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MEAN DERIVATIVES METHODOLOGY BY USING EULER INTEGRATOR IMPROVED TO ALLOW THE VARIATION IN THE SIZE OF INTEGRATION STEP

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Abstract. *In this paper, a theoretical approach is taken to propose how to find out a parable interpolation between two successive points derived from the solution of an autonomous nonlinear dynamic system without the need of precisely knowing a third point. However, it is necessary to represent the mean derivative functions of the solution of this dynamic system by a universal functions approximator that uses the simple Euler integrator structure to obtain the discrete solution for this same system. In this way, the parable interpolation allows the variability of the integration step size with relevant success, although each interpolation of that original discrete solution has a particular and different parable. Two important theorems of differential and integral mean values are used to reach this goal.*

Keywords: Autonomous ordinary nonlinear and linear differential equations, Neural numerical integrators, Neural networks, Fuzzy systems, Universal functions approximators, Direct mean derivatives methodology

1. **Introduction.** Neural networks and fuzzy logic have been used for the identification of dynamic system and application in control theory involving nonlinear autonomous dynamic systems for project and simulation of real world plans (e.g., [4,8,11,14,16,19-21]). The usual approach in control schemes is the one that uses NARMAX inputs/outputs (nonlinear autoregressive moving average with exogenous inputs) to approximate dynamic systems (see [4]).

An alternative to solve this kind of problem is to use a numerical integrator (e.g., [9,10,23]) for ordinary differential equations (ODE) together with feedforward network (see [12,13,15]) and therefore, when this set is used simultaneously, there are some advantages as listed as follows: (i) the network only needs to learn the instantaneous derivative functions or rates of change of system states that are only a static function in themselves, (ii) the network can predict the system behavior at any time [12] and not for a fixed time step as it happens in the NARMAX method, and (iii) local errors can be adjusted by evaluating methods that automatically vary the order of the integrator step size.

In this way, Wang and Lin [12] introduced the RKNN's (Runge-Kutta neural networks) term into artificial neural networks literature for mapping nonlinear dynamic systems. An application that uses neural numerical integration of fourth order with Adams-Bashforth structure for training dynamic systems and its respective application in control theory was performed in [22].

The control theory using neural numerical integrators was recently introduced (e.g., [18,22]) and the first important feature about it is the difficulty of using a numerical integrator with higher order because there are many inherent obstacles to determine the gradient function that combines feedforward network sets and numerical integrators which uses backpropagation concept.

In this way, mean derivatives functions were elaborated inside a Euler integrator or first order integrator with fixed time step size (e.g., [17,24,25]), where the neural network is used only as a mean derivative of the original system and not as instantaneous derivative anymore. This has the great advantage of simplifying the backpropagation that is present in the training of the neural network and also in the neural predictive control and adaptative theory. However, the mean derivative method has only a discrete solution with constant step size and for this reason, this paper has the purpose of allowing the variation of integration step size in the solution obtained by Euler integrator, which uses mean derivative functions through a parable interpolation between two successive points without the need of a third intermediate point.

As described below, in Section 2, Euler neural integrator structure that uses mean derivatives for the solution of a nonlinear dynamic system is presented in a concise way. In Section 3, the complete mathematical reasoning for the development of parable interpolation by using differential and integral mean values theorems is presented in order to change the mentioned method for the variable step size method. Section 4 elaborates the conclusion of this work.

2. Mean Derivatives Methodology.

In this section, it firstly presents the fundamental concepts that support the possibility of getting a nonlinear dynamic system model through Euler numerical integrators by using mean derivative functions. Qualitative properties of first order nonlinear autonomous dynamic systems are considered to prove the possibility of using some universal approximator of functions (e.g., [1-3,11,16]) as mean derivative functions inside Euler neural integrators.

These theoretical results support the possibility of obtaining a discrete model for a dynamic system by using a feedforward neural network that represents mean derivatives in the structure of a Euler integrator. The capacity of neural networks to represent nonlinear functions and the use of mean derivatives, instead of instantaneous derivatives, allows the accuracy of any high order integrator by using a first order Euler structure only. This means that there is a meaningfully less complex discrete model to be used in simulations.

Being the autonomous system of nonlinear ordinary differential equations,

$$\dot{y} = f(y) \tag{1.a}$$

where,

$$y = [y_1 \quad y_2 \quad \dots \quad y_n]^T \tag{1.b}$$

$$f(y) = [f_1(y) \quad f_2(y) \quad \dots \quad f_n(y)]^T \tag{1.c}$$

Consider also by definition, $y_j^i = y_j^i(t)$ for $j = 1, 2, \dots, n$ a trajectory of the family of solutions of nonlinear differential equations system $\dot{y} = f(y)$ going past $y_j^i(t_o)$ at the initial time t_o, starting from a domain of interest $\left[y_j^{\min}(t_o), y_j^{\max}(t_o)\right]^n$, where $y_j^{\min}(t_o)$ and $y_j^{\max}(t_o)$ are finite. It is appropriate to introduce the following vector notation concerning the possible sets of initial conditions and solutions of (1.a):

$$y_o^i = y^i(t_o) = [y_1^i(t_o) \quad y_2^i(t_o) \quad \dots \quad y_n^i(t_o)]^T \tag{2.a}$$

$$y^i = y^i(t) = [y_1^i(t) \quad y_2^i(t) \quad \dots \quad y_n^i(t)]^T \tag{2.b}$$

where $i = 1, 2, \dots, \infty$, and ∞ indicates that the number of points in the discretization loop can be as big as one wants. As, in general, $\dot{y}^i = f(y^i)$ does not have analytical solution, it is common to know only for $y^i = y^i(t)$ a discrete set of points $[y^i(t + k \cdot \Delta t) \quad y^i[t +$

$(k+1)\cdot\Delta t] \quad \ldots \quad y^i[t+(k+n)\cdot\Delta t]] \equiv [{}^k y^i \quad {}^{k+1}y^i \quad \ldots \quad {}^{k+n}y^i]$ for ${}^k y^i = y^i(t+k\cdot\Delta t)$ on the horizon $[t_k, t_{k+n}]$ and $\Delta t = (t_{k+n} - t_k)/n$ a constant. By definition (see [10]), the secant between two points ${}^k y^i$ and ${}^{k+1}y^i$ belonging to curve $y^i(t)$ is the straight segment which joins these two points. So the tangents of the secants between points ${}^k y_1^i$ and ${}^{k+1}y_1^i$, ${}^k y_2^i$ and ${}^{k+1}y_2^i$, ..., ${}^k y_n^i$ and ${}^{k+1}y_n^i$ are defined as:

$$\tan_{\Delta t}\alpha^i(t+k\cdot\Delta t) = \tan_{\Delta t}{}^k\alpha^i = [\tan_{\Delta t}{}^k\alpha_1^i \quad \tan_{\Delta t}{}^k\alpha_2^i \quad \ldots \quad \tan_{\Delta t}{}^k\alpha_n^i]^T \tag{3.a}$$

with,

$$\tan_{\Delta t}{}^k\alpha_j^i = \frac{{}^{k+1}y_j^i - {}^k y_j^i}{\Delta t}, \qquad \text{to } j = 1,2,\ldots,n \tag{3.b}$$

where α_j^i is the secant angle that joins both ${}^k y_j^i$ and ${}^{k+1}y_j^i$ points belonging to curve $y_j^i(t)$. Being so, some properties and theorems without demonstration are enunciated in the following. For more details concerning to the mathematical development that follows, see [24].

Property 2.1. (P1). *If ${}^k y^i$ is a discretization of solution $\dot{y}^i = f(y^i)$ and $\Delta t \neq 0$, $\tan_{\Delta t}{}^k\alpha^i$ exists and it is unique.*

Property 2.2. (P2). *Given the state vector $y^i(t_o)$ at t_o instant of the nonlinear differential equations $\dot{y} = f(y)$ and if the vector $[{}^k y^i \quad {}^{k+1}y^i \quad \ldots \quad {}^{k+n}y^i]$ is the discretization of a trajectory solution so that for ${}^k y^i = y^i(t+k\cdot\Delta t)$ it is a solution of the dynamic system, then $\tan_{\Delta t}{}^{k+l}\alpha^i$ for $l = 0,1,\ldots,(n-1)$ exist and are unique.*

Two other very important theorems that link the values of $\tan_{\Delta t}{}^k\alpha^i$ and $\tan_{\Delta t}{}^k\dot{\alpha}^i$, respectively, with the mean derivatives calculated from $[{}^k y^i \quad {}^{k+1}y^i \ldots {}^{k+n}y^i]$ and $[{}^k\dot{y}^i \quad {}^{k+1}\dot{y}^i \ldots {}^{k+n}\dot{y}^i]$ are the differential and integral mean value theorems (e.g., [5,7]) which are set out below without proof.

Theorem 2.1. (T1). *(The differential mean value theorem). If the function $y_j^i(t)$ for $j = 1,2,\ldots,n$ is defined and continuous function on the closed interval $[t_k, t_{k+1}]$ and differentiable on the open interval (t_k, t_{k+1}), there is at least one number t_k^* with $t_k < t_k^* < t_k + \Delta t$ such that*

$$\dot{y}_j^i(t_k^*) = \frac{{}^{k+1}y_j^i - {}^k y_j^i}{\Delta t} \tag{4}$$

Theorem 2.2. (T2). *(The integral mean value theorem). If a function $y_j^i(t)$ for $j = 1,2,\ldots,n$ is a continuous function on the closed interval $[t_k, t_{k+1}]$, then there is at least one t_k^x inner number to $[t_k, t_{k+1}]$ such that*

$$y(t_k^x)\cdot\Delta t = \int_{t_k}^{t_{k+1}} y^i(t)\cdot dt \tag{5}$$

Generally, t_k^* is different from t_k^x and it is also important to observe that the mean values theorems say nothing on how to determine the values t_k^* and t_k^x. The theorems simply state that t_k^* and t_k^x are contained in the interval $[t_k, t_{k+1}]$.

Property 2.3. (P3). *Applying the T2 theorem on curve $\dot{y}^i(t)$ is equivalent to applying the T1 theorem on the curve $y^i(t)$ both on the same closed interval $[t_k, t_{k+1}]$, that is, $\dot{y}^i(t_k^x) = \dot{y}^i(t_k^*)$.*

Property 2.4. (P4). *The mean derivative $\dot{y}^i(t_k^x)$ on the graph of $y^i(t)$ in the closed interval $[t_k, t_{k+1}]$ is equal to $\tan_{\Delta t}{}^k\alpha^i$, as an immediate consequence of the definition itself of the mean derivatives.*

Property 2.5. (P5). *Note that if $\dot{y}^i(t_k^x) = \dot{y}^i(t_k^*)$ (P3) and $\dot{y}^i(t_k^*)$ is the mean derivative of $y^i(t)$ (T1 theorem) in the closed interval $[t_k, t_{k+1}]$, $\dot{y}^i(t_k^x)$ will also be numerically equal to the mean derivative of $y^i(t)$ for the same close interval.*

Theorem 2.3. (T3). *The discrete general solution and exact ${}^{k+1}y_j^i$ for $j = 1, 2, \ldots, n$ of the nonlinear differential equations systems $\dot{y}^i = f(y^i)$ can be established through the relation ${}^{k+1}y_j^i = \tan_{\Delta t} {}^k\alpha^i \cdot \Delta t + {}^k y_j^i$ for a given ${}^k y^i$ and Δt.*

Proof: If $\dot{y}^i = \frac{dy^i}{dt} = f(y^i)$, $\int\limits_{{}^k y^i}^{{}^{k+1}y^i} dy^i = \int\limits_{t_k}^{t_{k+1}} f(y^i) \cdot dt$. Thus,

$$ {}^{k+1}y^i = \int_{t_k}^{t_{k+1}} f(y^i) \cdot dt + {}^k y^i \tag{6} $$

The application of the integral mean value theorem T2 on the curve $\dot{y}^i(t)$ in the closed interval $[t_k, t_{k+1}]$ implies that there is at least a number t_k^x in $[t_k, t_{k+1}]$ so that

$$ \dot{y}(t_k^x) \cdot \Delta t = \int_{t_k}^{t_{k+1}} \dot{y}^i(t) \cdot dt = \int_{t_k}^{t_k+1} f(y^i) \cdot dt \tag{7} $$

By P4 $\dot{y}^i(t_k^x) = \tan_{\Delta t} {}^k\alpha^i$. Therefore, replacing the greatness $\dot{y}^i(t_k^x)$ in Equation (7) and this in Equation (6), results in:

$$ {}^{k+1}y^i = \tan_{\Delta t} {}^k\alpha^i \cdot \Delta t + {}^k y^i \tag{8} $$

Equation (8) is the discrete general solution and exact to the ordinary differential equations system (1.a). Note that this is a fairly simple expression. It is nothing more than Euler simple step integration structure. However, instead of using instantaneous derivatives functions, it uses mean derivatives functions.

It has been mathematically proved that Equation (8) can be obtained from differential and integral mean values theorems by starting with the equation $\dot{y} = f(y)$. It is important to note that Equation (8) is a discrete equation and not a continuous one. Besides to this, if the discrete value of Δt varies, the mean derivatives functions values also vary. One should also note that Equation (8) can be used with control independent variables (e.g., $\dot{y} = f(y, u)$) in the function of $\tan_{\Delta t} {}^k\alpha^i$. In such a way, Equation (8) can have several practical applications in fuzzy control or neurocontrol (see [18]).

Nevertheless, empirically establishing mean derivatives functions is not a trivial task, because some kind of universal approximator of functions is needed for that, such as artificial neural networks or fuzzy systems. This section is ended with the enunciation of two more useful theorems (see [24]).

Theorem 2.4. (T4). *The definition of the secant between two distinct and any points, separated by the finite distance Δt on the continuous curves $y_j(t)$, for $j = 1, 2, \ldots, n$ is also an exact general solution and discrete to the system of autonomous ordinary differential and nonlinear equations $\dot{y} = f[y(t)]$.*

Theorem 2.5. (T5). *For sequences of discrete points represented by equations ${}^{k+1}y_j^i = \tan_{\Delta t} {}^k\alpha_j^i \cdot \Delta t + {}^k y_j^i$, for $j = 1, 2, \ldots, n$ there is always an instantaneous derivative function $\dot{y} = f(y)$ associated to them.*

3. **Mean Derivatives Methodology with Step Variation through a Parabolic Interpolation.** In this section, it is proposed of an approximate method to obtain an approximate continuous general solution for nonlinear dynamic systems. The continuous solution is obtained by the interpolation of endless parables, one for each discrete interval $[t_k, t_{k+1}]$. Theorems T6 and T7 are used with the purpose of demonstrating this.

Theorem 3.1. (T6). The value of $\dot{y}_j^i(t_k^*)$ *is given by* $\tan_{\Delta t}{}^k\alpha_j^i$.

Proof: According to Theorem T2, there is a value $\dot{y}_j^i(t_k^x)$ such that, $\dot{y}_j^i(t_k^x).\Delta t = \int_{t_k}^{t_{k+1}} \dot{y}_j(t)dt = \int_{t_k}^{t_{k+1}} \frac{dy_j(t)}{dt}dt = y_j^i(t_{k+1}) - y_j^i(t_k) = {}^{k+1}y_j^i - {}^k y_j^i$ and thus,

$$\dot{y}_j^i(t_k^x) = \frac{{}^{k+1}y_j^i - {}^k y_j^i}{\Delta t} = \tan_{\Delta t}{}^k\alpha_j^i \tag{9}$$

Corollary 3.1. (C1). *The following relation is valid,*

$$\ddot{y}_j^i\left(t_k^*\right) = \frac{d}{dt}\dot{y}_j^i\left(t_k^x\right) \tag{10}$$

Equation (10) simply states that the instantaneous variation of $\dot{y}_j^i(t_k^x)$ at t_k^x instant is equal to the instantaneous variation value of $\dot{y}_j^i(t_k^*)$ at t_k^* instant. This derives immediately from Property 2.3 (P3).

Theorem 3.2. (T7). *The value of* $\ddot{y}_j^i(t_k^x)$ *is given by* $\tan_{\Delta t}{}^k\Psi_j^i = \frac{d}{dt}\tan_{\Delta t}{}^k\alpha_j^i$.

Proof: By applying the differential mean value theorem on the curve $y(t)$, one has $\dot{y}_j^i(t_k^*) = \frac{y_j^i(t_{k+1}) - y_j^i(t_k)}{\Delta t} = \tan_{\Delta t}{}^k\alpha_j^i$. Differentiating both sides of this equation in relation the instant t, it will result in $\frac{d}{dt}\dot{y}_j^i(t_k^*) = \ddot{y}_j^i(t_k^*) = \ddot{y}_j^i(t_k^x) = \frac{d}{dt}\tan_{\Delta t}{}^k\alpha_j^i$. Only to simplify the notation, the definition $\tan_{\Delta t}{}^k\Psi_j^i = \frac{d}{dt}\tan_{\Delta t}{}^k\alpha_j^i$ is given.

With an immediate analysis of the theorems and corollary and definition shown in this section, it is possible to build Table 1 with the coordinated points in question.

TABLE 1. Four coordinate points within the interval $[t_k, t_{k+1}]$

n	Time	$y(t)$, $\dot{y}(t)$ and $\ddot{y}(t)$	Determining Form
1	t_k	${}^k y_j^i$	Instant initial
2	t_{k+1}	${}^{k+1}y_j^i$	Given by Euler integrator with mean derivatives
3	t_k^x	$\dot{y}_j^i(t_k^x) = \tan_{\Delta t}{}^k\alpha_j^i$	Provided by universal approximator used
4	t_k^*	$\ddot{y}_j^i(t_k^*) = \tan_{\Delta t}{}^k\Psi_j^i$	Given by Theorem 3.2

It is important to note that the instants t_k^x and t_k^* are confined in the interval $[t_k, t_{k+1}]$. However, it is impossible to determine where exactly these values are because Theorems T1 and T2 omit this information. Even so, they are not necessary if one desires to interpolate only a parable on the given interval $[t_k, t_{k+1}]$.

The following paragraphs develop an analytical method to find a parable on the interval $[t_k, t_{k+1}]$ considering the only points $\left(t_k, {}^k y_j^i\right)$, $\left(t_{k+1}, {}^{k+1}y_j^i\right)$ and $\left(t_k^*, \ddot{y}_j^i(t_k^*) = \tan_{\Delta t}{}^k\Psi_j^i\right)$, where the two main issues are: 1) it is demonstrated that it is not necessary to know the value of t_k^*; 2) how it is possible to find out the value of the variable $\tan_{\Delta t}{}^k\Psi_j^i$.

The first step is to find out the coefficients α_k, β_k, and γ_k of the equation of the parable given by $y(t) = \alpha_k t^2 + \beta_k t + \gamma_k$. In general, these coefficients are functions of the following variables: $\alpha_k = f_1\left(t_k, t_{k+1}, {}^k y_j^i, {}^{k+1}y_j^i, \tan_{\Delta t}{}^k\alpha_j^i, \tan_{\Delta t}{}^k\Psi_j^i\right)$, $\beta_k = f_2\big(t_k, t_{k+1}, {}^k y_j^i, {}^{k+1}y_j^i,$

$\tan_{\Delta t} {}^k\alpha_j^i, \tan_{\Delta t} {}^k\Psi_j^i\big)$ and $\gamma_k = f_3\left(t_k, t_{k+1}, {}^k y_j^i, {}^{k+1}y_j^i, \tan_{\Delta t} {}^k\alpha_j^i, \tan_{\Delta t} {}^k\Psi_j^i\right)$. The letter k pres-ent in the coefficients α_k, β_k, and γ_k means that it a parable is needed for each interval $[t_k, t_{k+1}]$. These coefficients can be obtained by simply solving the linear system given below.

$$ {}^k y_j^i = \alpha_k . t_k^2 + \beta_k . t_k + \gamma_k \tag{11.a} $$

$$ {}^{k+1} y_j^i = \alpha_k . t_{k+1}^2 + \beta_k . t_{k+1} + \gamma_k \tag{11.b} $$

$$ \tan_{\Delta t} {}^k\Psi_j^i = 2 \cdot \alpha_k \tag{11.c} $$

The equations from (11.a) to (11.c) can also be developed in the vector form, like:

$$ \left\{ \begin{array}{c} {}^k y_j^i \\ {}^{k+1} y_j^i \\ \tan_{\Delta t} {}^k\Psi_j^i \end{array} \right\} = \begin{bmatrix} t_k^2 & t_k & 1 \\ t_{k+1}^2 & t_{k+1} & 1 \\ 2 & 0 & 0 \end{bmatrix} \cdot \left\{ \begin{array}{c} \alpha_k \\ \beta_k \\ \gamma_k \end{array} \right\} \tag{12} $$

The linear system given by Equation (12) can be solved by Cramer's method, in this way, the coefficients are determined by $\alpha_k = D1/D$, $\beta_k = D2/D$ and $\gamma_k = D3/D$, where $D = \begin{vmatrix} t_k^2 & t_k & 1 \\ t_{k+1}^2 & t_{k+1} & 1 \\ 2 & 0 & 0 \end{vmatrix}$, $D_1 = \begin{vmatrix} {}^k y_j^i & t_k & 1 \\ {}^{k+1} y_j^i & t_{k+1} & 1 \\ \tan_{\Delta t} {}^k\Psi_j^i & 0 & 0 \end{vmatrix}$, $D_2 = \begin{vmatrix} t_k^2 & {}^k y_j^i & 1 \\ t_{k+1}^2 & {}^{k+1} y_j^i & 1 \\ 2 & \tan_{\Delta t} {}^k\Psi_j^i & 0 \end{vmatrix}$, and $D_3 = \begin{vmatrix} t_k^2 & t_k & {}^k y_j^i \\ t_{k+1}^2 & t_{k+1} & {}^{k+1} y_j^i \\ 2 & 0 & \tan_{\Delta t} {}^k\Psi_j^i \end{vmatrix}$. The expressions given by D, D_1, D_2 and D_3 can be simplified as follows:

$$ D = -2 \cdot \Delta t \tag{13.a} $$

$$ D_1 = -\Delta t \cdot \tan_{\Delta t} {}^k\Psi_j^i \tag{13.b} $$

$$ D_2 = \Delta t \cdot (t_{k+1} + t_k) \cdot \tan_{\Delta t} {}^k\Psi_j^i - 2 \cdot \left({}^{k+1} y_j^i - {}^k y_j^i \right) \tag{13.c} $$

$$ D_3 = \Delta t \cdot \left(-t_k \cdot t_{k+1} \cdot \tan_{\Delta t} {}^k\Psi_j^i + 2 \cdot t_k \cdot \tan_{\Delta t} {}^k\alpha_j^i - 2 \cdot {}^k y_j^i \right) \tag{13.d} $$

Thus, the solution to the linear system given by Equation (12) is given by:

$$ \alpha_k = \frac{1}{2} \cdot \tan_{\Delta t} {}^k\Psi_j^i \tag{14.a} $$

$$ \beta_k = \tan_{\Delta t} {}^k\alpha_j^i - \frac{1}{2} \cdot (t_{k+1} + t_k) \cdot \tan_{\Delta t} {}^k\Psi_j^i \tag{14.b} $$

$$ \gamma_k = \frac{1}{2} \cdot \left(t_k \cdot t_{k+1} \cdot \tan_{\Delta t} {}^k\Psi_j^i - 2 \cdot t_k \cdot \tan_{\Delta t} {}^k\alpha_j^i + 2 \cdot {}^k y_j^i \right) \tag{14.c} $$

where the value of $\dot{y}_j^i\left(t_k^x\right) = \tan_{\Delta t} {}^k\alpha_j^i$ can be previously known through a supervised training by using a universal approximator for functions. However, the most pertinent question related to equations from (14.a) to (14.c) is how to determine the value of $\tan_{\Delta t} {}^k\Psi_j^i$. This can be done from previous knowledge of the value of $\tan_{\Delta t} {}^k\alpha_j^i$. Because of Theorem 3.2 (T7), $\tan_{\Delta t} {}^k\Psi_j^i = \frac{d}{dt} \tan_{\Delta t} {}^k\alpha_j^i$ for $j = 1, 2, \ldots, n$. Nevertheless, if the chain rule (e.g., [5-7]) is used, in this latter expression, the following result is obtained:

$$
\begin{aligned}
\frac{d}{dt}\tan_{\Delta t}{}^{k}\alpha_j^i &= \frac{\partial \tan_{\Delta t}{}^{k}\alpha_j^i}{\partial^k y_1^i}\cdot\frac{d^k y_1^i}{dt}+\cdots+\frac{\partial \tan_{\Delta t}{}^{k}\alpha_j^i}{\partial^k y_n^i}\cdot\frac{d^k y_n^i}{dt} \\
&= \left\{\frac{\partial \tan_{\Delta t}{}^{k}\alpha_j^i}{\partial^k y_1^i}\cdots\frac{\partial \tan_{\Delta t}{}^{k}\alpha_j^i}{\partial^k y_n^i}\right\}\cdot\left\{\begin{array}{c}\frac{d^k y_1^i}{dt}\\ \vdots \\ \frac{d^k y_n^i}{dt}\end{array}\right\} \\
&= \left\{\frac{\partial \tan_{\Delta t}{}^{k}\alpha_j^i}{\partial^k y_1^i}\cdots\frac{\partial \tan_{\Delta t}{}^{k}\alpha_j^i}{\partial^k y_n^i}\right\}\cdot\left\{\begin{array}{c}\tan^k\theta_1^i\\ \vdots \\ \tan^k\theta_n^i\end{array}\right\} = \frac{\partial \tan_{\Delta t}{}^{k}\alpha_j^i}{\partial^k y^i}\circ\tan^k\theta^i
\end{aligned}
\tag{15}
$$

for $j=1,2,\ldots,n$. The last expression can be simply given by:

$$
\frac{d}{dt}\tan_{\Delta t}{}^{k}\alpha_j^i = \frac{\partial \tan_{\Delta t}{}^{k}\alpha_j^i}{\partial^k y^i}\circ\tan^k\theta^i \quad \text{for } j=1,2,\ldots,n \tag{16}
$$

The operator $\circ$ is just the usual scalar product. The vector form of Equation (16) can be expressed by:

$$
\tan_{\Delta t}{}^{k}\dot{\alpha}^i = \frac{\partial \tan_{\Delta t}{}^{k}\alpha^i}{\partial^k y^i}\cdot\tan^k\theta^i \tag{17.a}
$$

where,

$$
\tan_{\Delta t}{}^{k}\dot{\alpha}^i = \left\{\begin{array}{c}\tan_{\Delta t}{}^{k}\dot{\alpha}_1^i\\ \tan_{\Delta t}{}^{k}\dot{\alpha}_2^i\\ \vdots\\ \tan_{\Delta t}{}^{k}\dot{\alpha}_n^i\end{array}\right\} \tag{17.b}
$$

$$
\frac{\partial \tan_{\Delta t}{}^{k}\alpha^i}{\partial^k y^i} = \begin{bmatrix}
\frac{\partial \tan_{\Delta t}{}^{k}\alpha_1^i}{\partial^k y_1^i} & \frac{\partial \tan_{\Delta t}{}^{k}\alpha_1^i}{\partial^k y_2^i} & \cdots & \frac{\partial \tan_{\Delta t}{}^{k}\alpha_1^i}{\partial^k y_n^i}\\
\frac{\partial \tan_{\Delta t}{}^{k}\alpha_2^i}{\partial^k y_1^i} & \frac{\partial \tan_{\Delta t}{}^{k}\alpha_2^i}{\partial^k y_2^i} & \cdots & \frac{\partial \tan_{\Delta t}{}^{k}\alpha_2^i}{\partial^k y_n^i}\\
\vdots & \vdots & \ddots & \vdots\\
\frac{\partial \tan_{\Delta t}{}^{k}\alpha_n^i}{\partial^k y_1^i} & \frac{\partial \tan_{\Delta t}{}^{k}\alpha_n^i}{\partial^k y_2^i} & \cdots & \frac{\partial \tan_{\Delta t}{}^{k}\alpha_n^i}{\partial^k y_n^i}
\end{bmatrix} \tag{17.c}
$$

$$
\tan^k\theta^i = \left\{\begin{array}{c}\tan^k\theta_1^i\\ \tan^k\theta_2^i\\ \vdots\\ \tan^k\theta_n^i\end{array}\right\} \tag{17.d}
$$

Note that the vector $\tan^k\theta^i$ is the instantaneous derivatives function. However, there is another problem here, because it is also necessary to train a new universal approximator of functions to learn this instantaneous derivatives function. An explanation on how to determine empirically the instantaneous derivatives functions, that is, the vector $\tan^k\theta^i$, can be found in [12,13]. However, in order to avoid this new problem, it is possible to approximate these instantaneous derivatives through mean derivatives when the interval

$\Delta t = t_{k+1} - t_k$ tends to zero. For doing so, it is sufficient to simplify expressions (16), (17.a), and (17.d) as described below.

$$\frac{d}{dt}\tan_{\Delta t}{}^k\alpha_j^i \cong \frac{\partial \tan_{\Delta t}{}^k\alpha_j^i}{\partial^k y^i} \circ \tan_{\Delta t}{}^k\alpha^i \quad \text{for } j = 1, 2, \ldots, n \tag{18}$$

So, the vector form of Equation (18) is:

$$\tan_{\Delta t}{}^k\dot{\alpha}^i \cong \frac{\partial \tan_{\Delta t}{}^k\alpha^i}{\partial^k y^i} \cdot \tan_{\Delta t}{}^k\alpha^i \tag{19}$$

Expression (19) can be considered as the simplest equation to find the value of $\tan_{\Delta t}{}^k\Psi_j^i$. However, this expression can be applied only in the case where $\Delta t = t_{k+1} - t_k \to 0$. However, if two universal approximators are used, one to represent the mean derivatives function and the other to represent the instantaneous derivatives function, the integration step $\Delta t = t_{k+1} - t_k$ may be slightly bigger. For very large integration steps, the parable does not fit on the real solution. Furthermore, for $\Delta t > 1$, the supervised learning becomes more difficult. Another important issue is how to determine the values of the derivatives $\frac{\partial \tan_{\Delta t}{}^k\alpha^i}{\partial^k y^i}$, given by Equation (17.c). For example, if one uses a neural network with MLP (multilayer perceptron) architecture to represent the mean derivatives functions, then the calculation of the derivatives in (17.c) should be obtained by the backpropagation algorithm. Nevertheless, in the latter, instead of calculating the partial derivatives of the network output with respect to the connection weights, one should calculate the partial derivatives of the network output with respect to inputs. Figure 1 graphically illustrates the difference between mean derivatives functions and instantaneous derivatives functions.

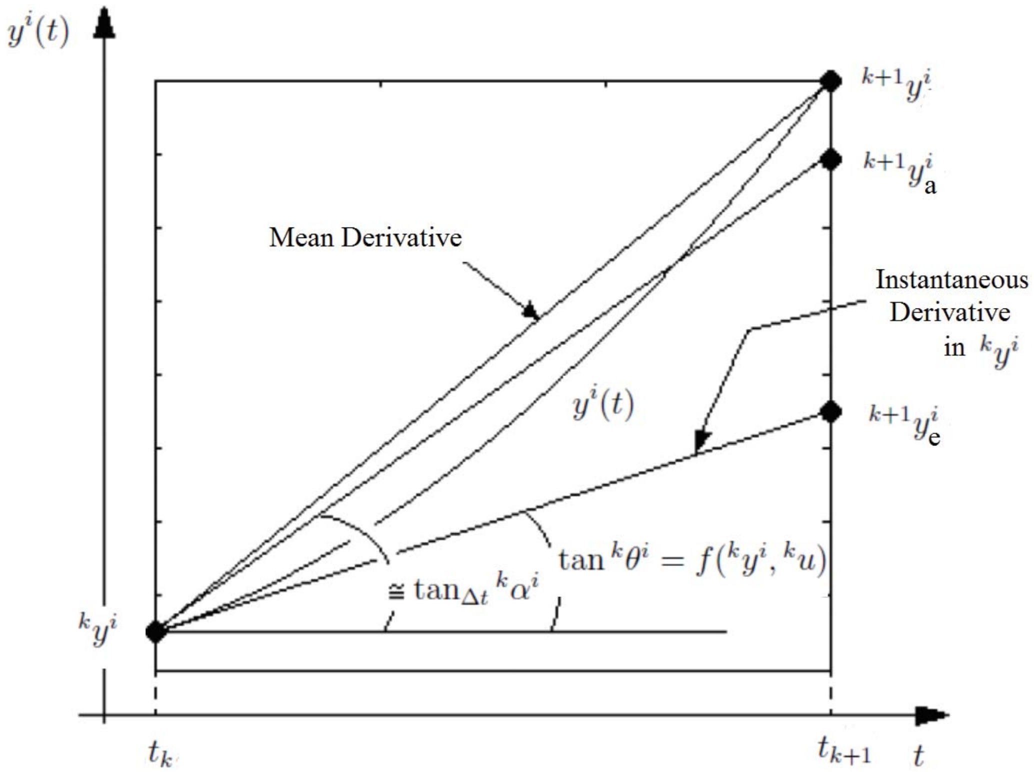

FIGURE 1. Graphical representation of the mean derivatives $\tan_{\Delta t}{}^k\alpha^i$ and of the instantaneous derivatives $\tan{}^k\theta^i = f({}^k y^i, {}^k u)$, for the dynamic system $\dot{y}^i = f(y^i)$

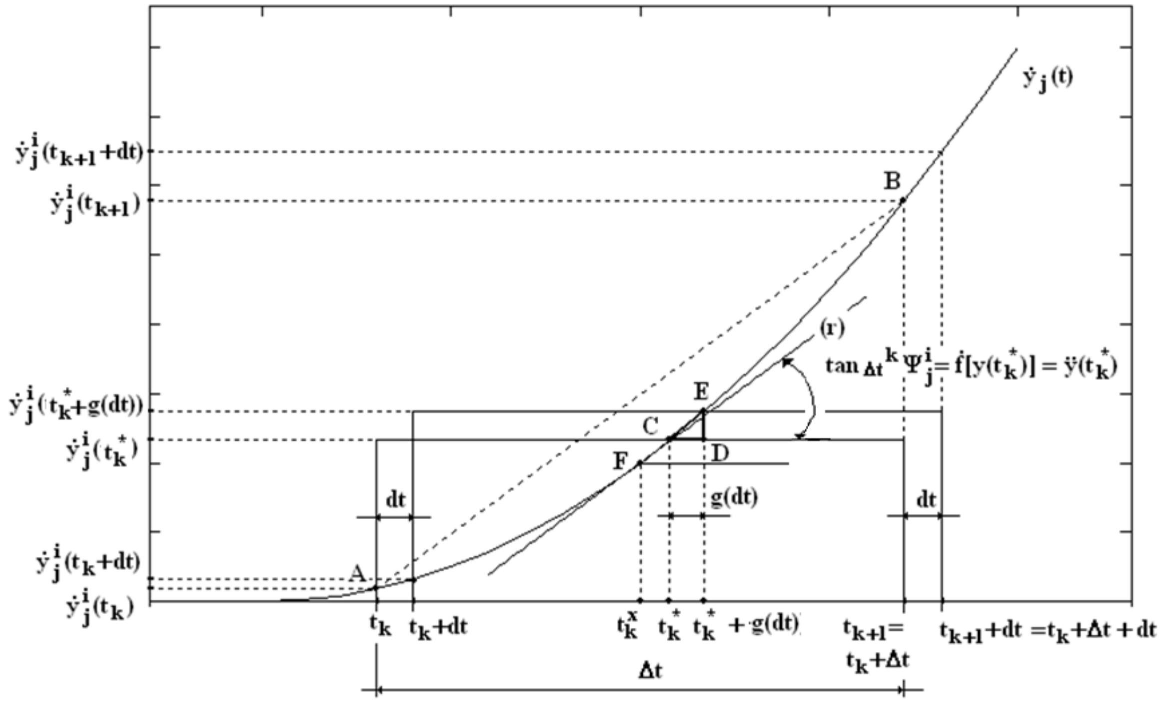

FIGURE 2. Geometrical meaning of the variable $\tan_{\Delta t}{}^{k}\Psi_j^i$

Another important information is the fact that the variable $\tan_{\Delta t}{}^{k}\Psi_j^i$ has a geometric interpretation. Figure 2 illustrates this. Based on this figure, one can establish the variable $\tan_{\Delta t}{}^{k}\Psi_j^i$ in the following way:

$$\tan_{\Delta t}{}^{k}\Psi_j^i = \lim_{dt\to 0}\frac{\bar{E}\bar{D}}{\bar{C}\bar{D}} = \lim_{dt\to 0}\frac{\dot{y}\left(t_k^*+dt\right)-\dot{y}\left(t_k^*\right)}{dt} \equiv \frac{d\dot{y}\left(t_k^*\right)}{dt} \equiv \ddot{y}\left(t_k^*\right), \quad \text{to } j=1,2,\ldots,n \tag{20}$$

In summary, the numerical determination of equation and computational (17.a) demands the knowledge of mean derivatives functions and of instantaneous derivatives ones. In the pioneer and important work of Wang and Lin [12], it presents a way of empirically establishing the instantaneous derivatives functions. In order to perform that, these authors have inserted a *Feedforward* neural network inside a Runge-Kutta 4-5 integration structure. By proceeding in such a way, they have proved that the neural training converges to instantaneous derivative functions. Nevertheless, the proposed method is a little complex from a mathematical and computational perspective.

4. **Conclusions.** From what has been stated in the previous sections, one takes up the following conclusions.

1) It can be shown that there is an *approximate continuous* solution for Euler discrete solution with mean derivatives proposed in this article. The continuous approach can be achieved by a parabolic interpolation on the interval $[t_k, t_{k+1}]$. There are endless parables, one for each initial condition on the interval $[t_k, t_{k+1}]$. It can be mathematically proven that the parabolic interpolating function passes through at least one internal exact point within the interval $[t_k, t_{k+1}]$ of the real solution $y(t)$. Unfortunately, this approximate continuous solution cannot be applied to dynamic systems with control variables. In addition to this, in order that the parable properly accommodate on the exact solution, the step of integration Δt cannot be too large.

2) The general solution given by the mean derivatives functions in Euler integrator (see Equation (8)) for the nonlinear autonomous ordinary differential equations system is discrete and exact. However, the empirical determination of mean derivatives functions through any universal approximator of functions (e.g., artificial neural networks and fuzzy systems) is approximated, but it is always within a desired error.

3) Interpolation or estimation of mean derivatives functions up to the infinite is impossible; therefore, there is a restriction imposed to finite domains.

4) We have that the values of $\tan_{\Delta t}{}^{k+l}\alpha^i$ to $1 = 0, 1, \ldots, (L-1)$ exist and are unique; thus, $\tan_{\Delta t}{}^{k+l}\alpha^i$ is a static function with the same qualitative properties of the instantaneous derivative function ${}^{k+l}\dot{y}^i = f({}^{k+l}y^i, {}^{k+l}u)$. In fact, it can be shown that $\lim\limits_{\Delta t \to 0} \tan_{\Delta t}{}^{k+l}y^i = f({}^{k+l}y^i, {}^{k+l}u)$. It is also important to realize that the instantaneous derivative function $\dot{y}^i = f(y^i, u)$ does not depend on Δt, but $\tan_{\Delta t}{}^{k+1}\alpha^i$ does. This latter property implies that the mean derivatives methodology has a fixed integration step while the method of instantaneous derivatives, firstly proposed by Wang and Lin [12] may have variable integration step.

5) Empirically establishing mean derivatives functions values and instantaneous derivatives function values for practical engineering problems, as well as to test the parable interpolation proposed here for the mean derivatives method is not part of the scope of this work. Therefore, the main focus of this work has a purely mathematical nature.

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