International Journal of Innovative Computing, Information and Control
Volume **12**, Number **6**, December **2016**
ICIC International ©2016 ISSN 1349-4198
pp. 1943–1960

STABILITY ANALYSIS OF 2-D DISCRETE SYSTEMS DESCRIBED BY ROESSER MODEL WITH TIME VARYING DELAY AND INPUT SATURATION USING ANTI-WINDUP STRATEGY

Vipin Chandra Pal[1], Richa Negi[1] and Haranath Kar[2]

[1]Department of Electrical Engineering
[2]Department of Electronics & Communication Engineering
Motilal Nehru National Institute of Technology Allahabad
Allahabad 211004, India
vipin.vchandra@gmail.com; richa@mnnit.ac.in; hnkar1@rediffmail.com

Received March 2016; revised July 2016

Abstract. *The multidimensional system finds various real time applications and modeling of these systems are very important to understand the system dynamics. This paper presents stability analysis of two-dimensional (2-D) discrete systems described by Roesser model with time varying state delay and input saturation. Dynamic output feedback compensator is used and static anti-windup gain is given which stabilizes the above system. An optimization algorithm is proposed to maximize the domain of attraction for 2-D discrete systems using linear matrix inequality (LMI). Some numerical examples are also provided to show the effectiveness of the proposed theory.*

Keywords: Two-dimensional discrete systems, Anti-windup, Delay-dependent stability, Linear matrix inequality

1. **Introduction.** Over the past few decades, considerable attention has been dedicated to the study of multidimensional systems (m-D) due to their interdisciplinary applications. The model of n-dimensional system has been studied in [1-6]. The real time applications of two-dimensional (2-D) systems have been investigated for several areas such as robot navigation [7], mapping in landslide areas [8], real time clinical experiments [9], visualization of magnetic field [10], video application [11], 2-D controller design [12], Roesser model based distributed grid sensor network [14], direction control in sheet metal process [15], self-purification phenomena with modeling the river pollution process [16,17], optical fiber network [18], magnetics [19] and photovoltaic applications [20]. A 2-D elevator traffic system is presented in [21] which is based on the assumption that the elevator vehicle can move in both directions (i.e., horizontal and vertical) on external façade of multistory buildings and effect of time delay to turn round the vehicle. The stability analysis of m-D systems has drawn the attention of many researchers [3-6,12-14,21,22]. Several case studies of m-D discrete systems like grid sensor networks, and robot manipulator are discussed in [22].

A common problem associated with physical systems is the nonlinearity in the form of saturation, dead-zone, backlash, overflow, etc., which degrade the systems performance. Most of the works have been carried out for continuous and discrete time systems subjected to input saturation nonlinearity in one dimensional (1-D) systems [23-29] whereas a few papers deal with the case of 2-D systems [30,31].

Time delay is another cause of system instability and poor behavior. Several global asymptotic stability conditions [32-40] have been given on digital filters with state delay combined with quantization/overflow nonlinearities. Stability analysis of 2-D state delayed systems with input saturation nonlinearities is a very challenging task. The stability problem of 1-D time delayed systems in presence of input saturation has been studied with constant time delay [41-44] and time varying delay [45-48]. The sufficient conditions for asymptotic stability of 2-D discrete systems with state delay and state saturation have been derived in [49-53]. The problem of stability analysis and l_2 gain control for 2-D nonlinear stochastic systems with time varying delays and actuator saturation have been investigated in [54]. The H_∞ stabilization problems of 2-D discrete switched delayed systems with input saturation has been studied in [55]. An approach for the design of observer based output feedback H_∞ controller for 2-D uncertain discrete systems with time-varying state delay and actuator saturation has been presented in [56]. The integrated fault detection problem for 2-D discrete systems with time varying state delays has been considered in [57]. [58] discusses the developments of a data acquisition system for 2-D position sensitive micro pattern gas detectors. By combining Kalman-Yakubovich-Popov lemma with frequency-partitioning idea, less conservative stability conditions have been derived for 2-D continuous-discrete systems [59]. The anti-windup approach is considered to be an effective and powerful tool to tackle the effect of actuator saturation in continuous and discrete time systems in more practical perspective [23-27,29,31]. The ant-windup problem for 2-D discrete system described by Fornasini-Marchesini second local state space (FMSLSS) model with saturating control has been investigated in [31].

Motivated by preceding discussion, in this paper, we consider the problem of anti-windup design for 2-D discrete system described by Roesser model with input saturation and time varying state delay. This paper aims at providing a technique to compute the anti-windup gain of the 2-D dynamic compensator such that closed loop system is asymptotically stable. The rest of the paper is organized as follows. In Section 2, problem is formulated. In Section 3, static anti-windup gains are calculated using dynamic output feedback compensator and an algorithm is proposed for enlarging the domain of attraction. To validate the presented results, several numerical examples are given in Section 4. In Section 5, a conclusion is given.

2. Problem Formulation.

The following notations are used throughout the paper:

$\Re^{m\times n}$	set of $m \times n$ real matrices
$\Re^{m}$	set of $m \times 1$ real matrices
$\mathbf{0}$	null matrix or null vector
$\boldsymbol{I}$	identity matrix of appropriate dimension
$\boldsymbol{I}_n$	$n \times n$ identity matrix
$\lambda_{\max}(\boldsymbol{\Omega})$	maximum eigenvalue of any given matrix $\boldsymbol{\Omega}$
$\boldsymbol{\Omega}^T$	transpose of matrix $\boldsymbol{\Omega}$
$*$	symmetric entries in a symmetric matrix
$diag\{a_1, a_2, \ldots, a_n\}$	diagonal matrix with diagonal elements $a_1, a_2, \ldots, a_n$
$\oplus$	direct sum

Consider a 2-D discrete system with delay given by Roesser model [13,51]

$$\begin{bmatrix} \boldsymbol{x}^h(i+1,j) \\ \boldsymbol{x}^v(i,j+1) \end{bmatrix} = \boldsymbol{A}_p \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} + \boldsymbol{A}_{dp} \begin{bmatrix} \boldsymbol{x}^h(i-d_h(i),j) \\ \boldsymbol{x}^v(i,j-d_v(j)) \end{bmatrix} + \boldsymbol{B}_p \boldsymbol{u}(i,j) \tag{1a}$$

$$\boldsymbol{y}(i,j) = \boldsymbol{C} \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} \tag{1b}$$

where $i \in z_+$, $j \in z_+$ and z_+ denotes the set of nonnegative integers. The $\boldsymbol{x}^h(i,j) \in \Re^n$ and $\boldsymbol{x}^v(i,j) \in \Re^m$ are the horizontal and the vertical states, respectively. The $\boldsymbol{u}(i,j) \in \Re^p$ and $\boldsymbol{y}(i,j) \in \Re^q$ are the input and measured output vectors, respectively. Matrices $\boldsymbol{A}_p =$

$\begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \in \Re^{(n+m)\times(n+m)}$, $\boldsymbol{A}_{dp} = \begin{bmatrix} \boldsymbol{A}_{d11} & \boldsymbol{A}_{d12} \\ \boldsymbol{A}_{d21} & \boldsymbol{A}_{d22} \end{bmatrix} \in \Re^{(n+m)\times(n+m)}$, $\boldsymbol{B}_p \in \Re^{(n+m)\times p}$ and $\boldsymbol{C} \in \Re^{q\times(n+m)}$ are known constant matrices representing a nominal plant. We assume that the delays involved in system (1) are interval-like time varying state delays such that

$$d_{hL} \leq d_h(i) \leq d_{hH}, \quad d_{vL} \leq d_v(j) \leq d_{vH} \tag{1c}$$

where d_{hL} and d_{vL} are constant positive integers representing the lower delay bounds along horizontal and vertical directions, respectively; d_{hH} and d_{vH} are constant positive integers representing the upper delay bounds along horizontal and vertical directions, respectively. It may be mentioned that such type of modeling of time varying delays has been widely used in the literature [38,49,50,54-57].

Let a linear 2-D dynamic compensator stabilizing the system given by (1a) and (1b) and meeting the desired performance specifications in absence of actuator saturation be

$$\begin{bmatrix} \boldsymbol{x}_c^h(i+1,j) \\ \boldsymbol{x}_c^v(i,j+1) \end{bmatrix} = \boldsymbol{A}_c \begin{bmatrix} \boldsymbol{x}_c^h(i,j) \\ \boldsymbol{x}_c^v(i,j) \end{bmatrix} + \boldsymbol{B}_c \boldsymbol{u}_c(i,j) \tag{2a}$$

$$\boldsymbol{V}_c(i,j) = \boldsymbol{C}_c \begin{bmatrix} \boldsymbol{x}_c^h(i,j) \\ \boldsymbol{x}_c^v(i,j) \end{bmatrix} + \boldsymbol{D}_c \boldsymbol{C} \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} \tag{2b}$$

where $\boldsymbol{x}_c^h(i,j) \in \Re^{n_c}$ and $\boldsymbol{x}_c^v(i,j) \in \Re^{m_c}$ are horizontal and vertical states of the controller respectively, $\boldsymbol{u}_c(i,j) = \boldsymbol{y}(i,j) \in \Re^q$ is a controller input vector and $\boldsymbol{V}_c(i,j) \in \Re^p$ is a controller output vector. The matrices $\boldsymbol{A}_c \in \Re^{(n_c+m_c)\times(n_c+m_c)}$, $\boldsymbol{B}_c \in \Re^{(n_c+m_c)\times q}$, $\boldsymbol{C}_c \in \Re^{p\times(n_c+m_c)}$ and $\boldsymbol{D}_c \in \Re^{p\times q}$ are constant matrices of the desired controller. The input vector $\boldsymbol{u}(i,j)$ is subjected to amplitude constraint defined as

$$-u_{0(l)} \leq u_{(l)}(i,j) \leq u_{0(l)} \tag{3}$$

where $u_{0(l)} > 0$, $l = 1, 2, \ldots, p$ denote the control amplitude bounds. Therefore, the actual control signal injected to the system (1) can be written as

$$\boldsymbol{u}(i,j) = sat(\boldsymbol{V}_c(i,j)) = sat\left(\boldsymbol{C}_c \begin{bmatrix} \boldsymbol{x}_c^h(i,j) \\ \boldsymbol{x}_c^v(i,j) \end{bmatrix} + \boldsymbol{D}_c \boldsymbol{C} \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix}\right) \tag{4}$$

where the saturation nonlinearities are characterized by

$$sat(V_c(i,j))_{(l)} = \begin{cases} -u_{0(l)} & \text{if } V_{c(l)} < -u_{0(l)} \\ V_{c(l)} & \text{if } -u_{0(l)} \leq V_{c(l)} \leq u_{0(l)} \\ u_{0(l)} & \text{if } V_{c(l)} > u_{0(l)} \end{cases}, \quad l = 1, 2, \ldots, p \tag{5}$$

The actuator saturation causes windup of the controller and to mitigate its undesirable effect an anti-windup term $\boldsymbol{E}_c(sat(\boldsymbol{V}_c(i,j)) - \boldsymbol{V}_c(i,j))$ can be added to the controller [27], where $\boldsymbol{E}_c$ is anti-windup gain. Thus, considering the dynamic controller and the anti-windup strategy, the closed loop 2-D system can be represented by

$$\begin{aligned} \begin{bmatrix} \boldsymbol{x}^h(i+1,j) \\ \boldsymbol{x}^v(i,j+1) \end{bmatrix} &= \boldsymbol{A}_p \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} + \boldsymbol{A}_{dp} \begin{bmatrix} \boldsymbol{x}^h(i-d_h(i),j) \\ \boldsymbol{x}^v(i,j-d_v(j)) \end{bmatrix} \\ &\quad + \boldsymbol{B}_p \left(sat\left(\boldsymbol{C}_c \begin{bmatrix} \boldsymbol{x}_c^h(i,j) \\ \boldsymbol{x}_c^v(i,j) \end{bmatrix} + \boldsymbol{D}_c \boldsymbol{C} \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix}\right)\right) \end{aligned} \tag{6a}$$

$$\boldsymbol{y}(i,j) = \boldsymbol{C} \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} \tag{6b}$$

$$\begin{bmatrix} \boldsymbol{x}_c^h(i+1,j) \\ \boldsymbol{x}_c^v(i,j+1) \end{bmatrix} = \boldsymbol{A}_c \begin{bmatrix} \boldsymbol{x}_c^h(i,j) \\ \boldsymbol{x}_c^v(i,j) \end{bmatrix} + \boldsymbol{B}_c \boldsymbol{C} \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} + \boldsymbol{E}_c(sat(\boldsymbol{V}_c(i,j)) - \boldsymbol{V}_c(i,j)) \tag{7a}$$

$$\boldsymbol{V}_c(i,j) = \boldsymbol{C}_c \begin{bmatrix} \boldsymbol{x}_c^h(i,j) \\ \boldsymbol{x}_c^v(i,j) \end{bmatrix} + \boldsymbol{D}_c \boldsymbol{C} \begin{bmatrix} \boldsymbol{x}^h(i,j) \\ \boldsymbol{x}^v(i,j) \end{bmatrix} \tag{7b}$$

Let us introduce the elementary matrix

$$\boldsymbol{\Pi} = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{n_c} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_m & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{m_c} \end{bmatrix} \tag{8}$$

Using (6)-(8) and the relation $\boldsymbol{\Pi}^{-1} = \boldsymbol{\Pi}^T$, the closed loop system of the plant and the controller can be expressed as

$$\begin{aligned} \begin{bmatrix} \boldsymbol{\xi}^h(i+1,j) \\ \boldsymbol{\xi}^v(i,j+1) \end{bmatrix} &= \boldsymbol{A} \begin{bmatrix} \boldsymbol{\xi}^h(i,j) \\ \boldsymbol{\xi}^v(i,j) \end{bmatrix} + \boldsymbol{A_d} \begin{bmatrix} \boldsymbol{\xi}^h(i-d_h(i),j) \\ \boldsymbol{\xi}^v(i,j-d_v(j)) \end{bmatrix} \\ &\quad - (\boldsymbol{B} + \boldsymbol{R}\boldsymbol{E}_c)\boldsymbol{\psi}\left(\boldsymbol{K}\begin{bmatrix} \boldsymbol{\xi}^h(i,j) \\ \boldsymbol{\xi}^v(i,j) \end{bmatrix}\right) \end{aligned} \tag{9}$$

where

$$\boldsymbol{\xi}^h(i,j) = \begin{bmatrix} \boldsymbol{x}^{hT}(i,j) & \boldsymbol{x}_c^{hT}(i,j) \end{bmatrix}^T, \quad \boldsymbol{\xi}^v(i,j) = \begin{bmatrix} \boldsymbol{x}^{vT}(i,j) & \boldsymbol{x}_c^{vT}(i,j) \end{bmatrix}^T$$

$$\boldsymbol{A}_d = \boldsymbol{\Pi}\begin{bmatrix} \boldsymbol{A}_{dp} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}\boldsymbol{\Pi}^T \tag{10}$$

$$\boldsymbol{A} = \boldsymbol{\Pi}\begin{bmatrix} \boldsymbol{A}_p + \boldsymbol{B}_p\boldsymbol{D}_c\boldsymbol{C} & \boldsymbol{B}_p\boldsymbol{C}_c \\ \boldsymbol{B}_c\boldsymbol{C} & \boldsymbol{A}_c \end{bmatrix}\boldsymbol{\Pi}^T, \quad \boldsymbol{B} = \boldsymbol{\Pi}\begin{bmatrix} \boldsymbol{B}_p \\ \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{R} = \boldsymbol{\Pi}\begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_{n_c+m_c} \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{D}_c\boldsymbol{C} & \boldsymbol{C}_c \end{bmatrix}\boldsymbol{\Pi}^T \tag{11}$$

$$\boldsymbol{\psi}(\boldsymbol{V}_c(i,j)) = \boldsymbol{V}_c(i,j) - sat(\boldsymbol{V}_c(i,j)) \tag{12}$$

The boundary conditions of system (9) are assumed to satisfy

$$\begin{aligned} &\begin{cases} \boldsymbol{\xi}^h(i,j) = \boldsymbol{h}_{ij}, & \forall\, 0 \le j \le r_1,\ -d_{hH} \le i \le 0 \\ \boldsymbol{\xi}^h(i,j) = \boldsymbol{0}, & \forall\, j \ge r_1,\ -d_{hH} \le i \le 0 \end{cases} \\ &\begin{cases} \boldsymbol{\xi}^v(i,j) = \boldsymbol{v}_{ij}, & \forall\, 0 \le i \le r_2,\ -d_{vH} \le j \le 0 \\ \boldsymbol{\xi}^v(i,j) = \boldsymbol{0}, & \forall\, i \ge r_2,\ -d_{vH} \le j \le 0 \end{cases} \end{aligned} \tag{13}$$

where r_1 and r_2 are finite positive integers, $\boldsymbol{h}_{ij}$ and $\boldsymbol{v}_{ij}$ are given vectors.

Remark 2.1. *The boundary conditions (13) play a key role in the derivation of the stability criterion for 2-D systems. With the suitable choice of r_1 and r_2, it is not difficult to define the boundary conditions of dynamic compensator such that (13) holds.*

The following definition and lemmas are needed in the proof of main result.

Definition 2.1. [12] *The system (9) with the boundary conditions (13) is asymptotically stable if*

$$\lim_{\ell\to\infty} \chi_\ell = 0 \tag{14}$$

where $\chi_\ell = \sup\{\|\boldsymbol{\xi}(i,j)\| : i+j=\ell,\ i,j \ge 1\}$ *and* $\boldsymbol{\xi}(i,j) = \begin{bmatrix} \boldsymbol{\xi}^{hT}(i,j) & \boldsymbol{\xi}^{vT}(i,j) \end{bmatrix}^T$.

Lemma 2.1. [60] *For any positive definite matrix $\boldsymbol{W} \in \Re^{m\times m}$ two positive integers l_1 and l_2 satisfying $1 \le l_1 \le l_2$, vector function $\boldsymbol{\omega} \in \Re^m$, one has*

$$(l_2 - l_1 + 1)\sum_{i=l_1}^{l_2} \boldsymbol{\omega}^T(i)\boldsymbol{W}\boldsymbol{\omega}(i) \ge \left(\sum_{i=l_1}^{l_2}\boldsymbol{\omega}(i)\right)^T \boldsymbol{W}\left(\sum_{i=l_1}^{l_2}\boldsymbol{\omega}(i)\right) \tag{15}$$

Consider a matrix $\boldsymbol{G} \in \Re^{p\times(n+m+n_c+m_c)}$ and define a polyhedral set

$$\ell \triangleq \left\{\boldsymbol{\xi} \in \Re^{(n+m+n_c+m_c)}; -u_{0(l)} \le \left(\boldsymbol{K}_{(l)} - \boldsymbol{G}_{(l)}\right)\boldsymbol{\xi}(i,j) \le u_{0(l)}, l = 1, 2, \ldots, p\right\}.$$

Lemma 2.2. *If $\boldsymbol{\xi} \in \ell$ then*

$$\delta_\ell = 2\boldsymbol{\psi}^T(\boldsymbol{K}\boldsymbol{\xi}(i,j))\boldsymbol{D}\left[\boldsymbol{\psi}(\boldsymbol{K}\boldsymbol{\xi}(i,j)) - \boldsymbol{G}\boldsymbol{\xi}(i,j)\right] \leq 0 \tag{16}$$

where $\boldsymbol{D}$ is positive definite diagonal matrix.

Proof: The proof of Lemma 2.2 directly follows from [31,46].

In the following section, we aim to determine the anti-windup gain matrix $\boldsymbol{E}_c$ such that the asymptotic stability of closed loop system (9) is guaranteed for time varying delays satisfying (1c). An optimization procedure is presented in order to maximize the estimate of the domain of attraction associated to it.

3. **Main Results.** The main result of the paper may be stated as follows.

Theorem 3.1. *For given positive integers d_{hL}, d_{vL}, d_{hH} and d_{vH}, if there exist matrices $\boldsymbol{G} \in \Re^{p\times(n+m+n_c+m_c)}$, $\boldsymbol{H} \in \Re^{(n_c+m_c)\times p}$, a diagonal positive definite matrix $\boldsymbol{L} \in \Re^{p\times p}$, positive definite symmetric matrices $\boldsymbol{P}^h \in \Re^{(n+n_c)\times(n+n_c)}$, $\boldsymbol{P}^v \in \Re^{(m+m_c)\times(m+m_c)}$, $\boldsymbol{Q}^h \in \Re^{(n+n_c)\times(n+n_c)}$, $\boldsymbol{Q}^v \in \Re^{(m+m_c)\times(m+m_c)}$, $\boldsymbol{W}^h \in \Re^{(n+n_c)\times(n+n_c)}$, $\boldsymbol{W}^v \in \Re^{(m+m_c)\times(m+m_c)}$, $\boldsymbol{R}_k = \boldsymbol{R}_k^h \oplus \boldsymbol{R}_k^v$, $k = 1,2,3$ and $\boldsymbol{X}_k$, $k = 1,2,3,4$ satisfying*

$$\begin{bmatrix}
\boldsymbol{\Upsilon}_{11} & \boldsymbol{R}_2\boldsymbol{D}_h^2\boldsymbol{A}_d & \boldsymbol{G}^T & \boldsymbol{R}_3^T & -\boldsymbol{R}_2^T & \boldsymbol{A}^T & \boldsymbol{A}^T & \boldsymbol{D}_h(\boldsymbol{A}-\boldsymbol{I})^T & \boldsymbol{D}_h^T \\
* & -\boldsymbol{Q} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_d^T & \boldsymbol{A}_d^T & \boldsymbol{D}_h\boldsymbol{A}_d^T & \boldsymbol{0} \\
* & * & -2\boldsymbol{L} & \boldsymbol{0} & \boldsymbol{0} & -(\boldsymbol{B}\boldsymbol{L}+\boldsymbol{R}\boldsymbol{H})^T & -(\boldsymbol{B}\boldsymbol{L}+\boldsymbol{R}\boldsymbol{H})^T & (-\boldsymbol{D}_h(\boldsymbol{B}\boldsymbol{L}+\boldsymbol{R}\boldsymbol{H}))^T & (-\boldsymbol{D}_h(\boldsymbol{B}\boldsymbol{L}+\boldsymbol{R}\boldsymbol{H}))^T \\
* & * & * & -\boldsymbol{W}-\boldsymbol{R}_3 & \boldsymbol{R}_2^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & -\boldsymbol{R}_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & -2\boldsymbol{X}_1+\boldsymbol{X}_1\boldsymbol{P}\boldsymbol{X}_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & * & -2\boldsymbol{X}_2+\boldsymbol{X}_2\boldsymbol{Q}\boldsymbol{X}_2 & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & * & * & -2\boldsymbol{X}_3+\boldsymbol{X}_3\boldsymbol{R}_3\boldsymbol{X}_3 & \boldsymbol{0} \\
* & * & * & * & * & * & * & * & -2\boldsymbol{X}_4+\boldsymbol{X}_4\boldsymbol{R}_2\boldsymbol{X}_4
\end{bmatrix} < \boldsymbol{0} \tag{17}$$

$$\begin{bmatrix} \boldsymbol{P} & \boldsymbol{K}_{(l)}^T - \boldsymbol{G}_{(l)}^T \\ * & \boldsymbol{u}_{0(l)}^2 \end{bmatrix} > \boldsymbol{0}, \quad l = 1,2,\ldots,p \tag{18}$$

where

$$\boldsymbol{P} = \boldsymbol{P}^h \oplus \boldsymbol{P}^v, \quad \boldsymbol{Q} = \boldsymbol{Q}^h \oplus \boldsymbol{Q}^v, \quad \boldsymbol{W} = \boldsymbol{W}^h \oplus \boldsymbol{W}^v$$

$$\boldsymbol{0} < \boldsymbol{R}^h = \begin{bmatrix} \boldsymbol{R}_1^h & \boldsymbol{R}_2^h \\ \boldsymbol{R}_2^{hT} & \boldsymbol{R}_3^h \end{bmatrix} \in \Re^{2(n+n_c)\times 2(n+n_c)}, \ \boldsymbol{0} < \boldsymbol{R}^v = \begin{bmatrix} \boldsymbol{R}_1^v & \boldsymbol{R}_2^v \\ \boldsymbol{R}_2^{vT} & \boldsymbol{R}_3^v \end{bmatrix} \in \Re^{2(m+m_c)\times 2(m+m_c)} \tag{19}$$

$$\boldsymbol{\Upsilon}_{11} = -\boldsymbol{P} + \boldsymbol{W} + \boldsymbol{D}_d\boldsymbol{Q} + \boldsymbol{D}_h^2\boldsymbol{R}_1 + (\boldsymbol{A}-\boldsymbol{I})^T\boldsymbol{D}_h^2\boldsymbol{R}_2 + \boldsymbol{D}_h^2\boldsymbol{R}_2(\boldsymbol{A}-\boldsymbol{I}) - \boldsymbol{R}_3$$

$$\boldsymbol{D}_d = \begin{bmatrix} (d_{hH}-d_{hL})\boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & (d_{vH}-d_{vL})\boldsymbol{I}_n \end{bmatrix}, \quad \boldsymbol{D}_h = \begin{bmatrix} d_{hH}\boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & d_{vH}\boldsymbol{I}_n \end{bmatrix} \tag{20}$$

then, for the gain matrix $\boldsymbol{E}_c = \boldsymbol{H}\boldsymbol{L}^{-1}$ the ellipsoid $\varepsilon(\boldsymbol{P}) = \left\{\boldsymbol{\xi} \in \Re^{n+m+n_c+m_c}; \boldsymbol{\xi}^T\boldsymbol{P}\boldsymbol{\xi} \leq 1\right\}$ is a region of asymptotic stability for the system (9). Further, an estimate of the domain of attraction for system (9) is

$$\begin{aligned}
\Gamma_{(\beta_{rh},\beta_{rv})} = {} & \Big[\beta_{rh}^2\left(\lambda_{\max}\left(\boldsymbol{P}^h\right) + (d_{hH}+1)\lambda_{\max}\left(\boldsymbol{Q}^h\right) + d_{hH}\lambda_{\max}\left(\boldsymbol{W}^h\right)\right. \\
& + 0.5(d_{hH}-d_{hL}+1)(d_{hH}+d_{hL})\lambda_{\max}\left(\boldsymbol{Q}^h\right) + 2d_{hH}^2(d_{hH}+1)\lambda_{\max}\left.\left(\boldsymbol{R}^h\right)\right) \\
& + \beta_{rv}^2\big(\lambda_{\max}(\boldsymbol{P}^v) + (d_{vH}+1)\lambda_{\max}(\boldsymbol{Q}^v) + d_{vH}\lambda_{\max}(\boldsymbol{W}^v) \\
& + 0.5(d_{vH}-d_{vL}+1)(d_{vH}+d_{vL})\lambda_{\max}(\boldsymbol{Q}^v) + 2d_{vH}^2(d_{vH}+1)\lambda_{\max}(\boldsymbol{R}^v)\big)\Big] \\
\leq {} & 1
\end{aligned} \tag{21}$$

where

$$\beta_{rh} = \max\left(\sum_{j=0}^{r_1}\left\|\boldsymbol{\xi}^h\left(-\sigma^h, j\right)\right\|\right)_{-d_{hH}\leq\sigma^h\leq 0}$$

and

$$\beta_{rv} = \max\left(\sum_{i=0}^{r_2} \|\boldsymbol{\xi}(i, -\sigma^v)\|\right)_{-d_{vH} \le \sigma^v \le 0}$$

Proof: The proof of Theorem 3.1 is based on standard Lyapunov theory. It consists of several steps. First, we select a 2-D quadratic Lyapunov function $v(\boldsymbol{\xi}(i,j))$. Second, we estimate the forward difference of the Lyapunov functional along the trajectories of the system (1) i.e., $\Delta v(\boldsymbol{\xi}(i,j))$. Third, the condition under which $\Delta v(\boldsymbol{\xi}(i,j)) < 0$ is determined.

Define

$$\boldsymbol{\delta}^h(r,j) = \boldsymbol{\xi}^h(r+1,j) - \boldsymbol{\xi}^h(r,j), \quad \boldsymbol{\delta}^v(i,t) = \boldsymbol{\xi}^v(i,t+1) - \boldsymbol{\xi}^v(i,t) \tag{22}$$

$$\boldsymbol{\eta}^h(r,j) = \begin{bmatrix} \boldsymbol{\xi}^{hT}(r,j) & \boldsymbol{\delta}^{hT}(r,j) \end{bmatrix}^T, \quad \boldsymbol{\eta}^v(i,t) = \begin{bmatrix} \boldsymbol{\xi}^{vT}(i,t) & \boldsymbol{\delta}^{vT}(i,t) \end{bmatrix}^T \tag{23}$$

Now, consider a 2-D quadratic Lyapunov function [51]

$$v(\boldsymbol{\xi}(i,j)) = v^h\left(\boldsymbol{\xi}^h(i,j)\right) + v^v(\boldsymbol{\xi}^v(i,j)) \tag{24}$$

where

$$v^h\left(\boldsymbol{\xi}^h(i,j)\right) = \sum_{k=1}^{5} v_k^h\left(\boldsymbol{\xi}^h(i,j)\right) \tag{25}$$

$$v_1^h\left(\boldsymbol{\xi}^h(i,j)\right) = \boldsymbol{\xi}^{hT}(i,j)\boldsymbol{P}^h\boldsymbol{\xi}^h(i,j) \tag{26}$$

$$v_2^h\left(\boldsymbol{\xi}^h(i,j)\right) = \sum_{r=i-d_h(i)}^{i} \boldsymbol{\xi}^{hT}(r,j)\boldsymbol{Q}^h\boldsymbol{\xi}^h(r,j) \tag{27}$$

$$v_3^h\left(\boldsymbol{\xi}^h(i,j)\right) = \sum_{r=i-d_{hH}}^{i-1} \boldsymbol{\xi}^{hT}(r,j)\boldsymbol{W}^h\boldsymbol{\xi}^h(r,j) \tag{28}$$

$$v_4^h\left(\boldsymbol{\xi}^h(i,j)\right) = \sum_{s=-d_{hH}}^{-d_{hL}} \sum_{r=i+s}^{i-1} \boldsymbol{\xi}^{hT}(r,j)\boldsymbol{Q}^h\boldsymbol{\xi}^h(r,j) \tag{29}$$

$$v_5^h\left(\boldsymbol{\xi}^h(i,j)\right) = d_{hH} \sum_{s=-d_{hH}}^{-1} \sum_{r=i+s}^{i-1} \boldsymbol{\eta}^{hT}(r,j)\boldsymbol{R}^h\boldsymbol{\eta}^h(r,j) \tag{30}$$

$$v^v(\boldsymbol{\xi}^v(i,j)) = \sum_{k=1}^{5} v_k^v(\boldsymbol{\xi}^v(i,j)) \tag{31}$$

$$v_1^v(\boldsymbol{\xi}^v(i,j)) = \boldsymbol{\xi}^{vT}(i,j)\boldsymbol{P}^v\boldsymbol{\xi}^v(i,j) \tag{32}$$

$$v_2^v(\boldsymbol{\xi}^v(i,j)) = \sum_{s=j-d_v(j)}^{j} \boldsymbol{\xi}^{vT}(i,s)\boldsymbol{Q}^v\boldsymbol{\xi}^v(i,s) \tag{33}$$

$$v_3^v(\boldsymbol{\xi}^v(i,j)) = \sum_{t=j-d_{vH}}^{j-1} \boldsymbol{\xi}^{vT}(i,t)\boldsymbol{W}^v\boldsymbol{\xi}^v(i,t) \tag{34}$$

$$v_4^v(\boldsymbol{\xi}^v(i,j)) = \sum_{s=-d_{vH}}^{-d_{vL}} \sum_{t=j+s}^{j-1} \boldsymbol{\xi}^{vT}(i,t)\boldsymbol{Q}^v\boldsymbol{\xi}^v(i,t) \tag{35}$$

$$v_5^v(\boldsymbol{\xi}^v(i,j)) = d_{vH} \sum_{s=-d_{vH}}^{-1} \sum_{t=j+s}^{j-1} \boldsymbol{\eta}^{vT}(i,t)\boldsymbol{R}^v\boldsymbol{\eta}^v(i,t) \tag{36}$$

Taking the forward difference of Lyapunov functional along trajectories of system (9), we obtain

$$\Delta v(\boldsymbol{\xi}(i,j)) = \sum_{k=1}^{5} \Delta v_k(\boldsymbol{\xi}(i,j)) \tag{37}$$

where

$$\Delta v_1(\boldsymbol{\xi}(i,j)) = \Delta v_1^h\left(\boldsymbol{\xi}^h(i,j)\right) + \Delta v_1^v(\boldsymbol{\xi}^v(i,j)) \tag{38}$$

$$\begin{aligned} \Delta v_1^h\left(\boldsymbol{\xi}^h(i,j)\right) &= v_1^h\left(\boldsymbol{\xi}^h(i+1,j)\right) - v_1^h\left(\boldsymbol{\xi}^h(i,j)\right) \\ \Delta v_1^v(\boldsymbol{\xi}^v(i,j)) &= v_1^v(\boldsymbol{\xi}^v(i,j+1)) - v_1^v(\boldsymbol{\xi}^v(i,j)) \end{aligned} \tag{39}$$

$$\Delta v_1(\boldsymbol{\xi}(i,j)) = \begin{bmatrix} \boldsymbol{\xi}^h(i+1,j) \\ \boldsymbol{\xi}^v(i,j+1) \end{bmatrix}^T \boldsymbol{P} \begin{bmatrix} \boldsymbol{\xi}^h(i+1,j) \\ \boldsymbol{\xi}^v(i,j+1) \end{bmatrix} - \begin{bmatrix} \boldsymbol{\xi}^h(i,j) \\ \boldsymbol{\xi}^v(i,j) \end{bmatrix}^T \boldsymbol{P} \begin{bmatrix} \boldsymbol{\xi}^h(i,j) \\ \boldsymbol{\xi}^v(i,j) \end{bmatrix} \tag{40}$$

$$\begin{aligned} &= \boldsymbol{\xi}^T(i,j)\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}\boldsymbol{\xi}(i,j) + \boldsymbol{\xi}^T(i,j)\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}_d \begin{bmatrix} \boldsymbol{\xi}^h(i-d_h(i),j) \\ \boldsymbol{\xi}^v(i,j-d_v(j)) \end{bmatrix} \\ &\quad + \boldsymbol{\xi}^T(i,j)\boldsymbol{A}^T\boldsymbol{P}(-\boldsymbol{B}-\boldsymbol{R}\boldsymbol{E}_c)\boldsymbol{\psi}(\boldsymbol{K}\boldsymbol{\xi}(i,j)) \\ &\quad + \begin{bmatrix} \boldsymbol{\xi}^{hT}(i-d_h(i),j) & \boldsymbol{\xi}^{vT}(i,j-d_v(j)) \end{bmatrix} \boldsymbol{A}_d^T\boldsymbol{P}\boldsymbol{A}\boldsymbol{\xi}(i,j) \\ &\quad + \begin{bmatrix} \boldsymbol{\xi}^{hT}(i-d_h(i),j) & \boldsymbol{\xi}^{vT}(i,j-d_v(j)) \end{bmatrix} \boldsymbol{A}_d^T\boldsymbol{P}\boldsymbol{A}_d \begin{bmatrix} \boldsymbol{\xi}^h(i-d_h(i),j) \\ \boldsymbol{\xi}^v(i,j-d_v(j)) \end{bmatrix} \\ &\quad + \begin{bmatrix} \boldsymbol{\xi}^{hT}(i-d_h(i),j) & \boldsymbol{\xi}^{vT}(i,j-d_v(j)) \end{bmatrix} \boldsymbol{A}_d^T\boldsymbol{P}(-\boldsymbol{B}-\boldsymbol{R}\boldsymbol{E}_c)\boldsymbol{\psi}(\boldsymbol{K}\boldsymbol{\xi}(i,j)) \\ &\quad + \boldsymbol{\psi}^T(\boldsymbol{K}\boldsymbol{\xi}(i,j))(-\boldsymbol{B}-\boldsymbol{R}\boldsymbol{E}_c)^T\boldsymbol{P}\boldsymbol{A}\boldsymbol{\xi}(i,j) \\ &\quad + \boldsymbol{\psi}^T(\boldsymbol{K}\boldsymbol{\xi}(i,j))(-\boldsymbol{B}-\boldsymbol{R}\boldsymbol{E}_c)^T\boldsymbol{P}\boldsymbol{A}_d \begin{bmatrix} \boldsymbol{\xi}^h(i-d_h(i),j) \\ \boldsymbol{\xi}^v(i,j-d_v(j)) \end{bmatrix} \\ &\quad + \boldsymbol{\psi}^T(\boldsymbol{K}\boldsymbol{\xi}(i,j))(-\boldsymbol{B}-\boldsymbol{R}\boldsymbol{E}_c)^T\boldsymbol{P}(-\boldsymbol{B}-\boldsymbol{R}\boldsymbol{E}_c)\boldsymbol{\psi}(\boldsymbol{K}\boldsymbol{\xi}(i,j)) \\ &\quad - \boldsymbol{\xi}^T(i,j)\boldsymbol{P}\boldsymbol{\xi}(i,j) \end{aligned} \tag{41}$$

$$\Delta v_2(\boldsymbol{\xi}(i,j)) = \Delta v_2^h\left(\boldsymbol{\xi}^h(i,j)\right) + \Delta v_2^v(\boldsymbol{\xi}^v(i,j)) \tag{42}$$

$$\begin{aligned} \Delta v_2^h\left(\boldsymbol{\xi}^h(i,j)\right) &= v_2^h\left(\boldsymbol{\xi}^h(i+1,j)\right) - v_2^h\left(\boldsymbol{\xi}^h(i,j)\right) \\ \Delta v_2^v\left(\boldsymbol{\xi}^v(i,j)\right) &= v_2^v\left(\boldsymbol{\xi}^v(i,j+1)\right) - v_2^v\left(\boldsymbol{\xi}^v(i,j)\right) \end{aligned} \tag{43}$$

$$\begin{aligned} \Delta v_2(\boldsymbol{\xi}(i,j)) = &\begin{bmatrix} \boldsymbol{\xi}^h(i+1,j) \\ \boldsymbol{\xi}^v(i,j+1) \end{bmatrix}^T \boldsymbol{Q} \begin{bmatrix} \boldsymbol{\xi}^h(i+1,j) \\ \boldsymbol{\xi}^v(i,j+1) \end{bmatrix} \\ &- \begin{bmatrix} \boldsymbol{\xi}^h(i-d_h(i),j) \\ \boldsymbol{\xi}^v(i,j-d_v(j) \end{bmatrix}^T \boldsymbol{Q} \begin{bmatrix} \boldsymbol{\xi}^h(i-d_h(i),j) \\ \boldsymbol{\xi}^v(i,j-d_v(j) \end{bmatrix} \end{aligned} \tag{44}$$

$$\begin{aligned} &= \boldsymbol{\xi}^T(i,j)\boldsymbol{A}^T\boldsymbol{Q}\boldsymbol{A}\boldsymbol{\xi}(i,j) + \boldsymbol{\xi}^T(i,j)\boldsymbol{A}^T\boldsymbol{Q}\boldsymbol{A}_d \begin{bmatrix} \boldsymbol{\xi}^h(i-d_h(i),j) \\ \boldsymbol{\xi}^v(i,j-d_v(j)) \end{bmatrix} \\ &\quad + \boldsymbol{\xi}^T(i,j)\boldsymbol{A}^T\boldsymbol{Q}(-\boldsymbol{B}-\boldsymbol{R}\boldsymbol{E}_c)\boldsymbol{\psi}(\boldsymbol{K}\boldsymbol{\xi}(i,j)) \\ &\quad + \begin{bmatrix} \boldsymbol{\xi}^{hT}(i-d_h(i),j) & \boldsymbol{\xi}^{vT}(i,j-d_v(j)) \end{bmatrix} \boldsymbol{A}_d^T\boldsymbol{Q}\boldsymbol{A}\boldsymbol{\xi}(i,j) \\ &\quad + \begin{bmatrix} \boldsymbol{\xi}^{hT}(i-d_h(i),j) & \boldsymbol{\xi}^{vT}(i,j-d_v(j)) \end{bmatrix} \boldsymbol{A}_d^T\boldsymbol{Q}\boldsymbol{A}_d \begin{bmatrix} \boldsymbol{\xi}^h(i-d_h(i),j) \\ \boldsymbol{\xi}^v(i,j-d_v(j)) \end{bmatrix} \\ &\quad + \begin{bmatrix} \boldsymbol{\xi}^{hT}(i-d_h(i),j) & \boldsymbol{\xi}^{vT}(i,j-d_v(j)) \end{bmatrix} \boldsymbol{A}_d^T\boldsymbol{Q}(-\boldsymbol{B}-\boldsymbol{R}\boldsymbol{E}_c)\boldsymbol{\psi}(\boldsymbol{K}\boldsymbol{\xi}(i,j)) \\ &\quad + \boldsymbol{\psi}^T(\boldsymbol{K}\boldsymbol{\xi}(i,j))(-\boldsymbol{B}-\boldsymbol{R}\boldsymbol{E}_c)^T\boldsymbol{Q}\boldsymbol{A}\boldsymbol{\xi}(i,j) \\ &\quad + \boldsymbol{\psi}^T(\boldsymbol{K}\boldsymbol{\xi}(i,j))(-\boldsymbol{B}-\boldsymbol{R}\boldsymbol{E}_c)^T\boldsymbol{Q}\boldsymbol{A}_d \begin{bmatrix} \boldsymbol{\xi}^h(i-d_h(i),j) \\ \boldsymbol{\xi}^v(i,j-d_v(j)) \end{bmatrix} \end{aligned}$$

$$+\boldsymbol{\psi}^T(\boldsymbol{K}\boldsymbol{\xi}(i,j))(-\boldsymbol{B}-\boldsymbol{R}\boldsymbol{E}_c)^T\boldsymbol{Q}(-\boldsymbol{B}-\boldsymbol{R}\boldsymbol{E}_c)\boldsymbol{\psi}(\boldsymbol{K}\boldsymbol{\xi}(i,j))$$
$$-\begin{bmatrix}\boldsymbol{\xi}^h(i-d_h(i),j)\\ \boldsymbol{\xi}^v(i,j-d_v(j))\end{bmatrix}^T\boldsymbol{Q}\begin{bmatrix}\boldsymbol{\xi}^h(i-d_h(i),j)\\ \boldsymbol{\xi}^v(i,j-d_v(j))\end{bmatrix} \tag{45}$$

$$\Delta v_3(\boldsymbol{\xi}(i,j))=\Delta v_3^h\left(\boldsymbol{\xi}^h(i,j)\right)+\Delta v_3^v(\boldsymbol{\xi}^v(i,j)) \tag{46}$$

$$\begin{aligned}\Delta v_3^h\left(\boldsymbol{\xi}^h(i,j)\right)&=v_3^h\left(\boldsymbol{\xi}^h(i+1,j)\right)-v_3^h\left(\boldsymbol{\xi}^h(i,j)\right)\\ \Delta v_3^v(\boldsymbol{\xi}^v(i,j))&=v_3^v(\boldsymbol{\xi}^v(i,j+1))-v_3^v(\boldsymbol{\xi}^v(i,j))\end{aligned} \tag{47}$$

$$\begin{aligned}\Delta v_3(\boldsymbol{\xi}(i,j))=&\begin{bmatrix}\boldsymbol{\xi}^h(i,j)\\ \boldsymbol{\xi}^v(i,j)\end{bmatrix}^T\boldsymbol{W}\begin{bmatrix}\boldsymbol{\xi}^h(i,j)\\ \boldsymbol{\xi}^v(i,j)\end{bmatrix}\\ &-\begin{bmatrix}\boldsymbol{\xi}^h(i-d_{hH},j)\\ \boldsymbol{\xi}^v(i,j-d_{vH})\end{bmatrix}^T\boldsymbol{W}\begin{bmatrix}\boldsymbol{\xi}^h(i-d_{hH},j)\\ \boldsymbol{\xi}^v(i,j-d_{vH})\end{bmatrix}\end{aligned} \tag{48}$$

$$=\boldsymbol{\xi}^T(i,j)\boldsymbol{W}\boldsymbol{\xi}(i,j)-\begin{bmatrix}\boldsymbol{\xi}^h(i-d_{hH},j)\\ \boldsymbol{\xi}^v(i,j-d_{vH})\end{bmatrix}^T\boldsymbol{W}\begin{bmatrix}\boldsymbol{\xi}^h(i-d_{hH},j)\\ \boldsymbol{\xi}^v(i,j-d_{vH})\end{bmatrix} \tag{49}$$

$$\Delta v_4(\boldsymbol{\xi}(i,j))=\Delta v_4^h\left(\boldsymbol{\xi}^h(i,j)\right)+\Delta v_4^v(\boldsymbol{\xi}^v(i,j)) \tag{50}$$

$$\begin{aligned}\Delta v_4^h\left(\boldsymbol{\xi}^h(i,j)\right)&=v_4^h\left(\boldsymbol{\xi}^h(i+1,j)\right)-v_4^h\left(\boldsymbol{\xi}^h(i,j)\right)\\ \Delta v_4^v(\boldsymbol{\xi}^v(i,j))&=v_4^v(\boldsymbol{\xi}^v(i,j+1))-v_4^v(\boldsymbol{\xi}^v(i,j))\end{aligned} \tag{51}$$

$$\begin{aligned}\Delta v_4(\boldsymbol{\xi}(i,j))=&\sum_{s=-d_{hH}}^{-d_{hL}}\sum_{r=i+1+s}^{i}\boldsymbol{\xi}^{hT}(r,j)\boldsymbol{Q}^h\boldsymbol{\xi}^h(r,j)-\sum_{s=-d_{hH}}^{-d_{hL}}\sum_{r=i+s}^{i-1}\boldsymbol{\xi}^{hT}(r,j)\boldsymbol{Q}^h\boldsymbol{\xi}^h(r,j)\\ &+\sum_{s=-d_{vH}}^{-d_{vL}}\sum_{t=j+1+s}^{j}\boldsymbol{\xi}^{vT}(i,t)\boldsymbol{Q}^v\boldsymbol{\xi}^v(i,t)-\sum_{s=-d_{vH}}^{-d_{vL}}\sum_{t=j+s}^{j-1}\boldsymbol{\xi}^{vT}(i,t)\boldsymbol{Q}^v\boldsymbol{\xi}^v(i,t)\end{aligned} \tag{52}$$

$$=\begin{bmatrix}\boldsymbol{\xi}^h(i,j)\\ \boldsymbol{\xi}^v(i,j)\end{bmatrix}^T\begin{bmatrix}(d_{hH}-d_{hL})\boldsymbol{I}_{nh} & \mathbf{0}\\ \mathbf{0} & (d_{vH}-d_{vL})\boldsymbol{I}_{nv}\end{bmatrix}\boldsymbol{Q}\begin{bmatrix}\boldsymbol{\xi}^h(i,j)\\ \boldsymbol{\xi}^v(i,j)\end{bmatrix} \tag{53}$$

$$\Delta v_5(\boldsymbol{\xi}(i,j))=\Delta v_5^h\left(\boldsymbol{\xi}^h(i,j)\right)+\Delta v_5^v(\boldsymbol{\xi}^v(i,j)) \tag{54}$$

$$\begin{aligned}\Delta v_5^h\left(\boldsymbol{\xi}^h(i,j)\right)&=v_5^h\left(\boldsymbol{\xi}^h(i+1,j)\right)-v_5^h\left(\boldsymbol{\xi}^h(i,j)\right)\\ \Delta v_5^v(\boldsymbol{\xi}^v(i,j))&=v_5^v(\boldsymbol{\xi}^v(i,j+1))-v_5^v(\boldsymbol{\xi}^v(i,j))\end{aligned} \tag{55}$$

$$\begin{aligned}&\Delta v_5(\boldsymbol{\xi}(i,j))\\ =&\ d_{hH}\sum_{s=-d_{hH}}^{-1}\sum_{r=i+1+s}^{i}\boldsymbol{\eta}^{hT}(r,j)\boldsymbol{R}^h\boldsymbol{\eta}^h(r,j)-d_{hH}\sum_{s=-d_{hH}}^{-1}\sum_{r=i+s}^{i-1}\boldsymbol{\eta}^{hT}(r,j)\boldsymbol{R}^h\boldsymbol{\eta}^h(r,j)\\ &+d_{vH}\sum_{s=-d_{vH}}^{-1}\sum_{t=j+1+s}^{j}\boldsymbol{\eta}^{vT}(i,t)\boldsymbol{R}^v\boldsymbol{\eta}^v(i,t)-d_{vH}\sum_{s=-d_{vH}}^{-1}\sum_{t=j+s}^{j-1}\boldsymbol{\eta}^{vT}(i,t)\boldsymbol{R}^v\boldsymbol{\eta}^v(i,t)\\ =&\begin{bmatrix}\boldsymbol{\eta}^h(i,j)\\ \boldsymbol{\eta}^v(i,j)\end{bmatrix}^T\boldsymbol{D}_h^2\begin{bmatrix}\boldsymbol{R}^h & \mathbf{0}\\ \mathbf{0} & \boldsymbol{R}^v\end{bmatrix}\begin{bmatrix}\boldsymbol{\eta}^h(i,j)\\ \boldsymbol{\eta}^v(i,j)\end{bmatrix}-d_{hH}\sum_{s=-d_{hH}}^{-1}\boldsymbol{\eta}^{hT}(i+s,j)\boldsymbol{R}^h\boldsymbol{\eta}^h(i+s,j)\\ &-d_{vH}\sum_{t=-d_{vH}}^{-1}\boldsymbol{\eta}^{vT}(i,j+t)\boldsymbol{R}^v\boldsymbol{\eta}^v(i,j+t)\end{aligned} \tag{56}$$

Using Lemma 2.1, it follows from (56) that

$$\begin{aligned}
&\Delta v_5(\boldsymbol{\xi}(i,j)) \\
&\le \begin{bmatrix} \boldsymbol{\eta}^h(i,j) \\ \boldsymbol{\eta}^v(i,j) \end{bmatrix}^T \boldsymbol{D}_h^2 \begin{bmatrix} \boldsymbol{R}^h & \mathbf{0} \\ \mathbf{0} & \boldsymbol{R}^v \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta}^h(i,j) \\ \boldsymbol{\eta}^v(i,j) \end{bmatrix} + \begin{bmatrix} \sum\limits_{r=i-d_{hH}}^{i-1} \boldsymbol{\eta}^h(r,j) \\ \sum\limits_{t=j-d_{vH}}^{j-1} \boldsymbol{\eta}^v(i,t) \end{bmatrix}^T \begin{bmatrix} -\boldsymbol{R}^h & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{R}^v \end{bmatrix} \begin{bmatrix} \sum\limits_{r=i-d_{hH}}^{i-1} \boldsymbol{\eta}^h(r,j) \\ \sum\limits_{t=j-d_{vH}}^{j-1} \boldsymbol{\eta}^v(i,t) \end{bmatrix} \quad (57) \\
&\le \boldsymbol{\xi}^T(i,j)\boldsymbol{D}_h^2\boldsymbol{R}_1\boldsymbol{\xi}(i,j) + \boldsymbol{\xi}^T(i,j)(\boldsymbol{A}-\boldsymbol{I})^T\boldsymbol{D}_h^2\boldsymbol{R}_2^T\boldsymbol{\xi}(i,j) + \begin{bmatrix} \boldsymbol{\xi}^h(i-d_h(i),j) \\ \boldsymbol{\xi}^v(i,j-d_v(j)) \end{bmatrix}^T \boldsymbol{A}_d^T\boldsymbol{D}_h^2\boldsymbol{R}_2^T\boldsymbol{\xi}(i,j) \\
&\quad + \Psi^T(\boldsymbol{K}\boldsymbol{\xi}(i,j))(-\boldsymbol{B}-\boldsymbol{R}\boldsymbol{E}_c)^T\boldsymbol{D}_h^2\boldsymbol{R}_2^T\boldsymbol{\xi}(i,j) + \boldsymbol{\xi}^T(i,j)\boldsymbol{D}_h^2\boldsymbol{R}_2(\boldsymbol{A}-\boldsymbol{I})\boldsymbol{\xi}(i,j) \\
&\quad + \boldsymbol{\xi}^T(i,j)\boldsymbol{D}_h^2\boldsymbol{R}_2\boldsymbol{A}_d \begin{bmatrix} \boldsymbol{\xi}^h(i-d_h(i),j) \\ \boldsymbol{\xi}^v(i,j-d_v(j)) \end{bmatrix} + \boldsymbol{\xi}^T(i,j)\boldsymbol{D}_h^2\boldsymbol{R}_2(-\boldsymbol{B}-\boldsymbol{R}\boldsymbol{E}_c)\Psi(\boldsymbol{K}\boldsymbol{\xi}(i,j)) + \boldsymbol{\delta}^T(i,j)\boldsymbol{D}_h^2\boldsymbol{R}_3\boldsymbol{\delta}(i,j) \\
&\quad + \begin{bmatrix} \sum\limits_{r=i-d_{hH}}^{i-1} \boldsymbol{\xi}^{hT}(r,j) & \sum\limits_{t=j-d_{vH}}^{j-1} \boldsymbol{\xi}^{vT}(i,t) \end{bmatrix} \begin{bmatrix} -\boldsymbol{R}_1^h & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{R}_1^v \end{bmatrix} \begin{bmatrix} \sum\limits_{r=i-d_{hH}}^{i-1} \boldsymbol{\xi}^h(r,j) \\ \sum\limits_{t=j-d_{vH}}^{j-1} \boldsymbol{\xi}^{vT}(i,t) \end{bmatrix} \\
&\quad + \begin{bmatrix} \boldsymbol{\xi}^{hT}(i,j)-\boldsymbol{\xi}^{hT}(i-d_{hH},j) & \boldsymbol{\xi}^{vT}(i,j)-\boldsymbol{\xi}^{vT}(i,j-d_{vH}) \end{bmatrix} \begin{bmatrix} -\boldsymbol{R}_2^{hT} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{R}_2^{vT} \end{bmatrix} \begin{bmatrix} \sum\limits_{r=i-d_{hH}}^{i-1} \boldsymbol{\xi}^h(r,j) \\ \sum\limits_{t=j-d_{vH}}^{j-1} \boldsymbol{\xi}^{vT}(i,t) \end{bmatrix} \\
&\quad + \begin{bmatrix} \sum\limits_{r=i-d_{hH}}^{i-1} \boldsymbol{\xi}^{hT}(r,j) & \sum\limits_{t=j-d_{vH}}^{j-1} \boldsymbol{\xi}^{vT}(i,t) \end{bmatrix} \begin{bmatrix} -\boldsymbol{R}_2^h & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{R}_2^v \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}^h(i,j)-\boldsymbol{\xi}^h(i-d_{hH},j) \\ \boldsymbol{\xi}^v(i,j)-\boldsymbol{\xi}^v(i,j-d_{vH}) \end{bmatrix} \\
&\quad + \begin{bmatrix} \boldsymbol{\xi}^{hT}(i,j)-\boldsymbol{\xi}^{hT}(i-d_{hH},j) & \boldsymbol{\xi}^{vT}(i,j)-\boldsymbol{\xi}^{vT}(i,j-d_{vH}) \end{bmatrix} \begin{bmatrix} -\boldsymbol{R}_3^h & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{R}_3^v \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}^h(i,j)-\boldsymbol{\xi}^h(i-d_{hH},j) \\ \boldsymbol{\xi}^v(i,j)-\boldsymbol{\xi}^v(i,j-d_{vH}) \end{bmatrix} \quad (58)
\end{aligned}$$

where

$$\boldsymbol{\delta} = \begin{bmatrix} \boldsymbol{\delta}^{hT}(i,j) & \boldsymbol{\delta}^{vT}(i,j) \end{bmatrix}^T \quad (59)$$

Employing (37)-(58) and Lemma 2.2, one has

$$\begin{aligned}
&\boldsymbol{\Delta} v(\boldsymbol{\xi}(i,j)) \\
&\le \boldsymbol{\mu}^T(i,j) \begin{bmatrix} \boldsymbol{\Upsilon}_{11} & \boldsymbol{R}_2\boldsymbol{D}_h^2\boldsymbol{A}_d & \boldsymbol{R}_2\boldsymbol{D}_h^2(-\boldsymbol{B}-\boldsymbol{R}\boldsymbol{E}_c)+\boldsymbol{G}^T\boldsymbol{D} & \boldsymbol{R}_3^T & -\boldsymbol{R}_2^T \\ * & -\boldsymbol{Q} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & -2\boldsymbol{D} & \mathbf{0} & \mathbf{0} \\ * & * & * & -\boldsymbol{W}-\boldsymbol{R}_3 & \boldsymbol{R}_2^T \\ * & * & * & * & -\boldsymbol{R}_1 \end{bmatrix} \boldsymbol{\mu}(i,j) \\
&\quad + \boldsymbol{\delta}^T(i,j)\boldsymbol{D}_H^2\boldsymbol{R}_3\boldsymbol{\delta}(i,j) + \begin{bmatrix} \boldsymbol{\xi}^h(i+1,j) \\ \boldsymbol{\xi}^v(i,j+1) \end{bmatrix}^T \boldsymbol{Q} \begin{bmatrix} \boldsymbol{\xi}^h(i+1,j) \\ \boldsymbol{\xi}^v(i,j+1) \end{bmatrix} \\
&\quad + \begin{bmatrix} \boldsymbol{\xi}^h(i+1,j) \\ \boldsymbol{\xi}^v(i,j+1) \end{bmatrix}^T \boldsymbol{P} \begin{bmatrix} \boldsymbol{\xi}^h(i+1,j) \\ \boldsymbol{\xi}^v(i,j+1) \end{bmatrix} \quad (60a)
\end{aligned}$$

where

$$\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\xi}^T(i,j) & \boldsymbol{\xi}_d^T(i,j) & \boldsymbol{\psi}^T(\boldsymbol{K}\boldsymbol{\xi}(i,j)) & \boldsymbol{\xi}_H^T(i,j) & \boldsymbol{\xi}_s^T(i,j) \end{bmatrix}^T$$

$$\begin{aligned}
\boldsymbol{\xi}_d(i,j) &= \begin{bmatrix} \boldsymbol{\xi}^{hT}(i-d(i),j) & \boldsymbol{\xi}^{vT}(i,j-d(j)) \end{bmatrix}^T \\
\boldsymbol{\xi}_H(i,j) &= \begin{bmatrix} \boldsymbol{\xi}^{hT}(i-d_{hH},j) & \boldsymbol{\xi}^{vT}(i,j-d_{vH}) \end{bmatrix}^T \quad (60b)
\end{aligned}$$

$$\boldsymbol{\xi}_s(i,j) = \begin{bmatrix} \sum\limits_{r=i-d_{hH}}^{i-1} \boldsymbol{\xi}^{hT}(r,j) & \sum\limits_{t=j-d_{vH}}^{j-1} \boldsymbol{\xi}^{vT}(i,t) \end{bmatrix}^T \quad (60c)$$

Inequality (60a) can be rewritten as

$$\Delta v\left(\boldsymbol{\xi}(i,j)\right) \le \boldsymbol{\mu}^T(i,j)\boldsymbol{\phi}\boldsymbol{\mu}(i,j) \quad (61)$$

where

$$\boldsymbol{\phi} = \begin{bmatrix} \boldsymbol{\phi}_{11} & \boldsymbol{\phi}_{12} & \boldsymbol{\phi}_{13} & \boldsymbol{R}_3^T & -\boldsymbol{R}_2^T \\ * & \boldsymbol{\phi}_{22} & \boldsymbol{\phi}_{23} & \mathbf{0} & \mathbf{0} \\ * & * & \boldsymbol{\phi}_{33} & \mathbf{0} & \mathbf{0} \\ * & * & * & -\boldsymbol{W} - \boldsymbol{R}_3 & \boldsymbol{R}_2^T \\ * & * & * & * & -\boldsymbol{R}_1 \end{bmatrix} \tag{62}$$

$$\boldsymbol{\phi}_{11} = \boldsymbol{\Upsilon}_{11} + \boldsymbol{A}^T\boldsymbol{Q}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} + (\boldsymbol{A} - \boldsymbol{I})^T\boldsymbol{D}_h^2\boldsymbol{R}_3(\boldsymbol{A} - \boldsymbol{I}) \tag{63a}$$

$$\boldsymbol{\phi}_{12} = \boldsymbol{R}_2\boldsymbol{D}_h^2\boldsymbol{A}_d + \boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A}_d + \boldsymbol{A}^T\boldsymbol{Q}\boldsymbol{A}_d + (\boldsymbol{A} - \boldsymbol{I})^T\boldsymbol{D}_h^2\boldsymbol{R}_3\boldsymbol{A}_d \tag{63b}$$

$$\begin{aligned} \boldsymbol{\phi}_{13} = {} & -\boldsymbol{R}_2\boldsymbol{D}_h^2(\boldsymbol{B} + \boldsymbol{R}\boldsymbol{E}_c) + \boldsymbol{G}^T\boldsymbol{D} - \boldsymbol{A}^T\boldsymbol{P}(\boldsymbol{B} + \boldsymbol{R}\boldsymbol{E}_c) - \boldsymbol{A}^T\boldsymbol{Q}(\boldsymbol{B} + \boldsymbol{R}\boldsymbol{E}_c) \\ & - (\boldsymbol{A} - \boldsymbol{I})^T\boldsymbol{D}_h^2\boldsymbol{R}_3(\boldsymbol{B} + \boldsymbol{R}\boldsymbol{E}_c) \end{aligned} \tag{63c}$$

$$\boldsymbol{\phi}_{22} = -\boldsymbol{Q} + \boldsymbol{A}_d^T\boldsymbol{P}\boldsymbol{A}_d + \boldsymbol{A}_d^T\boldsymbol{Q}\boldsymbol{A}_d + \boldsymbol{A}_d^T\boldsymbol{D}_h^2\boldsymbol{R}_3\boldsymbol{A}_d \tag{64a}$$

$$\boldsymbol{\phi}_{23} = \boldsymbol{A}_d^T\boldsymbol{P}(-\boldsymbol{B} - \boldsymbol{R}\boldsymbol{E}_c) + \boldsymbol{A}_d^T\boldsymbol{Q}(-\boldsymbol{B} - \boldsymbol{R}\boldsymbol{E}_c) + \boldsymbol{A}_d^T\boldsymbol{D}_h^2\boldsymbol{R}_3(-\boldsymbol{B} - \boldsymbol{R}\boldsymbol{E}_c) \tag{64b}$$

$$\begin{aligned} \boldsymbol{\phi}_{33} = {} & -2\boldsymbol{D} + (\boldsymbol{B} + \boldsymbol{R}\boldsymbol{E}_c)^T\boldsymbol{P}(\boldsymbol{B} + \boldsymbol{R}\boldsymbol{E}_c) + (\boldsymbol{B} + \boldsymbol{R}\boldsymbol{E}_c)^T\boldsymbol{Q}(\boldsymbol{B} + \boldsymbol{R}\boldsymbol{E}_c) \\ & + (\boldsymbol{B} + \boldsymbol{R}\boldsymbol{E}_c)^T\boldsymbol{D}_h^2\boldsymbol{R}_3(\boldsymbol{B} + \boldsymbol{R}\boldsymbol{E}_c) \end{aligned} \tag{65}$$

Note that, if $\boldsymbol{\phi} < \mathbf{0}$, then $\Delta v(\boldsymbol{\xi}(i,j)) < 0$ for $\boldsymbol{\mu}(i,j) \neq \mathbf{0}$. In view of Schur's complement, condition $\boldsymbol{\phi} < \mathbf{0}$ is equivalent to

$$\begin{bmatrix} \boldsymbol{\Upsilon}_{11} & \boldsymbol{R}_2\boldsymbol{D}_h^2\boldsymbol{A}_d & \boldsymbol{G}^T\boldsymbol{D} & \boldsymbol{R}_3^T & -\boldsymbol{R}_2^T & \boldsymbol{A}^T & \boldsymbol{A}^T & \boldsymbol{D}_h(\boldsymbol{A} - \boldsymbol{I})^T & \boldsymbol{D}_h^T \\ * & -\boldsymbol{Q} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{A}_d^T & \boldsymbol{A}_d^T & \boldsymbol{D}_h\boldsymbol{A}_d^T & \mathbf{0} \\ * & * & -2\boldsymbol{D} & \mathbf{0} & \mathbf{0} & -(\boldsymbol{B} + \boldsymbol{R}\boldsymbol{E}_c)^T & -(\boldsymbol{B} + \boldsymbol{R}\boldsymbol{E}_c)^T & (-\boldsymbol{D}_h(\boldsymbol{B} + \boldsymbol{R}\boldsymbol{E}_c))^T & (-\boldsymbol{D}_h(\boldsymbol{B} + \boldsymbol{R}\boldsymbol{E}_c))^T \\ * & * & * & -\boldsymbol{W} - \boldsymbol{R}_3 & \boldsymbol{R}_2^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & -\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & -\boldsymbol{P}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & -\boldsymbol{Q}^{-1} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & * & -\boldsymbol{R}_3^{-1} & \mathbf{0} \\ * & * & * & * & * & * & * & * & -\boldsymbol{R}_2^{-1} \end{bmatrix} < \mathbf{0} \tag{66}$$

Pre-multiplying and post-multiplying (66) by $diag(\boldsymbol{I}, \boldsymbol{I}, \boldsymbol{D}^{-1}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I}, \boldsymbol{I})$, we obtain

$$\begin{bmatrix} \boldsymbol{\Upsilon}_{11} & \boldsymbol{R}_2\boldsymbol{D}_h^2\boldsymbol{A}_d & \boldsymbol{G}^T & \boldsymbol{R}_3^T & -\boldsymbol{R}_2^T & \boldsymbol{A}^T & \boldsymbol{A}^T & \boldsymbol{D}_h(\boldsymbol{A} - \boldsymbol{I})^T & \boldsymbol{D}_h^T \\ * & -\boldsymbol{Q} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{A}_d^T & \boldsymbol{A}_d^T & \boldsymbol{D}_h\boldsymbol{A}_d^T & \mathbf{0} \\ * & * & -2\boldsymbol{L} & \mathbf{0} & \mathbf{0} & -(\boldsymbol{B}\boldsymbol{L} + \boldsymbol{R}\boldsymbol{H})^T & -(\boldsymbol{B}\boldsymbol{L} + \boldsymbol{R}\boldsymbol{H})^T & (-\boldsymbol{D}_h(\boldsymbol{B}\boldsymbol{L} + \boldsymbol{R}\boldsymbol{H}))^T & (-\boldsymbol{D}_h(\boldsymbol{B}\boldsymbol{L} + \boldsymbol{R}\boldsymbol{H}))^T \\ * & * & * & -\boldsymbol{W} - \boldsymbol{R}_3 & \boldsymbol{R}_2^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & -\boldsymbol{R}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & -\boldsymbol{P}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & -\boldsymbol{Q}^{-1} & \mathbf{0} & \mathbf{0} \\ * & * & * & * & * & * & * & -\boldsymbol{R}_3^{-1} & \mathbf{0} \\ * & * & * & * & * & * & * & * & -\boldsymbol{R}_2^{-1} \end{bmatrix} < \mathbf{0} \tag{67}$$

where

$$\boldsymbol{D}^{-1} = \boldsymbol{L}, \quad \boldsymbol{E}_c = \boldsymbol{H}\boldsymbol{L}^{-1} \tag{68}$$

For any matrices $\mathbf{0} < \boldsymbol{X}_k$, $k = 1, 2, 3, 4$, we have [46,62]

$$\begin{aligned} -\boldsymbol{P}^{-1} \leq -2\boldsymbol{X}_1 + \boldsymbol{X}_1\boldsymbol{P}\boldsymbol{X}_1, \quad -\boldsymbol{Q}^{-1} \leq -2\boldsymbol{X}_2 + \boldsymbol{X}_2\boldsymbol{Q}\boldsymbol{X}_2 \\ -\boldsymbol{R}_3^{-1} \leq -2\boldsymbol{X}_3 + \boldsymbol{X}_3\boldsymbol{R}_3\boldsymbol{X}_3, \quad -\boldsymbol{R}_2^{-1} \leq -2\boldsymbol{X}_4 + \boldsymbol{X}_4\boldsymbol{R}_2\boldsymbol{X}_4 \end{aligned} \tag{69}$$

In view of (69), it is clear that (67) holds if (17) is satisfied.

Next, it remains to show that the set $\varepsilon(\boldsymbol{P}) = \left\{\boldsymbol{\xi} \in \Re^{n+m+n_c+m_c};\ \boldsymbol{\xi}^T\boldsymbol{P}\boldsymbol{\xi} \leq 1\right\}$ is included in polyhedral set ℓ as defined in (16), if (18) holds. It can be proven that $\varepsilon(\boldsymbol{P}) = \left\{\boldsymbol{\xi} \in \Re^{n+m+n_c+m_c};\ \boldsymbol{\xi}^T\boldsymbol{P}\boldsymbol{\xi} \leq 1\right\}$ is equivalent to [61]

$$\boldsymbol{P} - \left(\boldsymbol{K}_{(l)} - \boldsymbol{G}_{(l)}\right)^T \left(\boldsymbol{K}_{(l)} - \boldsymbol{G}_{(l)}\right) u_{0(l)}^{-2} \geq \mathbf{0}, \quad l = 1, 2, \ldots, p \tag{70}$$

which implies

$$\boldsymbol{\xi}^T(i,j)\left(\boldsymbol{P}-\left(\boldsymbol{K}_{(l)}-\boldsymbol{G}_{(l)}\right)^T\left(\boldsymbol{K}_{(l)}-\boldsymbol{G}_{(l)}\right)u_{0(l)}^{-2}\right)\boldsymbol{\xi}(i,j)\geq\boldsymbol{0} \tag{71}$$

The equivalence between (71) and (18) follows trivially from Schur's complement. This completes the proof of Theorem 3.1.

The proof for the estimate of domain of attraction (21) directly follows from [46,54] and detail of the proof is omitted for brevity.

Remark 3.1. *Theorem 3.1 can be used to determine the asymptotic stability of 2-D discrete system described by Roesser model with actuator saturation and time varying state delay. Note that, the conditions (17) and (18) are in LMI setting, and hence, computationally tractable.*

Remark 3.2. *In Theorem 3.1, a sector condition (see (16)) is used to characterize the actuator saturation nonlinearities. By contrast, [54-56] deal with the design of state feedback controller where convex hull approach is adopted for the characterization of saturation nonlinearities. In [54-56], the stability conditions are expressed as a convex combination of 2^p (where input $\boldsymbol{u}(i,j)\in\Re^p$) LMIs. Therefore, as compared to [54-56], the present approach is beneficial in terms of computational complexity.*

As a direct consequence of Theorem 3.1, we have the following corollary for the global asymptotic stability of system (9).

Corollary 3.1. *The system (9) is globally asymptotically stable provided there exists a matrix $\boldsymbol{H}\in\Re^{(n_c+m_c)\times p}$, a diagonal positive definite matrix $\boldsymbol{L}\in\Re^{p\times p}$, positive definite symmetric matrices $\boldsymbol{P}^h\in\Re^{(n+n_c)\times(n+n_c)}$, $\boldsymbol{P}^v\in\Re^{(m+m_c)\times(m+m_c)}$, $\boldsymbol{Q}^h\in\Re^{(n+n_c)\times(n+n_c)}$, $\boldsymbol{Q}^v\in\Re^{(m+m_c)\times(m+m_c)}$, $\boldsymbol{W}^h\in\Re^{(n+n_c)\times(n+n_c)}$, $\boldsymbol{W}^v\in\Re^{(m+m_c)\times(m+m_c)}$, $\boldsymbol{R}_k=\boldsymbol{R}_k^h\oplus\boldsymbol{R}_k^v$, $k=1,2,3$ and $\boldsymbol{X}_k$, $k=1,2,3,4$ satisfying the following LMI*

$$\begin{bmatrix}
\boldsymbol{\Upsilon}_{11} & \boldsymbol{R}_2\boldsymbol{D}_h^2\boldsymbol{A}_d & \boldsymbol{K}^T & \boldsymbol{R}_3^T & -\boldsymbol{R}_2^T & \boldsymbol{A}^T & \boldsymbol{A}^T & \boldsymbol{D}_h(\boldsymbol{A}-\boldsymbol{I})^T & \boldsymbol{D}_h^T \\
* & -\boldsymbol{Q} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}_d^T & \boldsymbol{A}_d^T & \boldsymbol{D}_h\boldsymbol{A}_d^T & \boldsymbol{0} \\
* & * & -2\boldsymbol{L} & \boldsymbol{0} & \boldsymbol{0} & -(\boldsymbol{BL}+\boldsymbol{RH})^T & -(\boldsymbol{BL}+\boldsymbol{RH})^T & (-\boldsymbol{D}_h(\boldsymbol{BL}+\boldsymbol{RH}))^T & (-\boldsymbol{D}_h(\boldsymbol{BL}+\boldsymbol{RH}))^T \\
* & * & * & -\boldsymbol{W}-\boldsymbol{R}_3 & \boldsymbol{R}_2^T & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & -\boldsymbol{R}_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & -2\boldsymbol{X}_1+\boldsymbol{X}_1\boldsymbol{P}\boldsymbol{X}_1 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & * & -2\boldsymbol{X}_2+\boldsymbol{X}_2\boldsymbol{Q}\boldsymbol{X}_2 & \boldsymbol{0} & \boldsymbol{0} \\
* & * & * & * & * & * & * & -2\boldsymbol{X}_3+\boldsymbol{X}_3\boldsymbol{R}_3\boldsymbol{X}_3 & \boldsymbol{0} \\
* & * & * & * & * & * & * & * & -2\boldsymbol{X}_4+\boldsymbol{X}_4\boldsymbol{R}_2\boldsymbol{X}_4
\end{bmatrix}<0 \tag{72}$$

where

$$\begin{aligned}
&\boldsymbol{\Upsilon}_{11}=-\boldsymbol{P}+\boldsymbol{W}+\boldsymbol{D}_d\boldsymbol{Q}+\boldsymbol{D}_h^2\boldsymbol{R}_1+(\boldsymbol{A}-\boldsymbol{I})^T\boldsymbol{D}_h^2\boldsymbol{R}_2+\boldsymbol{D}_h^2\boldsymbol{R}_2(\boldsymbol{A}-\boldsymbol{I})-\boldsymbol{R}_3\\
&\boldsymbol{P}=\boldsymbol{P}^h\oplus\boldsymbol{P}^v,\quad \boldsymbol{Q}=\boldsymbol{Q}^h\oplus\boldsymbol{Q}^v,\quad \boldsymbol{W}=\boldsymbol{W}^h\oplus\boldsymbol{W}^v\\
&\boldsymbol{0}<\boldsymbol{R}^h=\begin{bmatrix}\boldsymbol{R}_1^h & \boldsymbol{R}_2^h\\ \boldsymbol{R}_2^{hT} & \boldsymbol{R}_3^h\end{bmatrix}\in\Re^{2(n+n_c)\times 2(n+n_c)}\\
&\boldsymbol{0}<\boldsymbol{R}^v=\begin{bmatrix}\boldsymbol{R}_1^v & \boldsymbol{R}_2^v\\ \boldsymbol{R}_2^{vT} & \boldsymbol{R}_3^v\end{bmatrix}\in\Re^{2(m+m_c)\times 2(m+m_c)}\\
&\boldsymbol{D}_d=\begin{bmatrix}(d_{hH}-d_{hL})\boldsymbol{I}_n & \boldsymbol{0}\\ \boldsymbol{0} & (d_{vH}-d_{vL})\boldsymbol{I}_n\end{bmatrix},\quad \boldsymbol{D}_h=\begin{bmatrix}d_{hH}\boldsymbol{I}_n & \boldsymbol{0}\\ \boldsymbol{0} & d_{vH}\boldsymbol{I}_n\end{bmatrix}
\end{aligned} \tag{73}$$

Proof: Choosing $\boldsymbol{G}=\boldsymbol{K}$, one can see that (16) is automatically met for all $\boldsymbol{\xi}$. Substituting $\boldsymbol{G}=\boldsymbol{K}$ into (17), we obtain the global stability condition (72). This completes the proof. □

Maximization of Domain of Attraction

An optimization procedure to maximize the estimate of domain of attraction may be stated in the form of following theorem.

Theorem 3.2. *Consider the closed loop system (9) with the boundary conditions (13), then the maximized domain of attraction can be estimated if the following convex optimization problem*
minimize r,
where

$$\begin{aligned} r = {} & r_1^h + (d_{hH}+1)r_2^h + d_{hH}r_3^h + 0.5(d_{hH}-d_{hL}+1)(d_{hH}+d_{hL})r_4^h \\ & + 2d_{hH}^2(d_{hH}+1)r_5^h + r_1^v + (1+d_{vH})r_2^v + d_{vH}r_3^v \\ & + 0.5(d_{vH}-d_{vL}+1)(d_{vH}+d_{vL})r_4^v + 2d_{vH}^2(d_{vH}+1)r_5^v \end{aligned} \tag{74}$$

subject to (17)-(19) and

$$\begin{aligned} & r_1^h\boldsymbol{I}-\boldsymbol{P}^h \geq \mathbf{0},\ r_1^v\boldsymbol{I}-\boldsymbol{P}^v \geq \mathbf{0},\ r_2^h\boldsymbol{I}-\boldsymbol{Q}^h \geq \mathbf{0},\ r_2^v\boldsymbol{I}-\boldsymbol{Q}^v \geq \mathbf{0},\ r_3^h\boldsymbol{I}-\boldsymbol{W}^h \geq \mathbf{0} \\ & r_3^v\boldsymbol{I}-\boldsymbol{W}^v \geq \mathbf{0},\ r_4^h\boldsymbol{I}-\boldsymbol{Q}^h \geq \mathbf{0},\ r_4^v\boldsymbol{I}-\boldsymbol{Q}^v \geq \mathbf{0},\ r_5^h\boldsymbol{I}-\boldsymbol{R}^h \geq \mathbf{0},\ r_5^v\boldsymbol{I}-\boldsymbol{R}^v \geq \mathbf{0} \end{aligned} \tag{75}$$

has a feasible solution for the weighting parameters $r_i^h > 0$, $i = 1, 2, \ldots, 5$, $r_i^v > 0$, $i = 1, 2, \ldots, 5$, *positive definite symmetric matrices* $\boldsymbol{P}^h$, $\boldsymbol{P}^v$, $\boldsymbol{Q}^h$, $\boldsymbol{Q}^v$, $\boldsymbol{W}^h$, $\boldsymbol{W}^v$, $\boldsymbol{R}^h$, $\boldsymbol{R}^v$, $\boldsymbol{X}_k$, $k = 1, 2, 3, 4$, *matrices* $\boldsymbol{H}$, $\boldsymbol{G}$ *and a diagonal positive definite matrix* $\boldsymbol{L}$.

In this situation, an anti-windup gain $\boldsymbol{E}_c = \boldsymbol{H}\boldsymbol{L}^{-1}$ *provides a maximized estimate of domain of attraction given by* $\Gamma_{(\beta_{rh},\beta_{rv})} = 1$.

Proof: If the conditions given by (75) hold true, then $r_1^h\boldsymbol{I} \geq \lambda_{\max}\left(\boldsymbol{P}^h\right)$, $r_1^v\boldsymbol{I} \geq \lambda_{\max}(\boldsymbol{P}^v)$, $r_2^h\boldsymbol{I} \geq \lambda_{\max}\left(\boldsymbol{Q}^h\right)$, $r_2^v\boldsymbol{I} \geq \lambda_{\max}(\boldsymbol{Q}^v)$, $r_3^h\boldsymbol{I} \geq \lambda_{\max}\left(\boldsymbol{W}^h\right)$, $r_3^v\boldsymbol{I} \geq \lambda_{\max}(\boldsymbol{W}^v)$, $r_4^h\boldsymbol{I} \geq \lambda_{\max}\left(\boldsymbol{Q}^h\right)$, $r_4^v\boldsymbol{I} \geq \lambda_{\max}(\boldsymbol{Q}^v)$, $r_5^h\boldsymbol{I} \geq \lambda_{\max}\left(\boldsymbol{R}^h\right)$, $r_5^v\boldsymbol{I} \geq \lambda_{\max}(\boldsymbol{R}^v)$.

Thus, if we minimize r given by (74), estimate of domain of attraction is being maximized. In other words, the optimization procedure in Theorem 3.2 orients the solutions of (17)-(19) in order to get the domain of attraction as large as possible. □

4. Numerical Examples.

In this section, several numerical examples are given to illustrate the effectiveness of the proposed approach.

Example 4.1. *Consider a closed loop 2-D system in Roesser model with time varying delay (1a) and (1b) and stabilizing controller (2a) and (2b) with the following parameters*

$$\boldsymbol{A}_p = \left[\begin{array}{cc|c} 0.215 & 0.08 & 0 \\ 0.2 & -0.04 & 0.01 \\ \hline 0.01 & 0.01 & -0.1 \end{array}\right],\quad \boldsymbol{A}_{dp} = \left[\begin{array}{cc|c} 0.1 & 0 & 0.05 \\ 0.1 & 0.02 & 0 \\ \hline 0.03 & 0.12 & 0.03 \end{array}\right],\quad \boldsymbol{B}_p = \left[\begin{array}{cc} 0.8 & 0 \\ -0.01 & 0.01 \\ \hline 0.001 & 0.002 \end{array}\right]$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.1 & 0.01 \end{bmatrix},\quad \boldsymbol{A}_c = \left[\begin{array}{c|c} -0.051 & 0 \\ \hline 0 & -0.501 \end{array}\right],\quad \boldsymbol{B}_c = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.6 \end{bmatrix}$$

$$\boldsymbol{C}_c = \begin{bmatrix} -0.4958 & 0 \\ 0 & -0.51 \end{bmatrix},\quad \boldsymbol{D}_c = \begin{bmatrix} -0.011 & 0 \\ 0 & -0.527 \end{bmatrix}$$

The control bound $\boldsymbol{u}_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ *and delay ranges are defined as* $d_{hL} = 1$, $d_{hH} = 3$, $d_{vL} = 1$, $d_{vH} = 3$ *in this example. Using MATLAB LMI toolbox [66], it is verified that (17)-(19) and (75) are feasible for following values of unknown parameters*

$$\boldsymbol{P}^h = \begin{bmatrix} 189.5312 & 0.5397 & -28.0442 \\ 0.5397 & 121.7596 & -13.0162 \\ -28.0442 & -13.0162 & 148.1501 \end{bmatrix},\quad \boldsymbol{P}^v = \begin{bmatrix} 104.0758 & -0.4987 \\ -0.4987 & 133.0361 \end{bmatrix}$$

$$\boldsymbol{H} = \begin{bmatrix} -0.0186 & -0.0175 \\ -0.0013 & 0.0038 \end{bmatrix}, \quad \boldsymbol{L} = \begin{bmatrix} 3.2415 & 0 \\ 0 & 3.1998 \end{bmatrix}$$

In this case, the gain of stabilizing anti-windup compensator is given by $\boldsymbol{E}_c = \boldsymbol{H}\boldsymbol{L}^{-1} = \begin{bmatrix} -0.0057 & 0.0055 \\ -0.0004 & 0.0012 \end{bmatrix}$.

Example 4.2. *The present approach (Theorem 3.1) can be applied to the control of several dynamical processes. It is known that some dynamical processes in heat exchangers, air drying, water stream, heating and gas absorption, etc. can be expressed by Darboux equation [63-65]. In this example, we shall demonstrate the application of Theorem 3.1 for the control of processes which can be expressed by Darboux equation.*

Consider the Darboux equation [63-65] given by

$$\frac{\partial^2 s(x,t)}{\partial x \partial t} = a_1 \frac{\partial s(x,t)}{\partial t} + a_2 \frac{\partial s(x,t)}{\partial x} + a_0 s(x,t) + a_d s(x,t-d) + bf(x,t) \tag{76}$$

$$y(x,t) = c_1 \left[\frac{\partial s(x,t)}{\partial t} - a_2 s(x,t) \right] + c_2 s(x,t) \tag{77}$$

with the initial conditions $s(x,0) = p(x)$, $s(0,t) = q(t)$, *where* $s(x,t)$ *is an unknown function at space* $x \in [0, x_f]$ *and time* $t \in [0, \infty]$; $f(x,t)$ *is the input function subjected to saturation;* $y(x,t)$ *is the measured output;* a_1, a_2, a_0, a_d, b, c_1 *and* c_2 *are real constants.*

Define

$$r(x,t) = \frac{\partial s(x,t)}{\partial t} - a_2 s(x,t) \tag{78}$$

then (76) can be transformed into an equivalent system of the form:

$$\begin{bmatrix} \dfrac{\partial r(x,t)}{\partial x} \\ \dfrac{\partial s(x,t)}{\partial t} \end{bmatrix} = \begin{bmatrix} a_1 & a_1 a_2 + a_0 \\ 1 & a_2 \end{bmatrix} \begin{bmatrix} r(x,t) \\ s(x,t) \end{bmatrix} + \begin{bmatrix} a_d \\ 0 \end{bmatrix} s(x,t-d) + \begin{bmatrix} b \\ 0 \end{bmatrix} f(x,t) \tag{79}$$

with initial condition

$$r(0,t) = \left. \frac{\partial s(x,t)}{\partial t} \right|_{x=0} - a_2 s(0,t) = \frac{dq(t)}{dt} - a_2 q(t) = z(t) \tag{80}$$

Let $r(i,j) = r(i\Delta x, j\Delta t) \triangleq x^h(i,j)$, $s(i,j) = s(i\Delta x, j\Delta t) \triangleq x^v(i,j)$, $f(x,t) = u(i,j)$ *and*

$$\frac{\partial r(x,t)}{\partial x} \approx \frac{r(i+1,j) - r(i,j)}{\Delta x}, \quad \frac{\partial s(x,t)}{\partial t} \approx \frac{s(i,j+1) - s(i,j)}{\Delta t} \tag{81}$$

Now, taking

$$\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} = \begin{bmatrix} r(i,j) \\ s(i,j) \end{bmatrix} \tag{82}$$

(79) leads to

$$\begin{aligned} \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} &= \begin{bmatrix} (1 + a_1 \Delta x) & (a_1 a_2 + a_0)\Delta x \\ \Delta t & (1 + a_2 \Delta t) \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & a_d \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x^h(i - d_h(i), j) \\ x^v(i, j - d_v(j)) \end{bmatrix} + \begin{bmatrix} b\Delta x \\ 0 \end{bmatrix} u(i,j) \end{aligned} \tag{83}$$

with the initial conditions

$$\boldsymbol{x}^h(0,j) = z(j\Delta t), \quad \boldsymbol{x}^v(i,0) = p(i\Delta x) \tag{84}$$

Note that, the discrepancy between the partial differential equation (PDE) model of (76) and its 2-D difference approximation depends on the step size Δx *and* Δt.

Let the 2-D plant with delay be expressed in the form of (6) with

$$\boldsymbol{A}_p = \begin{bmatrix} -0.05 & 0.08 \\ 0.2 & 0.04 \end{bmatrix}, \quad \boldsymbol{A}_{dp} = \begin{bmatrix} 0.520 & 0 \\ 0.1 & 0.05 \end{bmatrix}$$

$$\boldsymbol{B}_p = \begin{bmatrix} 0.08 & 0 \\ -0.01 & 0.01 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} -1 & -0.20 \\ 0 & -0.1 \end{bmatrix}$$

The dynamic output feedback controller for the above plant is given by

$$\boldsymbol{A}_c = \begin{bmatrix} 0.2521 & 0.53 \\ -0.16 & 0.084 \end{bmatrix}, \quad \boldsymbol{B}_c = \begin{bmatrix} -0.045 & -0.05 \\ 0.0836 & 0.10 \end{bmatrix}$$

$$\boldsymbol{C}_c = \begin{bmatrix} -1.014958 & 0.036 \\ 2.0 & 0 \end{bmatrix}, \quad \boldsymbol{D}_c = \begin{bmatrix} -0.15 & -0.09 \\ 18.05 & -30.05 \end{bmatrix}$$

It is verified that (17) and (18) are feasible for control bound $\boldsymbol{u}_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ *and delay range* $d_{hL} = 1$, $d_{hH} = 2$, $d_{vL} = 1$, $d_{vH} = 43$. *The values of unknown parameters are*

$$\boldsymbol{P}^h = \begin{bmatrix} 157.4179 & 10.1323 \\ 10.1323 & 117.9968 \end{bmatrix}, \quad \boldsymbol{P}^v = \begin{bmatrix} 170.4225 & 0.1865 \\ 0.1865 & 222.9363 \end{bmatrix}$$

$$\boldsymbol{L} = \begin{bmatrix} 0.0004 & 0 \\ 0 & 2.1349 \end{bmatrix}, \quad \boldsymbol{H} = \begin{bmatrix} -0.0004 & -0.0074 \\ 0.0001 & 0.0180 \end{bmatrix}$$

and the stabilizing anti-windup gain is $\boldsymbol{E}_c = \boldsymbol{H}\boldsymbol{L}^{-1} = \begin{bmatrix} -1.2264 & -0.0035 \\ 0.1581 & 0.0084 \end{bmatrix}$.

Example 4.3. *Consider the 2-D discrete time system in Roesser Model setting (1) and the stabilizing controller (2) with*

$$\boldsymbol{A}_p = \left[\begin{array}{cc|c} 0.05 & 0.08 & 0 \\ 0.2 & -0.04 & 0.01 \\ \hline 0.01 & 0.01 & -0.1 \end{array}\right], \quad \boldsymbol{A}_{dp} = \left[\begin{array}{cc|c} 0.1 & 0 & 0.05 \\ 0.1 & 0.02 & 0 \\ \hline 0.03 & 0.12 & 0.03 \end{array}\right], \quad \boldsymbol{B}_p = \left[\begin{array}{cc} 0.8 & 0 \\ -0.01 & 0.01 \\ \hline 0.001 & 0.002 \end{array}\right]$$

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.1 & 0.01 \end{bmatrix}, \quad \boldsymbol{A}_c = \left[\begin{array}{c|c} -0.051 & 0 \\ \hline 0 & -0.501 \end{array}\right], \quad \boldsymbol{B}_c = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.6 \end{bmatrix}$$

$$\boldsymbol{C}_c = \begin{bmatrix} -0.4958 & 0 \\ 0 & -0.51 \end{bmatrix}, \quad \boldsymbol{D}_c = \begin{bmatrix} -0.011 & 0 \\ 0 & -0.527 \end{bmatrix}$$

For given control bound $\boldsymbol{u}_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ *and delay range* $d_{hL} = 1$, $d_{hH} = 2$, $d_{vL} = 1$, $d_{vH} = 3$, *it is found that the condition stated in Corollary 3.1 is feasible for*

$$\boldsymbol{P}^h = \begin{bmatrix} 195.1879 & -4.3330 & -10.7384 \\ -4.3330 & 162.7806 & -19.6112 \\ -10.7384 & -19.6112 & 164.6883 \end{bmatrix}, \quad \boldsymbol{P}^v = \begin{bmatrix} 181.6109 & -0.8438 \\ -0.8438 & 184.7405 \end{bmatrix}$$

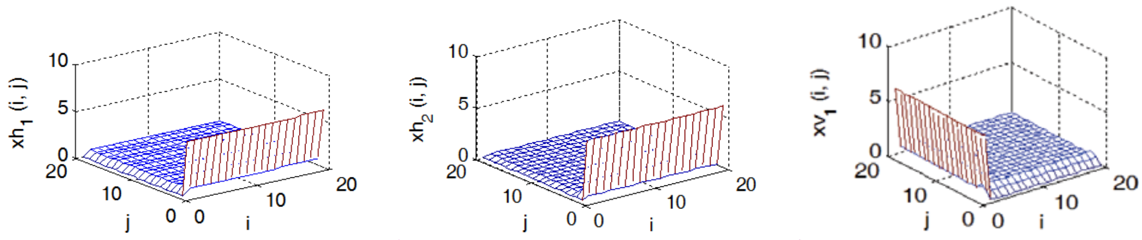
$$\boldsymbol{H} = \begin{bmatrix} 0.0208 & -0.0214 \\ 0.0047 & -0.0019 \end{bmatrix}, \quad \boldsymbol{L} = \begin{bmatrix} 12.8220 & 0 \\ 0 & 12.3038 \end{bmatrix}$$

and the anti-windup gain of stabilizing compensator is obtained as

$$\boldsymbol{E}_c = \boldsymbol{H}\boldsymbol{L}^{-1} = \begin{bmatrix} 0.0016 & -0.0017 \\ 0.0004 & -0.0002 \end{bmatrix}.$$

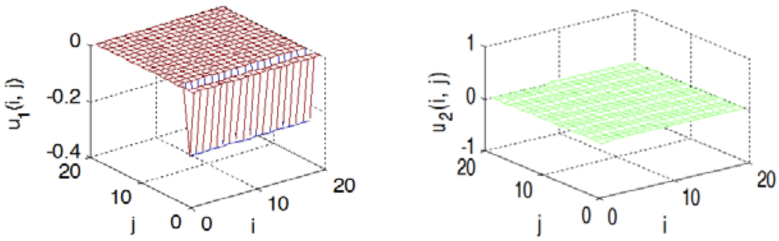
Therefore, Corollary 3.1 ensures the global asymptotic stability of closed loop system under consideration.

Figure 1 shows the trajectories of states variables of the closed loop system with two horizontal states and one vertical state of 2-D plant and it is seen that all the three states



(a) (b) (c)

FIGURE 1. Trajectories of the horizontal and vertical state variable (Example 4.3)



(a) (b)

FIGURE 2. Control effort (Example 4.3)

are converging to zero with $\boldsymbol{\xi}^h(i,j) = \begin{cases} 6 & 0 \le j \le 20 \\ 0 & j > 20 \end{cases}$, $\boldsymbol{\xi}^v(i,j) = \begin{cases} 6 & 0 \le j \le 20 \\ 0 & j > 20 \end{cases}$ *and time varying delay as* $d_h(i) = 1 + \sin\left(\frac{\pi i}{2}\right)$, $d_v(j) = 2 + \sin\left(\frac{\pi j}{2}\right)$. *The control efforts are bounded in Figure 2 for given values* $\boldsymbol{u}_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. *The effect of anti-windup controller can be seen from Figure 2 that the control efforts are bounded within defined limit, i.e.,* ± 1.

5. **Conclusion.** In this paper, an anti-windup strategy has been applied to 2-D discrete system in Roesser model when subjected to input saturation and time varying delay. Delay-dependent stability criterion for the said system has been established in LMI form. An algorithm is proposed to maximize the domain of attraction. Several examples are provided to demonstrate the applicability of the presented result.

Acknowledgments. The authors wish to thank the Editors and anonymous reviewers for their comments and suggestions to improve the manuscript.

REFERENCES

[1] K Galkowski, The Fornasini-Marchesini and the Roesser model: Algebraic methods for recasting, *IEEE Trans. Automatic Control*, vol.41, no.1, pp.107-112, 1996.
[2] S. A. Miri and J. D. Aplevich, Equivalence of n-dimensional Roesser and Fornasini-Marchesini models, *IEEE Trans. Automatic Control*, vol.43, no.3, pp.401-405, 1998.
[3] N. K. Bose, Implementation of a new stability test for n-D filters, *IEEE Trans. Acoustics, Speech, and Signal Processing*, vol.27, no.1, pp.1-4, 1979.
[4] P. K. Rajan and H. C. Reddy, On the necessary conditions for the BIBO stability of n-D filters, *IEEE Trans. Circuits and Systems*, vol.33, no.11, p.1143, 1986.
[5] J. Gregor, Stability of nonconvolutional n-D linear discrete systems, *IEEE Trans. Circuits and Systems-I: Fundamental Theory and Applications*, vol.49, no.12, pp.1773-1781, 2002.

[6] P. S Reddy, S. R. Palacherla and M. N. S. Swamy, Least squares inverse polynomials and a proof of practical-BIBO stability of n-D digital filters, *IEEE Trans. Circuits and Systems*, vol.37, no.12, pp.1565-1567, 1990.

[7] B. A. Merhy, P. Payeur and E. M. Petriu, Application of segmented 2-D probabilistic occupancy maps for robot sensing and navigation, *IEEE Trans. Instrumentation and Measurement*, vol.57, no.12, pp.2827-2837, 2008.

[8] V. V. Valenzuela, R. D. Lins and H. M. de Oliveira, Application of enhanced-2D-CWT in topographic images for mapping landslide risk areas, *Image Analysis and Recognition*, vol.79, no.50, pp.380-388, 2013.

[9] A. Cifor, L. Risser, D. Chung, E. M. Anderson and J. A. Schnabel, Hybrid feature-based diffeomorphic registration for tumor tracking in 2-D liver ultrasound images, *IEEE Trans. Medical Imaging*, vol.32, no.9, pp.1647-1656, 2013.

[10] D. Rivest-Hénault and M. Cheriet, 3-D curvilinear structure detection filter via structure-ball analysis, *IEEE Trans. Image Processing*, vol.22, no.7, pp.2849-2863, 2013.

[11] E. Tuncel and L. Onural, Utilization of the recursive shortest spanning tree algorithm for video-object segmentation by 2-D affine motion modelling, *IEEE Trans. Circuits and Systems for Video Technology*, vol.10, no.5, pp.776-781, 2000.

[12] T. Kaczorek, *Two-Dimensional Linear Systems*, Springer-Verlag, Berlin, 1985.

[13] R. P. Roesser, A discrete state-space model for linear image processing, *IEEE Trans. Automatic Control*, vol.20, no.1, pp.1-10, 1975.

[14] B. Sumanasena and P. H. Bauer, Realization using the Roesser model for implementations in distributed grid sensor networks, *Multidimensional Systems and Signal Processing*, vol.22, pp.131-146, 2011.

[15] M. B. Zarrop and P. E. Wellstead, Two-dimensional and EM technique for cross directional estimation and control, *IEE Proc. of Control Theory Appl.*, vol.149, no.5, pp.457-462, 2002.

[16] E. A Fornasini, 2-D system approach to river pollution modeling, *Multidimensional Systems and Signal Processing*, vol.2, pp.233-265, 1991.

[17] P. Costabile and F. Macchione, Enhancing river model set-up for 2-D dynamic flood modeling, *Environmental Modelling and Software, Environmental Modelling & Software*, vol.67, pp.89-107, 2015.

[18] H. Cai, H. Liang and X. Tang, Constructions of optimal 2-D optical orthogonal codes via generalized cyclotomic classes, *IEEE Trans. Information Theory*, vol.61, no.1, pp.688-695, 2015.

[19] O. Ueberschar, M. J. Almeida, P. Matthes et al., Optimized monolithic 2-D spin-valve sensor for high-sensitivity compass applications, *IEEE Trans. Magnetics Applications*, vol.51, no.1, 2015.

[20] L. J. Koduvelikulathu, V. D. Mihailetchi, S. Olibet et al., Two-dimensional modeling of the metallization-induced recombination losses of screen-printed solar cells, *IEEE Journal of Photovoltaics*, vol.5, no.1, pp.159-165, 2015.

[21] A. So, A. Al-Sharif and A. Hammoudeh, Concept design and derivation of the round trip time for a general two-dimensional elevator traffic system, *Journal of Building Engineering*, vol.5, pp.165-177, 2016.

[22] E. Rogers, K. Galkowski, W. Paszke, K. L. Moore, P. H. Bauer, L. Hladowski and P. Dabkowski, Multidimensional control systems: Case studies in design and evaluation, *Multidimensional Systems and Signal Processing*, vol.26, pp.895-939, 2015.

[23] G. Grimm, J. Hatfield, I. Postlethwaite, A. R. Teel, M. C. Turner and L. Zaccarian, Anti-windup for stable linear systems with input saturation: An LMI-based synthesis, *IEEE Trans. Automatic Control*, vol.48, no.9, pp.1509-1525, 2003.

[24] A. S. Rohman and R. H. Middleton, Anti-windup schemes for discrete time systems: An LMI-based design, *Asian Control Conference*, pp.554-561, 2004.

[25] M. Rehan, A. Ahmed and N. Iqbal, Static and low order anti-windup synthesis for cascade control systems with actuator saturation: An application to temperature-based process control, *ISA Transactions*, vol.49, pp.293-301, 2010.

[26] S.-K. Yang, Observer-based anti-windup compensator design for saturated control systems using an LMI approach, *Computers and Mathematics with Applications*, vol.64, pp.747-758, 2012.

[27] J. M. Gomes Da Silva Jr. and S. Tarbouriech, Anti-windup design with guaranteed regions of stability for discrete time linear systems, *System & Control Letters*, vol.55, no.3, pp.184-192, 2006.

[28] T. Hu, Z. Lin and B. M. Chen, Analysis and design for discrete-time linear systems subject to actuator saturation, *System & Control Letters*, vol.45, pp.97-112, 2002.

[29] J. M. Gomes Da Silva Jr., M. Z. Oliveira, D. Coutinho and S. Tarbouriech, Static anti-windup design for a class of nonlinear systems, *International Journal of Robust and Nonlinear Control*, vol.24, pp.793-810, 2014.

[30] A. Hmamed, F. Mesquine, F. Tadeo, M. Benhayoun and A. Benzaouia, Stabilization of 2-D saturated systems by state feedback control, *Multidimensional Systems and Signal Processing*, vol.21, pp.277-292, 2010.

[31] R. Negi, H. Kar and S. Purwar, Stability analysis of 2-D discrete linear system described by the Fornasini-Marchesini second model with actuator saturation, *ISRN Computational Mathematics*, Article ID 847178, 2012.

[32] V. K. Rao Kandanvli and H. Kar, Robust stability of discrete-time state-delayed systems employing generalized overflow nonlinearities, *Nonlinear Analysis*, vol.69, pp.2780-2787, 2008.

[33] V. K. Rao Kandanvli and H. Kar, An LMI condition for robust stability of discrete-time state-delayed systems using quantization/overflow nonlinearities, *Signal Processing*, vol.89, pp.2092-2102, 2009.

[34] V. K. Rao Kandanvli and H. Kar, Robust stability of discrete-time state-delayed systems with saturation nonlinearities: Linear matrix inequality approach, *Signal Processing*, vol.89, pp.161-173, 2009.

[35] V. K. Rao Kandanvli and H. Kar, Delay-dependent stability criterion for discrete-time uncertain state-delayed systems employing saturation nonlinearities, *Arabian Journal for Science and Engineering*, vol.38, pp.2911-2920, 2013.

[36] S.-F. Chen, Asymptotic stability of discrete-time systems with time-varying delay subject to saturation nonlinearities, *Chaos, Solitons and Fractals*, vol.42, pp.1251-1257, 2009.

[37] S. K. Tadepalli, V. K. Rao Kandanvli and H. Kar, Stability criteria for uncertain discrete-time systems under the influence of saturation nonlinearities and time-varying delay, *ISRN Applied Mathematics*, Article ID 861759, 2014.

[38] A. Dey and H. Kar, LMI-based criterion for robust stability of 2-D discrete systems with interval time-varying delays employing quantisation/overflow nonlinearities, *Multidimensional Systems and Signal Processing*, vol.25, pp.473-492, 2014.

[39] P. Kokil, A. Dey and H. Kar, Stability of 2-D digital filters described by the Roesser model using any combination of quantization and over flow nonlinearities, *Signal Processing*, vol.92, pp.2874-2880, 2012.

[40] N. Agarwal and H. Kar, An improved criterion for the global asymptotic stability of 2-D state-space digital filters with finite word length nonlinearities, *Signal Processing*, vol.105, pp.198-206, 2014.

[41] S.-I. Niculescu, J.-M. Dion and L. Dugard, Robust stabilization for uncertain time-delay systems containing saturating actuators, *IEEE Trans. Automatic Control*, vol.41, no.5, pp.742-747, 1996.

[42] S. Oucheriah, Global stabilization of a class of linear continuous time-delay systems with saturating controls, *IEEE Trans. Circuits and Systems-I. Fundamental Theory and Applications*, vol.43, no.12, pp.1012-1015, 1996.

[43] J. M. Gomes Da Silva Jr., S. Tarbouriech and G. Garcia, Anti-windup design for time-delay systems subject to input saturation: An LMI-based approach, *European Journal of Control*, vol.6, pp.622-634, 2006.

[44] A. Ahmed, M. Rehan and N. Iqbal, Delay-dependent anti-windup synthesis for stability of constrained state delay systems using pole-constraints, *ISA Transactions*, vol.50, pp.249-255, 2011.

[45] E. Fridman and M. Dambrine, Control under quantization, saturation and delay: An LMI approach, *Automatica*, vol.45, pp.2258-2264, 2009.

[46] R. Negi, S. Purwar and H. Kar, Delay-dependent stability analysis of discrete time delay systems with actuator saturation, *Intelligent Control and Automation*, vol.3, pp.34-43, 2012.

[47] G. Song, J. Lam and S. Xu, Robust quantized output feedback control for uncertain discrete time-delay systems with saturation nonlinearity, *Int. J. Robust Nonlinear Control*, 2014.

[48] X. Song, J. Lu, S. Xu, H. Shen and J. Lu, Robust stabilization of state delayed T-S fuzzy systems with input saturation via dynamic anti-windup fuzzy design, *International Journal of Innovative Computing, Information and Control*, vol.7, no.12, pp.6665-6676, 2011.

[49] S.-F. Chen, Stability analysis for 2-D systems with interval time-varying delays and saturation nonlinearities, *Signal Processing*, vol.90, pp.2265-2275, 2010.

[50] D. Chen and H. Yu, Stability analysis of state saturation 2-D discrete time-delay systems based on F-M Model, *Mathematical Problems in Engineering*, Article ID 749782, 2013.

[51] S.-F. Chen, Delay-dependent stability for 2-D systems with delays in the Roesser model, *American Control Conference*, pp.3470-3474, 2010.

[52] A. Dey and H. Kar, LMI-based criterion for the robust stability of 2-D discrete state-delayed systems using generalized overflow nonlinearities, *Journal of Control Science and Engineering*, Article ID 271515, 2011.

[53] A. Dey and H. Kar, An LMI based criterion for the global asymptotic stability of 2-D discrete state-delayed systems with saturation nonlinearities, *Digital Signal Processing*, vol.22, pp.633-639, 2012.

[54] S. Huang, Z. Xiang and H. R. Karimi, Robust l_2-gain control for 2-D nonlinear stochastic systems with time-varying delays and actuator saturation, *Journal of Franklin Institute*, vol.350, pp.1865-1885, 2013.

[55] I. Ghous and Z. Xiang, H_∞ Stabilization of 2-D discrete switched delayed systems represented by the Roesser model subject to actuator saturation, *Transactions of the Institute of Measurement and Control*, vol.37, no.10, pp.1242-1253, 2014.

[56] V. C. Pal and R. Negi, Robust output feedback control of 2D discrete systems with actuator saturation and time-varying delay, *Transactions of the Institute of Measurement and Control*, 2016.

[57] D. Zhao and Y. Wang, Integrated fault detection for two dimensional discrete systems with time varying delay, *IFAC Papers Online*, pp.1262-1267, 2015.

[58] A. R. Hanu, W. V. Prestwich and S. H. Byun, A data acquisition system for two-dimensional position sensitive micropattern gas detectors with delay-line readout, *Nuclear Instruments and Methods in Physics Research A*, vol.780, pp.33-39, 2015.

[59] L. Wang, W. W. J. Gao and W. Chen, Stability and robust stabilization of 2-D continuous-discrete systems in Roesser model based on KYP lemma, *Multidimensional Systems and Signal Processing*, 2015.

[60] J. Qiu, Y. Xia, H. Yang and J. Zhang, Robust stablisation for a class of discrete-time systems with time-varying delays via delta operators, *IET Control Theory and Applications*, vol.2, pp.87-93, 2008.

[61] S. Boyd, L. Ei-Ghaousi, E. Feron and V. Balakrishnan, Linear Matrix Inequalities in Systems and Control Theory, *SIAM Philadelphia*, 1994.

[62] K. F Chen and I. K. Fong, Stability analysis and output-feedback stabilization of discrete time systems with an interval time-varying state delay, *IET Control Theory and Applications*, vol.4, no.4, pp.563-572, 2010.

[63] C. Du, L. Xie and C. Zhang, H_∞ control and robust stabilization of two dimensional systems in Roesser models, *Automatica*, vol.37, no.2, pp.205-211, 2001.

[64] W. Marszalek, Two dimensional state-space discrete models for hyperbolic partial differential equations, *Applied Mathematical Modelling*, vol.8, no.1, pp.11-14, 1984.

[65] J. S. H. Tsai, J. S. Li and L. S. Sheih, Discretized quadratic optimal control for continuous-time two-dimensional systems, *IEEE Trans. Circuits and Systems I*, vol.49, no.1, pp.116-125, 2002.

[66] P. Gahinet, A. Nemirovski, A. J. Laub, M. Chilali, *LMI Control Toolbox for Use with MATLAB*, MATH Works Inc. Natic, MA, 1995.