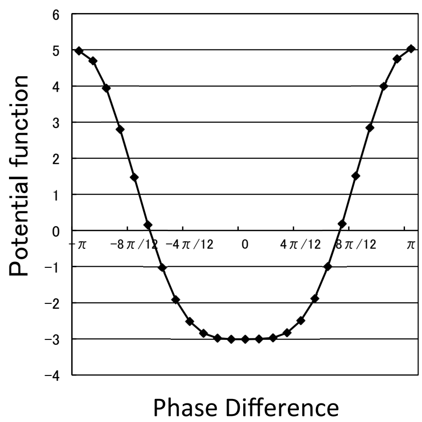

FIGURE 9. Value of potential function ($F=0.01$, $B=1$, $C=1.5$)

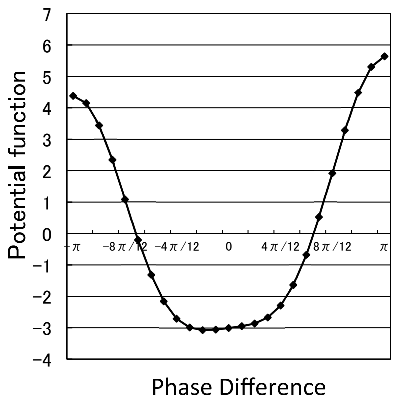

FIGURE 10. Value of potential function ($F=0.2$, $B=1$, $C=1.5$)

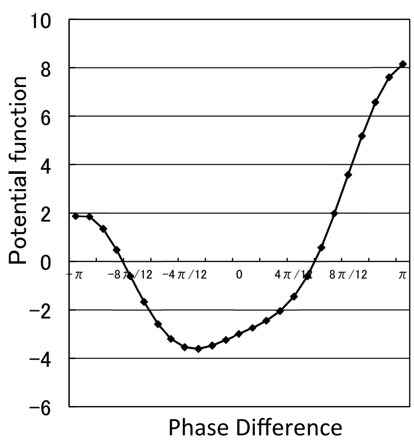

FIGURE 11. Value of potential function ($F=1$, $B=1$, $C=1.5$)

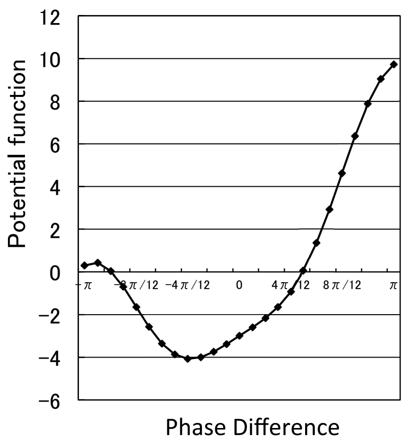

FIGURE 12. Value of potential function ($F=1.5$, $B=1$, $C=1.5$)

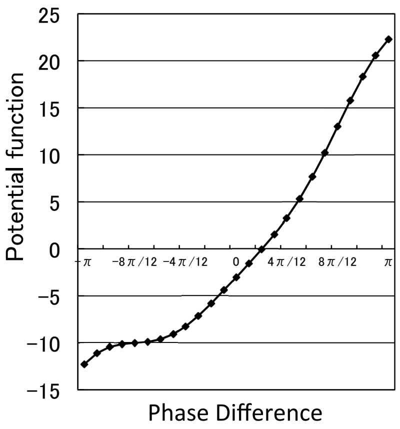

Figure 13. Value of potential function ($F = 5.5$, $B = 1$, $C = 1.5$)

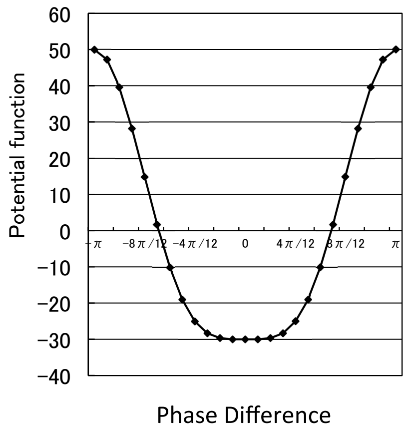

Figure 14. Value of potential function ($F = 0.01$, $B = 10$, $C = 1.5$)

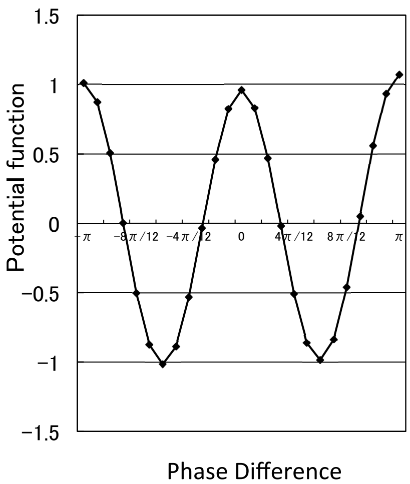

Figure 15. Value of potential function ($F = 0.01$, $B = 1$, $C = 0.01$)

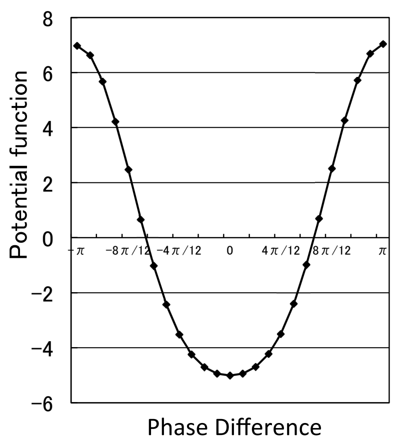

Figure 16. Value of potential function ($F = 0.01$, $B = 1$, $C = 1.5$)

the shape of the potential function. However, $F = 0.2$ or more do and the potential function's symmetry collapses. In other words, the production process cannot maintain the synchronous status. However, B does not significantly affect the symmetric potential function and neither does C; however, the stabilization period is shortened by setting a smaller value, i.e., $C = 0.01$. Please see our previous study in detail [15].

4.2. **Numerical results using the transition probability density function $W(\varphi, t)$.** The value of the transition probability density function $W(\varphi, t)$ changes along with the diffusion coefficient H ($\sim \mu^{-1}$). When μ is relatively small, $W(\varphi, t)$ becomes uniformly attenuated with respect to the value of the phase-difference variable. However, when μ attains a greater value, it can be seen that the slide into increased toward the one of boundary value of the phase difference variable. In other words, when the set of parameters is larger, the synchronous system becomes unstable due to the effects of μ^{-1}, which is equivalent to the diffusion coefficient. Similarly, when H is larger, the synchronization experiences a probable unstable status. In other words, the fluctuation width becomes larger near the origin, as shown in Figures 9 through 16. With respect to these figures, please see the data that are utilized in our research [11].

5. Verification of Actual Data.

5.1. **Production flow system.** Figure 17 shows a production process that is termed as a production flow process. This production process is employed in the production of control equipment. In this example, the production flow process consists of six stages. In each step S1-S6 of the production process, materials are being produced.

The direction of the arrows represents the direction of the production flow. In this process, production materials are supplied through the inlet and the end-product is shipped from the outlet.

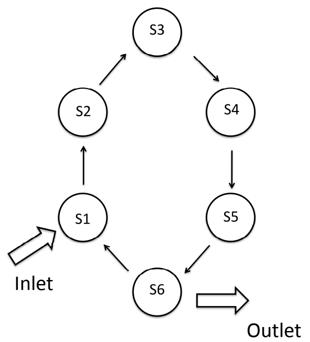

FIGURE 17. Production flow process

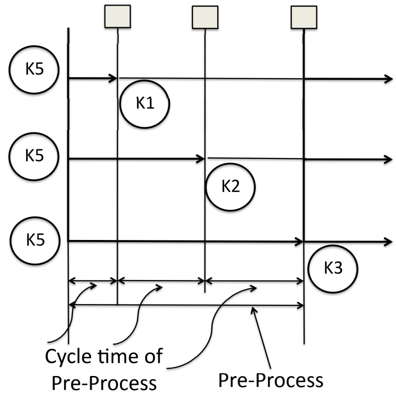

FIGURE 18. Previous process in production equipment

5.2. **Actual data example of production flow process.** The production throughput is evaluated using the number of equipment pieces in comparison with the target number of equipment pieces (production ranking) and simulating asynchronous and synchronous production (see Appendix A). The asynchronous method is prone to worker fluctuations imposed by various delays, whereas worker fluctuations in the synchronous method are small. The productivity ranking tests indicate that test run 3 > test run 2 > test run 1, where test run 1 is asynchronous and test runs 2 and 3 are synchronous.

Here, the throughput values calculated from the throughput probability in Test run 1 – Test run 3, are in Appendix A.

6. Dynamic Simulation of Production Processes.

We attempted to perform a dynamic simulation of the production process by utilizing the simulation system that NTT DATA Mathematical Systems Inc. (www.msi.co.jp) has developed. We conducted the simulation procedure in Figure 19. Please see the reference in detail [16].

With respect to the meaning of the individual parts in Figure 19, “record” calculates the worker’s operating time, which is obtained by multiplying the specified WE data for the log-normally distributed random numbers in the data. Please see the data used in our previous study for Figure 20 [16].

Figure 20 shows the operating time of process 1-6 (record1 – record6). As the working time of the synchronous process is less volatile, the work efficiency became higher than

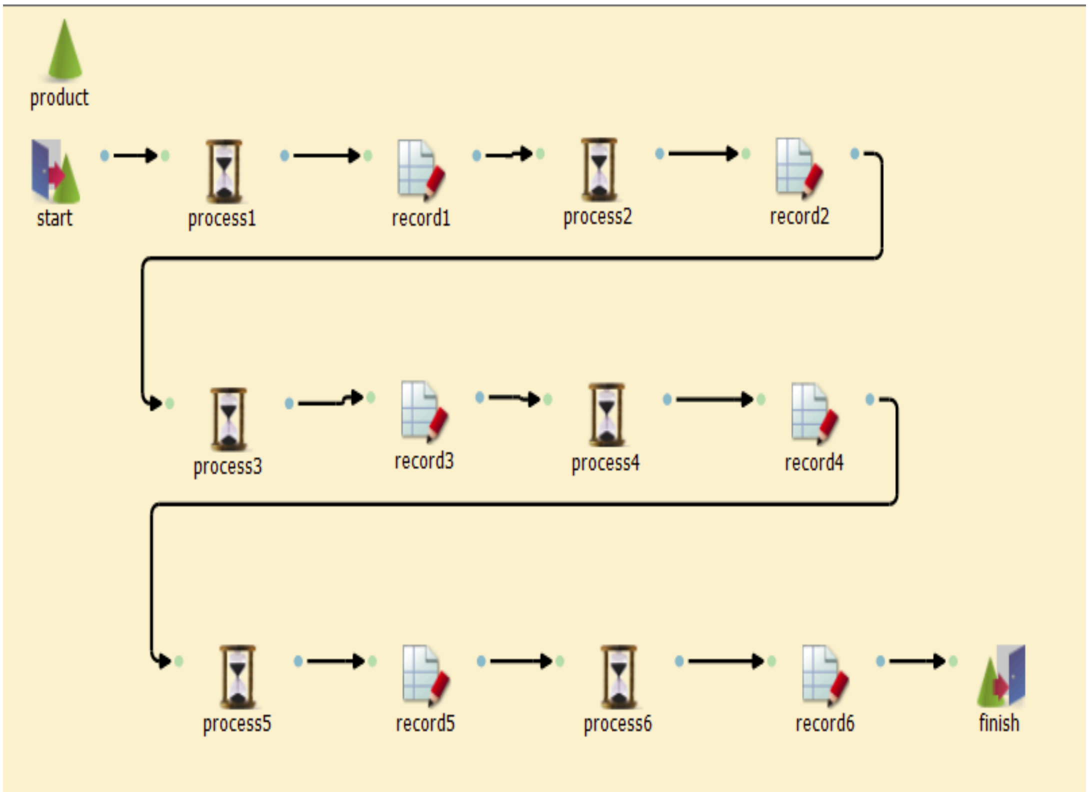

FIGURE 19. Simulation model of production flow system

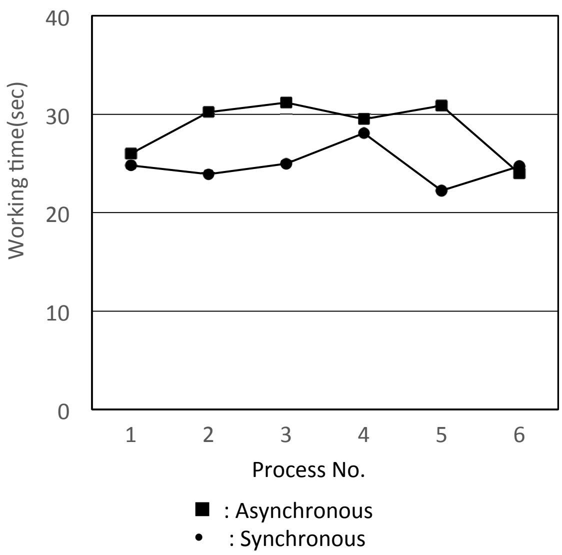

FIGURE 20. Working time for process number one through six

the asynchronous process. In Figure 20, the total working time of asynchronous and synchronous processes are 1241.7(sec) and 586.4(sec) respectively. The synchronous process shows more better production efficiency than the asynchronous process.

7. **Conclusions.** We have clarified the mathematical model using a Langevin-type equation for the propagation of throughput under noise and have also clarified that the

parameters of potential energy function exerted a great impact upon the synchronous status by means of varying the diffusion coefficient. Techniques for maintaining a stable state of the production process using analysis of fluctuations became more clear in this paper. We will report the autocorrelation function and the calculation of the power spectrum in a future study.

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Appendix A. **Analysis of the Test Run Results.** Please see the reference for the actual data used in these results [11, 16].

Table 1. Correspondence between the table labels and the test run number

	Production process	Working time	Volatility
test run 1	Asynchronous process	627(min)	0.29
test run 2	Synchronous process	500(min)	0.06
test run 3	"Synchronization with preprocess" method	470(min)	0.03

The results are as follows. Here, the trend coefficient, which is the actual number of pieces of equipment/the target number of equipment, represents a factor that indicates the degree of the number of pieces of manufacturing equipment.
test run 1: 4.4 (pieces of equipment)/6 (pieces of equipment) = 0.73,
test run 2: 5.5 (pieces of equipment)/6 (pieces of equipment) = 0.92,
test run 3: 5.7 (pieces of equipment)/6 (pieces of equipment) = 0.95.
Volatility data represent the average value of each test run.