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# INTERVAL TYPE-2 FUZZY INFORMATION AGGREGATION BASED ON EINSTEIN OPERATIONS AND ITS APPLICATION TO DECISION MAKING

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Abstract. *The aim of this paper is to investigate information aggregation methods under trapezoidal interval type-2 fuzzy environment. Some Einstein operational laws on trapezoidal interval type-2 fuzzy numbers are defined based on Einstein sum and Einstein product. Then, we present some trapezoidal interval type-2 fuzzy aggregation operators based on the Einstein operations: the trapezoidal interval type-2 fuzzy Einstein weighted averaging (TIT2FEWA) operator and the trapezoidal interval type-2 fuzzy Einstein weighted geometric (TIT2FEWG) operator. Based on the TIT2FEWA operator, TIT2FEWG operator and the fuzzy mean possibility degree, a new method of multi-attribute decision making with trapezoidal interval type-2 fuzzy information is proposed. Finally, an illustrative example is given to verify the developed approaches and to demonstrate its practicality and effectiveness.*
**Keywords:** Fuzzy multi-attributes decision making, Interval type-2 fuzzy sets, Einstein operator, Fuzzy mean possibility degree

1. **Introduction.** The concept of type-2 fuzzy sets (T2 FSs), initially introduced by Zadeh [1], can be regarded as an extension of the concept of type-1 fuzzy sets (T1 FSs). The main difference between the two kinds of fuzzy sets is that the memberships of T1 FSs are crisp numbers whereas the memberships of T2 FSs are T1 FSs [2]; hence, T2 FSs involve more uncertainties than T1 FSs. Since its introduction, T2 FSs are receiving more and more attention. Because the computational complexity of using general T2 FSs is very high, to date, interval type-2 fuzzy sets (IT2 FSs) [3] are the most widely used T2 FSs and have been successfully applied to many practical fields [4-7].

In recent years, some authors have applied IT2 FSs theory to the field of fuzzy multi-attributes decision making (FMADM). For example, Chen and Lee [8,9] presented TOPSIS method and ranking values method to handle FMADM problems based on IT2 FSs. Wang and Liu [19] developed an approach to handling the situations where the attribute values are characterized by IT2 FSs, and the information about attribute weights is partially known. Chen [10-14] presented some methods: ELECTRE method, LINMAP method, PROMETHEE method, Likelihoods method and TOPSIS method, to handle FMADM problems under IT2 fuzzy environment. Qin and Liu [15] presented an interval type-2 fuzzy decision making approach based on the combined ranking value. Gong et al. [16] presented a method based on geometric Bonferroni mean operator of trapezoidal interval type-2 fuzzy numbers.

However, it must be noticed that the above aggregation operations are all based on the arithmetic operations laws [8,9] of IT2 FSs for carrying the combination process. Up till now, only a small number of studies [17] have focused on the operations laws of IT2 FSs.

And it is well known that Einstein t-norms and Einstein t-conorms are two prototypical examples of the class of strict Archimedean t-norms and t-conorms [18]. Moreover, in literature there is still little research on aggregation operators using the Einstein operations for aggregating a collection of IT2 FSs. Wang and Liu [19,20] brought forward the intuitionistic fuzzy Einstein weighted geometric (IFEWG) operator, the intuitionistic fuzzy Einstein ordered weighted geometric (IFEOWG) operator, the intuitionistic fuzzy Einstein weighted averaging (IFEWA) operator and the intuitionistic fuzzy Einstein ordered weighted averaging (IFEOWA) operator successively. Zhao and Wei [21] developed the intuitionistic fuzzy Einstein hybrid averaging (IFEHA) operator and intuitionistic fuzzy Einstein hybrid geometric (IFEHG) operator. Zhang and Yu [22] proposed the Einstein based intuitionistic fuzzy Choquet geometric (EIFCG) operator and Einstein based interval-valued intuitionistic fuzzy Choquet geometric (EIIFCG) operator. Zhao et al. [23] also propose some intuitionistic trapezoidal fuzzy aggregation operators based on Einstein operations to aggregate the intuitionistic trapezoidal fuzzy numbers.

Therefore, the aim of this paper is to enrich trapezoidal T2 FSs theory by investigating information aggregation methods utilizing Einstein t-conorm and t-norm when the decision information takes the forms of trapezoidal interval type-2 fuzzy numbers (TIT2 FNs), and developing Einstein operations based on the operators. The Einstein operator laws have been extended to the interval type-2 fuzzy sets to organize and model the uncertainties better within multi-attribute decision analysis. The Einstein operations laws and some properties on TIT2 FNs are presented. Then a new method to deal with FMADM problems is presented based on Einstein operators and the fuzzy mean possibility degree of TIT2 FNs.

The remainder of this paper is organized as follows. Section 2 reviews basic concepts and arithmetic operations related to the trapezoidal interval type-2 fuzzy sets (TIT2 FSs). Some Einstein operations laws on TIT2 FSs are given in Section 3. Section 4 presents a new method for calculating the possibility degree of two TIT2 FSs based on the fuzzy possibility mean values. Section 5 introduces a procedure for FMADM problems based on TIT2FEWA operator, TIT2FEWG operator and the fuzzy mean possibility degree of two IT2 FSs. Section 6 uses global supplier selection problem to illustrate the proposed method. The conclusions are discussed in Section 7.

## 2. The Basic Concepts and Arithmetic Operations of IT2 FSs.

In this section, the basic concepts of IT2 FSs are introduced below to facilitate future discussions.

**Definition 2.1.** [4] *For a type-2 fuzzy set $\tilde{A}$, if all $\mu_{\tilde{A}}(x,u) = 1$, then $\tilde{A}$ is called an interval type-2 fuzzy set, i.e.,*

$$\tilde{A} = \int_{x\in X}\int_{u\in J_X} 1(x,u) \tag{1}$$

*where $J_X \subseteq [0,1]$.*

**Definition 2.2.** [4] *The upper membership function (UMF) and the lower membership function (LMF) of an IT2 FS are type-1 membership functions, respectively.*

**Definition 2.3.** [24] *An IT2 FS is called trapezoidal interval type-2 fuzzy numbers (TIT2 FSs) where the UMF and LMF are both trapezoidal fuzzy numbers, i.e., $\tilde{A} = \left(\tilde{A}^U, \tilde{A}^L\right) = \left(\left(a_1^U, a_2^U, a_3^U, a_4^U; H_1\left(\tilde{A}^U\right), H_2\left(\tilde{A}^U\right)\right), \left(a_1^L, a_2^L, a_3^L, a_4^L; H_1\left(\tilde{A}^L\right), H_2\left(\tilde{A}^L\right)\right)\right)$, where $H_j\left(\tilde{A}^U\right)$ and $H_j\left(\tilde{A}^L\right)$ $(j=1,2)$ denote membership values of the corresponding elements $a_{j+1}^U$ and $a_{j+1}^L$, respectively.*

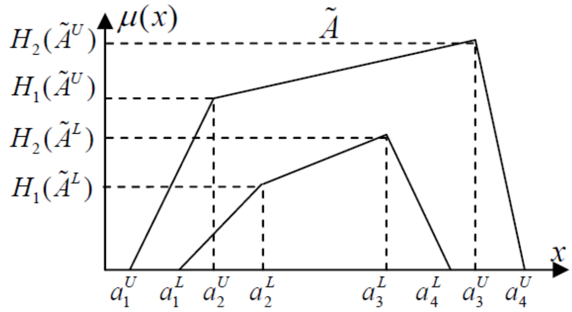


FIGURE 1. The trapezoidal interval type-2 fuzzy set $\tilde{A}$

In this paper, we briefly describe some basic operational laws related to trapezoidal interval type-2 fuzzy numbers, where the reference points and the heights of the upper and the lower membership functions of IT2 FSs are used to characterize T1 FSs. A TIT2 FS $\tilde{A}_i$ is shown in Figure 1.

**Definition 2.4.** [8] *Suppose that $\tilde{A}_1$ and $\tilde{A}_2$ are two TIT2 FSs, where* $\tilde{A}_1 = \left(\tilde{A}_1^U, \tilde{A}_1^L\right) = \left(\left(a_{11}^U, a_{12}^U, a_{13}^U, a_{14}^U; H_1\left(\tilde{A}_1^U\right), H_2\left(\tilde{A}_1^U\right)\right), \left(a_{11}^L, a_{12}^L, a_{13}^L, a_{14}^L; H_1\left(\tilde{A}_1^L\right), H_2\left(\tilde{A}_1^L\right)\right)\right)$ *and* $\tilde{A}_2 = \left(\tilde{A}_2^U, \tilde{A}_2^L\right) = \left(\left(a_{21}^U, {}^U_{22}, a_{23}^U, a_{24}^U; H_1\left(\tilde{A}_2^U\right), H_2\left(\tilde{A}_2^U\right)\right), \left(a_{21}^L, a_{22}^L, a_{23}^L, a_{24}^L; H_1\left(\tilde{A}_2^L\right), H_2\left(\tilde{A}_2^L\right)\right)\right)$. *The arithmetic operations between TIT2 FSs are defined as follows:*

(1)
$$\begin{aligned}\tilde{A}_1 \oplus \tilde{A}_2 &= \left(\tilde{A}_1^U, \tilde{A}_1^L\right) \oplus \left(\tilde{A}_2^U, \tilde{A}_2^L\right) \\ &= \Big(\Big(a_{11}^U + a_{21}^U, a_{12}^U + a_{22}^U, a_{13}^U + a_{23}^U, a_{14}^U + a_{24}^U; \\ &\quad \min\left(H_1\left(\tilde{A}_1^U\right), H_1\left(\tilde{A}_2^U\right)\right), \min\left(H_2\left(\tilde{A}_1^U\right), H_2\left(\tilde{A}_2^U\right)\right)\Big), \\ &\quad \Big(a_{11}^L + a_{21}^L, a_{12}^L + a_{22}^L, a_{13}^L + a_{23}^L, a_{14}^L + a_{24}^L; \\ &\quad \min\left(H_1\left(\tilde{A}_1^L\right), H_1\left(\tilde{A}_2^L\right)\right), \min\left(H_2\left(\tilde{A}_1^L\right), H_2\left(\tilde{A}_2^L\right)\right)\Big)\Big)\end{aligned} \tag{2}$$

(2)
$$\begin{aligned}\tilde{A}_1 \otimes \tilde{A}_2 &= \left(\tilde{A}_1^U, \tilde{A}_1^L\right) \otimes \left(\tilde{A}_2^U, \tilde{A}_2^L\right) \\ &= \Big(\Big(a_{11}^U \times a_{21}^U, a_{12}^U \times a_{22}^U, a_{13}^U \times a_{23}^U, a_{14}^U \times a_{24}^U; \\ &\quad \min\left(H_1\left(\tilde{A}_1^U\right), H_1\left(\tilde{A}_2^U\right)\right), \min\left(H_2\left(\tilde{A}_1^U\right), H_2\left(\tilde{A}_2^U\right)\right)\Big), \\ &\quad \Big(a_{11}^L \times a_{21}^L, a_{12}^L \times a_{22}^L, a_{13}^L \times a_{23}^L, a_{14}^L \times a_{24}^L; \\ &\quad \min\left(H_1\left(\tilde{A}_1^L\right), H_1\left(\tilde{A}_2^L\right)\right), \min\left(H_2\left(\tilde{A}_1^L\right), H_2\left(\tilde{A}_2^L\right)\right)\Big)\Big)\end{aligned} \tag{3}$$

(3)
$$\begin{aligned}k\tilde{A}_1 &= k\left(\tilde{A}_1^U, \tilde{A}_1^L\right) \\ &= \left(\left(ka_{11}^U, ka_{12}^U, ka_{13}^U, ka_{14}^U; H_1\left(\tilde{A}_1^U\right), H_2\left(\tilde{A}_1^U\right)\right), \right. \\ &\quad \left.\left(ka_{11}^L, ka_{12}^L, ka_{13}^L, ka_{14}^L; H_1\left(\tilde{A}_1^L\right), H_2\left(\tilde{A}_1^L\right)\right)\right)\end{aligned} \tag{4}$$

(4)
$$\begin{aligned}\left(\tilde{A}_1\right)^k &= \left(\tilde{A}_1^U, \tilde{A}_1^L\right)^k \\ &= \left(\left(a_{11}^U\right)^k, \left(a_{12}^U\right)^k, \left(a_{13}^U\right)^k, \left(a_{14}^U\right)^k; H_1\left(\tilde{A}_1^U\right), H_2\left(\tilde{A}_1^U\right)\right), \\ &\quad \left(\left(a_{11}^L\right)^k, \left(a_{11}^L\right)^k, \left(a_{12}^L\right)^k, \left(a_{13}^L\right)^k, \left(a_{14}^L\right)^k; H_1\left(\tilde{A}_1^L\right), H_2\left(\tilde{A}_1^L\right)\right)\end{aligned} \tag{5}$$

3. **The Einstein Operations Laws and Aggregation Operators of TIT2 FSs.** In fact, the theory of aggregation operators has an important role since the beginning of fuzzy set theory. All types of the particular operators were included in the general concepts of the t-norms and t-conorms, which satisfy the requirements of the conjunction and disjunction operators, respectively. Various t-norms and t-conorms families can be used to perform the corresponding intersections and unions of fuzzy numbers. Einstein operations include the Einstein sum$\oplus_\varepsilon$ and Einstein product$\otimes_\varepsilon$, which are examples of t-norms and t-conorms, respectively. They are defined as follows [18]: The Einstein sum$\oplus_\varepsilon$ use a t-conorm and Einstein product$\otimes_\varepsilon$ is a t-norm, where

$$a \oplus_\varepsilon b = \frac{a+b}{1+a\cdot b}, \quad a \otimes_\varepsilon b = \frac{a\cdot b}{1+(1-a)\cdot(1-b)}, \quad \forall(a,b)\in[0,1]^2 \tag{6}$$

3.1. **The Einstein operations laws of TIT2 FSs.** Hu et al. [17] pointed that some drawbacks in Definition 2.4 about arithmetic operations of TIT2 FSs. Thus, for an intersection of IT2 FSs, a good alternative to the algebraic product and sum is the Einstein product and Einstein sum, which typically gives the same smooth approximations as the algebraic product and sum. Moreover, the Einstein operations have been used in some application fields. Therefore, motivated by the operations on the IT2 FSs, the generalized intersection and union on two TIT2 FSs $\tilde{A}_1$ and $\tilde{A}_2$ become the Einstein product (denoted by $\tilde{A}_1 \otimes_\varepsilon \tilde{A}_2$) and Einstein sum (denoted by $\tilde{A}_1 \oplus_\varepsilon \tilde{A}_2$) of two TIT2 FSs, respectively, as follows.

**Definition 3.1.** *The Einstein sum and product between the two TIT2 FSs*

$$\begin{aligned}\tilde{A}_1 &= \left(\tilde{A}_1^U, \tilde{A}_1^L\right) \\ &= \left(\left(a_{11}^U, a_{12}^U, a_{13}^U, a_{14}^U; H_1\left(\tilde{A}_1^U\right), H_2\left(\tilde{A}_1^U\right)\right), \left(a_{11}^L, a_{12}^L, a_{13}^L, a_{14}^L; H_1\left(\tilde{A}_1^L\right), H_2\left(\tilde{A}_1^L\right)\right)\right)\end{aligned}$$

*and*

$$\begin{aligned}\tilde{A}_2 &= \left(\tilde{A}_2^U, \tilde{A}_2^L\right) \\ &= \left(\left(a_{21}^U, a_{22}^U, a_{23}^U, a_{24}^U; H_1\left(\tilde{A}_2^U\right), H_2\left(\tilde{A}_2^U\right)\right), \left(a_{21}^L, a_{22}^L, a_{23}^L, a_{24}^L; H_1\left(\tilde{A}_2^L\right), H_2\left(\tilde{A}_2^L\right)\right)\right)\end{aligned}$$

*is defined as follows:*

$$\begin{aligned}\tilde{A}_1 \oplus_\varepsilon \tilde{A}_2 = \Bigg(\Bigg(&\frac{a_{11}^U + a_{21}^U}{1+a_{11}^U\cdot a_{21}^U}, \frac{a_{12}^U + a_{22}^U}{1+a_{12}^U\cdot a_{22}^U}, \frac{a_{13}^U + a_{23}^U}{1+a_{13}^U\cdot a_{23}^U}, \frac{a_{14}^U + a_{24}^U}{1+a_{14}^U\cdot a_{24}^U}; \\ &\min\left(H_1\left(\tilde{A}_1^U\right), H_1\left(\tilde{A}_2^U\right)\right), \min\left(H_2\left(\tilde{A}_1^U\right), H_2\left(\tilde{A}_2^U\right)\right)\Bigg), \\ \Bigg(&\frac{a_{11}^L + a_{21}^L}{1+a_{11}^L\cdot a_{21}^L}, \frac{a_{12}^L + a_{22}^L}{1+a_{12}^L\cdot a_{22}^L}, \frac{a_{13}^L + a_{23}^L}{1+a_{13}^L\cdot a_{23}^L}, \frac{a_{14}^L + a_{24}^L}{1+a_{14}^L\cdot a_{24}^L}; \\ &\min\left(H_1\left(\tilde{A}_1^L\right), H_1\left(\tilde{A}_2^L\right)\right), \min\left(H_2\left(\tilde{A}_1^L\right), H_2\left(\tilde{A}_2^L\right)\right)\Bigg)\Bigg)\end{aligned} \tag{7}$$

$$\begin{aligned}\tilde{A}_1 \otimes_\varepsilon \tilde{A}_2 = & \left(\left(\frac{a_{11}^U \cdot a_{21}^U}{1+(1-a_{11}^U)\cdot(1-a_{21}^U)}, \frac{a_{12}^U \cdot a_{22}^U}{1+(1-a_{12}^U)\cdot(1-a_{22}^U)},\right.\right.\\ & \frac{a_{13}^U \cdot a_{23}^U}{1+(1-a_{13}^U)\cdot(1-a_{23}^U)}, \frac{a_{14}^U \cdot a_{24}^U}{1+(1-a_{14}^U)\cdot(1-a_{24}^U)};\\ & \left.\min\left(H_1\left(\tilde{A}_1^U\right), H_1\left(\tilde{A}_2^U\right)\right), \min\left(H_2\left(\tilde{A}_1^U\right), H_2\left(\tilde{A}_2^U\right)\right)\right),\\ & \left(\frac{a_{11}^L \cdot a_{21}^L}{1+(1-a_{11}^L)\cdot(1-a_{21}^L)}, \frac{a_{12}^L \cdot a_{22}^L}{1+(1-a_{12}^L)\cdot(1-a_{22}^L)},\right.\\ & \frac{a_{13}^L \cdot a_{23}^L}{1+(1-a_{13}^L)\cdot(1-a_{23}^L)}, \frac{a_{14}^L \cdot a_{24}^L}{1+(1-a_{14}^L)\cdot(1-a_{24}^L)};\\ & \left.\left.\min\left(H_1\left(\tilde{A}_1^L\right), H_1\left(\tilde{A}_2^L\right)\right), \min\left(H_2\left(\tilde{A}_1^L\right), H_2\left(\tilde{A}_2^L\right)\right)\right)\right)\end{aligned} \tag{8}$$

**Definition 3.2.** *The Einstein scalar multiplication operation between the TIT2 FSs* $\tilde{A} = \left(\tilde{A}^U, \tilde{A}^L\right) = \left(\left(a_1^U, a_2^U, a_3^U, a_4^U; H_1\left(\tilde{A}^U\right), H_2\left(\tilde{A}^U\right)\right), \left(a_1^L, a_2^L, a_3^L, a_4^L; H_1\left(\tilde{A}^L\right), H_2\left(\tilde{A}^L\right)\right)\right)$ *and the crisp value* $k$ *is defined as follows:*

$$\begin{aligned}k \cdot_\varepsilon \tilde{A} = & \left(\left(\frac{\left(1+a_1^U\right)^k - \left(1-a_1^U\right)^k}{\left(1+a_1^U\right)^k + \left(1-a_1^U\right)^k}, \frac{\left(1+a_2^U\right)^k - \left(1-a_2^U\right)^k}{\left(1+a_2^U\right)^k + \left(1-a_2^U\right)^k}, \frac{\left(1+a_3^U\right)^k - \left(1-a_3^U\right)^k}{\left(1+a_3^U\right)^k + \left(1-a_3^U\right)^k},\right.\right.\\ & \left.\frac{\left(1+a_4^U\right)^k - \left(1-a_4^U\right)^k}{\left(1+a_4^U\right)^k + \left(1-a_4^U\right)^k}; H_1\left(\tilde{A}^U\right), H_2\left(\tilde{A}^U\right)\right),\\ & \left(\frac{\left(1+a_1^L\right)^k - \left(1-a_1^L\right)^k}{\left(1+a_1^L\right)^k + \left(1-a_1^L\right)^k}, \frac{\left(1+a_2^L\right)^k - \left(1-a_2^L\right)^k}{\left(1+a_2^L\right)^k + \left(1-a_2^L\right)^k}, \frac{\left(1+a_3^L\right)^k - \left(1-a_3^L\right)^k}{\left(1+a_3^L\right)^k + \left(1-a_3^L\right)^k},\right.\\ & \left.\left.\frac{\left(1+a_4^L\right)^k - \left(1-a_4^L\right)^k}{\left(1+a_4^L\right)^k + \left(1-a_4^L\right)^k}; H_1\left(\tilde{A}^L\right), H_2\left(\tilde{A}^L\right)\right)\right)\end{aligned} \tag{9}$$

*where* $k > 0$.

**Definition 3.3.** *The Einstein scalar geometric operation between the TIT2 FSs*

$$\begin{aligned}\tilde{A} &= \left(\tilde{A}^U, \tilde{A}^L\right)\\ &= \left(\left(a_1^U, a_2^U, a_3^U, a_4^U; H_1\left(\tilde{A}^U\right), H_2\left(\tilde{A}^U\right)\right), \left(a_1^L, a_2^L, a_3^L, a_4^L; H_1\left(\tilde{A}^L\right), H_2\left(\tilde{A}^L\right)\right)\right)\end{aligned}$$

*and the crisp value* $k$ *is defined as follows:*

$$\begin{aligned}\tilde{A}^{\cdot_\varepsilon k} = & \left(\left(\frac{2\left(a_1^U\right)^k}{\left(2-a_1^U\right)^k + \left(a_1^U\right)^k}, \frac{2\left(a_2^U\right)^k}{\left(2-a_2^U\right)^k + \left(a_2^U\right)^k}, \frac{2\left(a_3^U\right)^k}{\left(2-a_3^U\right)^k + \left(a_3^U\right)^k},\right.\right.\\ & \left.\frac{2\left(a_4^U\right)^k}{\left(2-a_4^U\right)^k + \left(a_4^U\right)^k}; H_1\left(\tilde{A}^U\right), H_2\left(\tilde{A}^U\right)\right),\\ & \left(\frac{2\left(a_1^L\right)^k}{\left(2-a_1^L\right)^k + \left(a_1^L\right)^k}, \frac{2\left(a_2^L\right)^k}{\left(2-a_2^L\right)^k + \left(a_2^L\right)^k}, \frac{2\left(a_3^L\right)^k}{\left(2-a_3^L\right)^k + \left(a_3^L\right)^k},\right.\\ & \left.\left.\frac{2\left(a_4^L\right)^k}{\left(2-a_4^L\right)^k + \left(a_4^L\right)^k}; H_1\left(\tilde{A}^L\right), H_2\left(\tilde{A}^L\right)\right)\right)\end{aligned} \tag{10}$$

*where* $k > 0$.

**Proposition 3.1.** *Let $\tilde{A}$, $\tilde{A}_1$ and $\tilde{A}_2$ be three trapezoidal IT2 FSs, $k, k_1, k_2 > 0$, then we have*
*(1)* $\tilde{A}_1 \oplus_\varepsilon \tilde{A}_2 = \tilde{A}_2 \oplus_\varepsilon \tilde{A}_1$.
*(2)* $\tilde{A}_1 \otimes_\varepsilon \tilde{A}_2 = \tilde{A}_2 \otimes_\varepsilon \tilde{A}_1$.
*(3)* $k \cdot_\varepsilon \left(\tilde{A}_1 \oplus_\varepsilon \tilde{A}_2\right) = \left(k \cdot_\varepsilon \tilde{A}_1\right) \oplus_\varepsilon \left(k \cdot_\varepsilon \tilde{A}_2\right)$.
*(4)* $k_1 \cdot_\varepsilon \tilde{A} \oplus_\varepsilon k_2 \cdot_\varepsilon \tilde{A} = (k_1 + k_2) \cdot_\varepsilon \tilde{A}$
*(5)* $\left(\tilde{A}_1 \otimes_\varepsilon \tilde{A}_2\right)^{k \cdot \varepsilon} = \left(\tilde{A}_1\right)^{k \cdot \varepsilon} \otimes_\varepsilon \left(\tilde{A}_2\right)^{k \cdot \varepsilon}$
*(6)* $\left(\tilde{A}\right)^{k_1 \cdot \varepsilon} \otimes_\varepsilon \left(\tilde{A}\right)^{k_2 \cdot \varepsilon} = \left(\tilde{A}\right)^{(k_1+k_2) \cdot \varepsilon}$

3.2. **The trapezoidal interval type-2 fuzzy Einstein aggregation operators.** In this section, we will develop some aggregation operation for aggregating IT2 FSs with the help of the Einstein operations and study their desirable properties.

**Definition 3.4.** *Let $\tilde{A}_i = \left(\tilde{A}_i^U, \tilde{A}_i^L\right) = \left(\left(a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U; H_1\left(\tilde{A}_i^U\right), H_2\left(\tilde{A}_i^U\right)\right), \left(a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L; H_1\left(\tilde{A}_i^U\right), H_2\left(\tilde{A}_i^U\right)\right)\right)$ $(i = 1, 2, \ldots, n)$ be a collection of the TIT2 FNs, and weight vector is $w = (w_1, w_2, \ldots, w_n)$, which satisfies $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$. Then, we define the trapezoidal interval type-2 fuzzy Einstein weighted average (TIT2FEWA) operator as follows:*

$$TIT2FEWA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) = \mathop{\oplus_\varepsilon}_{j=1}^{n} \left(w_j \tilde{A}_j\right) = w_1 \cdot_\varepsilon \tilde{A}_1 \oplus_\varepsilon w_2 \cdot_\varepsilon \tilde{A}_2 \oplus_\varepsilon \cdots \oplus_\varepsilon w_n \cdot_\varepsilon \tilde{A}_n \quad (11)$$

Especially, if $w = (1/n, 1/n, \ldots, 1/n)$, then TIT2FEWA operator is reduced to a trapezoidal interval type-2 fuzzy Einstein averaging (TIT2FEA) operator of dimension $n$, which is defined as follows:

$$IT2TFEA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) = \frac{1}{n} \cdot_\varepsilon \left(\tilde{A}_1 \oplus_\varepsilon \tilde{A}_2 \oplus_\varepsilon \cdots \oplus_\varepsilon \tilde{A}_n\right) \quad (12)$$

According to the operations of the TIT2 FNs, we can get the following result.

**Theorem 3.1.** *Let $\tilde{A}_i = \left(\tilde{A}_i^U, \tilde{A}_i^L\right) = \left(\left(a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U; H_1\left(\tilde{A}_i^U\right), H_2\left(\tilde{A}_i^U\right)\right), \left(a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L; H_1\left(\tilde{A}_i^U\right), H_2\left(\tilde{A}_i^U\right)\right)\right)$ $(i = 1, 2, \ldots, n)$ be a collection of the trapezoidal IT2 FNs, then their aggregated value by using the TIT2FEWA operator is also a trapezoidal IT2 fuzzy number, and*

$$TIT2FEWA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) = \mathop{\oplus_\varepsilon}_{j=1}^{n} \left(w_j \tilde{A}_j\right) = \tilde{A} = \left(\tilde{A}^U, \tilde{A}^L\right) \quad (13)$$

*where*

$$\begin{aligned}\tilde{A}^U = \Bigg( & \frac{\prod_{j=1}^n \left(1 + a_{j1}^U\right)^{w_j} - \prod_{j=1}^n \left(1 - a_{j1}^U\right)^{w_j}}{\prod_{j=1}^n \left(1 + a_{j1}^U\right)^{w_j} + \prod_{j=1}^n \left(1 - a_{j1}^U\right)^{w_j}}, \frac{\prod_{j=1}^n \left(1 + a_{j2}^U\right)^{w_j} - \prod_{j=1}^n \left(1 - a_{j2}^U\right)^{w_j}}{\prod_{j=1}^n \left(1 + a_{j2}^U\right)^{w_j} + \prod_{j=1}^n \left(1 - a_{j2}^U\right)^{w_j}}, \\ & \frac{\prod_{j=1}^n \left(1 + a_{j3}^U\right)^{w_j} - \prod_{j=1}^n \left(1 - a_{j3}^U\right)^{w_j}}{\prod_{j=1}^n \left(1 + a_{j3}^U\right)^{w_j} + \prod_{j=1}^n \left(1 - a_{j3}^U\right)^{w_j}}, \frac{\prod_{j=1}^n \left(1 + a_{j4}^U\right)^{w_j} - \prod_{j=1}^n \left(1 - a_{j4}^U\right)^{w_j}}{\prod_{j=1}^n \left(1 + a_{j4}^U\right)^{w_j} + \prod_{j=1}^n \left(1 - a_{j4}^U\right)^{w_j}}; \\ & \min_{i=1,\cdots,n}\left(H_1\left(\tilde{A}_i^U\right)\right), \min_{i=1,\cdots,n}\left(H_2\left(\tilde{A}_i^U\right)\right) \Bigg)\end{aligned} \quad (14)$$

*and*

$$\tilde{A}^L = \left(\frac{\prod_{j=1}^n \left(1+a_{j1}^L\right)^{w_j} - \prod_{j=1}^n \left(1-a_{j1}^L\right)^{w_j}}{\prod_{j=1}^n \left(1+a_{j1}^L\right)^{w_j} + \prod_{j=1}^n \left(1-a_{j1}^L\right)^{w_j}}, \frac{\prod_{j=1}^n \left(1+a_{j2}^L\right)^{w_j} - \prod_{j=1}^n \left(1-a_{j2}^L\right)^{w_j}}{\prod_{j=1}^n \left(1+a_{j2}^L\right)^{w_j} + \prod_{j=1}^n \left(1-a_{j2}^L\right)^{w_j}}, \frac{\prod_{j=1}^n \left(1+a_{j3}^L\right)^{w_j} - \prod_{j=1}^n \left(1-a_{j3}^L\right)^{w_j}}{\prod_{j=1}^n \left(1+a_{j3}^L\right)^{w_j} + \prod_{j=1}^n \left(1-a_{j3}^L\right)^{w_j}}, \frac{\prod_{j=1}^n \left(1+a_{j4}^L\right)^{w_j} - \prod_{j=1}^n \left(1-a_{j4}^L\right)^{w_j}}{\prod_{j=1}^n \left(1+a_{j4}^L\right)^{w_j} + \prod_{j=1}^n \left(1-a_{j4}^L\right)^{w_j}}; \min_{i=1,\cdots,n}\left(H_1\left(\tilde{A}_i^L\right)\right), \min_{i=1,\cdots,n}\left(H_2\left(\tilde{A}_i^L\right)\right)\right) \tag{15}$$

We use mathematical induction to prove this theorem as follows.

**Proof:** Firstly, we prove that Equation (13) holds for $n = 2$.

By the operations of the TIT2 fuzzy variables defined in Equations (7)-(10)

$$w_j \cdot_\varepsilon \tilde{A}_j = \left(\left(\frac{\left(1+a_{j1}^U\right)^{w_j} - \left(1-a_{j1}^U\right)^{w_j}}{\left(1+a_{j1}^U\right)^{w_j} + \left(1-a_{j1}^U\right)^{w_j}}, \frac{\left(1+a_{j2}^U\right)^{w_j} - \left(1-a_{j2}^U\right)^{w_j}}{\left(1+a_{j2}^U\right)^{w_j} + \left(1-a_{j2}^U\right)^{w_j}}, \frac{\left(1+a_{j3}^U\right)^{w_j} - \left(1-a_{j3}^U\right)^{w_j}}{\left(1+a_{j3}^U\right)^{w_j} + \left(1-a_{j3}^U\right)^{w_j}}, \frac{\left(1+a_{j4}^U\right)^{w_j} - \left(1-a_{j4}^U\right)^{w_j}}{\left(1+a_{j4}^U\right)^{w_j} + \left(1-a_{j4}^U\right)^{w_j}}; H_1\left(\tilde{A}_j^U\right), H_2\left(\tilde{A}_j^U\right)\right), \left(\frac{\left(1+a_{j1}^L\right)^{w_j} - \left(1-a_{j1}^L\right)^{w_j}}{\left(1+a_{j1}^L\right)^{w_j} + \left(1-a_{j1}^L\right)^{w_j}}, \frac{\left(1+a_{j2}^L\right)^{w_j} - \left(1-a_{j2}^L\right)^{w_j}}{\left(1+a_{j2}^L\right)^{w_j} + \left(1-a_{j2}^L\right)^{w_j}}, \frac{\left(1+a_{j3}^L\right)^{w_j} - \left(1-a_{j3}^L\right)^{w_j}}{\left(1+a_{j3}^L\right)^{w_j} + \left(1-a_{j3}^L\right)^{w_j}}, \frac{\left(1+a_{j4}^L\right)^{w_j} - \left(1-a_{j4}^L\right)^{w_j}}{\left(1+a_{j4}^L\right)^{w_j} + \left(1-a_{j4}^L\right)^{w_j}}; H_1\left(\tilde{A}_1^L\right), H_2\left(\tilde{A}_1^L\right)\right)\right)$$

(a) When $n = 2$, we can get

$$w_1 \cdot_\varepsilon \tilde{A}_1 \oplus_\varepsilon w_2 \cdot_\varepsilon \tilde{A}_2 = \left(\left(\frac{\left(1+a_{11}^U\right)^{w_1}\left(1+a_{21}^U\right)^{w_2} - \left(1-a_{11}^U\right)^{w_1}\left(1-a_{21}^U\right)^{w_2}}{\left(1+a_{11}^U\right)^{w_1}\left(1+a_{21}^U\right)^{w_2} + \left(1-a_{11}^U\right)^{w_1}\left(1-a_{21}^U\right)^{w_2}}, \frac{\left(1+a_{12}^U\right)^{w_1}\left(1+a_{22}^U\right)^{w_2} - \left(1-a_{12}^U\right)^{w_1}\left(1-a_{22}^U\right)^{w_2}}{\left(1+a_{12}^U\right)^{w_1}\left(1+a_{22}^U\right)^{w_2} + \left(1-a_{12}^U\right)^{w_1}\left(1-a_{22}^U\right)^{w_2}}, \frac{\left(1+a_{13}^U\right)^{w_1}\left(1+a_{23}^U\right)^{w_2} - \left(1-a_{13}^U\right)^{w_1}\left(1-a_{23}^U\right)^{w_2}}{\left(1+a_{13}^U\right)^{w_1}\left(1+a_{23}^U\right)^{w_2} + \left(1-a_{13}^U\right)^{w_1}\left(1-a_{23}^U\right)^{w_2}}, \frac{\left(1+a_{14}^U\right)^{w_1}\left(1+a_{24}^U\right)^{w_2} - \left(1-a_{14}^U\right)^{w_1}\left(1-a_{24}^U\right)^{w_2}}{\left(1+a_{14}^U\right)^{w_1}\left(1+a_{24}^U\right)^{w_2} + \left(1-a_{14}^U\right)^{w_1}\left(1-a_{24}^U\right)^{w_2}}; \min\left(H_1\left(\tilde{A}_1^U\right), H_1\left(\tilde{A}_2^U\right)\right), \min\left(H_2\left(\tilde{A}_1^U\right), H_2\left(\tilde{A}_2^U\right)\right)\right), \left(\frac{\left(1+a_{11}^L\right)^{w_1}\left(1+a_{21}^L\right)^{w_2} - \left(1-a_{11}^L\right)^{w_1}\left(1-a_{21}^L\right)^{w_2}}{\left(1+a_{11}^L\right)^{w_1}\left(1+a_{21}^L\right)^{w_2} + \left(1-a_{11}^L\right)^{w_1}\left(1-a_{21}^L\right)^{w_2}}, \frac{\left(1+a_{12}^L\right)^{w_1}\left(1+a_{22}^L\right)^{w_2} - \left(1-a_{12}^L\right)^{w_1}\left(1-a_{22}^L\right)^{w_2}}{\left(1+a_{12}^L\right)^{w_1}\left(1+a_{22}^L\right)^{w_2} + \left(1-a_{12}^L\right)^{w_1}\left(1-a_{22}^L\right)^{w_2}}, \frac{\left(1+a_{13}^L\right)^{w_1}\left(1+a_{23}^L\right)^{w_2} - \left(1-a_{13}^L\right)^{w_1}\left(1-a_{23}^L\right)^{w_2}}{\left(1+a_{13}^L\right)^{w_1}\left(1+a_{23}^L\right)^{w_2} + \left(1-a_{13}^L\right)^{w_1}\left(1-a_{23}^L\right)^{w_2}}, \frac{\left(1+a_{14}^L\right)^{w_1}\left(1+a_{24}^L\right)^{w_2} - \left(1-a_{14}^L\right)^{w_1}\left(1-a_{24}^L\right)^{w_2}}{\left(1+a_{14}^L\right)^{w_1}\left(1+a_{24}^L\right)^{w_2} + \left(1-a_{14}^L\right)^{w_1}\left(1-a_{24}^L\right)^{w_2}}; \min\left(H_1\left(\tilde{A}_1^L\right), H_1\left(\tilde{A}_2^L\right)\right), \min\left(H_2\left(\tilde{A}_1^L\right), H_2\left(\tilde{A}_2^L\right)\right)\right)\right)$$

i.e., when $n = 2$, Equation (13) is right.

(b) Suppose when $n = k$, Equation (13) is right, then, when $n = k + 1$, we have

$$\begin{aligned} & TIT2FEWA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n, \tilde{A}_{n+1}\right) \\ = & \overset{n+1}{\underset{j=1}{\oplus_\varepsilon}}\left(w_j \tilde{A}_j\right) \\ = & w_1 \cdot_\varepsilon \tilde{A}_1 \oplus_\varepsilon w_2 \cdot_\varepsilon \tilde{A}_2 \oplus_\varepsilon \cdots \oplus_\varepsilon w_n \cdot_\varepsilon \tilde{A}_n \oplus_\varepsilon w_{n+1} \cdot_\varepsilon \tilde{A}_{n+1} \\ = & TIT2FEWA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) \oplus_\varepsilon w_{n+1} \cdot_\varepsilon \tilde{A}_{n+1} \\ = & \tilde{A} \oplus_\varepsilon w_{n+1} \cdot_\varepsilon \tilde{A}_{n+1} \end{aligned}$$

From (a), we can see (b) holds for $n = k + 1$. Therefore, Equation (13) holds for all $n$, which completes the proof of Theorem 3.1. □

**Definition 3.5.** *Let* $\tilde{A}_i = \left(\tilde{A}_i^U, \tilde{A}_i^L\right) = \left(\left(a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U; H_1\left(\tilde{A}_i^U\right), H_2\left(\tilde{A}_i^U\right)\right), \left(a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L; H_1\left(\tilde{A}_i^U\right), H_2\left(\tilde{A}_i^U\right)\right)\right)$ $(i = 1, 2, \ldots, n)$ *be a collection of the trapezoidal IT2 FNs, and weight vector is* $w = (w_1, w_2, \ldots, w_n)$*, which satisfies* $w_i \geq 0$ *and* $\sum_{i=1}^n w_i = 1$*. Then, we define the trapezoidal interval type-2 fuzzy Einstein weighted geometric (TIT2FEGA) operator as follows:*

$$\begin{aligned} TIT2FEGA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) &= \overset{n}{\underset{j=1}{\otimes_\varepsilon}}\left(\tilde{A}_j\right)^{w_j} \\ &= \left(\tilde{A}_1\right)^{w_1 \cdot \varepsilon} \otimes_\varepsilon \left(\tilde{A}_2\right)^{w_2 \cdot \varepsilon} \otimes_\varepsilon \cdots \otimes_\varepsilon \left(\tilde{A}_n\right)^{w_n \cdot \varepsilon} \end{aligned} \tag{16}$$

Especially, if $w = (1/n, 1/n, \ldots, 1/n)$, then *TIT2FEGA* operator is reduced to a trapezoidal interval type-2 fuzzy Einstein geometric (TIT2FEG) operator of dimension $n$, which is defined as follows:

$$IT2TFEG\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) = \left(\tilde{A}_1 \otimes_\varepsilon \tilde{A}_2 \otimes_\varepsilon \cdots \otimes_\varepsilon \tilde{A}_n\right)^{\frac{1}{n} \cdot \varepsilon} \tag{17}$$

According to the operations of the TIT2 fuzzy numbers, we can get the following result.

**Theorem 3.2.** *Let* $\tilde{A}_i = \left(\tilde{A}_i^U, \tilde{A}_i^L\right) = \left(\left(a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U; H_1\left(\tilde{A}_i^U\right), H_2\left(\tilde{A}_i^U\right)\right), \left(a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L; H_1\left(\tilde{A}_i^U\right), H_2\left(\tilde{A}_i^U\right)\right)\right)$ $(i = 1, 2, \ldots, n)$ *be a collection of the trapezoidal IT2 FNs, then their aggregated value by using the TIT2FEGA operator is also a trapezoidal IT2 fuzzy number, and*

$$TIT2FEGA\left(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\right) = \overset{n}{\underset{j=1}{\otimes_\varepsilon}}\left(\tilde{A}_j\right)^{w_j} = \tilde{A} = \left(\tilde{A}^U, \tilde{A}^L\right) \tag{18}$$

*where*

$$\begin{aligned} \tilde{A}^U = \Bigg( & \frac{2\prod_{j=1}^n \left(a_{j1}^U\right)^{w_j}}{\prod_{j=1}^n \left(2 - a_{j1}^U\right)^{w_j} + \prod_{j=1}^n \left(a_{j1}^U\right)^{w_j}}, \frac{2\prod_{j=1}^n \left(a_{j2}^U\right)^{w_j}}{\prod_{j=1}^n \left(2 - a_{j2}^U\right)^{w_j} + \prod_{j=1}^n \left(a_{j2}^U\right)^{w_j}}, \\ & \frac{2\prod_{j=1}^n \left(a_{j3}^U\right)^{w_j}}{\prod_{j=1}^n \left(2 - a_{j3}^U\right)^{w_j} + \prod_{j=1}^n \left(a_{j3}^U\right)^{w_j}}, \frac{2\prod_{j=1}^n \left(a_{j4}^U\right)^{w_j}}{\prod_{j=1}^n \left(2 - a_{j4}^U\right)^{w_j} + \prod_{j=1}^n \left(a_{j4}^U\right)^{w_j}}; \\ & \min_{i=1,\cdots,n}\left(H_1\left(\tilde{A}_i^U\right)\right), \min_{i=1,\cdots,n}\left(H_2\left(\tilde{A}_i^U\right)\right) \Bigg) \end{aligned} \tag{19}$$

*and*

$$\tilde{A}^L = \left(\frac{2\prod_{j=1}^n \left(a_{j1}^L\right)^{w_j}}{\prod_{j=1}^n \left(2-a_{j1}^L\right)^{w_j} + \prod_{j=1}^n \left(a_{j1}^L\right)^{w_j}}, \frac{2\prod_{j=1}^n \left(a_{j2}^L\right)^{w_j}}{\prod_{j=1}^n \left(2-a_{j2}^L\right)^{w_j} + \prod_{j=1}^n \left(a_{j2}^L\right)^{w_j}}, \right.$$
$$\frac{2\prod_{j=1}^n \left(a_{j3}^L\right)^{w_j}}{\prod_{j=1}^n \left(2-a_{j3}^L\right)^{w_j} + \prod_{j=1}^n \left(a_{j3}^L\right)^{w_j}}, \frac{2\prod_{j=1}^n \left(a_{j4}^L\right)^{w_j}}{\prod_{j=1}^n \left(2-a_{j4}^L\right)^{w_j} + \prod_{j=1}^n \left(a_{j4}^L\right)^{w_j}}; \tag{20}$$
$$\left.\min_{i=1,\cdots,n}\left(H_1\left(\tilde{A}_i^L\right)\right), \min_{i=1,\cdots,n}\left(H_2\left(\tilde{A}_i^L\right)\right)\right)$$

We can also use mathematical induction to prove this theorem.

4. **The Fuzzy Mean Possibility Degree of IT2 FSs.** In this section, we extended the concept of Carlsson and Fullér [25] about the possibilistic mean value of type-1 fuzzy numbers. We first introduce the lower and upper possibility mean value of IT2 FSs.

If an IT2 FS $\tilde{A} = \left(\tilde{A}^U, \tilde{A}^L\right)$ is a non-negative generalized trapezoidal fuzzy numbers, let $h^L$, $h^U$ denote the heights of the upper membership function $\tilde{A}^U$ and the lower membership function $\tilde{A}^L$. We have pseudo level sets with $\tilde{A}^U_\alpha = \left[a_1^U(\alpha), a_2^U(\alpha)\right]$, $\alpha \in \left[0, h^U\right]$ and $\tilde{A}^L_\beta = \left[a_1^L(\beta), a_2^L(\beta)\right]$, $\beta \in \left[0, h^L\right]$, and then we present the following concepts.

**Definition 4.1.** [26] *The lower possibility mean value of an IT2 FS $\tilde{A} = \left(\tilde{A}^U, \tilde{A}^L\right)$ is defined as*

$$M_*\left(\tilde{A}\right) = \int_0^{h^U} \alpha a_1^U(\alpha)d\alpha + \int_0^{h^L} \beta a_1^L(\beta)d\beta \tag{21}$$

Obviously, $M_*\left(\tilde{A}\right)$ is nothing else but the level-weight average of the arithmetic means of all pseudo level sets, that is, the weight of the arithmetic mean of $a_1^U(\alpha)$ and $a_1^L(\beta)$. In a similar manner, we introduce the upper possibility mean value $M^*\left(\tilde{A}\right)$ as follows.

**Definition 4.2.** [26] *The upper possibility mean value of a TIT2 FS $\tilde{A} = \left(\tilde{A}^U, \tilde{A}^L\right)$ is defined as*

$$M^*\left(\tilde{A}\right) = \int_0^{h^U} \alpha a_2^U(\alpha)d\alpha + \int_0^{h^L} \beta a_2^L(\beta)d\beta \tag{22}$$

Let us introduce the notation

$$M\left(\tilde{A}\right) = \left[M_*\left(\tilde{A}\right), M^*\left(\tilde{A}\right)\right] \tag{23}$$

That is, $M\left(\tilde{A}\right)$ is a closed interval bounded by the lower and upper possibility mean values of IT2 FS $\tilde{A}$.

**Definition 4.3.** [26] *Let*

$$M\left(\tilde{A}_1\right) = \left[M_*\left(\tilde{A}_1\right), M^*\left(\tilde{A}_1\right)\right]$$

*and*

$$M\left(\tilde{A}_2\right) = \left[M_*\left(\tilde{A}_2\right), M^*\left(\tilde{A}_2\right)\right]$$

*be interval-valued possibility mean values of TIT2 FSs $\tilde{A}_1$ and $\tilde{A}_2$, respectively, then we define the possibility degree formula of TIT2 FSs as follows:*

$$p\left(\tilde{A}_1 \succ \tilde{A}_2\right) = \min\left\{\max\left(\frac{M^*\left(\tilde{A}_1\right) - M_*\left(\tilde{A}_2\right)}{M^*\left(\tilde{A}_1\right) - M_*\left(\tilde{A}_1\right) + M^*\left(\tilde{A}_2\right) - M_*\left(\tilde{A}_2\right)}, 0\right), 1\right\} \tag{24}$$

**Theorem 4.1.** *The possibility degree $p\left(\tilde{A}_1 \succ \tilde{A}_2\right)$ of TIT2 FSs $\tilde{A}_1$ and $\tilde{A}_2$ has the following properties:*
*(1) $0 \le p\left(\tilde{A}_1 \succ \tilde{A}_2\right) \le 1$, $0 \le p\left(\tilde{A}_2 \succ \tilde{A}_1\right) \le 1$.*
*(2) If $M^*\left(\tilde{A}_1\right) \le M_*\left(\tilde{A}_2\right)$, then $p\left(\tilde{A}_1 \succ \tilde{A}_2\right) = 0$.*
*(3) If $M_*\left(\tilde{A}_1\right) \ge M^*\left(\tilde{A}_2\right)$, then $p\left(\tilde{A}_1 \succ \tilde{A}_2\right) = 1$.*
*(4) $p\left(\tilde{A}_1 \succ \tilde{A}_2\right) + p\left(\tilde{A}_2 \succ \tilde{A}_1\right) = 1$, specially $p\left(\tilde{A}_1 \succ \tilde{A}_1\right) = 0.5$.*
*(5) For IT2 FSs $\tilde{A}_1, \tilde{A}_2$ and $\tilde{A}_3$, if $p\left(\tilde{A}_1 \succ \tilde{A}_2\right) \ge 0.5$ and $p\left(\tilde{A}_2 \succ \tilde{A}_3\right) \ge 0.5$ then $p\left(\tilde{A}_1 \succ \tilde{A}_2\right) + p\left(\tilde{A}_2 \succ \tilde{A}_3\right) \ge p\left(\tilde{A}_1 \succ \tilde{A}_3\right)$.*

**Example 4.1.** *If the trapezoidal IT2 FSs*

$$\tilde{A} = \left(\tilde{A}^U, \tilde{A}^L\right) = \left(\left(a_1^U, a_2^U, a_3^U, a_4^U; h^U\right), \left(a_1^L, a_2^L, a_3^L, a_4^L; h^L\right)\right),$$

*then the lower possibility mean value can be calculated as*

$$\begin{aligned} M_*\left(\tilde{A}\right) &= \int_0^{h^U} \alpha\left(a_1^U + \frac{a_2^U - a_1^U}{h^U}\alpha\right) d\alpha + \int_0^{h^L} \beta\left(a_1^L + \frac{a_2^L - a_1^L}{h^L}\beta\right) d\beta \\ &= \frac{1}{6}\left(a_1^U + 2a_2^U\right)\left(h^U\right)^2 + \frac{1}{6}\left(a_1^L + 2a_2^L\right)\left(h^L\right)^2 \end{aligned}$$

*Similarly, the upper possibility mean value can be calculated as*

$$\begin{aligned} M^*\left(\tilde{A}\right) &= \int_0^{h^U} \alpha\left(a_4^U + \frac{a_3^U - a_4^U}{h^U}\alpha\right) d\alpha + \int_0^{h^U} \beta\left(a_4^L + \frac{a_3^L - a_4^L}{h^L}\beta\right) d\beta \\ &= \frac{1}{6}\left(a_4^U + 2a_3^U\right)(h^U)^2 + \frac{1}{6}\left(a_4^L + 2a_3^L\right)(h^L)^2 \end{aligned}$$

Let $\tilde{A}_i = \left(\tilde{A}_i^U, \tilde{A}_i^L\right)$ be a TIT2 FS, the fuzzy preference matrix $P = \left(p\left(\tilde{A}_i \succ \tilde{A}_j\right)\right)_{n,n}$ can be obtained. Then, the ranking value of interval type-2 fuzzy set $Rank\left(\tilde{A}_i\right)$ is calculated as follows [27]:

$$Rank\left(\tilde{A}_i\right) = \frac{1}{n(n-1)}\left(\sum_{k=1}^{n} p\left(\tilde{A}_i \succ \tilde{A}_k\right) + \frac{n}{2} - 1\right) \tag{25}$$

where $1 \le i \le n$ and $\sum_{i=1}^{n} Rank\left(\tilde{A}_i\right) = 1$. The larger ranking value $Rank\left(\tilde{A}_i\right)$, the greater the IT2 FSs $\tilde{A}_i$.

5. **A New Method for Fuzzy Multi-Attributes Group Decision Making under Interval Type-2 Fuzzy Environment.** In this section, we apply the proposed aggregation operators to developing a method for dealing with fuzzy multi-attributes decision making problems under interval type-2 fuzzy environment. For a decision-making problem, let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives, and $F = \{f_1, f_2, \ldots, f_m\}$ be a finite set of attributes, and $w = (w_i)_{1\times m}$ is the weighting vector of the attributes, such that $\sum_{i=1}^{m} w_i = 1$ and $w_i \in [0, 1]$. Assume that there are $l$ decision-makers $D_1$, $D_2$, $\ldots$, and $D_l$. Let $\tilde{R}^{(k)} = \left(\tilde{A}_{ij}^{(k)}\right)_{n\times m}$ be an IT2 fuzzy decision matrix, where $\tilde{A}_{ij}^{(k)}$ is an IT2 FS, provided by the DM $D_k$ for the alternative $x_i$ with respect to the attribute $f_j$. We aggregate all individual normalized decision matrices $\tilde{R}^{(k)} = \left(\tilde{A}_{ij}^{(k)}\right)_{n\times m}$ into the collective normalized decision matrix $\tilde{R} = \left(\tilde{A}_{ij}\right)_{n\times m}$ where $\tilde{A}_{ij} = \sum_{k=1}^{l} \lambda_k \tilde{A}_{ij}^{k}$. The attribute set $F$ can generally be classified into two sets $F_1$ and $F_2$, where $F_1$ denotes the set of benefit attributes, $F_2$ denotes the set of cost attributes, $F_1 \cap F_2 = \emptyset$, and $F_1 \cup F_2 = F$.

Suppose the information about attribute weights is completely known, that is, the weight vector $w = (w_i)_{1\times m}$ of the attributes $f_j$ $(j = 1, 2, \ldots, m)$ can be completely determined in advance. Then, we utilize the TIT2FEWA operator and the TIT2FEGA operator to develop an approach to FMADM problems with interval IT2 fuzzy information, which can be described as follows.

**Step 1.** Utilize the normalized decision matrix $\tilde{R} = \left(\tilde{A}_{ij}\right)_{n\times m}$ and the weight vector $w = (w_i)_{1\times m}$, the TIT2FEWA operator and the TIT2FEGA operator are shown as follows:

$$\begin{aligned} \tilde{d}_k^A &= \mathrm{TIT2FEWA}\left(\tilde{A}_{k,1}, \tilde{A}_{k,2}, \ldots, \tilde{A}_{k,m}\right) \\ &= \mathop{\oplus_\varepsilon}_{j=1}^{m} \left(w_j \tilde{A}_j\right) \\ &= w_1 \cdot_\varepsilon \tilde{A}_{k,1} \oplus_\varepsilon w_2 \cdot_\varepsilon \tilde{A}_{k,2} \oplus_\varepsilon \cdots \oplus_\varepsilon w_m \cdot_\varepsilon \tilde{A}_{k,m} \end{aligned} \tag{26}$$

and

$$\begin{aligned} \tilde{d}_k^G &= \mathrm{TIT2FEGA}\left(\tilde{A}_{k,1}, \tilde{A}_{k,2}, \ldots, \tilde{A}_{k,m}\right) \\ &= \mathop{\otimes_\varepsilon}_{j=1}^{m} \left(\tilde{A}_{k,j}\right)^{w_j} \\ &= \left(\tilde{A}_{k,1}\right)^{w_1 \cdot \varepsilon} \otimes_\varepsilon \left(\tilde{A}_{k,2}\right)^{w_2 \cdot \varepsilon} \otimes_\varepsilon \cdots \otimes_\varepsilon \left(\tilde{A}_{k,m}\right)^{w_n \cdot \varepsilon} \end{aligned} \tag{27}$$

where $k = (1, 2, \ldots, n)$.

**Step 2.** Utilize fuzzy possibility degree Equation (24) to calculate the fuzzy preference matrix $P = (p_{ij})_{n\times n}$.

**Step 3.** Utilize the ranking Formula (25) to calculate the ranking value $Rank\left(\tilde{d}_k^A\right)$ and $Rank\left(\tilde{d}_k^G\right)$ of the TIT2 FSs $\tilde{d}_k^A$ and $\tilde{d}_k^G$, where $1 \le k \le n$. The larger the value of $Rank\left(\tilde{d}_k^A\right)$ and $Rank\left(\tilde{d}_k^G\right)$, the more the preference of the alternative $x_k$, $1 \le k \le n$.

6. **Numerical Example.** In this section, we use an example to illustrate the FMADM process of the proposed method. Table 1 shows the linguistic terms "Very Low" (VL), "Low" (L), "Medium Low" (ML), "Medium" (M), "Medium High" (MH), "High" (H), "Very High" (VH) and their corresponding interval type-2 fuzzy sets, respectively [8,9].

Assume that the problem discussed here is concerned with a manufacturing company, searching the best global supplier for one of its most critical parts used in assembling process [19]. There are three potential global suppliers $x_1$, $x_2$ and $x_3$ to be evaluated

TABLE 1. Linguistic terms and their corresponding interval type-2 fuzzy sets

| Linguistic terms | Interval type-2 fuzzy sets |
|---|---|
| Very Low (VL) | $((0, 0, 0, 0.1; 1, 1), (0, 0, 0, 0.05; 0.9, 0.9))$ |
| Low (L) | $((0, 0.1, 0.1, 0.3; 1, 1), (0.05, 0.1, 0.1, 0.2; 0.9, 0.9))$ |
| Medium Low (ML) | $((0.1, 0.3, 0.3, 0.5; 1, 1), (0.2, 0.3, 0.3, 0.4; 0.9, 0.9))$ |
| Medium (M) | $((0.3, 0.5, 0.5, 0.7; 1, 1), (0.4, 0.5, 0.5, 0.6; 0.9, 0.9))$ |
| Medium High (MH) | $((0.5, 0.7, 0.7, 0.9; 1, 1), (0.6, 0.7, 0.7, 0.8; 0.9, 0.9))$ |
| High (H) | $((0.7, 0.9, 0.9, 1; 1, 1), (0.8, 0.9, 0.9, 0.95; 0.9, 0.9))$ |
| Very High (VH) | $((0.9, 1, 1, 1; 1, 1), (0.95, 1, 1, 1; 0.9, 0.9))$ |

TABLE 2. Evaluating values of alternatives with respect to different attributes

| Attributes | Alternatives | Decision-makers | | |
|---|---|---|---|---|
| | | $D_1$ | $D_2$ | $D_3$ |
| Quality of the product ($f_1$) | $x_1$ | MH | H | MH |
| | $x_2$ | H | MH | H |
| | $x_3$ | VH | H | MH |
| Risk factor ($f_2$) | $x_1$ | H | VH | H |
| | $x_2$ | MH | H | VH |
| | $x_3$ | VH | VH | H |
| Service performance of supplier ($f_3$) | $x_1$ | VH | H | H |
| | $x_2$ | H | VH | VH |
| | $x_3$ | M | MH | MH |
| Supplier's profile ($f_4$) | $x_1$ | VH | H | H |
| | $x_2$ | H | VH | H |
| | $x_3$ | H | VH | VH |

with four attributes $f_1$: quality of the product, $f_2$: risk factor, $f_3$: service performance of supplier, $f_4$: supplier's profile (whose weight vector $w = (0.30, 0.15, 0.20, 0.35)$). An expert group is formed which consists of three experts $D_1$, $D_2$ and $D_3$ (whose weight vector is $\lambda = (0.30, 0.45, 0.25)$) from each strategic decision area. The experts $D_1$, $D_2$ and $D_3$ use the linguistic terms shown in Table 1 to represent the characteristics of the potential global suppliers $x_1$, $x_2$ and $x_3$ with respect to different attributes $f_i$ $(i = 1, 2, 3, 4)$, respectively, listed in Table 2.

Considering that the attributes are the benefit attributes except the attribute $f_2$ (risk factor), then based on Table 2, the decision matrices $\tilde{R}^{(k)} = \left(\tilde{A}_{ij}^{(k)}\right)_{3\times 4}$ $(k = 1, 2, 3)$ can be updated to the following normalized matrices respectively, listed in Table 3.

Based on Table 1 we aggregate all individual normalized IT2 fuzzy decision matrices $\tilde{R}^{(k)} = \left(\tilde{A}_{ij}^{(k)}\right)_{3\times 4}$ $(k = 1, 2, 3)$ into a collective normalized IT2 fuzzy decision matrix $\tilde{R} = \left(\tilde{A}_{ij}\right)_{3\times 4}$, where

$\tilde{A}_{11} = ((0.590, 0.790, 0.790, 0.945; 1, 1), (0.690, 0.790, 0.790, 0.868; 0.9, 0.9))$,
$\tilde{A}_{12} = ((0, 0.055, 0.055, 0.210; 1, 1), (0.028, 0.055, 0.055, 0.133; 0.9, 0.9))$,
$\tilde{A}_{13} = ((0.760, 0.930, 0.930, 1; 1, 1), (0.845, 0.930, 0.930, 0.965; 0.9, 0.9))$,
$\tilde{A}_{14} = ((0.760, 0.930, 0.930, 1; 1, 1), (0.845, 0.930, 0.930, 0.965; 0.9, 0.9))$,
$\tilde{A}_{21} = ((0.610, 0.810, 0.810, 0.955; 1, 1), (0.710, 0.810, 0.810, 0.883; 0.9, 0.9))$,
$\tilde{A}_{22} = ((0.030, 0.135, 0.135, 0.31; 1, 1), (0.083, 0.135, 0.135, 0.223; 0.9, 0.9))$,
$\tilde{A}_{23} = ((0.840, 0.970, 0.970, 1; 1, 1), (0.905, 0.970, 0.970, 0.985; 0.9, 0.9))$,

$\tilde{A}_{24} = ((0.790, 0.945, 0.945, 1; 1, 1), (0.868, 0.945, 0.945, 0.973; 0.9, 0.9)),$
$\tilde{A}_{31} = ((0.710, 0.880, 0.880, 0.975; 1, 1), (0.795, 0.880, 0.880, 0.928; 0.9, 0.9)),$
$\tilde{A}_{32} = ((0, 0.025, 0.025, 0.150; 1, 1), (0.013, 0.025, 0.025, 0.088; 0.9, 0.9)),$
$\tilde{A}_{33} = ((0.440, 0.640, 0.640, 0.840; 1, 1), (0.540, 0.640, 0.640, 0.740; 0.9, 0.9)),$
$\tilde{A}_{34} = ((0.840, 0.970, 0.970, 1; 1, 1), (0.905, 0.970, 0.970, 0.985; 0.9, 0.9)).$

Table 3. The normalized evaluating values of alternatives

| Attributes | Alternatives | Decision-makers | | |
|---|---|---|---|---|
| | | $D_1$ | $D_2$ | $D_3$ |
| Quality of the product ($f_1$) | $x_1$ | MH | H | MH |
| | $x_2$ | H | MH | H |
| | $x_3$ | VH | H | MH |
| Risk factor ($f_2$) | $x_1$ | L | VL | L |
| | $x_2$ | ML | L | VL |
| | $x_3$ | VL | VL | L |
| Service performance of supplier ($f_3$) | $x_1$ | VH | H | H |
| | $x_2$ | H | VH | VH |
| | $x_3$ | M | MH | MH |
| Supplier's profile ($f_4$) | $x_1$ | VH | H | H |
| | $x_2$ | H | VH | H |
| | $x_3$ | H | VH | VH |

**Step 1.** By Equation (26), Equation (27) and Matlab software, we can get an overall performance value based on the two Einstein operators as follows

$$\begin{aligned}
\tilde{d}_1^A &= TIT2FEWA\left(\tilde{A}_{1,1}, \tilde{A}_{1,2}, \tilde{A}_{1,3}, \tilde{A}_{1,4}\right) \\
&= ((0.636, 0.846, 0.846, 1; 1, 1), (0.735, 0.846, 0.846, 0.910; 0.9, 0.9)), \\
\tilde{d}_2^A &= TIT2FEWA\left(\tilde{A}_{2,1}, \tilde{A}_{2,2}, \tilde{A}_{2,3}, \tilde{A}_{2,4}\right) \\
&= ((0.684, 0.886, 0.886, 1; 1, 1), (0.779, 0.886, 0.886, 0.934; 0.9, 0.9)), \\
\tilde{d}_3^A &= TIT2FEWA\left(\tilde{A}_{3,1}, \tilde{A}_{3,2}, \tilde{A}_{3,3}, \tilde{A}_{3,4}\right) \\
&= ((0.657, 0.862, 0.862, 1; 1, 1), (0.750, 0.862, 0.862, 0.914; 0.9, 0.9)),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{d}_1^G &= TIT2FEGA\left(\tilde{A}_{1,1}, \tilde{A}_{1,2}, \tilde{A}_{1,3}, \tilde{A}_{1,4}\right) \\
&= ((0, 0.646, 0.646, 0.825; 1, 1), (0.537, 0.646, 0.646, 0.748; 0.9, 0.9)), \\
\tilde{d}_2^G &= TIT2FEGA\left(\tilde{A}_{2,1}, \tilde{A}_{2,2}, \tilde{A}_{2,3}, \tilde{A}_{2,4}\right) \\
&= ((0.504, 0.727, 0.727, 0.860; 1, 1), (0.628, 0.727, 0.727, 0.799; 0.9, 0.9)), \\
\tilde{d}_3^G &= TIT2FEGA\left(\tilde{A}_{3,1}, \tilde{A}_{3,2}, \tilde{A}_{3,3}, \tilde{A}_{3,4}\right) \\
&= ((0, 0.578, 0.578, 0.776; 1, 1), (0.483, 0.578, 0.578, 0.699; 0.9, 0.9)).
\end{aligned}$$

**Step 2.** Based on Equations (22) and (23), calculate interval-valued possibility mean values of the weighted decision matrix $D = \left(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3\right)$, shown as follows:

$$M\left(\tilde{d}_1^A\right) = [0.716, 0.780], \quad M\left(\tilde{d}_2^A\right) = [0.753, 0.827], \quad M\left(\tilde{d}_3^A\right) = [0.731, 0.810]$$

and

$$M\left(\tilde{d}_1^G\right) = [0.462, 0.628], \quad M\left(\tilde{d}_2^G\right) = [0.608, 0.690], \quad M\left(\tilde{d}_3^G\right) = [0.414, 0.573]$$

Based on Equation (23), we can construct the fuzzy possibility degree preference matrix $P$, shown as follows:

$$P^A = \begin{bmatrix} 0.500 & 0.294 & 0.423 \\ 0.706 & 0.500 & 0.630 \\ 0.577 & 0.370 & 0.500 \end{bmatrix}, \quad P^G = \begin{bmatrix} 0.500 & 0.084 & 0.660 \\ 0.917 & 0.500 & 1.000 \\ 0.340 & 0.000 & 0.500 \end{bmatrix}$$

**Step 3.** Based on Equation (25), the ranking values $Rank\left(\tilde{d}_j^A\right)$ and $Rank\left(\tilde{d}_j^G\right)$ of the IT2 FS $\tilde{d}_j$ can be calculated, shown as follows:

$$Rank\left(\tilde{d}_1^A\right) = 0.286, \quad Rank\left(\tilde{d}_2^A\right) = 0.389, \quad Rank\left(\tilde{d}_3^A\right) = 0.325$$

and

$$Rank\left(\tilde{d}_1^G\right) = 0.291, \quad Rank\left(\tilde{d}_2^G\right) = 0.486, \quad Rank\left(\tilde{d}_3^G\right) = 0.223$$

Rank all the suppliers in accordance with $Rank\left(\tilde{d}_2^A\right) > Rank\left(\tilde{d}_3^A\right) > Rank\left(\tilde{d}_1^A\right)$ and $Rank\left(\tilde{d}_2^G\right) > Rank\left(\tilde{d}_1^G\right) > Rank\left(\tilde{d}_3^G\right)$, the preference orders of the alternatives $x_1$, $x_2$ and $x_3$ are: $x_2 > x_3 > x_1$ and $x_2 > x_1 > x_3$.

From the above analysis, it is easily seen that although the overall rating values of the alternatives are the same by using two operators respectively, the ranking orders of the alternatives are slightly different. However, the best desirable global supplier among $x_1$, $x_2$ and $x_3$ is $x_2$. The proposed method does not require complicated computations in the implementation procedure for evaluating global supplier. It provides us with a useful way for dealing with FMADM problems based on IT2 FSs.

A comparative study was conducted to validate the results of the proposed method with those from another approach. Using Chen and Lee's fuzzy ranking method [8,9], Hu's possibility degree method [17], and Gong's geometric Bonferroni mean operator method [16], the ranking order is consistent with the one by ours. (1) Comparing with the existing fuzzy ranking method, the main advantage of our method is that the values in UMF and LMF are considered simultaneously, and the possibility degree is calculated only once instead of twice in Chen's method, resulting in reduced computing time. Moreover, it is much easier to obtain the wrong order by Chen's method when the trapezoidal interval type-2 fuzzy numbers are closer. (2) Compared with the possibility degree method, first, the computation in our possibility degree formula is simpler than the possibility degree formula of Hu's method. Second, the TIT2FEWA operator and TIT2FEGA operator can be very good to aggregate TIT2 FS information compared with TIT2-WAA operator. (3) Compared with the geometric Bonferroni mean operator method, our Einstein operator method does not require a given parameter value, and Gong's method requires the parameter $(p, q)$ values of a given Bonferroni mean operator.

7. **Conclusions.** In this paper, we give some new operations laws of TIT2 FSs based on Einstein t-norm and t-conorm. The Einstein operator and possibility degree have been extended to the interval type-2 fuzzy environment to organize and model the uncertainties better within multi-attribute decision analysis. We have presented a new method for FMADM based on the TIT2FEWA operator and TIT2FEGA operator and the possibility degree of IT2 FS. Compared with trapezoidal type-1 fuzzy numbers, trapezoidal interval type-2 fuzzy number better represents the uncertainties of decision-maker. We also use

one example to illustrate the FMADM process of the proposed method. The result shows that the proposed method provides us with a useful way to deal with FMADM problems based on IT2 FSs.

In future studies, we will further consider the trapezoidal interval type-2 fuzzy aggregation operators which take account of the various interactions or priority among the decision criteria based on the Einstein t-conorm and t-norm operation laws. At the same time, we will apply the developed procedures to some other decision-making problems where the information about attribute weights is incomplete, such as making investment choices, hierarchical decision-making and hierarchical and distributed decision making [28,29].

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