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CHARACTERIZING WEAK CONVERGENCE ERROR OF SPARSE GRAPH COLORING PROCESS

Lu Liu and Zhenting Hou

Department of Mathematics and Statistics
Central South University
Central South University New Campus Shuli Building, Xiaoxiang Rd., Changsha 410205, P. R. China
g.jiayi.liu@gmail.com; zthou@csu.edu.cn

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Abstract. *This paper is an ongoing study of weak convergence properties of local coloring strategies which is a fundamental question about local coloring processes on large networks. Consider a piece of information spreading on a social network. Suppose 90% of the nodes receive the information. Is it possible that the consequence is by accident? i.e., is it possible that the next time the same information is only received by less than 10% of the nodes? The classic result of Sullivan says no. However, another question remains. Is the consequence determined by the local structure of the graph? The question can be phrased as whether the empirical process induced by an independent jump finite range process (local coloring process) converges if the graph structure converges. Here the local structure of a graph, G, refers to the limit of the empirical distribution of all subgraphs of G. An independent jump finite range process (local coloring process) refers to a jump process with each node jumping independently of each other (independent jump), and the intensity measure of a node, c, depends on the current state of nodes within a fixed distance to c (finite range). We give a condition for the local structure of the graph that garantees the independent jump finite range process (local coloring process) converges provided the local graph structure converges. The result improves our previous result.*
Keywords: Weak convergence, Graph coloring, Interacting particle system, Local structure of a graph

1. **Introduction.** Consider a piece of information spreading on a social network. Each node of the network represents a person and jumps among three states "receive, retweet", "receive, does not retweet" and "does not receive". Each node jumps independently of each other's jump. The intensity measure of the jump of a node c depends on the current states of nodes within certain distance to c. Say the more his/her friends retweet the information, the more likely he/she retweets the information. Such a process is, in general, an independent jump finite range process (see Definition 2.2). A natural question is whether the phenomenon about such a process is by accident. For N independent random variables, where N is large, the shape of the empirical measure of them is not by accident. Say tossing a fair coin a billion times with 0.50002 portion of positive sides is not an accident, i.e., the next time you toss it a billion times the consequence would be close to 0.50002. However, for a mean field model, synergetic effect does exist, i.e., it is possible that in one trial 90% of nodes receive the information while in another trial only 10% of nodes receive it. Sullivan's classic result [11] says no, the consequence is not by accident if the graph (social network) is sparse in the sense that every node is only connected to a small portion of the other nodes. Actually, Sullivan's result says that for the independent jump finite range process, two nodes' correlation decays away as their

distance is large. Note that the graph of the mean field model is a complete graph which clearly fails under the condition that each node is connected with only a small portion of the other nodes. By calculating the variance of the integration of a given function wrt the empirical distribution, Sullivan's result clearly implies that the distribution of the empirical process of subgraphs with states (see 2.4) *concentrate weakly* as the graph size tends to infinity while the local structure of the graph (see Definition 2.1) converges.

However, a fundamental question still remains in whether the empirical process *converges weakly* provided the local structure of the graph converges and the graph size tends to infinity[1]; or, equivalently whether the limit point is determined by the local structure of the graph sequence. It is the same to ask whether the local structure of a graph determines the empirical process induced by an independent jump finite range process on that graph. In [8] the authors provide a condition of the local graph structure that garantees the weak convergence of the empirical process, namely $\sum_{k=1}^{\infty} \frac{1}{\left(\bar{D}_k^{d,d_0}\right)^2} = \infty$, where $\bar{D}_k^{d,d_0}$ is the maximum number of nodes within a distance less than $d_0 + (k+1)d + 1$ and larger than $d_0 + kd$ to some node (see Section 3 for more details). In this paper we improve the result by showing that the condition $\sum_{k=1}^{\infty} \frac{1}{\bar{D}_k^{d,d_0}\sqrt{k}} = \infty$ suffices. Therefore, for a local graph structure with $\bar{D}_k^{d,d_0}\sqrt{k} \asymp \sqrt{k}\log k$ the condition in [8] fails while the condition in this paper holds. Here, for two sequences of reals $a_n, b_n, n \in \mathbb{N}$, $a_n \asymp b_n$ means there exists $c, C > 0$ such that $cb_n \leq a_n \leq Cb_n$ for all n. Considering the unfamiliarity of the readers to jump process, we only give the result for local coloring process.

Research on graph coloring process also arises in ecology study. [5, 7] study the limit behavior (with respect to time) of various contact processes. The contact process simulating the survival of a species is a 2-colored graph process. The state of each node evolves according to the states of its neighbors. [2, 4, 10] study the limit behaviors of the multitype contact process. The central problem is this: when can two species coexist. However, to our knowledge, there is no research on the relation of convergence of general graph coloring process with graph structure. [8] proved that weak convergence holds if the graph is not so interactive. This improved the known result that on a single chain the weak convergence of finite range (local coloring) holds.

In Section 2, we present definitions such as the local structure of a graph, the independent jump finite range process, local coloring, and the empirical distribution of subgraphs. We also introduce some notations. In Section 3 we briefly point out the application of the result in large deviation theory of probability model induced by the graph. We give the main result in Section 3 and its proof in Section 4. Finally, an application on large deviation theory is given in Section 5.

2. **Preliminaries.**

A centered graph $G = (V, E, x)$ is a directed graph (V, E) with a specified center $x \in V$. (We write G_x to indicate x as its center.) For a directed graph $G = (V, E)$ and a node $i \in V$, let $G_{i,d}$ denote a subgraph consisting of vertices within distance d to i, specifying i as its center. Let S be a set. A stated (or colored) directed graph $G^S = (V, E, f)$ is a graph $G = (V, E)$ together with a *partial* function $f : V \to S$ and for $i \in V$ $f(i)$ represents the state of node i. An isomorphism between two centered directed graph G_x, $\tilde{G}_{x'}$, is an isomorphism between G, $\tilde{G}$ which preserves the center. An isomorphism between two stated directed graph G^S, $\tilde{G}^S$ is an isomorphism between

[1] The distribution of a sequence of random variable $X_n, n \in \mathbb{N}$, concentrate if $\lim_{n\to\infty} Var(X_n) = 0$. However, this does not imply that X_n converges in probability to a constant since their mean may not converge.

G, $\tilde{G}$ which preserves states (colors). For a directed graph, a centered directed graph, and a centered directed graph with state G, G_x, G_x^S, denote by $[G]$, $[G_x]$ and $[G_x^S]$ their isomorphism classes respectively. Denote the set of isomorphism class of centered directed graphs of radius d, centered directed graphs with states of radius d by

$$\begin{aligned}\mathcal{G}_d &= \{[G] : G = (V, E, center = x), (\forall i \in V) d^G(x, i) \leq d\} \\ \mathcal{G}_d^S &= \{[G^S] : G^S = (V, E, f, center = x), (\forall i \in V) d^G(x, i) \leq d\}\end{aligned} \tag{1}$$

respectively. Here, $d^G(x, i)$ denotes the distance between x and i on the graph G. i.e., $\inf\{d : (\exists x_1, \cdots, x_d \in V) x_1 = x \wedge x_d = i \wedge (x_j, x_{j+1}) \in E\}$. As far as we know, the local structure of a graph is proposed by [1].

Definition 2.1 (Local structure of a graph [1])**.** *The* local structure *of a directed graph G refers to the empirical distribution of all subgraphs namely*

$$L_d^G = \frac{1}{|V|} \sum_{i \in V} \delta_{[G_{i,d}]}, \quad d \in \mathbb{N}$$

Here δ denotes the Dirac measure. The pattern *of a stated directed graph G^S refers to the empirical distribution of all stated directed subgraphs*

$$L_d^{G^S} = \frac{1}{|V|} \sum_{i \in V} \delta_{[G_{i,d}^S]}, \quad d \in \mathbb{N}$$

A directed graph sequence G^N, $N \in \mathbb{N}$ (a stated directed graph sequence $G^{N,S}$, $N \in \mathbb{N}$) is said to be convergent locally *if and only if for every $d \in \mathbb{N}$ there exists $L_d^\infty \in \mathcal{P}(\mathcal{G}_d)$ ($L_d^{\infty,S} \in \mathcal{P}(\mathcal{G}_d^S)$) such that*

$$\lim_{N\to\infty} L_d^{G^N} = L_d^\infty \quad \left(\lim_{N\to\infty} L_d^{G^{N,S}} = L_d^{\infty,S}\right)$$

For equivalent definitions, see also [9]. Examples of a convergent sequence of graphs include a sequence of increasingly longer single chains, a sequence of increasingly larger d-dimension grids, or a sequence of increasingly higher binary trees, among others.

Let $G^{S,t}, t \in [0, T]$ be a stochastic process with state space $S^{|V|}$. For a stated directed graph $G^S = (V, E, f)$, $x \in V$, let $G^S(x, s) = (V, E, f_{x,s})$ denote the stated directed graph with identical graph structure as G^S and $f_{x,s}(x) = s$, $f_{x,s}(i) = f(i)$ if $i \neq x$.

Definition 2.2. *An* independent jump finite range process *on G is a Markov process whose generator is of form*

$$\Omega_t f\left(G^S\right) = \sum_{i \in V} \sum_{s \in S} c_t\left(s, [G_{i,d}^S]\right)\left(f\left(G^S(i, s)\right) - f\left(G^S\right)\right)$$

where $d \in \mathbb{N}$ is the "range".

From the definition of independent jump finite range process, it is clear that the probability of two nodes jumping simultaneously is zero. And, the intensity measure of node i, $c_t(\cdot, G_{i,d}^S)$, depends only on a subgraph centered at i with radials d and time t. Examples of finite range process (not necessarily independent jump) include contact process, Ising process, voter model, and exclusion process, etc. [6]. Considering the unfamiliarity of the jump process to readers, we instead study the so called local coloring process.

Definition 2.3. *A* coloring strategy *is a function with input of a stated centered directed graph and time $c_t(\cdot|G_i^S) : t \times G_i^S \mapsto c_t(\cdot|G_i^S)\mathcal{P}(S)$. Denote by $\mathcal{C}$ the set of coloring strategies.*

The coloring process we study is induced by a directed graph G and a coloring strategy c, denoted by $G^{S,c}(t)$, $t \in [0,1]$ as follows. Let $\pi : N \to N$ be a uniformly random permutation of $\{1, 2, \cdots, N\}$. At time $t \in [0,1]$, a random node that has not yet been colored is picked up, namely $\pi([t|V|])$ (here, for a real r $[r]$ denote the largest integer that does not exceed r) and colored randomly according to distribution $c_t\left(\cdot|G^{S,c}_{\pi([t|V|])}(t)\right)$.

Definition 2.4 (Graph coloring process).

$$G^{S,c}(0) = (V, E, f_0), \quad dom(f_0) = \emptyset. \tag{2}$$

For $i = 1, 2, \cdots, |V|$

$$X_{\pi(i)} \sim c_{i/|V|}\left(\cdot|G^{S,c}_{\pi(i)}(i/|V|)\right)$$

$$let\ f_i(\pi(i)) = X_{\pi(i)}, \quad f_i(k) = f_{i-1}(k)\ if\ k \neq \pi(i);$$

The coloring process is $G^{S,c}(t) = (V, E, f_{[t|V|]}), \quad t \in [0,1]$

The empirical process of subgraphs with states (or empirical process of subgraphs for short) refers to the stochastic process $L^{G^{S,c}(t)}_d$, $t \in [0,1]$ *for some* $d \in \mathbb{N}$.

When no ambiguity is made, we write $\pi(t)$ for $\pi([tN])$. A local coloring process is induced by a local coloring strategy.

Definition 2.5 (Local coloring strategy). *A coloring strategy c is a local coloring strategy iff there exists $d \in \mathbb{N}$ such that for all $t \in [0,1]$, any stated centered directed graph G^S_i, it holds that $c_t(\cdot|G^S_i) = c_t(\cdot|G^S_{i,d})$. We call d the range of c. Let $\mathcal{C}_d$ denote the set of coloring strategy with range d.*

3. **Main Result.** Now we can phrase our problem as follows. For a sequence of directed graph G^N let $G^{N,S,c}(t)$, $t \in [0,1]$ denote the graph coloring process as defined in 2.4 induced by coloring strategy c. Does local convergence of G^N imply that for all $c_{d',t} \in \mathcal{C}_{d'}$, $d'' \in \mathbb{N}$, there exist function $L^{\infty,c_{d'}}_{d''}(t) : [0,1] \ni t \mapsto L^{\infty,c_{d'}}_{d''}(t) \in \mathcal{P}\left(\mathcal{G}^S_{d''}\right)$ such that the sequence of empirical process $\left(L^{G^{N,S,c_{d'}}(t)}_{d''},\ t \in [0,1]\right)_{N=1}^{\infty}$ converges in probability to $L^{\infty,c_{d'}}_{d''}(t)$, $t \in [0,1]$[2]? We give a partial positive answer in this section. If the local convergence of G^N implies convergence of $L^{G^{N,S,c_{d'}}(\cdot)}_{d''}$, then it clearly means that the limit point $L^{\infty,c_{d'}}_{d''}$ is determined by the local structure L^∞_d, $d \in \mathbb{N}$. Otherwise, there exist two sequences G^N, $\tilde{G}^N$, $N \in \mathbb{N}$ converging to the same limit L^∞_d, $d \in \mathbb{N}$ while $L^{G^{N,S,c_{d'}}(\cdot)}_{d''}$, $L^{\tilde{G}^{N,S,c_{d'}}(\cdot)}_{d''}$ converge to different limits. However, simply merging the two sequences into one, we obtain a locally convergent directed graph sequence such that the empirical process induced by $c_{d'}$ fails to converge, a contradiction.

Before we introduce the main result we specify the following notations. For a directed graph G, a set of nodes A, let $\partial_k A = \{i \in G : d^G(i, A) \leq k\}$ (that is, the neighbor of A that is of distance less than k to A), and let $\partial_{k_1,k_2} A = \{i \in G : k_1 \leq d^G(i, A) \leq k_2\}$. For a sequence of graph, G^N, let

$$N_k = \max_{N, x \in V^N} \{|\partial_k\{x\}|\} \tag{3}$$
$$D_k = \max_{N, x \in V^N} \{|\partial_k\{c\} - \partial_{k-1}\{x\}|\}$$

[2]Note that the set of sample paths of $L^{G^{N,S,c_{d'}}(t)}_{d''}$, $t \in [0,1]$ is uniformly continuous in t so there is no need to specify the manner in which it converges.

$$D_{k_1,k_2} = \max_{N,x\in V^N}\{|\partial_{k_2}\{c\} - \partial_{k_1}\{x\}|\}$$

$$D_k^{d,d_0} = \max_{N\in\mathbb{N},x\in V^N}\left\{|\partial_{(k+1)d+d_0}\{x\} - \partial_{kd+d_0}\{x\}|\right\}$$

Fast growing N_k, D_k with respect to k clearly means more interactive. Our main result is as follows.

Theorem 3.1. *Let G^N be a sequence of locally convergent uniformly bounded degree directed graph, i.e., for every d' there exists $L_{d'}^\infty$ s.t., $\lim_{N\to\infty} L_{d'}^{G^N} = L_{d'}^\infty$ and $N_1 < \infty$.*

Let $c_{d,\cdot} \in \mathcal{C}_d$, $c \in \mathcal{C}$ be such that,

$$\left(\forall t \in [0,1], G_x^S\right) \left\|c_t\left(|G_x^S\right) - c_{d,t}\left(|G_{x,d}^S\right)\right\|_1 \le \epsilon \tag{4}$$

For every $d' \in \mathbb{N}$, let $L_{d'}^{G^{N,S,c}(t)}$, $L_{d'}^{G^{N,S,c_d}(t)}$, $t \in [0,1]$ be two independent empirical processes induced by c, $c_{d,\cdot}$ respectively.

We have for all d_0, $K \in \mathbb{N}$, any $\varepsilon > 0$, there exists function $f_k(t) : t \mapsto \mathbb{R}^+$, $k \le K$ such that

- $(\forall t \le 1-\varepsilon) \lim_{N\to\infty} \mathbf{Pr}\left[\left\|L_{kd+d_0}^{G^{N,S,c}(t)} - L_{kd+d_0}^{G^{N,S,c_d}(t)}\right\| \le f_k(t)\right] = 1$
- $f_K(t) \equiv 2$. *And for all $k \le K-1$, $t \le 1-\varepsilon$*

$$\frac{df_k(t)}{dt} = \begin{cases} \frac{2}{\varepsilon}D_k^{d,d_0} f_{k+1}(t) + \frac{1}{\varepsilon}\epsilon N_{(k+1)d+d_0} & \text{if } f_k(t) < 2 \\ 0 & \text{if } f_k(t) = 2 \end{cases} \tag{5}$$

$$f_k(0) = 0$$

Intuitively, Theorem 3.1 says that deviation of $L_{kd+d_0}^{G^{N,S,c}(t)}$, $t \in [0,1]$ to $L_{kd+d_0}^{G^{N,S,c_d}(t)}$, $t \in [0,1]$ is governed by $f_k(t)$, $t \in [0,1]$, $k \le K$ which satisfy linear differential Equation (5). Note that $f_k(0) = 0$ is guaranteed by the convergence of local law of G^N. From Theorem 3.1 it is obvious that the following conditions guarantee weak convergence.

Corollary 3.1. *Let G^N be a sequence of convergent directed graph, i.e., for any d' there exists some $L_{d'}^\infty \in \mathcal{P}\left(\mathcal{G}_{d'}^S\right)$*

$$(\forall d' \in \omega) \lim_{N\to\infty} L_{d'}^{G^N} = L_{d'}^\infty$$

Let D_k, N_k, D_k^{d,d_0} be defined as 3 for G^N. Suppose for all $d, d_0 \in \mathbb{N}$, any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ and a group of functions f_k, $k \le K$ satisfying differential Equation (5) with $\epsilon = 0$ such that $(\forall t \le 1-\varepsilon) f_0(t) \le \varepsilon$.

Then we have for all local strategy $c_{d,\cdot} \in \mathcal{C}_d$, the empirical process sequence $L_{d_0}^{G^{N,S,c_d}(\cdot)}$ converges, i.e., there exists $L_{d_0}^{\infty,S,c_d}(t)$, s.t.,

$$\lim_{N\to\infty} L_{d_0}^{G^{N,S,c_d}(\cdot)} \to^{\mathcal{D}} L_{d_0}^{\infty,S,c_d}(\cdot) \tag{6}$$

By analyzing differential Equation (5), we give the following conditions on local graph law to guarantee weak convergence.

Corollary 3.2. *For a convergent graph sequence G^N, suppose $N_k < \infty$ for all $k \in \mathbb{N}$ (see Definition 2.1) and let $\overline{D}_k^{d,d_0} = \max_{j\le k}\left\{D_j^{d,d_0}\right\}$. If for all $d, d_0, M, k_0 \in \mathbb{N}$ there exists $K \in \mathbb{N}$ and a sequence $k_0 = \alpha_1 < \alpha_2 < \cdots < \alpha_m = K$ such that*

$$\sum_{i=1}^{K-1} \frac{1}{\overline{D}_{\alpha_i}^{d,d_0}}\left[1 - \left(\frac{1}{\overline{D}_{\alpha_{i+1}}^{d,d_0}}\right)^{\alpha_{i+1}-\alpha_i} \prod_{\alpha_i < j \le \alpha_{i+1}} \overline{D}_j^{d,d_0}\right] \ge M \tag{7}$$

And suppose for all $i \in \mathbb{N}$, $\varepsilon > 0$ *there exists* j *with* $\frac{\overline{D}_i^{d,d_0}}{\overline{D}_j^{d,d_0}} < \varepsilon$.

Then weak convergence holds for the local structure $\lim_{N\to\infty} L_d^{G^N}$, $d \in \mathbb{N}$.

Proof: To derive weak convergence for a local strategy, we let $c = c_{d,\cdot}$; thus $\epsilon = 0$. Suppose g_k, $k \le K$ satisfy Equation (5) with D_k^{d,d_0} replaced by $\overline{D}_k^{d,d_0}$. Clearly, by induction $f_k(t) \le g_k(t)$ for all $t \in [0,1]$, $k \le K$. We will prove that $(\forall t \in [0,1]) g_{k_0}(t) < 2$ and $g_0(t) < \frac{\overline{D}_0^{d,d_0}}{\overline{D}_{k_0}^{d,d_0}} g_{k_0}(t) < 2\frac{\overline{D}_0^{d,d_0}}{\overline{D}_{k_0}^{d,d_0}}$. Since k_0 is arbitrary, we can choose k_0 such that $\frac{\overline{D}_0^{d,d_0}}{\overline{D}_{k_0}^{d,d_0}} < \varepsilon$ for any $\varepsilon > 0$. Therefore, we can show that for any $\varepsilon > 0$ there exists K and a group of functions g_k, $k \le K$ such that g_k, $k \le K$ satisfy Equation (5) and $(\forall t \in [0,1]) g_0(t) < \varepsilon$. Thus, the condition of Corollary 3.1 is verified. It remains to prove the assertion $(\forall t \in [0,1]) g_{k_0}(t) < 2$ and $g_0(t) < \frac{\overline{D}_0^{d,d_0}}{\overline{D}_{k_0}^{d,d_0}} g_{k_0}(t) < 2\frac{\overline{D}_0^{d,d_0}}{\overline{D}_{k_0}^{d,d_0}}$.

Denote by $\tau_k = \inf\{t : g_k(t) = 2\}$. Note that for $t \le \tau_K$, $g_{K-1}(t) \le g_K(t)$. Since for $t \le \tau_K$, $\frac{g'_{k+1}(t)}{g'_k(t)} = \frac{\overline{D}_{k+1}^{d,d_0} g_{k+2}(t)}{\overline{D}_k^{d,d_0} g_{k+1}(t)}$, we can show by induction that $(\forall t \le \tau_K,\ k \le K-2)$, $\frac{g_k(t)}{g_{k+1}(t)} \le \frac{\overline{D}_k^{d,d_0}}{\overline{D}_{K-1}^{d,d_0}}$. Thus, for all $t \le \tau_K$, $\frac{g_k(t)}{g_{K-1}(t)} \le \left(\frac{1}{\overline{D}_{K-1}^{d,d_0}}\right)^{K-k} \prod_{j=k}^{K-1} \overline{D}_j^{d,d_0}$. So $g_k(\tau_K) \le 2\left(\frac{1}{\overline{D}_{K-1}^{d,d_0}}\right)^{K-k} \prod_{j=k}^{K-1} \overline{D}_j^{d,d_0}$. Since $g'_k(t) \le 2\overline{D}_k^{d,d_0}$ for all k, t, we can conclude that $\tau_k \ge \frac{2}{\overline{D}_k^{d,d_0}}\left(1 - \left(\frac{1}{\overline{D}_{K-1}^{d,d_0}}\right)^{K-k} \prod_{j=k}^{K-1} \overline{D}_j^{d,d_0}\right)$. In another words, the existence of a sequence α_j satisfying condition (7) implies that $(\forall t \in [0,1]) g_{k_0}(t) < 2$. Thus, the proof is accomplished.

Using Corollary 3.2, a more simple form of conditions can be derived as follows.

Corollary 3.3. *Let* $\overline{D}_k^{d,d_0}$ *be defined as Corollary 3.2. Suppose for all* d, $d_0 \in \mathbb{N}$, *there exists* r *such that for all function* $c(k) : k \to \mathbb{N}$ *with* $c(k) \le \sqrt{k}$ *we have,*

- $\lim_{k\to\infty} \frac{\overline{D}_k^{d,d_0}}{\overline{D}_{k+c(k)}^{d,d_0}} \le 1 - r\frac{c(k)}{k}$;
- $\sum_{k=0}^{\infty} \frac{1}{\overline{D}_k^{d,d_0}\sqrt{k}} = \infty$

Then, conditions of Corollary 3.2 hold.

Proof: For any $M > 0$, $k_0 \in \mathbb{N}$, we will construct a sequence α_i, $i \le m$, $K \in \mathbb{N}$ such that inequality (7) holds. Without loss of generality assume for all $k \ge k_0$, $c \le \sqrt{k}$, $\frac{\overline{D}_k^{d,d_0}}{\overline{D}_{k+c}^{d,d_0}} \le 1 - r\frac{c}{k}$. For some M' sufficiently large, suppose $\sum_{k=k_0}^{K'} \frac{1}{\overline{D}_k^{d,d_0}\sqrt{k}} = M'$. By condition on $\overline{D}_k^{d,d_0}$, $k \in \mathbb{N}$ such K' exists. Define α_i inductively as the following: $\alpha_1 = k_0$, $\alpha_{k+1} = \alpha_k + \sqrt{\alpha_k}$; $\alpha_{m-1} \le K' \le \alpha_m$. Let $K = \alpha_m$. By condition on $\overline{D}_k^{d,d_0}$, $k \in \mathbb{N}$, there exists $R > 0$ such that for all $i \le m-1$,

$$1 - \left(\frac{1}{\overline{D}_{\alpha_{i+1}}^{d,d_0}}\right)^{\alpha_{i+1}-\alpha_i} \prod_{\alpha_i < j \le \alpha_{i+1}} \overline{D}_j^{d,d_0} \ge 1 - \prod_{0 \le j \le \alpha_{i+1}-\alpha_i-1} \left(1 - r\frac{j}{\alpha_{i+1}}\right) \ge R \qquad (8)$$

Furthermore, R clearly does not depend on K. Therefore, the left-hand side of inequality (7) becomes,

$$\sum_{i=1}^{K-1} \frac{1}{\overline{D}_{\alpha_i}^{d,d_0}} \left[1 - \left(\frac{1}{\overline{D}_{\alpha_{i+1}}^{d,d_0}} \right)^{\alpha_{i+1}-\alpha_i} \prod_{\alpha_i < j \le \alpha_{i+1}} \overline{D}_j^{d,d_0} \right] \ge \sum_{i=1}^{m-1} \frac{R}{\overline{D}_{\alpha_i}^{d,d_0}} \tag{9}$$

However, it is easy to see that there exists R' depending only on k_0 such that

$$\sum_{i=1}^{m-1} \frac{R}{\overline{D}_{\alpha_i}^{d,d_0}} \ge \sum_{k=\alpha_1}^{\alpha_m} \frac{RR'}{\overline{D}_k^{d,d_0}\sqrt{k}} \ge RR'M' \tag{10}$$

Since $R, R' > 0$ does not depend on K and thus, does not depend on M'. Therefore, we can choose M' sufficiently large such that $RR'M' > M$. So the proof is completed.

4. **Proof of Theorem 3.1.** The proof is reduced to the following lemma.

Lemma 4.1. *Assume G^N satisfies condition of Lemma 3.1. Let $c_{d,\cdot} \in \mathcal{C}_d$, $c \in \mathcal{C}$ be such that,*

$$\left(\forall t, G_x^S\right) ||c_t\left(|G_x^S\right) - c_{d,t}\left(|G_{x,d}^S\right)||_1 \le \epsilon$$

Then we have for all $\varepsilon > 0$ there exists $\delta' > 0$ such that for any $\delta < \delta'$, any $t \in [0,1]$, $t + \delta < 1 - \varepsilon$ any $d'' \ge d$,

$$\lim_{N\to\infty} \left\| L_{d''}^{G^{N,S,c}(t+\delta)} - L_{d''}^{G^{N,S,c_d}(t+\delta)} \right\|_1 \tag{11}$$

$$\le^{\mathcal{D}} \left\| L_{d''}^{G^{N,S,c}(t)} - L_{d''}^{G^{N,S,c_d}(t)} \right\|_1 + \delta\frac{1}{\varepsilon}\epsilon N_{d''+d} + \delta\frac{2}{\varepsilon}D_{d''+d,d''} \left\| L_{d''+d}^{G^{N,S,c}(t)} - L_{d''+d}^{G^{N,S,c_d}(t)} \right\|_1 + \frac{N_{d''}^2}{\varepsilon^2}\delta^2$$

Here for two sequence of random variable X^N, Y^N we write $\lim NX^N \le^{\mathcal{D}} Y^N$ for $\lim N\mathbf{Pr}[X^N \le Y^N] = 1$. Similarly for $=^{\mathcal{D}}$, $\ge^{\mathcal{D}}$. It is obvious that Theorem 4.1 follows from Lemma 4.1. By induction from K to 1. For K, it is clear $(\forall t) f_K(t) \equiv 2 \ge \left\| L_{Kd+d_0}^{G^{N,S,c}(t)} - L_{Kd+d_0}^{G^{N,S,c_d}(t)} \right\|_1$. For assuming Theorem 3.1 holds for $f_K, \cdots, f_k$, then for $k-1$ simply note that in Lemma 4.1 δ can be arbitrary small; thus the δ^2 term can be ignored.

4.1. **Proof of Lemma 4.1.** The proof is not hard but tedious. Within time segment $[t, t+\delta)$, for every d_2, the coloring strategy c, c_d, each induce a transition probability on space $\mathcal{G}_{d_2}^S$. To define such transition probability, for $t < t'$, G_{x,d_1}^S, $G_{x,d_2}'^S$, $d_1 \ge d_2$ let,

$$L^{G^N,c,[t,t+\delta)}\left(G_{x,d_1}^S, G_{x,d_2}'^S\right) = \frac{\left|\left\{ i \in V^N : G^{N,S,c}(t)_{i,d_1} \cong G_{x,d_1}^S \wedge G_{i,d_2}^{N,S,c,t+\delta} \cong G_{x,d_2}'^S \right\}\right|}{|V^N|} \tag{12}$$

and let,

$$L^{G^N,c,[t,t+\delta)}\left(G_{x,d_2}'^S | G_{x,d_1}^S\right) = \frac{L^{G^N,c,[t,t+\delta)}\left(G_{x,d_1}^S, G_{x,d_2}'^S\right)}{L_{d_1}^{G^{N,S,c}(t)}\left(G_{x,d_1}^S\right)} \tag{13}$$

The propagation of deviation between $L_{d''}^{G^{N,S,x,t+\delta}}$ and $L_{d''}^{G^{N,S,c_d}(t+\delta)}$ is dissolved into

$$\left\| L_{d''+d}^{G^{N,S,c}(t)} - L_{d''+d}^{G^{N,S,c_d}(t)} \right\|$$

and difference between $L^{G^N,c,[t,t+\delta)}\left(G'^S_{x,d''}|G^S_{x,d''}\right)$ and $L^{G^N,c_d,[t,t+\delta)}\left(G'^S_{x,d''}|G^S_{x,d''}\right)$. More specifically,

$$\begin{aligned}
&\left\|L_{d''}^{G^{N,S,c}(t+\delta)} - L_{d''}^{G^{N,S,c_d}(t+\delta)}\right\|_1 \qquad (14)\\
=& \int_{G'^S_{x,d''}\in\mathcal{G}^S_{d''}} \left| \int_{G^S_{x,d''}\in\mathcal{G}^S_{d''}} L_{d''}^{G^{N,S,c}(t)}\left(dG^S_{x,d''}\right) L^{G^N,c,[t,t+\delta)}\left(dG'^S_{x,d''}|G^S_{x,d''}\right)\right.\\
&\left. - L_{d''}^{G^{N,S,c_d}(t)}\left(dG^S_{x,d''}\right) L^{G^N,c_d,[t,t+\delta)}\left(dG'^S_{x,d''}|G^S_{x,d''}\right)\right|\\
\le& \left\|L_{d''}^{G^{N,S,c}(t)} - L_{d''}^{G^{N,S,c_d}(t)}\right\|_1\\
&+\int_{G^S_{x,d''}\in\mathcal{G}^S_{d''}} L_{d''}^{G^{N,S,c}(t)}\left(G^S_{x,d''}\right)\left\|L_{d''}^{G^N,c,[t,t+\delta)}\left(\cdot|G^S_{x,d''}\right) - L_{d''}^{G^N,c_d,[t,t+\delta)}\left(\cdot|G^S_{x,d''}\right)\right\|_1
\end{aligned}$$

It remains to look at the second term of the right hand side of (14) which is the difference between the transition probability induced by c, $c_{d,\cdot}$.

Definition 4.1. *For $i\in V^N$, let $path(i,d',[t,t'))$ denote the vertex sequence within distance d' to i, chosen by π during $[t,t')$ with time point recorded, i.e.,*

$$path(i,d',[t,t')) = (n_1,\tau_1)\times(n_2,\tau_2)\times\cdots\times(n_l,\tau_l) \qquad (15)$$

where $\{n_1,\cdots,n_l\} = \{j\in\partial_{d'}\{i\} : \exists\tau\in[t,t'),\pi(\tau)=j\}$, $t\le\tau_1<\tau_2<\cdots<\tau_l<t'$, $\tau_j=\inf\{\tau:\pi(\tau)=n_j\}$. We write $PATH(G^S_{x,d'})$ for the path space of $G^S_{x,d'}$, $PATH(\mathcal{G}^S_{d'})$ for union of path space of $G^S_{x,d'}\in\mathcal{G}^S_{d'}$.

When no ambiguity is made, we always omit $G^S_{x,d'}$ and write $PATH$ instead. We write $\theta(G^S_{x,d'}) = G'^S_{x',d'}$ to denote θ is an isomorphism between $G^S_{x,d'}$, $G'^S_{x',d'}$. Note that an isomorphism between two graphs, also induces an isomorphism between path space between two graphs, i.e., $\theta((n_1,\tau_1)\times\cdots\times(n_l,\tau_l)) = (\theta(n_1),\tau_1)\times\cdots\times(\theta(n_l),\tau_l)$.

The coloring process on G^N within time $[t,t')$ induces nature random measure on $\mathcal{G}^S_{d'}\times PATH(\mathcal{G}^S_{d'})\times\mathcal{G}^S_{d''}$, with $d''\le d'$.

Definition 4.2. *For, any $Pa\subseteq PATH$, $G^S_{x,d'}, G^S_{x,d''}$,*

$$\begin{aligned}
&L^{G^N,c,[t,t')}\left(G^S_{x,d'},Pa,G^S_{x,d''}\right) \qquad (16)\\
=&\frac{|\{i\in V^N : \exists\theta,\theta\left(G^S_{x,d'}\right)=G^{N,S,c}_{i,d'}(t)\wedge\theta^{-1}\left(path(i,d',[t,t'))\right)\in Pa\wedge G^{N,S,c}_{i,d''}(t)\cong G^S_{x,d''}\}|}{|V^N|}
\end{aligned}$$

Similarly,

$$\begin{aligned}
&L^{G^N,c,[t,t')}\left(G^S_{x,d'},Pa\right) \qquad (17)\\
=&\frac{\{i\in V^N : \exists\theta,\theta(G^S_{x,d'})=G^{N,S,c}_{i,d'}(t)\wedge\theta^{-1}(path(i,d',[t,t')))\in Pa\}}{|V^N|}
\end{aligned}$$

Note that, since π is independent from $G^{N,S,c}(t)$, for any $path$, any coloring strategy c', any d', d'', let

$$L^{G^N,c',[t,t+\delta)}\left(path|G^S_{x,d''+d'}\right) = \frac{L^{G^N,c',[t,t+\delta)}\left(G^S_{x,d''+d'},path\right)}{L^{G^N,c',t}\left(G^S_{x,d''+d'}\right)} \qquad (18)$$

we have,

$$L^{G^N,c,[t,t+\delta)}\left(path|G^S_{x,d''+d}\right) =^{\mathcal{D}} L^{G^N,c_d,[t,t+\delta)}\left(path|G_{x,d''+d}\right) \qquad (19)$$

i.e., $L^{G^N,c,[t,t+\delta)}\left(path|G^S_{x,d''+d}\right)$ depends only on graph structure of $G^S_{x,d''+d}$ and $path$ and is independent of c, state structure of $G^S_{x,d''+d}$.

Now fix $G^S_{x,d''}$ and let us look at $L^{G^N,c,[t,t+\delta)}_{d''}\left(\cdot|G^S_{x,d''}\right) - L^{G^N,c_d,[t,t+\delta)}_{d''}\left(\cdot|G^S_{x,d''}\right)$.

$$\begin{aligned}
&\int_{G'^S_{x,d''}\in\mathcal{G}^S_{d''}}\left|L^{G^N,c,[t,t+\delta)}_{d''}\left(dG'^S_{x,d''}|G^S_{x,d''}\right) - L^{G^N,c_d,[t,t+\delta)}_{d''}\left(dG'^S_{x,d''}|G^S_{x,d''}\right)\right| \qquad (20)\\
&=\int_{G'^S_{x,d''}\in\mathcal{G}^S_{d''},|Pa|\le 1}\left|L^{G^N,c,[t,t+\delta)}_{d''}\left(dG'^S_{x,d''},Pa|G^S_{x,d''}\right) - L^{G^N,c_d,[t,t+\delta)}_{d''}\left(dG'^S_{x,d''},Pa|G^S_{x,d''}\right)\right|\\
&\quad+\int_{G'^S_{x,d''}\in\mathcal{G}^S_{d''},|Pa|>1}\left|L^{G^N,c,[t,t+\delta)}_{d''}\left(dG'^S_{x,d''},Pa|G^S_{x,d''}\right) - L^{G^N,c_d,[t,t+\delta)}_{d''}\left(dG'^S_{x,d''},Pa|G^S_{x,d''}\right)\right|\\
&=I+II
\end{aligned}$$

Since π is independent from $G^{N,S,c}(t), G^{N,S,c_d}(t)$, every vertex in $G^S_{x,d''}$ has probability $\delta/(1-t-\delta)$ to be chosen mutually indepenently by π during $[t,t+\delta)$. Therefore, as $N\to\infty$

$$II \le^{\mathcal{D}} 2\int_{|Pa|\ge 2} L^{G^N,c,[t,t+\delta)}_{d''}\left(dPa|G^S_{x,d''}\right) \le^{\mathcal{D}} \frac{N^2_{d''}}{(1-t-\delta)^2}\delta^2 \le \frac{N^2_{d''}}{\varepsilon^2}\delta^2 \qquad (21)$$

For the first term I, we further dissolve the path space into two disjoint subsets,

$$\{path\in PATH(G^S_{x,d''}) : |path| = 1,\ path\subseteq G_{x,d''-d}\},$$

$$\{path\in PATH(G^S_{x,d''}) : |path| = 1,\ path\cap G_{x,d''-d} = \emptyset\}$$

and calculate the integration (20) respectively.

$$\begin{aligned}
I =& \int_{Pa=\emptyset,G'^S_{x,d''}\in\mathcal{G}^S_{d''}}\left|L^{G^N,c,[t,t+\delta)}_{d''}\left(dG'^S_{x,d''}|G^S_{x,d''},Pa\right)\right. \qquad (22)\\
&\left.- L^{G^N,c_d,[t,t+\delta)}_{d''}\left(dG'^S_{x,d''}|G^S_{x,d''},Pa\right)\right|\cdot L^{G^N,c,[t,t+\delta)}_{d''}\left(Pa|G^S_{x,d''}\right)\\
&+\int_{|Pa|=1\wedge Pa\subseteq G^S_{x,d''-d},G'^S_{x,d''}\in\mathcal{G}^S_{d''}}\left|L^{G^N,c,[t,t+\delta)}_{d''}\left(dG'^S_{x,d''}|G^S_{x,d''},Pa\right)\right.\\
&\left.- L^{G^N,c_d,[t,t+\delta)}_{d''}\left(dG'^S_{x,d''}|G^S_{x,d''},Pa\right)\right|\cdot L^{G^N,c,[t,t+\delta)}_{d''}\left(Pa|G^S_{x,d''}\right)\\
&+\int_{|Pa|=1\wedge Pa\cap G^S_{x,d''-d}=\emptyset,G'^S_{x,d''}\in\mathcal{G}^S_{d''}}\left|L^{G^N,c,[t,t+\delta)}_{d''}\left(dG'^S_{x,d''}|G^S_{x,d''},Pa\right)\right.\\
&\left.- L^{G^N,c_d,[t,t+\delta)}_{d''}\left(dG'^S_{x,d''}|G^S_{x,d''},Pa\right)\right|\cdot L^{G^N,c,[t,t+\delta)}_{d''}\left(Pa|G^S_{x,d''}\right)\\
=& I_1+I_2+I_3
\end{aligned}$$

Clearly

$$I_1 =^{\mathcal{D}} 0$$

since $Pa=\emptyset$ implies

$$L^{G^N,c,[t,t+\delta)}_{d''}\left(dG'^S_{x,d''}\middle|G^S_{x,d''},Pa\right) =^{\mathcal{D}} L^{G^N,c_d,[t,t+\delta)}_{d''}\left(dG'^S_{x,d''}\middle|G^S_{x,d''},Pa\right)$$

As to I_2, clearly we have as $N \to \infty$,

$$I_2 \le^{\mathcal{D}} \sum_{x \in G^S_{x,d''-d}} \sup_{s \in [t,t+\delta)} \left\{ \mathbf{Pr}[\exists t'' \in [t,t+\delta)\pi(t'') = x] \cdot ||c_s(\cdot|G^S_{x,d},x) - c_{d,s}(\cdot|G^S_{x,d})||_1 \right\} \quad (23)$$

Note that since $x \in G_{x,d''-d}$, $G_{x,d} \subseteq G^S_{x,d''}$. Therefore, $||c_s(\cdot|G^S_{x,d},x) - c_{d,s}(\cdot|G^S_{x,d})||_1 \le \epsilon$. Thus,

$$I_2 \le^{\mathcal{D}} \delta \frac{1}{\varepsilon} \epsilon N_{d''-d} \quad (24)$$

It remains to analyze I_3. Note that the reason $\left\| c_s\left(\cdot|G^S_{x,d},x\right) - c_{d,s}\left(\cdot|G^S_{x,d}\right)\right\|_1 \le \epsilon$ does not hold for $x \in G_{x,d''} - G_{x,d''-d}$ is due to unknown information of $G^S_{x,d''+d}$. We write $L^{c,t}_{d''}$, $L^{c_d,t}_{d''}$ for $L^{G^{N,S,c}(t)}_{d''}$, $L^{G^{N,S,c_d}(t)}_{d''}$. For $G^S_{x,d''} \subseteq G^S_{x,d''+d}$, let

$$L^{c,t}_{d''+d}\left(G^S_{x,d''+d}|G^S_{x,d''}\right) = \frac{L^{c,t}_{d''+d}\left(G^S_{x,d''+d}\right)}{L^{c,t}_{d''}\left(G^S_{x,d''}\right)} \quad (25)$$

We have as $N \to \infty$,

$$\begin{aligned}
I_3 \le^{\mathcal{D}} & \sum_{x \in G^S_{x,d''-d}} \sup_{s \in [t,t+\delta)} \{\mathbf{Pr}[\exists t'' \in [t,t+\delta)\pi(t'') = x]\} \quad (26)\\
& \cdot \int_{G^S_{x,d''} \subseteq G^S_{x,d''+d}} \left| L^{c,t}_{d''+d}\left(dG^S_{x,d''+d}\right) c_t\left(\cdot|G^S_x\right) - L^{c_d,t}_{d''+d}\left(dG^S_{x,d''+d}\right) c_{d,t}\left(\cdot|G^S_{x,d}\right) \right| \\
\le & \sum_{x \in G^S_{x,d''-d}} \sup_{s \in [t,t+\delta)} \{\mathbf{Pr}[\exists t'' \in [t,t+\delta)\pi(t'') = x] \cdot \left\| c_{d,s}\left(\cdot|G^S_{x,d}\right) - c_s\left(\cdot|G^S_{x,d}\right)\right\|_1 \\
& + \sum_{x \in G^S_{x,d''-d}} \mathbf{Pr}[\exists t'' \in [t,t+\delta)\pi(t'') = x] \\
& \cdot \int_{G^S_{x,d''} \subseteq G^S_{x,d''+d}} \left| L^{c,t}_{d''+d}\left(dG^S_{x,d''+d}|G^S_{x,d''}\right) - L^{c_d,t}_{d''+d}\left(dG^S_{x,d''+d}|G^S_{x,d''}\right) \right| \\
\le & \delta \frac{1}{\varepsilon} \epsilon D_{d''+d,d''} + \delta \frac{1}{\varepsilon} D_{d''+d,d''} \\
& \cdot \int_{G^S_{x,d''} \subseteq G^S_{x,d''+d}} \left| L^{c,t}_{d''+d}\left(dG^S_{x,d''+d}|G^S_{x,d''}\right) - L^{c_d,t}_{d''+d}\left(dG^S_{x,d''+d}|G^S_{x,d''}\right) \right|
\end{aligned}$$

Now noting the relation between (14) and (20), and using estimation of I, II, I_1, I_2, I_3, we have,

$$\begin{aligned}
& \left\| L^{G^{N,S,c}(t+\delta)}_{d''} - L^{G^{N,S,c_d}(t+\delta)}_{d''} \right\|_1 \quad (27)\\
\le^{\mathcal{D}} & \left\| L^{G^{N,S,c}(t)}_{d''} - L^{G^{N,S,c_d}(t)}_{d''} \right\|_1 + \frac{N^2_{d''}}{\varepsilon^2}\delta^2 + \delta\frac{1}{\varepsilon}\epsilon N_{d''-d} + \delta\frac{1}{\varepsilon} D_{d''+d,d''} \\
& \cdot \int_{G^S_{x,d''} \in \mathcal{G}^S_{d''}} \left[\int_{G^S_{x,d''} \subseteq G^S_{x,d''+d}} \left| L^{c,t}_{d''+d}\left(dG^S_{x,d''+d}|G^S_{x,d''}\right) \right.\right. \\
& \left.\left. - L^{c_d,t}_{d''+d}\left(dG^S_{x,d''+d}|G^S_{x,d''}\right) \right| \right] L^{c,t}_{d''+d}\left(dG^S_{x,d''}\right) \\
\le & \left\| L^{G^{N,S,c}(t)}_{d''} - L^{G^{N,S,c_d}(t)}_{d''} \right\|_1 + \frac{N^2_{d''}}{\varepsilon^2}\delta^2 + \delta\frac{1}{\varepsilon}\epsilon N_{d''+d} + \delta\frac{1}{\varepsilon} D_{d''+d,d''} \cdot III
\end{aligned}$$

Now it remains to analyze

$$III = \int_{G^S_{x,d''}\in\mathcal{G}^S_{d''}} \left[\int_{G^S_{x,d''}\subseteq G^S_{x,d''+d}} \left|L^{c,t}_{d''+d}\left(dG^S_{x,d''+d}|G^S_{x,d''}\right) - L^{c_d,t}_{d''+d}\left(dG^S_{x,d''+d}|G^S_{x,d''}\right)\right|\right] L^{c,t}_{d''+d}\left(dG^S_{x,d''}\right).$$

Note that,

$$\begin{aligned}
&\int_{G^S_{x,d''+d}\in\mathcal{G}^S_{d''+d}} \left|L^{c,t}_{d''+d}\left(dG^S_{x,d''+d}\right) - L^{c_d,t}_{d''+d}\left(dG^S_{x,d''+d}\right)\right| \qquad (28)\\
&= \int_{G^S_{x,d''}\in\mathcal{G}^S_{d''}} \int_{G^S_{x,d''}\subseteq G^S_{x,d''+d}} \left|L^{c,t}_{d''+d}\left(dG^S_{x,d''+d}|G^S_{x,d''}\right) L^{c,t}_{d''}\left(dG^S_{x,d''}\right)\right.\\
&\quad \left. - L^{c_d,t}_{d''+d}\left(dG^S_{x,d''+d}|G^S_{x,d''}\right) L^{c_d,t}_{d''}\left(dG^S_{x,d''}\right)\right|\\
&= \int_{G^S_{x,d''}\in\mathcal{G}^S_{d''}} \int_{G^S_{x,d''}\subseteq G^S_{x,d''+d}} \left|\left[L^{c,t}_{d''+d}\left(dG^S_{x,d''+d}|G^S_{x,d''}\right) - L^{c_d,t}_{d''+d}\left(dG^S_{x,d''+d}|G^S_{x,d''}\right)\right] L^{c,t}_{d''}\left(dG^S_{x,d''}\right)\right.\\
&\quad \left. - L^{c_d,t}_{d''+d}\left(dG^S_{x,d''+d}|G^S_{x,d''}\right)\left[L^{c_d,t}_{d''}\left(dG^S_{x,d''}\right) - L^{c,t}_{d''}\left(dG^S_{x,d''}\right)\right]\right|
\end{aligned}$$

Therefore,

$$\begin{aligned}
III &\le \int_{G^S_{x,d''}\in\mathcal{G}^S_{d''}} \int_{G^S_{x,d''}\subseteq G^S_{x,d''+d}} L^{c_d,t}_{d''+d}\left(dG^S_{x,d''+d}|G^S_{x,d''}\right)\left|L^{c_d,t}_{d''}\left(dG^S_{x,d''}\right)\right. \qquad (29)\\
&\quad \left. - L^{c,t}_{d''}\left(dG^S_{x,d''}\right)\right| + \left\|L^{c,t}_{d''+d} - L^{c_d,t}_{d''+d}\right\|_1\\
&= \left\|L^{c,t}_{d''} - L^{c_d,t}_{d''}\right\|_1 + \left\|L^{c,t}_{d''+d} - L^{c_d,t}_{d''+d}\right\|_1\\
&\le 2\left\|L^{c,t}_{d''+d} - L^{c_d,t}_{d''+d}\right\|_1
\end{aligned}$$

Combining with Equation (27), the proof is finished.

5. **Application on Large Deviation Theory.**

The type of weak convergence result 3.1 is needed for the large deviation theory of a generalized version of Gibbs random field. The model is usually written as,

$$X_1, \cdots, X_N | G^N \sim \frac{\exp\left\{NF\left(L^{G^{N,S}}_d\right)\right\}\prod_{i\le N}\rho\left(dX_i\right)}{\int_{S^N}\exp\left\{NF\left(L^{G^{N,S}}_d\right)\right\}\prod_{i\le N}\rho\left(dx_i\right)} \qquad (30)$$

where $X_1, \cdots, X_N|G^N$ is the joint distribution of $X_1, \cdots, X_N$ given graph G^N. And $F \in \mathbf{C}_b\left(\mathcal{P}\left(\mathcal{G}^S_d\right), \mathbb{R}\right)$.

Example 5.1 (Ising model)**.** *The state space is $S = \{-1, 1\}$. Fix $\rho \in \mathcal{P}(S)$ to be $\rho(\{-1\}) = \rho(\{1\}) = 1/2$. For a graph of radius 1, $G^S_1 = (V, E, f, center = v) \in \mathcal{G}^S_1$, let $q\left(G^S_{x,1}\right) = |\{u \in V : (x,u) \in E, f(x) \neq f(u)\}|$. For $L^S_1 \in \mathcal{P}\left(\mathcal{G}^S_1\right)$, let $F_\beta\left(L^S_1\right) = \beta\int_{\mathcal{G}^S_1} q\left(G^S_{x,1}\right) L^S_1\left(dG^S_1\right)$. Ising model on graph G is,*

$$X_1, \cdots, X_N | G \sim \frac{1}{Z}\exp\left\{NF_\beta\left(L^{G^S}_{x,1}\right)\right\} \qquad (31)$$

In order to establish the large deviation principle (LDP) of model (30), by the framework in [3], the rate function in LDP (if it exists) for $L_d^{G^{N,S}}$ (empirical distribution induced by model (30)) is related to a control problem's objective function. The control problem seeks to minimize the following objective by choosing optimal coloring strategy c,

$$J\left(c; F, D_{KL}, G^N\right) = \mathrm{E}\left[\frac{1}{N}\sum_{i=1}^{N} D_{KL}\left(c_{i/N}\left(\cdot|G^{N,S,c}_{\pi(i/N)}(i/N)\right)\|\rho\right) + F\left(L_d^{G^{N,S,c}(1)}\right)\right] \quad (32)$$

i.e., the performance of c depends on function F, cost function (which is D_{KL}, Kullback-Leibler divergence, in the control problem derived from LDP), and graph G^N. Then the rate function is,

$$I\left(L_d^S\right) = \liminf_{N\to\infty} \inf_{c:\lim\limits_{N\to\infty} L_d^{G^{N,S,c}(1)}=L_d^S} \left\{\mathrm{E}\left[\int_0^1 D_{KL}\left(c_t\left(\cdot|G^{N,S,c}_{\pi(t)}(t)\right)\Big\|\rho\right)dt\right] + F(L_d^S)\right\} \quad (33)$$

Provided the strategy c with $\lim\limits_{N\to\infty} L_d^{G^{N,S,c}(1)} = L_d^S$ in probability exists and the sequence $\mathrm{E}\left[\int_0^1 D_{KL}\left(c_t(\cdot|G^{N,S,c}_{\pi(t)}(t))||\rho\right)dt\right]$ converges.

It is known that the optimal strategy is actually the conditional probability, i.e., the optimal strategy colors a vertex $\pi(i)$ at time i/N by the conditional (on the current state of the whole graph $G^{N,S,c}(i/N)$) probability of $X_{\pi(i)}$. However, it is reasonable to conjecture that local strategy (recall Definition 2.5) can approximate to an arbitrary degree of the performance of the optimal strategy. Roughly speaking, if a strategy c is "very close" to a local strategy, then obviously both limits $\lim\limits_{N\to\infty} L_d^{G^{N,S,c}(1)} = L_d^S$, $\lim\limits_{N\to\infty} \mathrm{E}\left[\int_0^1 D_{KL}\left(c_t(\cdot|G^{N,S,c}_{\pi(t)}(t))||\rho\right)dt\right]$ exist by the weak convergence result of local coloring processes (finite range processes).

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