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ROBUST SUBOPTIMAL MPC FOR NONLINEAR SYSTEMS WITH BOUNDED DISTURBANCES

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Abstract. *This paper presents a suboptimal model predictive control (MPC) algorithm for constrained nonlinear discrete-time systems with bounded disturbances. By adopting the existing theorem of "Feasibility implies stability", the conditions for the novel theorem of "Feasibility implies input-to-state stability" are given. The nonlinear systems are controlled by suboptimal control laws (not the optimal ones), which are feasible solutions of optimization problem in the designed MPC algorithm. By incorporating an appropriate Lyapunov equation as a constraint in optimization problem, the conditions for "Feasibility implies input-to-state stability" are constructed, and then the input-to-state stability of closed-loop system is proven. In order to verify the efficiency of the proposed algorithm, a numerical example of a 2-order disturbed nonlinear system is given.*
Keywords: Suboptimal control, Model predictive control (MPC), Nonlinear systems, Stability

1. **Introduction.** Model predictive control is an efficient control strategy for the general nonlinear systems with constraints and disturbances. However, the computational load generated from solving the optimization is the main obstacle to enlarge its application areas [1, 2, 3, 4]. Generally, the computational time can be decreased by improving the basic hardware configuration of a computer. Nevertheless, for systems like aircraft, it is costly to improve the hardware configuration because of the restriction of space, weight and cost. The only way to control these plants (with fast dynamics) is adopting the fast MPC algorithms promoting the efficiency of on-line computation. The existing fast MPC algorithms include the off-line MPC [5], explicit MPC [6], set-membership MPC [7], suboptimal MPC [8, 9, 10, 11] and so on.

In this note, we focus on the MPC algorithms based on the suboptimal control laws, i.e., suboptimal MPC. For nominal nonlinear discrete-time systems, a suboptimal dual-mode MPC algorithm and a suboptimal MPC algorithm with terminal equality constraint are presented respectively in the Ph.D. Thesis [8]. The corresponding journal version was published in [12], which introduces a critical theorem called "Feasibility implies stability" for proving the closed-loop stability. By virtue of this new theorem, the closed-loop system under suboptimal MPC is proven to be asymptotically stable in [12]. Comparing the suboptimal MPC algorithm with general MPC algorithms, the most attractive character is its short computational time, which allows for controlling the nonlinear systems with fast dynamics. In [13], a suboptimal MPC based on Euler auxiliary system is presented.

In [14], a suboptimal MPC algorithm without terminal constraint is proposed, and the continuous-time nonlinear systems under input constraints are considered. In [15], the input-affine nonlinear systems with input constraint are considered, and a suboptimal MPC algorithm resorted to the control Lyapunov function is given. Moreover, the conditions under which the suboptimal MPC is inherently robust is introduced in [16, 17]. Unlike the works in [16, 17], authors in [18] present a novel suboptimal MPC algorithm that achieves the input-to-state stability for discontinuous nonlinear systems. This paper considers the disturbed nonlinear systems, but it needs to know the optimal value function of previous time step. Also in [19], a suboptimal min-max MPC algorithm for constrained nonlinear systems with disturbances is proposed. Most of extant papers mentioned above consider the nominal nonlinear systems, i.e., the systems with no disturbances and/or un-modeled dynamics. In this note, the often faced discrete-time nonlinear systems with additive disturbances are controlled.

The main contribution of this paper is that a new suboptimal MPC algorithm for discrete-time nonlinear systems with bounded disturbances is proposed. Unlike the general MPC algorithms, the proposed algorithm only requires the feasible solutions of an optimization problem. Moreover, by virtue of the definition of input-to-state stability, the theorem of "Feasibility implies input-to-state stability" is given. Then, according to this theorem, the nonlinear closed-loop system under the proposed suboptimal MPC is proven to be input-to-state stable. By comparison, the computational time of the presented suboptimal MPC algorithm is much shorter than the general optimal MPC algorithms. In this way, we can expand the application areas of MPC.

This paper is organized as follows. Section 2 describes the controlled discrete-time nonlinear system with constraints and bounded disturbances. Moreover, the related preliminaries are introduced in Section 2. Section 3 introduces a novel theorem of "Feasibility implies input-to-state stability", and the detailed proof is given. In Section 4, by virtue of the proposed theorem in Section 3, a new suboptimal MPC algorithm is presented, which achieves the input-to-stability for closed-loop systems. In Section 5, a 2-order nonlinear system and the corresponding simulation results are given. Section 6 concludes this paper.

Notations: $\mathbb{R}$ is the real number, and $\mathbb{R}_{\geq 0}$ is the real number which equals and is bigger than zero. $\mathbb{Z}_+$ represents the set of positive integers. I is the identity matrix with appropriate dimension. $x(j|k)$ is the value of vector x at a future time $k+j$ predicted at time k. For a vector x and positive-definite matrix W, $\|x\|_W^2 = x^{\mathrm{T}} W x$. $\sup(x)$ is the function which obtains the supremum of x. The Id denotes the identity function from $\mathbb{R}$ onto $\mathbb{R}$. Given two sets $\mathbb{X}$ and $\mathbb{Y}$, then $\mathbb{X} \sim \mathbb{Y} := \{x | x + y \in \mathbb{X}, y \in \mathbb{Y}\}$. $\mathbb{B}_r^n := \{x \in \mathbb{R}^n \,|\, \|x\| \leq r\}$ identifies a sphere with radius $r \geq 0$.

2. **Problem Statement and Preliminaries.**

We consider the following discrete-time nonlinear systems:

$$x(k+1) = f(x(k), u(k)) + w(k) \tag{1}$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$ are state and input vector at time k, respectively; $w(k) \in \mathbb{R}^n$ is the vector of additive disturbances. The system state and input are constrained by

$$x(k) \in \mathbb{X} := \left\{ x(k) \in \mathbb{R}^n \,\middle|\, -\underline{x} \leq x(k) \leq \bar{x} \right\} \subseteq \mathbb{R}^n, \tag{2}$$

$$u(k) \in \mathbb{U} := \left\{ u(k) \in \mathbb{R}^m \,\middle|\, -\underline{u} \leq u(k) \leq \bar{u} \right\} \subseteq \mathbb{R}^m \tag{3}$$

where $\bar{x} \in \mathbb{R}^n$ and $\bar{u} \in \mathbb{R}^m$ are the upper bounds of x and u, respectively. $\underline{x} \in \mathbb{R}^n$ and $\underline{u} \in \mathbb{R}^m$ are the lower bounds of x and u, respectively. The disturbance $w(k)$ is restricted by

$$w(k) \in \mathbb{W} := \left\{ w(k) \in \mathbb{R}^n \,\middle|\, \|w(k)\|_p \leq \eta \right\} \tag{4}$$

where $p \in [1, \infty]$ identifies the type of vector norm, and $\eta > 0$ is an appropriate constant.

The following assumptions hold in the paper.

Assumption 2.1. *$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is twice continuously differentiable and $f(0, 0) = 0$. Thus, $o \in \mathbb{R}^n$ is an equilibrium point of the system with $u = 0$.*

Assumption 2.2. *$\mathbb{X}$ is a closed set, and $\mathbb{U}$ a compact, convex set, both of them containing the origin as interior point.*

For disturbed nonlinear system (1), the corresponding nominal model is given by

$$x(k + 1) = f(x(k), u(k)) \tag{5}$$

Remark 2.1. *The nominal model defined above is without any disturbances and uncertainty terms. Also in nominal model (5), the state and input are variables only depending on the time k. For system (1), by excluding the additive disturbances $w(k)$, the nominal model (5) is obtained.*

Define the local linearization of (5) at origin

$$A = \frac{\partial f}{\partial x}(0, 0),\ B = \frac{\partial f}{\partial u}(0, 0) \tag{6}$$

Assumption 2.3. *(A, B) is stabilizable.*

Assumption 2.4. *The function $f(x, u)$ in nominal model (5) satisfies the Lipschitz condition, i.e., there exists a Lipschitz constant $L_f > 0$ such that*

$$\|f(x_1, u) - f(x_2, u)\|_p \le \|x_1 - x_2\|_p, \quad \forall x_1, x_2 \in \mathbb{X},\ \forall u \in \mathbb{U} \tag{7}$$

In the sequel, the basic definitions of three comparison functions, which are the critical tools for analyzing the stability and robustness of nonlinear control systems, are introduced. For more knowledge about them, readers can refer to [22] for details.

Definition 2.1. *A function $\gamma : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\gamma(0) = 0$; it is a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and also $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is a $\mathcal{KL}$-function if, for each fixed $t \ge 0$, the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function, and for each fixed $s \ge 0$, the function $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$.*

Consider the following nonlinear system

$$x(k + 1) = f(x(k), w(k)) \tag{8}$$

where $x(k)$ is the vector of system states, and $w(k)$ is the vector of disturbances. In addition, it is assumed that $f(0, 0) = 0$.

Remark 2.2. *Under a certain explicit feedback control law $u(k) = g(x(k))$, the disturbed nonlinear model (1) can be equivalently transformed into Equation (8). On the other hand, $w(k)$ can be viewed as a new input vector, and the nonlinear system (8) can be viewed as a special nominal model (5) at the same time.*

The basic definitions of input-to-state stable (ISS) and the corresponding ISS-Lyapunov function are given below [22]. Moreover, a known lemma in [22] is introduced to identify the sufficient condition for ISS. The following lemmas and definitions on ISS are critical for proving the new theorem presented in Section 3, namely "Feasibility implies input-to-state stability".

Definition 2.2. *The nonlinear system* (8) *is input-to-state stable if there exists a $\mathcal{KL}$-function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a $\mathcal{K}$-function γ such that, for each disturbance $w \in \mathbb{W}$ and each $x(0) \in \mathbb{R}^n$, the following inequality holds:*

$$\|x(k, x(0), w)\| \leq \beta(\|x(0)\|, k) + \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \tag{9}$$

where $x(k, x(0), w)$ is the solution of dynamic model (8) *at time $k \in \mathbb{Z}_+$, $x(0)$ is the starting point of state, and $w(t)$ represents the w at time t.*

Definition 2.3. *A continuous function $\hat{V} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called the ISS-Lyapunov function for nonlinear system* (8) *if the following conditions hold:*

1) *There exist $\mathcal{K}_\infty$-functions α_1, α_2, such that*

$$\alpha_1(\|x\|) \leq \hat{V}(\xi) \leq \alpha_2(\|x\|),\ \forall x \in \mathbb{R}^n \tag{10}$$

2) *There exists a $\mathcal{K}_\infty$-function α_3 and a $\mathcal{K}$-function σ, such that*

$$\hat{V}(f(x, w)) - \hat{V}(x) \leq -\alpha_3(\|x\|) + \sigma(\|w\|) \tag{11}$$

for all $x \in \mathbb{R}^n$ and for all $w \in \mathbb{R}^n$.

Lemma 2.1. *If the nonlinear system* (8) *admits a continuous ISS-Lyapunov function, then it is ISS.*

Proof: The proof can be found in [22].

3. **Feasibility Implies Input-to-State Stability.**

In this section, a new theorem of "Feasibility implies input-to-state stability", which corresponds to the robust case of [12], is given. In [12], the theorem of "Feasibility implies asymptotical stability" is proposed and the nominal nonlinear system alike to (5) is considered. Resort to the proposed theorem "Feasibility implies input-to-state stability", the only feasible solutions of optimization problem are enough to guarantee the input-to-state stability of closed-loop system. The proposed theorem is depicted as follows.

Theorem 3.1. *If there exists:*

1) *A continuous function $V(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^{Nm} \to \mathbb{R}$, which satisfies $V(0, 0) = 0$, and a $\mathcal{K}$-function $\alpha_1(\cdot)$, such that*

$$V(x(k), \boldsymbol{u}(k)) \leq \alpha_1(\|x(k)\|), \quad \forall x(k) \in \mathbb{R}^n,\ \forall \boldsymbol{u}(k) \in \mathbb{R}^{Nm} \tag{12}$$

2) *A set $\mathbb{X}_F \subseteq \mathbb{R}^n$, which is an open set and also a neighborhood of origin, a $\mathcal{K}_\infty$-function $\alpha_2(\cdot)$ and a $\mathcal{K}$-function $\gamma(\cdot)$, such that $x(k) \in \mathbb{X}_F$ and*

$$V(x(k), \boldsymbol{u}(k)) - V(x(k-1), \boldsymbol{u}(k-1)) \leq -\alpha_2(\|x(k-1)\|) + \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right)$$

where $\{x(k), \boldsymbol{u}(k)\}$ is the realization of controlled system (1) *under initial condition $x(0) \in \mathbb{X}_F$;*

3) *A constant $r > 0$ and a $\mathcal{K}$-function $\alpha_3(\cdot)$, such that each realization $\{x(k), \boldsymbol{u}(k)\}$ of controlled system* (1) *under arbitrary $x(k) \in \mathbb{X}_F \subseteq \mathbb{B}_r^n$ satisfies:*

$$\|\boldsymbol{u}(k)\| \leq \alpha_3(\|x(k)\|) \tag{13}$$

then the controlled system (1) *is ISS in $\mathbb{X}_F$.*

Proof: By adopting "Lemma 3.5" in [22] and "Theorem 1" in [12], the proof can be given as below. Under initial condition $x(0) \in \mathbb{X}_F$, control input $u(k)$ and the term of disturbances $w(k)$, we define the state trajectory of controlled system (1) as $x(k) = x(k, x(0), w)$. It is easy seen from inequality (12) that there exists $V(\cdot,\cdot)$ and α_1 such that $V(x(k), \mathbf{u}(k)) \leq \alpha_1(\|x(k)\|)$, and then inequality (13) can be modified into

$$\begin{aligned} &V(x(k), \mathbf{u}(k)) - V(x(k-1), \mathbf{u}(k-1)) \\ &\leq -\alpha_4(V(x(k-1), \mathbf{u}(k-1))) + \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \end{aligned} \tag{14}$$

where $\alpha_4 = \alpha_2 \circ \alpha_1^{-1}$.

Since inequality (14) holds for all $k \in \mathbb{Z}_+$, it can be equivalently written as

$$\begin{aligned} &V(x(k+1), \mathbf{u}(k+1)) - V(x(k), \mathbf{u}(k)) \\ &\leq -\alpha_4(V(x(k), \mathbf{u}(k))) + \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \end{aligned} \tag{15}$$

According to (15), the norm $\|(x(k), \mathbf{u}(k))\|$ of controlled system (1) will not decrease when $\triangle_k = V(x(k+1), \mathbf{u}(k+1) - V(x(k), \mathbf{u}(k)) = 0$, and then we have

$$\begin{aligned} &-\alpha_4(V(x(k), \mathbf{u}(k))) + \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) = 0 \\ &\Rightarrow V(x(k), \mathbf{u}(k)) = \alpha_4^{-1} \circ \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \end{aligned} \tag{16}$$

By virtue of (16), the following set is given as

$$\mathbb{D}_s := \{(x(k), \mathbf{u}(k)) | V(x(k), \mathbf{u}(k)) \leq c\} \tag{17}$$

where $c = \alpha_4^{-1} \circ \gamma(\sup_{t\in\mathbb{Z}_+} \|w(t)\|)$. It is assumed that $\|w(k)\| \leq \eta_p$, where $\eta_p = \eta$ when $p = 2$, and then we have $\sup_{t\in\mathbb{Z}_+} \|w(t)\| = \eta_p$ and $\gamma(\sup_{t\in\mathbb{Z}_+} \|w(t)\|) = \gamma(\eta_p)$. Finally, $c = \alpha_4^{-1} \circ \gamma(\eta_p)$ is confirmed.

The proceeding steps focus on two cases, i.e., $(x(0), \mathbf{u}(0)) \in \mathbb{D}_s$ and $(x(0), \mathbf{u}(0)) \notin \mathbb{D}_s$. For the first case of $(x(0), \mathbf{u}(0)) \in \mathbb{D}_s$, we need to prove that $(x(k), \mathbf{u}(k)) \in \mathbb{D}_s$ for all $k \geq 0$. The set $\mathbb{D}_s$ satisfying this condition means that it is an invariant set for the combined vector $(x(k), \mathbf{u}(k))$. Since $(x(0), \mathbf{u}(0)) \in \mathbb{D}_s$, we have $\alpha_4(V(x(0), \mathbf{u}(0))) \leq \gamma(\sup_{t\in\mathbb{Z}_+} \|w(t)\|)$. Also from inequality (15), we have

$$V(x(1), \mathbf{u}(1)) - V(x(0), \mathbf{u}(0)) \leq -\alpha_4(V(x(0), \mathbf{u}(0))) + \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \tag{18}$$

and then we get

$$\begin{aligned} &V(x(1), \mathbf{u}(1)) \\ &\leq V(x(0), \mathbf{u}(0)) - \alpha_4(V(x(0), \mathbf{u}(0))) + \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \\ &= (\mathrm{Id} - \alpha_4)(V(x(0), \mathbf{u}(0))) + \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \\ &\leq (\mathrm{Id} - \alpha_4)(c) + \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \\ &= c - \alpha_4(c) + \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \end{aligned}$$

$$= c \tag{19}$$

By recursion, the inequality $V(x(k), \mathbf{u}(k)) \leq c$ holds for all $k \geq 0$, and then we have $(x(k), \mathbf{u}(k)) \in \mathbb{D}_s, \forall k \geq 0$. Consider the case of $(x(0), \mathbf{u}(0)) \notin \mathbb{D}_s$, that is $V(x(0), \mathbf{u}(0)) > c$. From inequality (13), we have $\alpha_4(V(x(0), \mathbf{u}(0))) > \gamma(\sup_{t\in\mathbb{Z}_+} \|w(t)\|)$, and then

$$\begin{aligned} & V(x(1), \mathbf{u}(1)) - V(x(0), \mathbf{u}(0)) \\ \leq & -\alpha_4(V(x(0), \mathbf{u}(0))) + \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \\ \leq & -\alpha_4(c) + \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \\ = & \ 0 \end{aligned} \tag{20}$$

Also by recursion, it is obvious that the value of $V(x(k), \mathbf{u}(k))$ decreases as the time k increases. That is to say, there exists a time instant j such that the equality $V(x(j), \mathbf{u}(j)) = c$ holds, and for any time $k \in [0, j]$, the following inequality holds:

$$V(x(k+1), \mathbf{u}(k+1)) - V(x(k), \mathbf{u}(k)) \leq 0 \tag{21}$$

As we can see from "Lemma 3.5" in [22], there exists a $\mathcal{KL}$-function β for $0 \leq k \leq j+1$, such that

$$V(x(k), \mathbf{u}(k)) \leq \beta(V(x(0), \mathbf{u}(0)), k) \tag{22}$$

Hence, for any $k \in \mathbb{Z}_+$, we have

$$V(x(k), \mathbf{u}(k)) \leq \max\left\{\alpha_4^{-1} \circ \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right), \beta(V(x(0), \mathbf{u}(0)), k)\right\} \tag{23}$$

Equation (23) is equivalent to

$$\|(x(k), \mathbf{u}(k))\| \leq \beta\left(\|(x(0), \mathbf{u}(0))\|, k\right) + \alpha_4^{-1} \circ \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \tag{24}$$

$V(x(k), \mathbf{u}(k))$ can be verified as an ISS-Lyapunov function by referring to the concept of ISS in Definition 2.2. However, it is an ISS-Lyapunov function towards the combination of state $x(k)$ and input sequence $\mathbf{u}(k)$. The next step is to prove the ISS property still holds for the state $x(k)$ only.

Combining (13) with (24), the following equation is obtained:

$$\begin{aligned} & \|(x(k), \mathbf{u}(k))\| \\ & \leq \beta(\|(x(0), \mathbf{u}(0))\|, k) + \alpha_4^{-1} \circ \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \\ & \leq \beta\left(\|x(0)\| + \|\mathbf{u}(0)\|, k\right) + \alpha_4^{-1} \circ \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \beta(\|x(0)\| \\ & \quad + \alpha_3(\|x(0)\|), k) + \alpha_4^{-1} \circ \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \\ & = \beta(\mathrm{Id} + \alpha_3(x(0)), k) + \alpha_4^{-1} \circ \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \end{aligned} \tag{25}$$

Letting $\hat{\beta}(x(0), k) = \beta(\mathrm{Id} + \alpha_3)(x(0), k) \in \mathcal{KL}$, we have

$$\|(x(k), \mathbf{u}(k))\| \leq \hat{\beta}(x(0), k) + \alpha_4^{-1} \circ \gamma\left(\sup_{t\in\mathbb{Z}_+} \|w(t)\|\right) \tag{26}$$

Since $\|(x(k), \mathbf{u}(k))\| \geq \|x(k)\|$, the following equation holds:

$$\|x(k)\| \leq \hat{\beta}(x(0), k) + \alpha_4^{-1} \circ \gamma \left(\sup_{t \in \mathbb{Z}_+} \|w(t)\| \right) \tag{27}$$

By virtue of Definition 2.2, the controlled system (1) is proven to be ISS.

4. **Robust Suboptimal Model Predictive Control Algorithm.** Before giving the proposed suboptimal MPC algorithm, the robust MPC algorithm with tighten constraints (we call it the constraint-tighten MPC in this note) is introduced. For more details, readers can refer to [20]. The constraint-tighten MPC is to solve the following optimization problem:

$$\min_{u(0|k),\cdots,u(N-1|k)} V_N(x(k), \mathbf{u}(\cdot)) \tag{28}$$

$$\text{s.t. } x(i+1|k) = f(x(i|k), u(i|k)) \tag{29}$$

$$x(i|k) \in \mathbb{X}_i, \ \ i = 0, \cdots, N-1 \tag{30}$$

$$u(i|k) \in \mathbb{U}, \ \ i = 0, \cdots, N-1 \tag{31}$$

$$x(N|k) \in \mathbb{X}_f \tag{32}$$

with the objective function

$$V_N(x(k), \mathbf{u}(\cdot)) = \sum_{i=0}^{N-1} L((x(i|k), u(i|k)) + V(x(N|k))) \tag{33}$$

where $x(i|k)$ is the prediction of x at the future time $k+i$, predicted at time k. $x(0|k)$ is the current state, which equals $x(k)$. $\mathbb{X}_f$ is the terminal constraint set which contains the equilibrium point. The running cost $L(\cdot,\cdot)$ and the terminal cost $V(\cdot)$ in objective function $V_N(\cdot,\cdot)$ are defined by

$$L(x(i|k), u(i|k)) = ||x(i|k)||_Q^2 + ||u(i|k)||_R^2 \tag{34}$$

$$V(x(N|k) = ||x(N|k)||_P^2 \tag{35}$$

where the corresponding weighting matrices Q, P, R are positive definite.

The tighten constraint on state $x(i|k)$ is given by [20]:

$$\mathbb{X}_i = \mathbb{X} \sim \mathbb{C}_\eta^i \tag{36}$$

where $\mathbb{C}_\eta^i$ is chosen as

$$\mathbb{C}_\eta^i := \left\{ x \in \mathbb{R}^n \,\middle|\, \|x\|_p \leq \frac{L_f^i - 1}{L_f - 1} \cdot \eta \right\} \tag{37}$$

By solving the optimization problem (28), the following optimal control sequence is obtained

$$\mathbf{u}^*(k) = \{u^*(0|k), u^*(1|k), \ldots, u^*(N-1|k)\} \tag{38}$$

and the first element $u^*(0|k)$ of $\mathbf{u}^*(k)$ is applied to the system (1).

Proposition 4.1. *Suppose Assumption 2.4 holds and the predicted state $x(i|k)$ subjects to the tighten constraints* (36) *and* (37)*, then the solution of optimization problem* (28) *is the admissible control sequence, that is, the real state of closed-loop system satisfies the state constraints. Moreover, the difference between the predicted state $x(i|k)$ and the corresponding real state $x(k+i)$ is bounded by*

$$\|x(i|k) - x(k+i)\|_p \leq \frac{L_f^i - 1}{L_f - 1}\eta \tag{39}$$

Proof: The proof can be found in [20].

Assumption 4.1. *The terminal constraint set* $\mathbb{X}_f$ *is a closed set, which contains the origin in its interior.*

Assumption 4.2. *Suppose that the stage cost function* $L(x, u)$ *and the objective function* $V_N(x, \boldsymbol{u})$ *fulfill the following conditions:*

1) $|L(x_1, u) - L(x_2, u)| \le L_c\|x_1 - x_2\|_p,\ L_c \in (0, \infty),\ \forall x_1, x_2 \in \mathbb{X}, u \in \mathbb{U}$;
2) *There exist constants* $a > 0$ *and* $\delta > 1$*, such that* $L(x, u) \ge a\,\|(x, u)\|^\delta$;
3) $V_N(x, \boldsymbol{u}) \le a_2\|x\|$.

Assumption 4.3. *For nominal nonlinear model* (5)*, there exists a local controller* $u = h_L(x) \in \mathbb{U}$ *and a Lyapunov function* $V_f(x(k)) = x(k)^{\mathrm{T}}Px(k)$*, such that the following conditions are fulfilled in the set* $\Omega = \{x \in \mathbb{R}^n | V_f(x) \le \zeta\}$*:*

1) $V_f(f(x(k), h_L(x(k)))) - V_f(x(k)) \le -L(x(k), h_L(x(k))),\ \forall x(k) \in \Omega$;
2) *There exists a Lpschitz constant* $L_v \in (0, \infty)$*, such that* $|V_f(x_1) - V_f(x_2)| \le L_v\|x_1 - x_2\|_p,\ \forall x_1, x_2 \in \Omega$;
3) *For all* $x \in \Omega$*, we have* $x \in \mathbb{X},\ h_L(x) \in \mathbb{U}$.

Define

$$u(k) = h_L(x(k)) = Fx(k) \tag{40}$$

where F is determined such that $A + BF$ is Schur stable. For the nominal nonlinear system (5), appropriate $\hat{\alpha}$ and P should be chosen such that

$$(A + BF)^{\mathrm{T}}P(A + BF) - P = -\hat{\alpha}P - Q - F^{\mathrm{T}}RF. \tag{41}$$

where a procedure for selecting P, ζ, $\hat{\alpha}$ can be found in [21].

Definition 4.1. *For nominal nonlinear system* (5)*, the terminal constraint set* $\mathbb{X}_f := \{x \in \mathbb{R}^n | V_f(x) \le \alpha\}$ *is defined as a set in which the arbitrary* $x(k) \in \Omega$*, it follows that* $f(x(k), h_L(x(k))) \in \mathbb{X}_f$.

The terminal constraint set $\mathbb{X}_f$ can be chosen as the one-step forward contractive set of Ω. By assuming the contraction rate $\rho \in (0, 1]$, i.e., $\mathbb{X}_f = \rho\Omega$. Thus, for all $x(k) \in \Omega$, we have

$$\begin{aligned}
&x(k+1)^{\mathrm{T}}Px(k+1) - x(k)^{\mathrm{T}}Px(k) \\
=\ &(Ax(k) + Bu(k) + \theta(x))^{\mathrm{T}}\, P\,(Ax(k) + Bu(k) + \theta(x)) - x(k)^{\mathrm{T}}Px(k) \\
\le\ &x(k)^{\mathrm{T}}\left(A_K^{\mathrm{T}}PA_K - P + \varsigma P\right)x(k) \\
\le\ &-(1-\rho)x(k)^{\mathrm{T}}Px(k)
\end{aligned}$$

where $A_K = A - BK$ and $\theta(x) = f(x, Fx) - (A + BF)x$.

In order to fulfill the condition $x(k+1)^{\mathrm{T}}Px(k+1) - x(k)^{\mathrm{T}}Px(k) \le 0$, the linear matrix inequality $A_K^{\mathrm{T}}PA_K - P + \varsigma P - \rho P \le 0$ must be given. By solving the following semidefinite programming (SDP):

$$\begin{aligned}
&\min_{\rho}\ \rho \\
\text{s.t. }\ &0 < \rho \le 1 \\
&A_K^{\mathrm{T}}PA_K + \varsigma P - \rho P \le 0
\end{aligned}$$

the optimal contraction rate ρ^* is given. Hence, the terminal constraint set $\mathbb{X}_f$ is confirmed as

$$x(k+1) \in \mathbb{X}_f = \rho^*\Omega := \left\{x \in \mathbb{R}^n | x^{\mathrm{T}}Px \le \rho^*\zeta\right\} \tag{42}$$

For disturbed nonlinear system (1), a robust nonlinear MPC algorithm based on the suboptimal control laws is presented. Unlike the algorithm in [12], the controller does not need to switch between the local controller $h_L(x)$ and model prediction controller. Moreover, the proposed algorithm also fits for controlling the nonlinear systems with bounded disturbances.

Algorithm 1. *(Robust Suboptimal MPC Algorithm)*

- **step 0**. *Choose the prediction horizon N, admissible invariant set Ω, terminal cost function $V_f(x)$, local controller $u = -Kx$ and terminal constraint set $\mathbb{X}_f$.*
- **step 1**. *Let $k := 0$. Calculate a feasible solution which satisfies the constraints* (30)-(32)*, and then the feasible control sequence $\tilde{\mathbf{u}}(0) = (\tilde{u}(0|0), \tilde{u}(1|0), \ldots, \tilde{u}(N-1|0))$ is obtained; apply the control law $u(0) = \tilde{u}(0|0)$ to the nonlinear system* (1)*. Let $k := k + 1$.*
- **step 2**. *if $k > 0$, then let $\hat{\mathbf{u}}(k) = (\tilde{u}(1|k-1), \ldots, \tilde{u}(N-1|k-1), h_L(\hat{x}(N-1|k)))$. Calculate a feasible solution which satisfies the constraints* (30)-(32) *and $V_N(x(k), \tilde{\boldsymbol{u}}(k)) \leq V_N(x(k-1), \tilde{u}(k-1)) + \varphi \cdot \eta - a \cdot \|x(k-1)\|^\sigma$ (where $\varphi > 0$ is an appropriate constant), and then the feasible control sequence $\tilde{\boldsymbol{u}}(k) = (\tilde{u}(0|k), \tilde{u}(1|k), \ldots, \tilde{u}(N-1|k))$ is obtained; apply the control law $u(k) = \tilde{u}(0|k)$ to the nonlinear system* (1)*. Let $k := k + 1$ and go back to step 2.*

Proposition 4.2. *For controlled system* (1) *subjected to the constraints* (2)-(4)*, the presented Algorithm 1 admits a feasible set $\mathbb{X}_F$, in which the sufficient condition for the system state to fulfill the constraints* (2)-(4) *is*

$$\eta \leq \frac{\zeta - \alpha}{L_v \cdot L_f^{N-1}} \tag{43}$$

Proof: The proof can be found in [20].

Theorem 4.1. *By applying Algorithm 1 to the nonlinear system* (1) *subjected to the constraints* (2)-(4)*, and assuming that there exists a feasible solution $\hat{\mathbf{u}}(k)$ which satisfies* (2)-(4) *and $V_N(x(k), \tilde{\boldsymbol{u}}(k)) \leq V_N(x(k-1), \tilde{u}(k-1)) + \varphi \cdot \eta - a \cdot \|x(k-1)\|^\sigma$ for all $x \in \mathbb{X}_F$, then the closed-loop system is ISS in $\mathbb{X}_F$.*

Proof: According to Assumption 4.3, we have that the inequality $V_N(x(k), \mathbf{u}(k)) \leq a_2\|x(k)\|$ can be fulfilled for any x and $\mathbf{u}$, and then the first term in Theorem 3.1 is proven. In addition, the objective function $V_N(x(k), \mathbf{u}(k))$ satisfies

$$V_N(x(k), \mathbf{u}(k)) - V_N(x(k-1), \mathbf{u}(k-1)) \leq \varphi \cdot \eta - a \cdot \|x(k-1)\|^\sigma$$

and then the term 2) in Theorem 3.1 is proven. For the third term of Theorem 3.1, the condition $\|\mathbf{u}(k)\| \leq \alpha_3(\|x(k)\|)$ can be fulfilled by adding this condition into optimization as in [12].

By virtue of Theorem 3.1, the closed-loop system is proven to be ISS in feasible region $\mathbb{X}_F$.

5. **Numerical Example.** In order to verify the effectiveness of the proposed control algorithm, the following disturbed nonlinear system is considered [20].

$$\begin{cases} x_1(k+1) = 0.55x_1(k) + 0.12x_2(k) + (0.01 - 0.6x_1(k) + x_2(k))u(k) + w_1(k) \\ x_2(k+1) = 0.67x_2(k) + (0.15 + x_1(k) - 0.8x_2(k))u(k) + w_2(k) \end{cases} \tag{44}$$

where $x(k) = [x_1(k), x_2(k)]^{\mathrm{T}}$ is the vector of state variables, $u(k)$ is the input variable, and $w(k) = [w_1(k), w_2(k)]^{\mathrm{T}}$ is the vector of disturbances. $x(k)$ and $u(k)$ are confined by

$$-2 \leq x_1(k) \leq 2, \quad -2 \leq x_2(k) \leq 2, \quad -0.1 \leq u(k) \leq 0.1 \tag{45}$$

Removing the disturbances in Equation (44), the corresponding nominal model is given by

$$\begin{cases} x_1(k+1) = 0.55x_1(k) + 0.12x_2(k) + (0.01 - 0.6x_1(k) + x_2(k))u(k) \\ x_2(k+1) = 0.67x_2(k) + (0.15 + x_1(k) - 0.8x_2(k))u(k) \end{cases} \tag{46}$$

Choosing $p = \infty$, the Lipscitz constant L_f of the nominal model (46) is confirmed by

$$\begin{aligned} \left\| \frac{\partial f}{\partial x} \right\|_\infty &= \left\| \begin{bmatrix} 0.55 - 0.6u & 0.12 + u \\ u & 0.67 - 0.8u \end{bmatrix} \right\|_\infty \\ &= \max\left\{ |0.55 - 0.6u| + |0.12 + u|, |u| + |0.67 - 0.8u| \right\} \\ &\leq \max\left\{ 0.55 + 0.6|u| + 0.12 + |u|, |u| + 0.67 + 0.8|u| \right\} \\ &\leq 0.85 \end{aligned}$$

Then one has $L_f = 0.85$.

Selecting the stage cost function $L(x, u) = x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru$, where the weighting matrices Q and R are chosen as

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1$$

then we have $L(x, u) = x_1^2 + x_2^2 + u^2$. Deducting from the following equation:

$$\left\| \frac{\partial L}{\partial x} \right\|_\infty = \|[2x_1, 2x_2]\|_\infty = \max\left\{ |2x_1|, |2x_2| \right\} \leq 4$$

we have $L_c = 4$. According to the value of $L(x, u)$ and letting $\delta = 2$, $a\|(x, u)\|^2 = a(x_1^2 + x_2^2 + u^2)$. To satisfy the inequality $L(x, u) \geq a\|(x, u)\|^\delta$, one can choose $a \in (0, 1]$.

Linearize the nominal nonlinear model (45) around origin, the system matrices A and B are

$$A = \begin{bmatrix} 0.55 & 0.12 \\ 0.67 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.01 \\ 0.15 \end{bmatrix}$$

Solve the LQR problem based on the selected weighting matrices Q and R, the state feedback gain $K = [0.1273 \quad 0.0054]$ is given. Thus, the local controller can be identified as $h_L(x) = -Kx$. Moreover, the terminal weighting matrix P is given by

$$P = \begin{bmatrix} 2.8561 & 0.2319 \\ 0.2319 & 1.1715 \end{bmatrix}$$

By choosing the terminal cost function as $V_f(x) = x^{\mathrm{T}}Px$, the positive admissible invariant set $\Omega := \left\{ x \in \mathbb{R}^n | x^{\mathrm{T}}Px \leq 0.6 \right\}$. Thus, $\zeta = 0.6$ and we have that

$$\begin{aligned} \left\| \frac{\partial V_f}{\partial x} \right\|_\infty &= \|[5.9122x_1 + 0.4638x_2, 0.4638x_1 + 2.3430x_2]\|_\infty \\ &\leq \max\left\{ 5.9122|x_1| + 0.4638|x_2|, 0.4638|x_1| + 2.3430|x_2| \right\} \\ &\leq 2.7023 \end{aligned}$$

Then $L_v = 2.7023$. The terminal set $\mathbb{X}_f$ is given by

$$\mathbb{X}_f = \left\{ x \in \mathbb{R}^N | x^{\mathrm{T}}Px \leq 0.3889 \right\}$$

that is, $\alpha = 0.3889$.

By selecting the prediction horizon $N = 5$, the upper bound on disturbances is

$$\eta \leq \frac{\zeta - \alpha}{L_v \cdot L_f^{N-1}} = \frac{0.6 - 0.3889}{2.7023 \times 0.85^4} = 0.1497$$

For comparison, the initial state is set as $x(0) = [-2, 2]^{\mathrm{T}}$ and the parameter $\varphi = 20$. Since the essence of disturbances is uncertain, the Mento-Carlo method is adopted to verify the efficiency of the proposed algorithm. Since the optimal solution can be viewed as the special form of suboptimal solution, the suboptimal solver "FSQP" in [23] is adopted to get the suboptimal solution and the optimal solver "FMINCON" in Matlab is utilized for comparison at the same time. It can be seen from Figures 1-3 that the system, controlled by the proposed robust suboptimal MPC algorithm, is input-to-state stable. From Figures 1, 2 and 4, all the hard constraints on state and input variables are not violated at all. Comparing the proposed suboptimal MPC algorithm based on FSQP with the FMINCON-based algorithm, we can conclude from Table 1 that the computational time is extremely decreased.

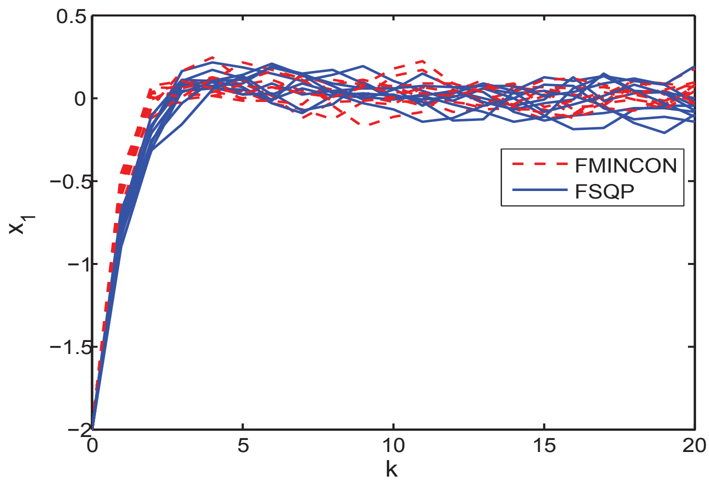

FIGURE 1. Trajectories of x_1 under suboptimal solutions and optimal solutions

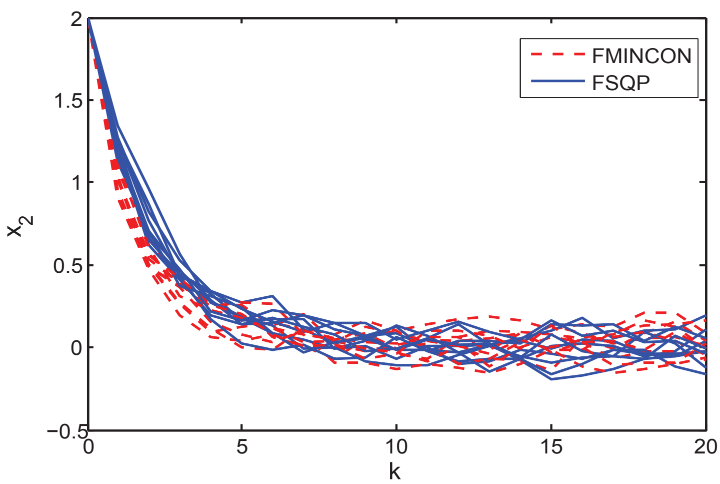

FIGURE 2. Trajectories of x_2 under suboptimal solutions and optimal solutions

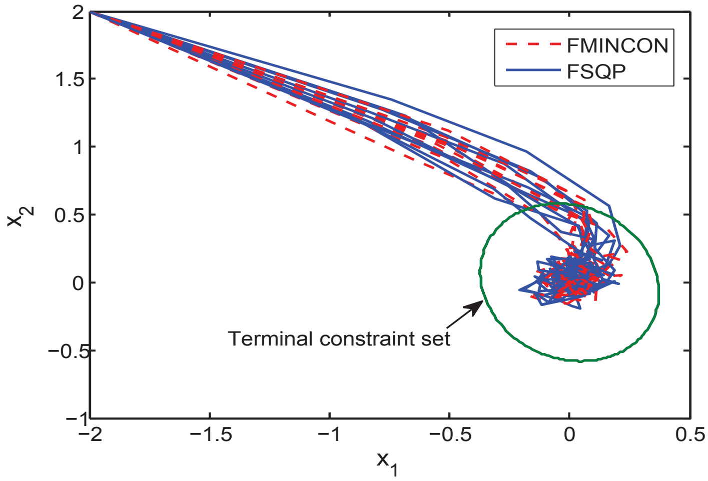

Figure 3. Phase trajectories under suboptimal solutions and optimal solutions

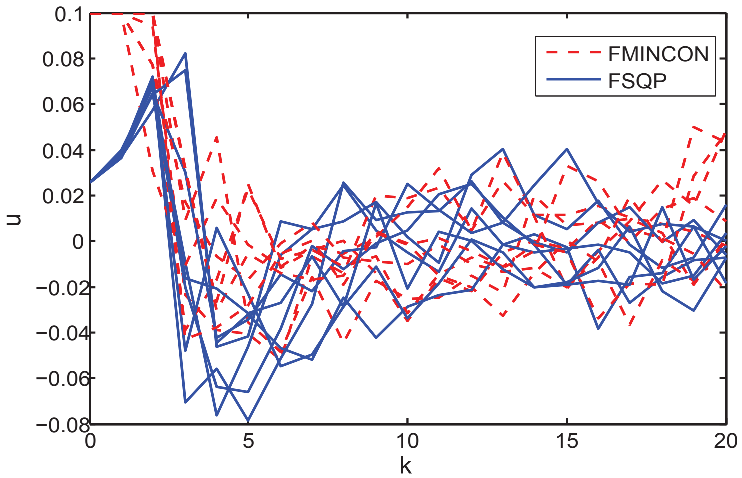

Figure 4. Control inputs u under suboptimal solutions and optimal solutions

Table 1. Comparison of suboptimal solutions by FSQP and optimal solutions by FMINCON

x	u(FSQP)/time(s)	u(FMINCON)/time(s)
$(3,3)$	$1.048 \times 10^{-5}/0.0051$	$-3.3857 \times 10^{-8}/0.0106$
$(2,2)$	$1.5986 \times 10^{-6}/0.0051$	$2.5749 \times 10^{-8}/0.0137$
$(1,1)$	$1.5986 \times 10^{-6}/0.0057$	$-4.0531 \times 10^{-9}/0.0102$
$(-1,-1)$	$1.5986 \times 10^{-6}/0.0048$	$-4.0531 \times 10^{-8}/0.0103$
$(-2,-2)$	$1.5986 \times 10^{-6}/0.0061$	$2.5749 \times 10^{-8}/0.0107$
$(-3,-3)$	$1.0480 \times 10^{-5}/0.0068$	$-3.3857 \times 10^{-8}/0.0103$

6. **Conclusions.** In this paper, a robust suboptimal MPC algorithm for nonlinear discrete-time systems with bounded disturbances is presented. By satisfying a certain of conditions, the theorem of "Feasibility implies input-to-state stability" is given. That is, by virtue of this theorem, the closed-loop system under suboptimal control laws is proven to be input-to-state stable. The advantage of the proposed suboptimal MPC algorithm is its low computational time. The simulation example confirms the control performance and shows the advantages. Future work may include the simplification of conditions for the theorem of "Feasibility implies input-to-state stability", and the extension of this work to large-scale systems.

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REFERENCES

[1] F. Allgöwer, R. Findeisen and Z. K. Nagy, Nonlinear model predictive control: From theory to application, *Journal of the Chinese Institute of Chemical and Engineers*, vol.36, no.3, pp.299-315, 2004.

[2] H. Chen and F. Allgöwer, A quasi-infinte horizon nonlinear model predictive control scheme with guaranteed stability, *Automatica*, vol.34, no.10, pp.1205-1217, 1998.

[3] Y. Du, Q. Wu, C. Jiang and Y. Wang, Adaptive robust predictive control for hypersonic vehicles using recurrent functional link artificial neural network, *International Journal of Innovative Computing, Information and Control*, vol.6, no.12, pp.5351-5365, 2010.

[4] R. Hedjar, Adaptive neural network model predictive control, *International Journal of Innovative Computing, Information and Control*, vol.9, no.3, pp.1245-1257, 2013.

[5] Z. Y. Wan and M. V. Kothare, An efficient off-line formulation of robust model predictive control using linear matrix inequalities, *Automatica*, vol.39, no.5, pp.837-846, 2003.

[6] A. Grancharova and T. A. Johansen, Explicit approaches to constrainted model predictive control: A survey, *Modeling, Identification and Control*, vol.25, no.3, pp.131-157, 2004.

[7] M. Canale, L. Fagiano and M. Milanese, Set membership approximation theory for fast implementation of model predictive control laws, *Automatica*, vol.45, no.1, pp.45-54, 2009.

[8] R. H. Findeisen, *Suboptimal Nonlinear Model Predictive Control*, Ph.D. Thesis, University of Wisconsin-Madison, USA, 1997.

[9] X. Liu, D. Constantinescu and Y. Shi, Multistage suboptimal model predictive control with improved computational efficiency, *Journal of Dynamic Systems Measurement & Control*, vol.136, no.3, pp.833-846, 2014.

[10] S. Ahmed and E. Kerrigan, Suboptimal predictive control for satellite detumbling, *Journal of Guidance Control & Dynamics*, vol.37, no.3, pp.850-859, 2014.

[11] D. Axehill, T. Besselmann, D. Raimondo and M. Morari, A parametric branch and bound approach to suboptimal explicit hybrid MPC, *Automatica*, vol.50, no.1, pp.240-246, 2014.

[12] P. O. M. Scokaert, D. Q. Mayne and J. B. Rawlings, Suboptimal model predictive control (Feasibility implies stability), *IEEE Trans. Automatic Control*, vol.44, no.3, pp.648-654, 1999.

[13] F. Blanchini, S. Miani and F. A. Pellegrino, Suboptimal receding horizon control for continuous-time systems, *IEEE Trans. Automatic Control*, vol.48, no.6, pp.1081-1086, 2003.

[14] K. Graichen and A. Kugi, Stability and incremental improvement of suboptimal MPC without terminal constraints, *IEEE Trans. Automatic Control*, vol.55, no.11, pp.2576-2580, 2010.

[15] M. Sznaier, R. Suárez and J. Cloutier, Suboptimal control of constrained nonlinear systems via receding horizon constrained control Lyapunov functions, *International Journal of Robust and Nonlinear Control*, vol.13, nos.3-4, pp.247-259, 2003.

[16] G. Pannocchia, J. B. Rawlings and S. J. Wright, Inherently robust suboptimal nonlinear MPC: Theory and application, *Proc. of the 50th IEEE Conference on Decision and Control and European Control Conference*, 2011.

[17] G. Pannocchia, J. B. Rawlings and S. J. Wright, Conditions under which suboptimal nonlinear MPC is inherently robust, *Systerms & Control Letters*, vol.60, no.9, pp.747-755, 2011.

[18] M. Lazar and W. P. M. H. Heemels, Predictive control of hybrid systems: Input-to-state stability results for sub-optimal solutions, *Automatica*, vol.45, no.1, pp.180-185, 2009.

[19] D. F. He, H. B. Ji and T. Zheng, On robustness of suboptimal min-max model predictive control, *Wseas Trans. Systems and Control*, vol.28, no.2, pp.428-433, 2007.

[20] D. L. Marruedo, T. Álamo and E. F. Camacho, Input-to-state stable MPC for constrained discrete-time nonlinear systems with bounded additive uncertainties, *Proc. of the 41st IEEE Conference on Decision and Control*, 2002.

[21] M. Zhao and B. Ding, Distributed model predictive control for constrained nonlinear systems with decoupled local dynamics, *ISA Transactions*, vol.54, no.2, pp.1-12, 2015.

[22] Z. P. Jiang and Y. Wang, Input-to-state stability for discrete-time nonlinear systems, *Automatica*, vol.37, no.6, pp.857-869, 2001.

[23] Z. B. Zhu and K. C. Zhang, A new SQP method of feasible directions for nonlinear programming, *Applied Mathematics and Computation*, vol.148, no.1, pp.121-134, 2004.