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IMPLEMENTING LIFETIME PERFORMANCE INDEX OF BURR XII PRODUCTS WITH PROGRESSIVELY TYPE II RIGHT CENSORED SAMPLE

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Abstract. *In this paper, we will construct the maximum likelihood estimator (MLE) of larger-the-better type process capability index (or lifetime performance index) C_L for the two-parameter Burr XII distribution with progressively type II right censored sample on the condition of known L. Furthermore, we propose the asymptotic normal distribution of the MLE for C_L in order to develop the hypothesis testing procedure for assessing the lifetime performance of products. Moreover, the hypothesis testing procedure not only can effectively evaluate the lifetime performance of products but also is the supplier selection criteria of the customers. Finally, two examples and Monte Carlo simulation are given to illustrate the application of the results.*
Keywords: Process capability index, Lifetime performance index, Burr XII distribution, Maximum likelihood estimator, Progressively type II right censored sample, Asymptotic normal distribution

1. **Introduction.** It is very important that enterprises emphasize effective management and assessment of quality performance for products in the competitive market. Process capability analysis is an effective means to measure the capability and performance of a manufacturing process. During the last thirty years, process capability indices (*PCIs*) have received much attention in the statistical literature. For instance, Montgomery [30] (or Kane [21]) proposed that the process capability index C_L (or C_{PL}) for evaluating the lifetime performance of electronic components, where L is the lower specification limit, since the lifetime of electronic components exhibits the larger-the-better quality characteristic of time orientation. Pearn and Chen [32], and Pearn and Shu [33] have developed a procedure and confidence intervals for the process capability index C_{PU} and C_{PL}, and presented extensive tables to test for practitioners when applying these methods. All of the above *PCIs* are assumed to be under normal distribution. The assumption of normality is commonly used in process capability analysis. Nevertheless, the normality is very questionable in manufactures, service process and business operation process. The lifetime model of many products may generally follow a non-normal distribution which

include exponential, Rayleigh, Weibull, gamma, Burr XII or other distributions and so forth. For example, Tong et al. [37] constructed a uniformly minimum variance unbiased estimator (*UMVUE*) of C_L based on the complete sample from an exponential distribution. Moreover, the *UMVUE* of C_L is then utilized to develop the hypothesis testing procedure. Chen et al. [11] also used the *UMVUE* of C_L to develop the confidence interval under an exponential distribution with the complete sample. Then, the customers can employ the testing procedure to determine whether the lifetime of electronic components attain to the required level. Suppliers can also utilize this procedure to enhance process capability. The hypothesis testing procedure not only can effectively evaluate the lifetime performance of products but also is the supplier selection decision criteria of the customers. The selection of Supplier is very important in customers' business operation. Product management is a basic function, which enables a firm to quickly and economically deliver products that are requested by customers. Suppliers' production management decisions determine their product performances, which are the critical supplier selection criteria and influence both suppliers' performances and the manufacturing firm's performance (see Vonderembse and Tracey [39]).

The Burr XII distribution has been applied in the field of quality control, reliability studies, and failure time modeling (see Soliman [35]). The probability density function (p.d.f.) and cumulative distribution function (c.d.f.) of the Burr XII distribution are given, respectively, by

$$f_X(x|\boldsymbol{\theta}) = ckx^{c-1}(1+x^c)^{-(k+1)}, \quad x > 0, \quad c > 0, \quad k > 0, \tag{1}$$

and

$$F_X(x|\boldsymbol{\theta}) = 1 - (1+x^c)^{-k}, \quad x > 0, \quad c > 0, \quad k > 0, \tag{2}$$

where the vector parameters $\boldsymbol{\theta} = (c, k)^T$, both c and k are shape parameters. For $c > 1$, the p.d.f. as Equation (1) is unimodal and is L-shaped for $c \leq 1$. We will use the notation $X \sim BXII(c, k)$ to indicate that a random variable X has the distribution given by Equation (1). Its capacity to assume various shapes often permits a good fit when used to describe biological, clinical, or other experimental data (see Wu and Yu [42]). The Burr XII distribution has been recognized as a useful model for the analysis of lifetime data. For instance, Wang and Keats [40] used the maximum likelihood method for obtaining point and interval estimators of the parameters of the Burr XII distribution. Abdel-Ghaly et al. [1] applied the Burr XII distribution to measure software reliability. Zimmer et al. [47] also presented statistical and probabilistic properties of the Burr XII distribution and described its relationship to other distributions used in reliability analyses. Moore and Papadopoulos [31] derived Bayesian estimators of the parameter k and the reliability function for the Burr XII distribution under three different loss functions. Wu and Yu [42] proposed m pivotal quantities to test the shape parameter and establish confidence interval of the shape parameter of the two-parameter Burr type XII distribution under the failure-censored plan. Liu and Chen [29] proposed a novel modification of Clements's method (see Clements [12]) using the Burr XII distribution to improve the accuracy of estimates of indices associated with one-sided specification limits for non-normal process data. Li et al. [27] proposed the empirical estimators of reliability performances for Burr XII distribution under LINEX error loss. Wu et al. [44] used the maximum likelihood method to derive the point estimators of the parameters for Burr XII distribution.

In life testing experiments, the experimenter may not always be in a position to observe the life times of all the products (or items) put on test. This may be because of time limitation and/or other restrictions (such as material resources, cost limitation, artificial negligence of recorder or typist, experimental or mechanical difficulties) on data collection. Therefore, censored samples often arise in practice. Suppose that out of n

items put on life test, for instance, the m life times $x_{1:n} \leq x_{2:n} \leq \cdots \leq x_{m:n}$ have only been observed and the life times for the rest $n-m$ components remain unobserved or missing. This type of censoring is known as type II right censoring. In type II right censoring scheme, Hong et al. [19] constructed a maximum likelihood estimator (*MLE*) of C_L under the Pareto distribution with the type II right censored sample. Moreover, the *MLE* of C_L is then utilized to develop a hypothesis testing procedure. The managers can then employ the testing procedure to assess the business performance. Wu et al. [41] also proposed a computational testing procedure to evaluate the lifetime performance of products under two-parameter exponential distribution with the type II right censored sample. Hong et al. [20] constructed a *MLE* of C_L, and developed a confidence interval for the lifetime performance index of businesses under the Pareto distribution with the type II right censored sample. Because there are many scenarios in life-testing and reliability experiments in which units are lost or removed from experimentation before failure. The loss may occur unintentionally, or it may have been designed so in the study. Unintentional loss may occur, for example, in the case of accidental breakage of an experimental unit (Consider a number of lamps placed simultaneously on life-test. One of the lamps might be accidentally broken after the start of the test but before all the lamps had burned out.), or if an individual under study drops out, or if the experimentation itself must cease due to some unforeseen circumstances such as depletion of funds, unavailability of testing facilities. More often, however, the removal of units from experimentation is pre-planned and intentional, and is done so in order to free up testing facilities for other experimentation, to save time and cost, or to exploit the straightforward analysis that often results (see Balakrishnan and Aggarwala [3]). In additional, the drop-out of patients may be caused by migration, lack of interest or by ethical decisions in clinical trails (see Balakrishnan et al. [5]). Therefore, products (or items) may break accidentally in an industrial experiment. Moreover, the experimenter can remove items which is preplanned prior to failure from a life test at various stages during the experiments, possibly resulting in a saving of costs and time of testing (see Sen [34], and Asgharzadeh [2]). The progressively type II right censored samples also often arise in practice. So, in this paper, we consider the condition of progressively type II censoring. Let m be the number of failures observed before termination and $x_{1:m:n} \leq x_{2:m:n} \leq \cdots \leq x_{m:m:n}$ be the observed ordered lifetimes. Let r_i denote the number of items removed at the time of the ith failure, $0 \leq r_i \leq n - \sum\limits_{j=1}^{i-1} r_j - i$, $i = 2, 3, \cdots, m-1$, with $0 \leq r_1 \leq n-1$ and $r_m = n - \sum\limits_{j=1}^{m-1} r_j - m$, where r_i's and m are pre-specified integers (see Viveros and Balakrishnan [38]). Notice that the complete sample ($r_1 = r_2 = \cdots = r_m = 0$) and type II right censored samples ($r_1 = r_2 = \cdots = r_{m-1} = 0,\ r_m = n-m$) are special cases of progressively type II right censored samples. The use of progressively censoring has been investigated, among others, by Cohen ([13-15]); Sen [34]; Balakrishnan and Cohen [4]; Viveros and Balakrishnan [38]; Balakrishnan and Sandhu [8]; Balakrishnan and Aggarwala [3]; Balakrishnan et al. [7]; Balakrishnan and Lin [6]; Fernández [17], Wu et al. [43], and Lio et al. [28]. Moreover, in order to evaluate the quality performance of products under no-normal distribution with progressively type II right censored samples, Lee et al. [24] proposed a testing procedure to evaluate the lifetime performance of products under the exponential distribution with progressively type II right censored samples. Lee et al. [26] constructed a Bayesian estimator of C_L based on the conjugate prior distribution and squared-error loss function under the Rayleigh distribution with the progressively type II right censored sample. Lee et al. [25] applied data transformation technology to constructs a maximum likelihood estimator (*MLE*) of C_L under the Burr XII distribution with the progressively type II right censored sample. Moreover, the *MLE* of C_L is then utilized to develop a hypothesis testing procedure. The managers can then employ the testing procedure to evaluate the quality performance of products under Burr XII distribution with progressively type II right censored samples.

Large sample also often arises in practice. In order to evaluate the quality performance of product under no-normal distribution with large sample and progressively type II right censored sample in this study. This study proposed an innovative approach to evaluate the quality performance of product under no-normal distribution with large sample and progressively type II right censored sample. Large sample theory is the cornerstone of statistical inference for quality performance evaluation model. The limiting distribution of a statistic gives approximate distributional results that are often straightforward to derive, even in complicate quality performance evaluation models. These distributions are useful for approximate inference, including constructing approximate confidence intervals and hypothesis testing. Therefore, the main aim of this study will apply the large sample theory to construct the asymptotic normal distribution of the *MLE* of C_L under the two-parameter Burr XII distribution with the progressively type II right censored sample. The asymptotic normal distribution of *MLE* of C_L is then utilized to construct a confidence interval. Furthermore, we utilize the confidence interval to develop the innovative hypothesis testing procedure for evaluating the lifetime performance of products. The innovative hypothesis testing procedure can evaluate the quality performance of products under no-normal distribution with large sample and progressively type II right censored sample. Moreover, the customers can then employ the innovative hypothesis testing procedure to determine whether the lifetime of products adheres to the required level. Suppliers can also utilize the innovative hypothesis testing procedure to enhance process capability.

The rest of this study is organized as follows. Section 2 introduces some properties of the lifetime performance index for lifetime of product under the Burr XII distribution. Section 3 discusses the relationship between the lifetime performance index C_L and conforming rate. Section 4 then presents *MLE* of C_L and its statistical properties under the Burr XII distribution with large sample and the progressively type II right censored sample. Section 5 proposes the asymptotic normal distribution of the *MLE* for C_L in order to develop the hypothesis testing procedure for evaluating the lifetime performance of products. Section 6 discusses two numerical examples. A Monte Carlo simulation algorithm of confidence level and concluding remarks are made in Section 7, and Section 8, respectively.

2. The Lifetime Performance Index.

Montgomery [30] has developed a process capability index C_L to measure the larger-the-better quality characteristic. Then, C_L is defined by

$$C_L = \frac{\mu - L}{\sigma}, \tag{3}$$

where μ, σ, and L are the process mean, the process standard deviation and the lower specification limit, respectively.

To evaluate the product performance of products, C_L can be defined as the lifetime performance index. If X comes from the Burr XII distribution, then there are several important properties, as follows:

(I) The lifetime performance index C_L can be rewritten as

$$\begin{aligned} C_L &= \frac{\mu - L}{\sigma} = \frac{kB\left(k - 1/c, 1 + 1/c\right) - L}{\sqrt{kB\left(k - 2/c, 1 + 2/c\right) - k^2B^2\left(k - 1/c, 1 + 1/c\right)}} \\ &= \frac{1}{M}\left[kB\left(k - 1/c, 1 + 1/c\right) - L\right], \quad -\infty < C_L < \frac{kB\left(k - 1/c, 1 + 1/c\right)}{\mathrm{M}}, \end{aligned} \tag{4}$$

where $M = \sqrt{kB\left(k-2/c, 1+2/c\right) - k^2B^2\left(k-1/c, 1+1/c\right)}$, $ck > 2$, $B(a, b)$ denotes the beta function, the process mean $\mu = kB(k-1/c, 1+1/c)$, the process standard deviation $\sigma = M$, and L is the lower specification limit.

(II) The failure rate function $R(x)$ is

$$R(x) = \frac{f_X(x|c,k)}{1-F_X(x|c,k)} = \frac{ckx^{c-1}}{1+x^c}, \quad x > 0, \quad c > 0, \quad k > 0. \tag{5}$$

For various values of c and k, some of the possible shapes of the failure rate function given by Equation (5) are illustrated in Figure 1. Furthermore, we can also see that for $c > 1$ the failure rate function is also unimodal and its critical point (single maximum) is $x = (c-1)^{1/c}$; and the failure rate function is L-shaped for $c \le 1$.

When the process mean $kB(k-1/c, 1+1/c) (> L)$, then the lifetime performance index $C_L > 0$. From Figure 1(b), and Figure 2, for $c > 1$, and $x > (c-1)^{1/c}$, if x is large, and k is small then the lifetime performance index C_L is relatively large and the failure rate is relatively small. Therefore, the lifetime performance index C_L reasonably and accurately describes the lifetime performance of products.

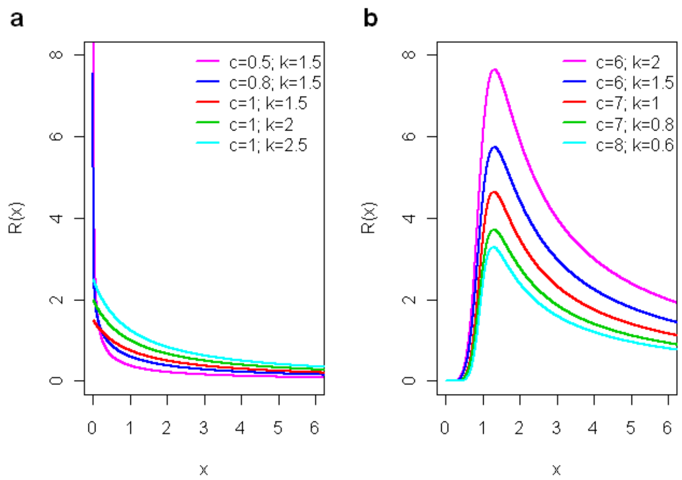

FIGURE 1. Plots of the failure rate function (5) for various values of c and k

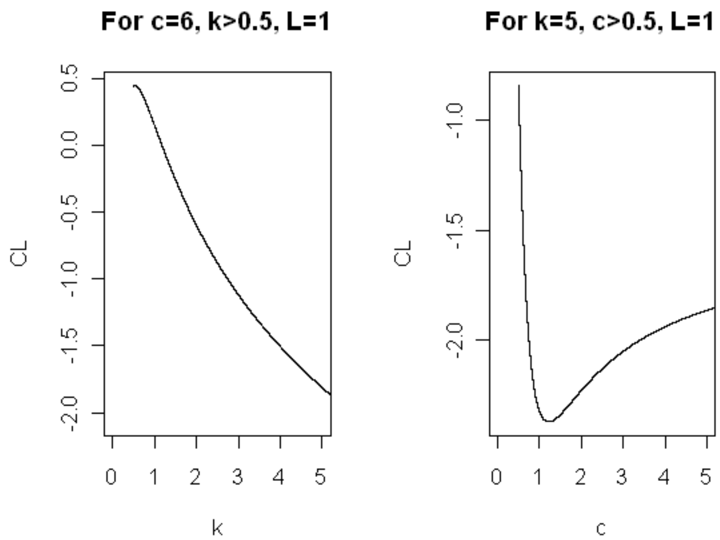

FIGURE 2. A comparison of various parameters for C_L

3. **The Conforming Rate.** If the lifetime of a product X exceeds the lower specification limit L, then the product is defined as a conforming product. The ratio of conforming products is known as the conforming rate and can be defined as

$$\begin{aligned} P_r &= P(X \geq L) \\ &= \left\{1 + [kB(k-1/c, 1+1/c) - M \cdot C_L]^c\right\}^{-k}, \quad -\infty < C_L < \frac{kB\,(k-1/c, 1+1/c)}{M}, \end{aligned} \tag{6}$$

where $M = \sqrt{kB(k-2/c, 1+2/c) - k^2B^2(k-1/c, 1+1/c)}$, $ck > 2$, c, and k are the shape parameters.

TABLE 1. The lifetime performance index C_L v.s. the conforming rate P_r for Burr XII distribution with $(\hat{c}, \hat{k}) = (3.070429, 2.48687)$

C_L	P_r	C_L	P_r	C_L	P_r
$-\infty$	0.00000	-0.10	0.40411	0.83	0.80214
-10.0	0.00002	0.00	0.44536	0.85	0.80932
-5.00	0.00117	0.10	0.48829	0.90	0.82706
-2.50	0.01981	0.20	0.53247	0.95	0.84396
-1.50	0.07360	0.30	0.57735	1.00	0.85999
-1.00	0.14171	0.40	0.62235	1.10	0.88932
-0.80	0.18249	0.50	0.66683	1.25	0.92618
-0.40	0.29372	0.60	0.71015	1.50	0.96884
-0.20	0.36492	0.70	0.75167	1.75	0.99108
-0.15	0.38424	0.80	0.79081	2.00	0.99900

Note: $C_L \to \frac{\hat{k}B\left(\hat{k}-1/\hat{c}, 1+1/\hat{c}\right)}{M} \approx 2.239725 \Rightarrow P_r \to 1.0$.

Obviously, there is a strictly increasing relationship between the conforming rate P_r, and the lifetime performance index C_L for given c, and k. Tables 1 and 2 list various C_L values and the corresponding conforming rates P_r with the given values of parameters c, and k, respectively. Moreover, we will also need Tables 1 and 2 to help for assessing the lifetime performance of products in two practical examples of Section 6, respectively. For the C_L values which are not listed in Tables 1 and 2, the conforming rate P_r can be obtained by using Equation (6).

4. **Maximum Likelihood Estimator of Lifetime Performance Index.** Suppose that $X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}$ are the corresponding progressive type II right censored sample from a life test of n products (or items) whose lifetimes follow Burr XII distribution with the p.d.f. of X given by Equation (1), and $r = (r_1, r_2, \cdots, r_m)$ denotes the corresponding numbers of products (or items) removed from the life test. Then the joint p.d.f. of all m progressively type II right censored order statistics (see Soliman [36]) is given by

$$A \prod_{i=1}^{m} \left\{ f_X(x_{i:m:n}|\boldsymbol{\theta})[1 - F_X(x_{i:m:n}|\boldsymbol{\theta})]^{r_i} \right\}, \tag{7}$$

where $A = n(n-r_1-1)\cdots(n-r_1-r_2-\cdots-r_{m-1}-m+1)$, $f_X(x_{i:m:n}|\boldsymbol{\theta})$, and $F_X(x_{i:m:n}|\boldsymbol{\theta})$ are respectively the p.d.f. and c.d.f. of X given by Equations (1) and (2). Substituting Equations (1) and (2) into Equation (7), the likelihood function of $X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}$

TABLE 2. The lifetime performance index C_L v.s. the conforming rate P_r for Burr XII distribution with $(\hat{c}, \hat{k}) = (5.927297, 2.103976)$

C_L	P_r	C_L	P_r	C_L	P_r
$-\infty$	0.00000	0.15	0.54708	1.00	0.85039
-6.00	0.00009	0.30	0.61038	1.10	0.87405
-4.00	0.00127	0.40	0.65123	1.15	0.88484
-1.50	0.06799	0.50	0.69044	1.23	0.90069
-1.00	0.14640	0.60	0.72759	1.25	0.90439
-0.60	0.25272	0.80	0.79451	1.50	0.94242
-0.30	0.35959	0.82	0.80061	2.00	0.98294
-0.10	0.44069	0.85	0.80955	2.50	0.99646
0.00	0.48297	0.90	0.82388	3.00	0.99959
0.10	0.52571	0.95	0.83749	3.50	0.99999

Note: $C_L \to \frac{\hat{k}B\left(\hat{k}-1/\hat{c},1+1/\hat{c}\right)}{M} \approx 4.134517 \Rightarrow P_r \to 1.0$.

is given as

$$\begin{aligned} L(\boldsymbol{\theta}) &= A\prod_{i=1}^{m}\left\{f_X(x_{i:m:n}|\boldsymbol{\theta})\left[1-F_X(x_{i:m:n}|\boldsymbol{\theta})\right]^{r_i}\right\} \\ &= A\prod_{i=1}^{m}\left\{\left[ckx_{i:m:n}^{c-1}(1+x_{i:m:n}^{c})^{-(k+1)}\right]\left[1-\left(1-(1+x_{i:m:n}^{c})^{-k}\right)\right]^{r_i}\right\} \\ &= A(ck)^m\prod_{i=1}^{m}x_{i:m:n}^{c-1}\left(1+x_{i:m:n}^{c}\right)^{(-k(r_i+1)+1)}, \end{aligned} \tag{8}$$

where $A = n(n-r_1-1)\cdots(n-r_1-r_2-\cdots-r_{m-1}-m+1)$. Then, the natural logarithm of the likelihood function may be written as

$$\ell(\theta) \propto m\ln(ck) + (c-1)\sum_{i=1}^{m}\ln(x_{i:m:n}) - \sum_{i=1}^{m}\left(k(r_i+1)+1\right)\ln(1+x_{i:m:n}^{c}) \tag{9}$$

The *MLE* $\hat{\boldsymbol{\theta}} = \left(\hat{c}, \hat{k}\right)^T$ of $\boldsymbol{\theta}$ can be obtained by setting the first partial derivatives of Equation (9) to zero with respect to c, and k. These likelihood equations for the parameters c and k are given by

$$\frac{\partial\ell(\boldsymbol{\theta})}{\partial c} = \frac{m}{c} + \sum_{i=1}^{m}\ln(x_{i:m:n}) - \sum_{i=1}^{m}(k(r_i+1)+1)\frac{x_{i:m:n}^{c}\ln(x_{i:m:n})}{(1+x_{i:m:n}^{c})} = 0 \tag{10}$$

and

$$\frac{\partial\ell(\boldsymbol{\theta})}{\partial k} = \frac{m}{k} - \sum_{i=1}^{m}\left(r_i+1\right)\ln\left(1+x_{i:m:n}^{c}\right) = 0. \tag{11}$$

Equation (11) yields the *MLE* of k as given by

$$\hat{k} = \frac{m}{\sum\limits_{i=1}^{m}(r_i+1)\ln(1+x_{i:m:n}^{\hat{c}})} \tag{12}$$

By using the *MLE* of k given by Equation (12), Equation (10) can reduce to

$$\frac{m}{\hat{c}}+\sum_{i=1}^{m}\ln\left(x_{i:m:n}\right)-\left[\frac{m}{\sum\limits_{i=1}^{m}\left(r_i+1\right)\ln\left(1+x_{i:m:n}^{\hat{c}}\right)}\right] \times\sum_{i=1}^{m}(r_i+1)\frac{x_{i:m:n}^{\hat{c}}\ln(x_{i:m:n})}{\left(1+x_{i:m:n}^{\hat{c}}\right)}-\sum_{i=1}^{m}\frac{x_{i:m:n}^{\hat{c}}\ln(x_{i:m:n})}{\left(1+x_{i:m:n}^{\hat{c}}\right)}=0. \quad (13)$$

Since the closed form solutions of Equation (13) is hard to be analytically solved of $\hat{c}$, we will solve the non-linear equation by using the subroutine ZREAL of IMSL from the mathematical software Compaq Visual Fortran version 6.6 and IMSL (2000) (see [16]). The subroutine ZREAL is to find the real zeros of a real function using Müller's method. The Müller's method is based on linear approximations to the function whose zero we are seeking is to approximate the function by a quadratic function (see Laurene ([22])).

According to the invariance property of the *MLE* (see Zehna [46]), the *MLE* of C_L can be written as

$$\hat{C}_L=\frac{\hat{k}B\left(\hat{k}-1/\hat{c},1+1/\hat{c}\right)-L}{\sqrt{\hat{k}B\left(\hat{k}-2/\hat{c},1+2/\hat{c}\right)-\hat{k}^2B^2\left(\hat{k}-1/\hat{c},1+1/\hat{c}\right)}}. \quad (14)$$

Moreover, the asymptotic normal distribution for the *MLE*s can be expressed in the following way (also see Soliman [36]), and Wu and Kus [45]). From the natural logarithm of the likelihood function in Equation (8), we have

$$-\frac{\partial^2\ell(\boldsymbol{\theta})}{\partial c^2}=\frac{m}{c^2}+\sum_{i=1}^{m}(k(r_i+1)+1)\frac{x_{i:m:n}^{c}\ln^2(x_{i:m:n})}{(1+x_{i:m:n}^{c})^2}=\nu_{11}(\boldsymbol{\theta}), \quad (15)$$

$$-\frac{\partial^2\ell(\boldsymbol{\theta})}{\partial k^2}=\frac{m}{k^2}=\nu_{22}(\boldsymbol{\theta}), \quad (16)$$

and

$$-\frac{\partial^2\ell(\boldsymbol{\theta})}{\partial c\partial k}\equiv-\frac{\partial^2\ell(\boldsymbol{\theta})}{\partial k\partial c}=\sum_{i=1}^{m}(r_i+1)\frac{x_{i:m:n}^{c}\ln(x_{i:m:n})}{(1+x_{i:m:n}^{c})}=\nu_{12}(\boldsymbol{\theta}). \quad (17)$$

Based on the result of Soliman [36], under some regularity conditions, the asymptotic normality results of the *MLE* of $\boldsymbol{\theta}$ can be defined as

$$\hat{\boldsymbol{\theta}}\sim N(\boldsymbol{\theta},I(\boldsymbol{\theta})^{-1}). \quad (18)$$

The Fisher information matrix $I(\boldsymbol{\theta})$ for $\boldsymbol{\theta}=(c,k)^T$ is defined by taking expectations of Equations (15)-(17). However, it is difficult to directly obtain the exact mathematical form of the above expectations. Therefore, we construct the approximate (observed) information matrix $I_0\left(\hat{\boldsymbol{\theta}}\right)$, which is given by dropping the expectation operator. The approximate (observed) information matrix $I_0\left(\hat{\boldsymbol{\theta}}\right)$ is given by

$$I_0\left(\hat{\boldsymbol{\theta}}\right)=\begin{bmatrix}-\dfrac{\partial^2\ell(\boldsymbol{\theta})}{\partial c^2} & -\dfrac{\partial^2\ell(\boldsymbol{\theta})}{\partial c\partial k}\\ \dfrac{\partial^2\ell(\boldsymbol{\theta})}{\partial k\partial c} & \dfrac{\partial^2\ell(\boldsymbol{\theta})}{\partial k^2}\end{bmatrix}_{\hat{\boldsymbol{\theta}}}=\begin{bmatrix}\nu_{11}(\boldsymbol{\theta}) & \nu_{12}(\boldsymbol{\theta})\\ \nu_{12}(\boldsymbol{\theta}) & \nu_{22}(\boldsymbol{\theta})\end{bmatrix}_{\hat{\boldsymbol{\theta}}}. \quad (19)$$

Moreover, we use the approximate (observed) asymptotic variance-covariance matrix $\left(I_0(\hat{\boldsymbol{\theta}})^{-1}\right)$ of $\boldsymbol{\theta}$ to estimate $I\left(\boldsymbol{\theta}\right)^{-1}$, where $I_0\left(\hat{\boldsymbol{\theta}}\right)^{-1}$ is expressed as

$$I_0\left(\hat{\boldsymbol{\theta}}\right)^{-1} = \begin{bmatrix} \nu_{11}(\boldsymbol{\theta}) & \nu_{12}(\boldsymbol{\theta}) \\ \nu_{12}(\boldsymbol{\theta}) & \nu_{22}(\boldsymbol{\theta}) \end{bmatrix}^{-1}_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}} = \begin{bmatrix} \operatorname{var}(\hat{c}) & \operatorname{cov}\left(\hat{c}, \hat{k}\right) \\ \operatorname{cov}\left(\hat{c}, \hat{k}\right) & \operatorname{var}\left(\hat{k}\right) \end{bmatrix}_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}. \tag{20}$$

Now, we will use $C_L \equiv h(\boldsymbol{\theta})$, and the multivariate delta method (see Casella and Berger [10], Theorem 5.5.28) stated that the asymptotic normal distribution of $h\left(\hat{\boldsymbol{\theta}}\right)$ can be defined as

$$\hat{C}_L \equiv h\left(\hat{\boldsymbol{\theta}}\right) \sim N\left(h(\boldsymbol{\theta}), \Sigma_{\boldsymbol{\theta}}\right), \tag{21}$$

i.e., $\hat{C}_L \sim N\left(C_L, \Sigma_{\boldsymbol{\theta}}\right)$. Moreover, we use the approximate (observed) asymptotic variance-covariance matrix $(\Sigma_{\hat{\boldsymbol{\theta}}})$ of $h(\boldsymbol{\theta})$ to estimate $\Sigma_{\boldsymbol{\theta}}$, where $\Sigma_{\hat{\boldsymbol{\theta}}}$ is expressed as

$$\Sigma_{\hat{\boldsymbol{\theta}}} = \left(\frac{\partial h(\boldsymbol{\theta})}{\partial c} \ \frac{\partial h(\boldsymbol{\theta})}{\partial k} \right) I_0(\boldsymbol{\theta})^{-1} \left(\begin{array}{c} \dfrac{\partial h(\boldsymbol{\theta})}{\partial c} \\ \dfrac{\partial h(\boldsymbol{\theta})}{\partial k} \end{array} \right)\Bigg|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}, \tag{22}$$

$\partial h(\boldsymbol{\theta})/\partial c$, and $\partial h(\boldsymbol{\theta})/\partial k$ are the first partial derivates of $h(\boldsymbol{\theta})$ with respect to c, and k.

5. **Testing Procedure for the Lifetime Performance Index.** In this subsection, we will apply the statistical testing procedure to evaluate whether the lifetime performance index adheres to the required level. The one-sided hypothesis testing and one-sided confidence interval for C_L can be derived by taking $\hat{C}_L$ to be asymptotic normal distribution with mean C_L, and asymptotic variance-covariance matrix $\Sigma_{\boldsymbol{\theta}}$ given by Equation (21). Assuming that the required index value of lifetime performance is larger than c^*, where c^* denotes the target value, the null hypothesis $H_0 : C_L \le c^*$ and the alternative hypothesis $H_1 : C_L > c^*$ are performed. Since the MLE of C_L is used as the test statistic, the rejection region can be obtained as $\left\{\hat{C}_L \,\middle|\, \hat{C}_L > C_0\right\}$. For a given the specified significance level α, we calculate the critical value C_0 as follows:

$$\begin{aligned} &\underset{\{C_L \le c^*\}}{Sup}\ P\left(\hat{C}_L > C_0\right) \le \alpha \\ &\Rightarrow P\left(\hat{C}_L > C_0 | C_L = c^*\right) = \alpha \\ &\Rightarrow P\left(\hat{C}_L - C_L \le C_0 - C_L | C_L = c^*\right) = 1 - \alpha \\ &\Rightarrow P\left(\frac{\hat{C}_L - C_L}{\sqrt{\Sigma_{\hat{\boldsymbol{\theta}}}}} \le \frac{C_0 - c^*}{\sqrt{\Sigma_{\hat{\boldsymbol{\theta}}}}}\right) = 1 - \alpha, \end{aligned} \tag{23}$$

where $\left(\hat{C}_L - C_L\right) \Big/ \sqrt{\Sigma_{\hat{\boldsymbol{\theta}}}} \sim N(0,1)$ and $\Sigma_{\hat{\boldsymbol{\theta}}}$ is shown in Equation (22). From Equation (23), utilizing z_α which is the percentile of the standard normal distribution with right-tail probability α, then

$$\frac{C_0 - c^*}{\sqrt{\Sigma_{\hat{\boldsymbol{\theta}}}}} = z_\alpha$$

is obtained. Thus, the critical value can be written as

$$C_0 = c^* + z_\alpha \sqrt{\Sigma_{\hat{\boldsymbol{\theta}}}} \tag{24}$$

where c^*, α and $\Sigma_{\boldsymbol{\theta}}$ denote the target value, the specified significance level and the approximate (observed) asymptotic variance-covariance matrix given by Equation (22), respectively. Moreover, we also find that C_0 is independent of n and r_i, $i = 1, 2, \cdots, m$.

In addition, the level $100(1-\alpha)\%$ one-sided confidence interval of C_L can be derived as follows:

With the pivotal quantity is $\left(\hat{C}_L - C_L\right) \Big/ \sqrt{\Sigma_{\hat{\boldsymbol{\theta}}}} \sim N(0,1)$ and z_α which is the percentile of the standard normal distribution with right-tail probability α, then

$$\begin{aligned} &P\left(\frac{\hat{C}_L - C_L}{\sqrt{\Sigma_{\hat{\boldsymbol{\theta}}}}} \le z_\alpha\right) = 1-\alpha \\ \Rightarrow &P\left(C_L \ge \hat{C}_L - z_\alpha\sqrt{\Sigma_{\hat{\boldsymbol{\theta}}}}\right) = 1-\alpha \end{aligned} \tag{25}$$

From Equation (25), we have that

$$C_L \ge \hat{C}_L - z_\alpha\sqrt{\Sigma_{\hat{\boldsymbol{\theta}}}} \tag{26}$$

is the level $100(1-\alpha)\%$ one-sided confidence interval of C_L. Thus, the $100(1-\alpha)\%$ lower confidence bound for C_L can be written as

$$\underline{LB} = \hat{C}_L - z_\alpha\sqrt{\Sigma_{\hat{\boldsymbol{\theta}}}} \tag{27}$$

where $\hat{C}_L$, α and $\Sigma_{\hat{\boldsymbol{\theta}}}$ denote the *MLE* of C_L given by Equation (14), the specified significance level, and the approximate (observed) asymptotic variance- covariance matrix given by Equation (22), respectively.

The managers can employ with the one-sided confidence interval to determine whether the product performance attains to the required level. The proposed testing procedure about C_L can be organized as follows:

Step 1: The *MLE* of the parameters c and k of the Burr XII distribution are solved by Equations (12) and (13) with the progressively type II right censored sample $X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}$ and the censoring scheme $r = (r_1, r_2, \cdots, r_m)$. We will solve the non-linear Equation (13) by using the subroutine ZREAL of IMSL from the mathematical software Compaq Visual Fortran version 6.6 and IMSL (2000) [16].

Step 2: The goodness of fit test based on the Gini statistic (see Gail and Gastwirth [18]) is applied for the progressively type II right censored sample $X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}$.

Step 3: Determine the lower lifetime limit L for products and performance index value c^*, then the testing null hypothesis $H_0 : C_L \le c^*$ and the alternative hypothesis $H_1 : C_L > c^*$ is constructed.

Step 4: Specify a significance level α.

Step 5: Given the number of observed failures before termination m, the censoring scheme $r = (r_1, r_2, \cdots, r_m)$, the lower lifetime limit L and the significance level α, then we can calculate the $100(1-\alpha)\%$ one-sided confidence interval $[\underline{LB}, \infty)$ for C_L, as

$$\underline{LB} = \hat{C}_L - z_\alpha\sqrt{\Sigma_{\hat{\boldsymbol{\theta}}}},$$

where $\hat{C}_L$, α and $\Sigma_{\hat{\boldsymbol{\theta}}}$ denote the *MLE* of C_L given by Equation (14), the specified significance level and the approximate (observed) asymptotic variance- covariance matrix given by Equation (22), respectively.

Step 6: The decision rule of statistical test is provided as:

If the performance index value $c^* \notin [\underline{LB}, \infty)$, then we will reject H_0. That is, there is an evidence to indicate that the lifetime performance index of products meets the required level.

Based on the above mentioned innovative hypothesis testing procedure, the innovative hypothesis testing procedure can evaluate easily the quality performance of products under no-normal distribution with large sample and progressively type II right censored sample. Moreover, the hypothesis testing procedure not only can effectively evaluate the lifetime performance of products but also is the supplier selection criteria of the customers. Two numerical examples illustrate the use of the testing procedure in Section 6.

6. **Numerical Examples.** We propose two procedures of test which are based on a one-sided confidence interval. Under large sample, Burr XII distribution and progressively type II right censored sample, these innovative hypothesis testing procedures can be used to determine whether the lifetime performance of products adheres to the required level. We will apply the following two examples to illustrate the use of these hypothesis testing procedures. In Example 6.1, we give data on the failure times of 25 ball bearings in endurance test from Lee et al. [26]. In Example 6.2, we present simulated data, the simulated data are generated from the Burr XII distribution with $c = 6$, $k = 2$, $n = 30$, $m = 20$, and the given censoring scheme $r = (0, 0, 0, 2, 0, 0, 0, 2, 0, 0, 0, 2, 2, 0, 0, 0, 0, 0, 0, 2)$.

Example 6.1. *(Real life data)*

Lee et al. [26] considered a type II right censored sample of size $m = 23$ *from the original data set of 25 observations which are the number of million revolutions before failure for each of ball bearings in endurance test (see Caroni [9]). The observations (in hundreds of millions)* $\{x_{i:23:25}, i = 1, \cdots, 23\} = \{0.1788, 0.2892, 0.3300, 0.4152, 0.4212, 0.4560, 0.4848, 0.5184, 0.5196, 0.5412, 0.5556, 0.6780, 0.6780, 0.6780, 0.6864, 0.6864, 0.6888, 0.8412, 0.9312, 0.9864, 1.0512, 1.0584, 1.2792\}$*, and the censoring scheme* $r = (r_1, r_2, \cdots, r_{23}) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2)$*, i.e., the 25 ball bearings placed simultaneously in endurance test. Two ball bearings are accidentally broken after the time of the 23th failure. The two ball bearings must be removed at the time of the 23th failure.*

Then, the proposed testing procedure of C_L based on a confidence interval is stated as follows:

Step 1: Consider the progressively type II right censored sample $\{x_{i:23:25}, i = 1, \cdots, 23\}$ and the censoring scheme $r = (r_1, r_2, \cdots, r_{23})$. Solve Equations (12) and (13) by using the subroutine ZREAL of IMSL from the mathematical software Compaq Visual Fortran version 6.6 and IMSL (2000) [16]. The*MLE* of the parameters of the Burr XII distribution are $\hat{c} = 3.070429$ and $\hat{k} = 2.48687$.

Step 2: We propose the goodness of fit test based on the Gini statistic for the progressively type II right censored sample $\{x_{i:23:25},\ i = 1, \cdots, 23\}$, and the censoring scheme $r = (r_1, r_2, \cdots, r_{23})$. To apply this Gini statistic to test whether the failures of ball bearings data come from the Burr XII distribution with the p.d.f. is

$$f_X\left(x|\hat{\boldsymbol{\theta}}\right) = 7.635758x^{2.070429}\left(1 + x^{3.070429}\right)^{-3.48687}, \quad x > 0 \tag{28}$$

where $\hat{\boldsymbol{\theta}} = (3.070429, 2.48687)^T$.

At a $\alpha = 0.05$ significance level, the hypothesis test is

$$H_0 : X \sim BXII(3.070429, 2.48687) \quad \text{v.s.}$$
$$H_1 : X \nsim BXII(3.070429, 2.48687).$$

The Gini statistic is given as (see Gail and Gastwirth [18])

$$G_{23} = \frac{\sum_{i=1}^{23-1} iW_{i+1}}{(23-1)\sum_{i=1}^{23} W_i},$$

where $W_i = (23 - i + 1)(Z_{(i)} - Z_{(i-1)})$, $i = 1, \cdots, 23$, $Z_{(0)} \equiv 0$, $Z_{(1)} < Z_{(2)} < \cdots < Z_{(23)}$ are the corresponding order statistics for $Z_1 = nY_1$, $Z_i = \left[n - \sum_{j=1}^{i-1}(r_j + 1)\right](Y_i - Y_{i-1})$, $i = 2, \ldots, 23$, and the data transformation $Y_i = \ln(1 + X_{i:23:25}^{3.070429})$, $i = 1, \ldots, 23$, $Y_0 = 0$ (see Lee et al. [25]; Lawless ([23])).

For $m > 20$ the rejection region is $\left\{\left|(G_m - 0.5)[12(m-1)]^{1/2}\right| > z_{\alpha/2}\right\}$, where the critical value $z_{\alpha/2}$ is the percentile of the standard normal distribution with right-tail probability $\alpha/2$. The Gini statistic is

$$G_{23} = \frac{\sum_{i=1}^{22} iW_{i+1}}{(23-1)\sum_{i=1}^{23} W_i} = 0.600728.$$

Then we get that $\left|(G_{23} - 0.5)[12(23-1)]^{1/2}\right| = 1.636636 < z_{0.025} = 1.96$, so we cannot reject H_0 at the 0.05 level of significance. That is, there is an evidence to indicate that the failures of ball bearings data come from the Burr XII distribution with the p.d.f. is

$$f_X\left(x|\hat{\boldsymbol{\theta}}\right) = 7.635758x^{2.070429}(1 + x^{3.070429})^{-3.48687}, \quad x > 0,$$

where $\hat{\boldsymbol{\theta}} = (3.070429, 2.48687)^T$.

Step 3: The lower lifetime limit L is assumed to be 0.3236569, i.e., if the lifetime of a ball bearing exceeds 0.3236569 number of million revolutions, then the ball bearing is defined as a conforming product. To deal with the product purchasers' concerns regarding operational performance, the conforming rate P_r of products is required to exceed 80 percent. Referring to Table 1, the C_L value is required to exceed 0.83. Thus, the performance index value is set at $c^* = 0.83$. The testing hypothesis: $H_0 : C_L \le 0.83$ v.s. $H_1 : C_L > 0.83$ is constructed.

Step 4: Specify a significance level $\alpha = 0.05$.

Step 5: With Equations (14), (22), and (27), we can calculate the 95% lower confidence interval bound for C_L, as

$$\begin{aligned}\underline{LB} &= \hat{C}_L - z_\alpha\sqrt{\Sigma_{\hat{\boldsymbol{\theta}}}} \\ &= 1.250000 - 1.645\sqrt{0.0505586} \\ &= 0.8801178.\end{aligned}$$

So, the 95% one-sided confidence interval for C_L is $[\underline{LB}, \infty) = [0.8801178, \infty)$.

Step 6: Because of the performance index value $c^* = 0.83 \notin [\underline{LB}, \infty)$, we reject $H_0 : C_L \le 0.83$. Thus, there is an evidence to indicate that the lifetime performance index of 25 ball bearings operation does meet the required level.

In addition, by using Equations (14) and (24), we calculate $\hat{C}_L = 1.250000 > C_0 = c^* + z_\alpha\sqrt{\Sigma_{\hat{\boldsymbol{\theta}}}} = 0.83 + 1.645\sqrt{0.0505586} \approx 1.19988$, so we also reject $H_0 : C_L \le 0.83$. Hence, it is concluded that the lifetime performance index of 25 ball bearings operation meets the required level.

Example 6.2. *(Simulated data)*

We consider a simulated data, the simulated data is a progressively type II censored sample from the Burr XII distribution with $(c = 6, k = 2)$*. The progressively type II censored sample* $\{x_{i:20:30}, i = 1, \cdots, 20\} = \{0.3177954, 0.5651290, 0.5961775, 0.6106820, 0.6232408, 0.6729921, 0.6734749, 0.7266641, 0.7712947, 0.7737789, 0.8211182, 0.8539740, 0.8625961, 0.9103306, 0.9191059, 0.9421834, 0.9600189, 0.9997976, 1.016076, 1.038353\}$ *and the censoring scheme* $r = (r_1, r_2, \cdots, r_{20}) = (0, 0, 0, 2, 0, 0, 0, 2, 0, 0, 0, 2, 2, 0, 0, 0, 0, 0, 0, 2)$*, i.e., the 30 experimental units placed simultaneously in lifetime test. Two experimental units are accidentally broken after the time of the 4th failure. The two experimental units must be removed at the time of the 4th failure. Two experimental units are accidentally broken after the time of the 8th failure. The two experimental units must be removed at the time of the 8th failure. Two experimental units are accidentally broken after the time of the 12th failure. The two experimental units must be removed at the time of the 12th failure. Two experimental units are accidentally broken after the time of the 13th failure. The two experimental units must be removed at the time of the 13th failure. Two experimental units are accidentally broken after the time of the 20th failure. The two experimental units must be removed at the time of the 20th failure.*

Then, the proposed testing procedure of C_L based on a confidence interval is stated as follows:

Step 1: Consider the progressively type II right censored sample $\{x_{i:20:30}, i = 1, \cdots, 20\}$ and the censoring scheme $r = (r_1, r_2, \cdots, r_{20})$. Solve Equations (12) and (13) by using the subroutine ZREAL of IMSL from the mathematical software Compaq Visual Fortran version 6.6 and IMSL (2000) [16]. The *MLE* of the parameters of the Burr XII distribution are $\hat{c} = 5.927297$ and $\hat{k} = 2.103976$.

Step 2: We propose the goodness of fit test based on the Gini statistic for the progressively type II right censored data $\{x_{i:20:30},\ i = 1, \cdots, 20\}$ and the censoring scheme $r = (r_1, r_2, \cdots, r_{20})$. To apply this Gini statistic to test whether the failures of the simulated data come from the Burr XII distribution with the p.d.f. given by

$$f_X\left(x|\hat{\boldsymbol{\theta}}\right) = 12.47089x^{4.927297}\left(1 + x^{5.927297}\right)^{-3.103976}, \quad x > 0, \tag{29}$$

where $\hat{\boldsymbol{\theta}} = (5.927297, 2.103976)^T$.

At a $\alpha = 0.05$ significance level, the hypothesis test is

$$\begin{aligned} &H_0 : X \sim BXII(5.927297, 2.103976) \quad \text{v.s.} \\ &H_1 : X \nsim BXII(5.927297, 2.103976) \end{aligned}$$

The Gini statistic is given as (see Gail and Gastwirth [18])

$$G_{20} = \frac{\sum_{i=1}^{20-1} iW_{i+1}}{(20-1)\sum_{i=1}^{20} W_i},$$

where $W_i = (20 - i + 1)(Z_{(i)} - Z_{(i-1)})$, $i = 1, \cdots, 20$, $Z_{(0)} \equiv 0$, $Z_{(1)} < Z_{(2)} < \cdots < Z_{(20)}$ are the corresponding order statistics for $Z_1 = 30Y_1$, $Z_i = \left[30 - \sum_{j=1}^{i-1}(r_j + 1)\right](Y_i - Y_{i-1})$, $i = 2, \ldots, 20$, and the data transformation $Y_i = \ln(1 + X_{i:20:30}^{2.57416})$.

For $m = 3, \cdots, 20$, the rejection region is $\{G_m > \xi_{1-\alpha/2}$ or $G_m < \xi_{\alpha/2}\}$, where the critical value $\xi_{\alpha/2}$ is the percentile of the G_m statistic with right-tail probability $\alpha/2$. The

Gini statistic is

$$G_{20} = \frac{\sum_{i=1}^{19} iW_{i+1}}{(20-1)\sum_{i=1}^{20} W_i} = 0.465109.$$

Since $G_{20} = 0.465109$, which is between $\xi_{0.025} = 0.37048$, and $\xi_{0.975} = 0.62952$, so we cannot reject H_0 at the 0.05 level of significance. That is, there is an evidence to indicate that the simulated data come from the Burr XII distribution with the p.d.f. is

$$f_X\left(x|\hat{\boldsymbol{\theta}}\right) = 12.47089x^{4.927297}(1+x^{5.927297})^{-3.103976}, \quad x > 0,$$

where $\hat{\boldsymbol{\theta}} = (5.927297, 2.103976)^T$.

Step 3: The lower lifetime limit L is assumed to be 0.5822911, i.e., if the lifetime of a product exceeds 0.5822911, then the product is defined as a conforming product. To deal with the product purchasers' concerns regarding operational performance, the conforming rate P_r of products is required to exceed 80 percent. Referring to Table 2, the C_L value is required to exceed 0.82. Thus, the performance index value is set at $c^* = 0.82$. The testing hypothesis: $H_0 : C_L \le 0.82$ v.s. $H_1 : C_L > 0.82$ is constructed.

Step 4: Specify a significance level $\alpha = 0.05$.

Step 5: With Equations (14), (22), and (27), we can calculate the 95% lower interval bound for C_L, as

$$\begin{aligned}\underline{LB} &= \hat{C}_L - z_\alpha\sqrt{\Sigma_{\hat{\boldsymbol{\theta}}}} \\ &= 1.340002 - 1.645\sqrt{0.0713563} \\ &= 0.9005797\end{aligned}$$

So, the 95% one-sided confidence interval for C_L is $[\underline{LB}, \infty) = [0.9005797, \infty)$.

Step 6: Because of the performance index value $c^* = 0.82 \notin [\underline{LB}, \infty)$, we reject the null hypothesis $H_0 : C_L \le 0.82$. Thus, there is an evidence to indicate that the lifetime performance index of products does meet the required level.

In addition, by using Equations (14), and (24), we calculate $\hat{C}_L = 1.340002 > C_0 = c^* + z_\alpha\sqrt{\Sigma_{\hat{\boldsymbol{\theta}}}} = 0.82 + 1.645\sqrt{0.0713563} \approx 1.25942$, so we also reject $H_0 : C_L \le 0.82$. Hence, it is concluded that the lifetime performance index of products meets the required level.

7. The Monte Carlo Simulation Study.

7.1. The Monte Carlo simulation algorithm of confidence level.

In this section, we will discuss the results of a simulation study for confidence level $(1-\alpha)$ based on a $100(1-\alpha)$% one-sided confidence interval of the lifetime performance index C_L. We used $\alpha = 0.05$, and then generated different sample sizes from Burr XII distribution with p.d.f. given by Equation (1) with respect to progressively type II right censored sample.

The Monte Carlo simulation algorithm of confidence level $(1-\alpha)$ is given in the following steps:

Step 1: Given n, m, c, k, L, and $r = (r_1, r_2, \cdots, r_m)$, where $c > 0$, $k > 0$, $ck > 2$, $m \le n$.

Step 2: (a) The generation of data $Z_1, Z_2, \cdots, Z_m$ is by the standard exponential distribution.

(b) Set $Y_1 = \frac{Z_1}{n}$, and $Y_i = \frac{Z_1}{n} + \frac{Z_2}{n-r_1-1} + \cdots + \frac{Z_i}{n-r_1-r_2-\cdots-r_{i-1}-i+1}$, for $i = 2, \cdots, m$. $Y_1, Y_2, \cdots, Y_m$ are the progressively type II right censored sample from a standard exponential distribution.

(c) Using the given values of parameters c, k, and the data transformation of $X_{i:m:n} = \left[\exp\left(\frac{Y_i}{k}\right) - 1\right]^{\frac{1}{c}}$, $i = 1, \cdots, m$, we obtain that $X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}$ are the corresponding progressively type II right censored sample from Burr XII distribution with p.d.f. given by Equation (1).

(d) The value of $\underline{LB}$ is calculated by

$$\underline{LB} = \hat{C}_L - z_\alpha \sqrt{\Sigma_{\hat{\boldsymbol{\theta}}}},$$

where $\hat{C}_L$, α, and $\Sigma_{\hat{\boldsymbol{\theta}}}$ denote the *MLE* of C_L given by Equation (14), the specified significance level and the estimated asymptotic variance- covariance matrix given by Equation (22), respectively.

(e) If $C_L \geq \underline{LB}$ then Count = 1, else Count = 0.

Step 3: (a) Step 2 is repeated 100 times.

(b) The estimation of confidence level $(1-\alpha)$ is $\widehat{(1-\alpha)}$ = total count/100.

Step 4: (a) Repeated Steps 2-4 with 100 times, then we can get the 100 estimations of confidence level as follows: $\widehat{(1-\alpha)}_1, \widehat{(1-\alpha)}_2, \cdots, \widehat{(1-\alpha)}_{100}$.

(b) The average empirical confidence level $\overline{1-\alpha}$ of $\widehat{(1-\alpha)}_i$, $i = 1, \cdots, 100$, i.e., $\overline{1-\alpha} = (1/100)\sum_{i=1}^{100} \widehat{(1-\alpha)}_i$.

(c) The sample mean square error (*SMSE*) of $\widehat{(1-\alpha)}_1, \widehat{(1-\alpha)}_2, \cdots, \widehat{(1-\alpha)}_{100}$, i.e., $SMSE = (1/100)\sum_{I=1}^{100} \left[\widehat{(1-\alpha)}_i - (1-\alpha)\right]^2$.

The results of simulation are illustrated in Tables 3 and 4 of Appendix based on $L = 1.0$, the different values of sample size n, observed number m $(n \geq m)$, shape parameters (c, k), and the censoring scheme $r = (r_1, r_2, \cdots, r_m)$, respectively. From Tables 3 and 4, based on $L = 1.0$ and $\alpha = 0.05$, the following points can be drawn:

(I) All of the average empirical confidence level $\overline{1-\alpha}$ close to confidence level $(1-\alpha)$ for any observed number m, $m \leq n$, and $r_m = n - \sum_{j=1}^{m-1} r_j - m$.

(II) As shape parameter $(c, k) = (6, 2), (6, 1.5)$, and $(7, 1.5)$, all of the *SMSE* are enough small. Moreover, the scope of *SMSE* is between 0.000240 and 0.000605.

(III) As shape parameter $(c, k) = (1.9, 2), (2, 2)$, and $(2, 2.1)$, all of the *SMSE* are enough small. Moreover, the scope of *SMSE* is between 0.000436 and 0.001428.

(IV) For any sample size n, fix the observed number m, the *SMSE* for $c > k$ are smaller than the *SMSE* for $c \leq k$.

Hence, these results from simulation studies illustrate that the performance of our proposed method is acceptable.

7.2. **The Monte Carlo simulation algorithm of the estimated risks.** In this section, we will discuss the results of a simulation study for the estimated risks of the *MLE*s and the asymptotic normal distribution of the lifetime performance index C_L. Using the similar algorithm described in Lee et al. [26], we generated different sample sizes from Burr XII distribution with p.d.f. given by Equation (1) with respect to progressively type II right censored sample.

The Monte Carlo simulation algorithm of the estimated risks is given in the following steps:

Step 1: Given n, m, c, k, L, and $r = (r_1, r_2, \cdots, r_m)$, where $c > 0$, $k > 0$, $ck > 2$, $m \leq n$.

Step 2: (a) The generation of data $Z_1, Z_2, \cdots, Z_m$ is by the standard exponential distribution.

(b) Set $Y_1 = \frac{Z_1}{n}$, and $Y_i = \frac{Z_1}{n} + \frac{Z_2}{n-r_1-1} + \cdots + \frac{Z_i}{n-r_1-r_2-\cdots-r_{i-1}-i+1}$, for $i = 2, \cdots, m$. $Y_1, Y_2, \cdots, Y_m$ are the progressively type II right censored sample from a standard exponential distribution.

(c) Using the given values of parameters c, k, and the data transformation of $X_{i:m:n} = \left[\exp\left(\frac{Y_i}{k}\right) - 1\right]^{\frac{1}{c}}$, $i = 1, \cdots, m$, we obtain that $X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}$ are the corresponding progressively type II right censored sample from Burr XII distribution with p.d.f. given by Equation (1).

Step 3: The *MLE* of the lifetime performance index C_L, and approximate (observed) asymptotic variance-covariance matrix $\Sigma_{\hat{\boldsymbol{\theta}}}$ are computed by using Equations (14), and (22), respectively.

Step 4: (a) Repeat Steps 2 and 3 10000 times, then we can get the 10000 estimations of $\hat{C}_L$, and $\Sigma_{\hat{\boldsymbol{\theta}}}$ as follows: $(\hat{C}_L)_i$, and $(\Sigma_{\hat{\boldsymbol{\theta}}})_i$, $i = 1, \cdots, 10000$.

(b) The *SMSE* of $\left(\hat{C}_L\right)_1, \left(\hat{C}_L\right)_2, \ldots, \left(\hat{C}_L\right)_{10000}$, i.e., $SMSE = (1/10000) \sum_{i=1}^{10000} \left[\left(\hat{C}_L\right)_i - C_L\right]^2$.

(c) The mean of variance-covariance matrix (*MVCM*) of $(\Sigma_{\hat{\boldsymbol{\theta}}})_1, (\Sigma_{\hat{\boldsymbol{\theta}}})_2, \ldots, (\Sigma_{\hat{\boldsymbol{\theta}}})_{100000}$, i.e., $MVCM = (1/10000) \sum_{i=1}^{10000} \left[(\Sigma_{\hat{\theta}})_i\right]$.

In addition, the Monte Carlo simulation algorithms of the estimated risks for $\hat{c}$, and $\hat{k}$ are completely analogous to the above algorithm of the estimated risks for $\hat{C}_L$; hence, they are omitted in here. The results of simulation are illustrated in Tables 5 and 6 of Appendix based on $L = 1.0$, the different values of sample size n, observed number m $(n \geq m)$,shape parameters (c, k), and the censoring scheme $r = (r_1, r_2, \cdots, r_m)$, respectively. From Tables 5 and 6 of Appendix, based on $L = 1.0$, the following points can be drawn:

(I) All of the mean variance-covariance matrix (*MVCM*) are smaller than the *SMSE* for any observed number m, $m \leq n$, and $r_m = n - \sum_{j=1}^{m-1} r_j - m$. It is indicated that the approximate (observed) asymptotic variance-covariance matrix is better than their corresponding *SMSE* for the considered cases.

(II) Fix the sample size n, if the observed number m increases, then it can be seen that the estimated risks of $\hat{C}_L, \hat{c}, \hat{k}$ will decrease for the shape parameter $(c, k) = (6, 2), (6, 1.5)$, and $(7, 1.5)$.

(III) Fix the sample size n, if the observed number m increases, then it can be seen that the estimated risks of $\hat{C}_L, \hat{c}, \hat{k}$ will decrease for the shape parameter $(c, k) = (1.9, 2), (2, 2)$, and $(2, 2.1)$.

Hence, these results from simulation studies illustrate that the performance of our proposed method is acceptable.

8. **Conclusions.** As the standard of living and economic development in Taiwan get increasingly higher, customer's demands for production quality become more critically requested. In order to satisfy customer needs, the merchant should control and promote their quality of processes and products by using statistical methods. Therefore, process capability indices are widely used to determine whether product quality meets the required level in the service (or manufacturing) industry. Lifetime performance index C_L is one of most well-known capability indexes, introduced by Montgomery [30], for larger-the-better type quality characteristic. The assumption of normality is commonly used in process

capability analysis, but, it is very questionable in most process such as manufactures, service, and business operation. Moreover, in life testing experiments, the experimenter may not always be in a position to observe the life times of all the products (or items) on test. This may be because of time limitation and/or other restrictions (such as material resources, cost limitation, artificial negligence of recorder or typist, experimental or mechanical difficulties) on data collection. Therefore, censored samples often arise in practice. Progressive censoring is quite useful in many practical situations where budget constraints are in place or there is a demand for rapid testing or in the case of accidental breakage of an experimental unit. Moreover, large sample often arises in life testing experiments. This study constructs the *MLE* of C_L under the two-parameter Burr XII distribution with the progressively type II right censored sample by using multivariate delta method and large sample theory. The *MLE* of C_L can be utilized to develop a confidence interval of C_L in the condition of known L. Further, the confidence interval of C_L is utilized to develop the innovative hypothesis testing procedure for evaluating the lifetime performance of products. The innovative hypothesis testing procedure is a quality performance assessment system in Enterprise Resource Planning (ERP). The innovative hypothesis testing procedure can assess the lifetime performance of products under Burr XII distribution with large sample and progressively type II right censored sample. For example, the innovative hypothesis testing procedure is utilized to evaluate the quality performance of products in the large sample quality data of biological, clinical, or other lifetime experiments, and in many practical situations where budget constraints are in place or there is a demand for rapid testing or in the case of accidental breakage of an experimental unit. The innovative hypothesis testing procedure not only can be easily applied and can effectively evaluate whether the lifetime of products adheres to the required level but also is the supplier selection criteria of the customers. The selection of Supplier is very important in customers' business operation. In addition, this study provides a table of the lifetime performance index with its corresponding conforming rate. Hence, for any specified conforming rate, a corresponding C_L can be obtained, and the hypotheses of the innovative testing procedure can also be expressed in terms of the conforming rate under L is known limit.

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Appendix A.

TABLE 3. Average empirical confidence level $(1-\alpha)$ for C_L under $\alpha = 0.05$

n	m	$r=(r_1, r_2, \cdots, r_m)$	$c=6,\ k=2$	$c=6,\ k=1.5$	$c=7,\ k=1.5$
30	15	$(15, 14*0)$	0.95420 (0.000432)	0.94510 (0.000340)	0.94500 (0.000470)
	20	$(10, 19*0)$	0.95310 (0.000444)	0.94620 (0.000585)	0.94910 (0.000477)
	25	$(5, 24*0)$	0.95430 (0.000482)	0.94630 (0.000451)	0.95150 (0.000361)
40	25	$(15, 24*0)$	0.95430 (0.000482)	0.94630 (0.000451)	0.95150 (0.000361)
	30	$(10, 29*0)$	0.95470 (0.000541)	0.947500 (0.000463)	0.95039 (0.000388)
	35	$(5, 34*0)$	0.95390 (0.000549)	0.94870 (0.000423)	0.94580 (0.000582)
50	30	$(20, 29*0)$	0.95470 (0.000541)	0.947500 (0.000463)	0.95039 (0.000388)
	35	$(15, 34*0)$	0.95390 (0.000549)	0.94870 (0.000423)	0.94580 (0.000582)
	40	$(10, 39*0)$	0.95320 (0.000464)	0.94600 (0.000452)	0.94810 (0.000511)
100	80	$(20, 79*0)$	0.95460 (0.000466)	0.94850 (0.000515)	0.94690 (0.000555)
	85	$(15, 84*0)$	0.94670 (0.000240)	0.95110 (0.000367)	0.94970 (0.000605)
	90	$(10, 89*0)$	0.95200 (0.000380)	0.94710 (0.000453)	0.94930 (0.000519)

Note:

1. n and m denote the sample size and the observed number, respectively.
2. $r = (r_1, r_2, \cdots, r_m)$ denotes censoring scheme and $r = (15, 14*0) = (15, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ for $m = 15$.
3. The value in parentheses are sample mean square error (*SMSE*).

Table 4. Average empirical confidence level $(1-\alpha)$ for C_L under $\alpha = 0.05$

n	m	$r = (r_1, r_2, \cdots, r_m)$	$c = 1.9,\ k = 2$	$c = 2,\ k = 2$	$c = 2,\ k = 2.1$
30	15	$(15, 14 * 0)$	0.94570 (0.001151)	0.95420 (0.000992)	0.95560 (0.001072)
	20	$(10, 19 * 0)$	0.94810 (0.001273)	0.95850 (0.001331)	0.96280 (0.001314)
	25	$(5, 24 * 0)$	0.94190 (0.001175)	0.94890 (0.000933)	0.95210 (0.000893)
40	25	$(15, 24 * 0)$	0.94790 (0.001121)	0.95870 (0.00136)	0.96200 (0.001428)
	30	$(10, 29 * 0)$	0.95070 (0.001253)	0.95960 (0.001226)	0.96290 (0.001403)
	35	$(5, 34 * 0)$	0.95010 (0.000865)	0.95870 (0.000879)	0.96160 (0.001064)
50	30	$(20, 29 * 0)$	0.94630 (0.001159)	0.95570 (0.001153)	0.95950 (0.001317)
	35	$(15, 34 * 0)$	0.94020 (0.000722)	0.94780 (0.000698)	0.95070 (0.000735)
	40	$(10, 39 * 0)$	0.95560 (0.000824)	0.96590 (0.001281)	0.96750 (0.001211)
100	80	$(20, 79 * 0)$	0.94830 (0.000661)	0.95350 (0.000745)	0.95550 (0.000715)
	85	$(15, 84 * 0)$	0.94950 (0.000707)	0.95260 (0.000550)	0.95430 (0.000637)
	90	$(10, n89 * 0)$	0.94540 (0.000436)	0.94980 (0.000568)	0.95180 (0.000566)

Note:

1. n and m denote the sample size and the observed number, respectively.
2. $r = (r_1, r_2, \cdots, r_m)$ denotes censoring scheme and $r = (15, 14 * 0) = (15, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ for $m = 15$.
3. The value in parentheses are sample mean square error (*SMSE*).

TABLE 5. Estimated risks of the *MLE*s $\hat{C}_L$, $\hat{c}$, $\hat{k}$ for C_L, c, k

n	m	$r=(r_1,r_2,\cdots,r_m)$	$c=6,\ k=2$		$c=6,\ k=1.5$		$c=7,\ k=1.5$	
			SMSE	*MVCM*	*SMSE*	*MVCM*	*SMSE*	*MVCM*
30	15	$(15, 14*0)$	0.094117 (1.467, 0.506)	0.082164 (1.350, 0.388)	0.089152 (1.599, 0.248)	0.076544 (1.492, 0.199)	0.084368 (2.235, 0.248)	0.071254 (2.030, 0.199)
	20	$(10, 19*0)$	0.080166 (1.251, 0.333)	0.069274 (1.119, 0.259)	0.071231 (1.384, 0.166)	0.063194 (1.256, 0.137)	0.071171 (1.884, 0.166)	0.060147 (1.710, 0.137)
	25	$(5, 24*0)$	0.069855 (1.085, 0.237)	0.062527 (0.939, 0.194)	0.061214 (1.223, 0.123)	0.050227 (1.063, 0.106)	0.056125 (1.664, 0.123)	0.048257 (1.446, 0.106)
40	25	$(15, 24*0)$	0.069855 (0.971, 0.238)	0.062527 (0.866, 0.195)	0.061214 (1.076, 0.123)	0.050227 (0.968, 0.106)	0.056125 (1.464, 0.123)	0.048257 (1.318, 0.106)
	30	$(10, 29*0)$	0.054103 (0.820, 0.186)	0.053545 (0.748, 0.156)	0.048657 (0.919, 0.099)	0.041705 (0.842, 0.086)	0.047951 (1.251, 1.146)	0.040765 (0.099, 0.086)
	35	$(5, 34*0)$	0.048120 (0.730, 0.148)	0.045422 (0.661, 0.130)	0.039822 (0.820, 0.081)	0.035832 (0.748, 0.072)	0.037412 (1.116, 0.081)	0.034672 (1.018, 0.072)
50	30	$(20, 29*0)$	0.054103 (0.762, 0.187)	0.053545 (0.707, 0.156)	0.048657 (0.845, 0.099)	0.041705 (0.791, 0.086)	0.047951 (1.151, 0.099)	0.040765 (1.076, 0.086)
	35	$(15, 34*0)$	0.048120 (0.681, 0.149)	0.045422 (0.626, 0.130)	0.039822 (0.757, 0.081)	0.035832 (0.703, 0.072)	0.037412 (1.031, 0.081)	0.034672 (0.957, 0.072)
	40	$(10, 39*0)$	0.040835 (0.594, 0.560)	0.038275 (0.129, 0.112)	0.034648 (0.666, 0.070)	0.030887 (0.630, 0.063)	0.034121 (0.906, 0.070)	0.030168 (0.857, 0.063)
100	80	$(20, 79*0)$	0.023271 (0.284, 0.056)	0.018772 (0.274, 0.053)	0.015626 (0.318, 0.032)	0.015584 (0.307, 0.030)	0.015425 (0.433, 0.032)	0.015250 (0.418, 0.030)
	85	$(15, 84*0)$	0.020561 (0.277, 0.052)	0.017638 (0.260, 0.049)	0.016153 (0.311, 0.029)	0.014720 (0.293, 0.028)	0.014489 (0.423, 0.029)	0.014403 (0.398, 0.028)
	90	$(10, 89*0)$	0.016836 (0.254, 0.048)	0.016827 (0.248, 0.047)	0.015014 (0.287, 0.027)	0.013963 (0.279, 0.027)	0.013769 (0.391, 0.027)	0.013583 (0.379, 0.027)

Note:

1. n and m denote the sample size and the observed number, respectively.
2. $r=(r_1,r_2,\cdots,r_m)$ denotes censoring scheme and $r=(15,14*0)=(15,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$ for $m=15$.
3. The value in parentheses are sample mean square error (*SMSE*).

TABLE 6. Estimated risks of the *MLE*s $\hat{C}_L$, $\hat{c}$, $\hat{k}$ for C_L, c, k

n	m	$r=(r_1,r_2,\cdots,r_m)$	$c=1.9,\ k=2$		$c=2,\ k=2$		$c=2,\ k=2.1$	
			SMSE	*MVCM*	*SMSE*	*MVCM*	*SMSE*	*MVCM*
30	15	$(15,14*0)$	0.240069 (0.168, 0.563)	0.223462 (0.140, 0.402)	0.233766 (0.187, 0.563)	0.193225 (0.155, 0.402)	0.257669 (0.185, 0.649)	0.212223 (0.153, 0.458)
	20	$(10,19*0)$	0.159916 (0.125, 0.333)	0.135950 (0.112, 0.259)	0.156942 (0.139, 0.333)	0.130877 (0.124, 0.259)	0.173420 (0.137, 0.379)	0.147204 (0.122, 0.291)
	25	$(5,24*0)$	0.119993 (0.109, 0.237)	0.104215 (0.094, 0.194)	0.118173 (0.121, 0.237)	0.100540 (0.104, 0.194)	0.130882 (0.119, 0.268)	0.113364 (0.103, 0.217)
40	25	$(15,24*0)$	0.120859 (0.097, 0.238)	0.105303 (0.087, 0.195)	0.119017 (0.108, 0.238)	0.101574 (0.096, 0.195)	0.131765 (0.107, 0.269)	0.114524 (0.095, 0.218)
	30	$(10,29*0)$	0.095367 (0.082,0.186)	0.084874 (0.075, 0.156)	0.094157 (0.091, 0.186)	0.082036 (0.083, 0.156)	0.104217 (0.090, 0.209)	0.092632 (0.082, 0.174)
	35	$(5,34*0)$	0.077225 (0.073, 0.148)	0.070719 (0.066, 0.130)	0.076455 (0.081, 0.148)	0.068514 (0.073, 0.130)	0.084679 (0.080, 0.166)	0.077543 (0.072, 0.144)
50	30	$(20,29*0)$	0.095752 (0.076, 0.187)	0.085419 (0.071, 0.156)	0.094536 (0.085, 0.187)	0.082555 (0.079, 0.156)	0.104615 (0.084, 0.210)	0.093215 (0.077, 0.174)
	35	$(15,34*0)$	0.077503 (0.068, 0.149)	0.071631 (0.063, 0.130)	0.076726 (0.076, 0.149)	0.069319 (0.070, 0.130)	0.084968 (0.075, 0.166)	0.078356 (0.069, 0.145)
	40	$(10,39*0)$	0.067902 (0.060, 0.129)	0.061542 (0.056, 0.112)	0.067225 (0.066 , 0.129)	0.059650 (0.062, 0.112)	0.074479 (0.065, 0.145)	0.067494 (0.061, 0.124)
100	80	$(20,79*0)$	0.030240 (0.028, 0.056)	0.029357 (0.027, 0.053)	0.030148 (0.032, 0.056)	0.028568 (0.030, 0.053)	0.033484 (0.031, 0.062)	0.032409 (0.030, 0.058)
	85	$(15,84*0)$	0.028275 (0.028, 0.052)	0.027488 (0.026, 0.049)	0.028159 (0.031, 0.052)	0.026759 (0.029, 0.049)	0.031295 (0.030, 0.057)	0.030361 (0.028, 0.055)
	90	$(10,89*0)$	0.026475 (0.026 , 0.048)	0.025833 (0.025, 0.047)	0.026378 (0.028, 0.048)	0.025156 (0.028, 0.047)	0.029320 (0.028, 0.053)	0.028546 (0.027, 0.051)

Note:

1. n and m denote the sample size and the observed number, respectively.
2. $r=(r_1,r_2,\cdots,r_m)$ denotes censoring scheme and $r=(15,14*0)=(15,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$ for $m=15$.
3. The values in parentheses are estimated risks of the maximum likelihood estimators $(\hat{c},\hat{k})$ for $(c,\ k)$.