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# A HYBRID APPROACH FOR CONTROL OF A CLASS OF INPUT-AFFINE NONLINEAR SYSTEMS

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Abstract. *In this paper, a methodology to control a class of input-affine nonlinear systems is presented. The method is particularly suitable for systems for which classical feedback-linearization and linear parameter-varying-based control methods are not applicable or difficult to apply. It borrows concepts from both these methods and results in a combined technique, more widely applicable and yet efficient, for the control of certain nonlinear systems, such as those typically encountered in flow control. The proposed method essentially consists in applying a step similar to feedback linearization to reduce the nonlinearities of the system and next converts the resulting system into a linear parameter-varying form. The method is demonstrated on a set of numerical examples, including a modified viscous Burgers' equation and an inverted pendulum on a cart.*

**Keywords:** Flow control, Nonlinear control, Hybrid control, Feedback linearization, Linear parameter-varying control

1. **Introduction.** While linear control is appealing and efficient, many, if not most, systems of practical interest are nonlinear. Considering the example of flow control, a turbulent fluid flow exhibits a strong nonlinearity. Fluid flows are usually modeled by the Navier-Stokes equations which lead after discretization to a very high-dimensional nonlinear system intractable for online optimization and control.

The control of general nonlinear systems as encountered in flow control is hence more challenging than linear control since there is no control method applicable to a general nonlinear system. Typically, the followed strategy is to develop methods for a specific class of nonlinear problems. These control strategies include adaptive nonlinear control [1], neural networks-based control [2], sum of squares-based control [3], sliding mode control [4], integrator backstepping [5], nonlinear optimal control [6], feedback linearization [7] and Linear Parameter Varying [8], each with its own pros and cons.

We here combine feedback linearization and Linear Parameter-Varying (LPV) control to develop a powerful and widely applicable method for a class of input-affine nonlinear systems. Feedback linearization is one of the model-based nonlinear control methods. It is a geometric method where the central idea is the algebraic transformation of the underlying nonlinear system into a fully or partially linear one. In this regard, its name is a bit misleading since it is not an approximate method like Jacobian linearization. Feedback linearization has been attracting the attention of researchers for many years and has been extensively studied in the literature [9, 10, 11]. It finds applications in biomedical devices, robotic systems, motor drivers, aircraft and automotive industry, among others. After transformation of the system into an equivalent linear system, the machinery of linear control theory can be used. Most of the time, it requires that state variables or process disturbances are available either through measurements or via an estimation scheme. As a consequence, the method may require a lot of information about the underlying process to be controlled, which may not be easy to get in some practical applications.

LPV-based control is another nonlinear system control strategy, arguably more advanced and complicated than feedback linearization. In a nutshell, it is a control technique for systems where the system matrices depend linearly on a time-varying parameter vector. The time-varying parameter vector may include external or user-defined parameters involving system states and/or control inputs. In the latter case, to distinguish from the solely external counterpart, the name *quasi-LPV* is used. LPV control is formulated as a semi-definite optimization problem where a control objective function is minimized under the constraint of a system of Linear Matrix Inequalities (LMIs). The solution of the optimization problem guarantees the existence of a controller such that the $\mathcal{L}_2$-gain from disturbance to the controlled output is bounded from above for all admissible parameter trajectories. Conceptually, it is possible to put a wide class of nonlinear systems into an LPV form by a suitable definition of parameters. There exists a large body of literature on control of systems based on LPV approaches. In [12, 13, 14, 15] parameter-dependent Lyapunov functions are used, the range of parameters is gridded and, for each grid point, a set of LMIs is solved. In [8], the followed strategy does not require gridding of the parameter space since a constant Lyapunov function is assumed. In all LPV frameworks, the derived controller consists of a Linear Time-Invariant (LTI) controller scheduled by the parameters. LPV-based control technique is a powerful nonlinear control method and, like feedback linearization, it has been widely used in many industrial applications, [16], [17] or [18] to cite just a few.

However, for some nonlinear systems, there exist situations where neither feedback linearization nor LPV-based control approaches can be applied or, at least, are difficult to apply. For example, feedback linearization-based control requires transformations which may not be easy to find or some required geometric conditions may not be satisfied. On the other hand, the optimization problem to synthesize an LPV controller may not have a feasible solution, mainly due to a large number of scheduling parameters. In this paper, we propose a method which combines feedback linearization and LPV control to develop a hybrid method for a class of input-affine nonlinear systems. The main idea of the proposed method is to reduce the nonlinearity in the original system so that the rest of the system can be put in LPV form with a minimum number of scheduling parameters. To this end, we add-subtract a subset of inputs to/from the system, apply a step similar as feedback linearization to weaken the nonlinearities of the system and finally convert the resulting model into an LPV form. As a result, the nonlinearities in the model are reduced through feedback linearization and the system control is achieved based on LPV control. The developed method is thus targetting applications requiring nonlinear control.

The paper is structured as follows. In Section 2, feedback linearization approaches are briefly introduced and their shortcomings are discussed. The proposed method is presented in its core concepts in Section 3 for input-affine nonlinear systems with equal number of control inputs ($u$) and states ($x$), hereafter denoted as *u-x-square* systems. In Section 4, the method is extended to a more general class of input-affine nonlinear systems using results from Section 3. Since the method heavily relies on the LPV control theory, a brief summary of the multiplier-based LPV control theory is given in Section 5. A discussion on the advantages/disadvantages of the proposed method is given in Section

6. Section 7 presents various case studies to demonstrate the strength and usefulness of the proposed method. Finally, we conclude with closing remarks in Section 8.

## 2. Overview of Feedback Linearization for MIMO Systems.

In geometric nonlinear control, mainly two kinds of feedback linearization methods exist: input-output and input-state linearization. Next, we will briefly summarize each of these methods.

### 2.1. Input-output linearization.

Consider the following class of nonlinear systems with equal number of inputs ($u$) and outputs ($y$), denoted as *u-y-square* systems.

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)u_1 + \cdots + g_p(x)u_p, \\ y_1 &= h_1(x), \\ &\vdots \\ y_p &= h_p(x), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u = [u_1 \cdots u_p]^T \in \mathbb{R}^p$, $y = [y_1 \cdots y_p]^T \in \mathbb{R}^p$, $f$, $\{g_i\}_{i=1}^p$, are assumed to be smooth vector fields and $\{h_j\}_{j=1}^p$ to be smooth functions. Starting to differentiate $y_j$ in (1) $\gamma_j$ times until at least one of the inputs appear on the right hand side, we obtain

$$y_j^{(\gamma_j)} = L_f^{\gamma_j} h_j + \sum_{i=1}^{p} L_{g_i}(L_f^{\gamma_j - 1} h_j)u_i, \tag{2}$$

where $L_f h_j(x) : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto \dfrac{\partial h_j}{\partial x} f(x)$, is called the *Lie derivative* of $h_j$ with respect to $f$ and $L_f^k h_j(x) \triangleq L_f(L_f^{k-1} h_j(x))$. Next, define $H(x) \in \mathbb{R}^{p \times p}$ as

$$H(x) \triangleq \begin{bmatrix} L_{g_1} L_f^{\gamma_1 - 1} h_1 & \cdots & L_{g_p} L_f^{\gamma_1 - 1} h_1 \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{\gamma_p - 1} h_p & \cdots & L_{g_p} L_f^{\gamma_p - 1} h_p \end{bmatrix}. \tag{3}$$

We now introduce the definition of *vector relative degree* for *u-y-square* MIMO systems.

**Definition 2.1.** *The system* (1) *is said to have* ***vector relative degree*** $\gamma = [\gamma_1\, \gamma_2 \cdots \gamma_p]^T$ *at* $x_0$ *if* $L_{g_i} L_f^k h_j(x) = 0$, $0 \le k \le \gamma_j - 2$, $1 \le i$, $j \le p$ *and* $H(x_0)$ *is nonsingular in a neighborhood of* $x_0$.

If the system (1) has vector relative degree $\gamma$, then (2) may be written as

$$\begin{bmatrix} y_1^{(\gamma_1)} \\ \vdots \\ y_p^{(\gamma_p)} \end{bmatrix} = \begin{bmatrix} L_f^{\gamma_1} h_1 \\ \vdots \\ L_f^{\gamma_p} h_p \end{bmatrix} + H(x) \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} \triangleq v. \tag{4}$$

Since $H(x)$ is invertible in a ball around $x_0$, the linearizing state feedback control law is given as

$$u = -H^{-1}(x) \begin{bmatrix} L_f^{\gamma_1} h_1 \\ \vdots \\ L_f^{\gamma_p} h_p \end{bmatrix} + H^{-1}(x)v. \tag{5}$$

Many control objectives such as pole placement and reference tracking can be achieved using this class of methods. The main disadvantages of the input-output feedback linearization are that *i*) the system states may evolve in a region where $H(x)$ is not invertible and *ii*) the zero-dynamics (which exists when $\gamma_1 + \gamma_2 + \cdots + \gamma_p < n$) may not be stable. In such cases, the method is not applicable.

2.2. **Input-state linearization.** Alternatively, we may consider (1) without output:

$$\mathcal{S} : \dot{x} = f(x) + g_1(x)u_1 + \cdots + g_p(x)u_p = f(x) + g(x)u, \tag{6}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $g(x) \in \mathbb{R}^{n\times p}$. Further, assume that $f$ and $\{g_i\}_{i=1}^p$ are smooth. Now, let

$$\hat{\mathcal{S}} : \dot{\hat{x}} = \hat{f}(\hat{x}) + \hat{g}_1(\hat{x})\hat{u}_1 + \cdots + \hat{g}_p(\hat{x})\hat{u}_p = \hat{f}(\hat{x}) + \hat{g}(\hat{x})\hat{u} \tag{7}$$

be another system with $\hat{x} \in \mathbb{R}^n$, $\hat{u} \in \mathbb{R}^p$, $\hat{g}(\hat{x}) \in \mathbb{R}^{n\times p}$, where again the smoothness conditions on the corresponding vector fields $\hat{f}$ and $\{\hat{g}_i\}_{i=1}^p$ are assumed.

**Definition 2.2.** *The systems $\mathcal{S}$ and $\hat{\mathcal{S}}$ are said to be **feedback equivalent** if there exists a transformation*

$$\Phi : \begin{cases} \hat{x} = \mathscr{R}(x) \\ u = \alpha(x) + \beta(x)\hat{u} \end{cases}$$

*that maps $\mathcal{S}$ into $\hat{\mathcal{S}}$. In case that $\alpha(x) = 0$, $\beta(x) = I_p$, the systems $\mathcal{S}$ and $\hat{\mathcal{S}}$ are said to be **state equivalent**.*

The mapping $\Phi$ is determined through the following set of partial differential equations [19]:

$$\frac{\partial \mathscr{R}}{\partial x}\big[f(x) + g(x)\alpha(x)\big] = \hat{f}(\mathscr{R}(x)), \tag{8a}$$

$$\frac{\partial \mathscr{R}}{\partial x}\big[g(x)\beta(x)\big] = \hat{g}(\mathscr{R}(x)). \tag{8b}$$

When the system $\mathcal{S}$ is transformed into

$$\hat{\mathcal{S}}_{\text{sl}} : \dot{z} = Az + Bu = Az + \sum_{i=1}^{p} B_i u_i \tag{9}$$

via $z = \mathscr{R}(x)$, it is said to be "*state-linearizable*". On the other hand, when $\mathcal{S}$ is transformed into

$$\hat{\mathcal{S}}_{\text{fl}} : \dot{z} = Az + B\hat{u} = Az + \sum_{i=1}^{p} B_i \hat{u}_i \tag{10}$$

via $z = \mathscr{R}(x)$ and $u = \alpha(x) + \beta(x)\hat{u}$, it is said to be "*feedback-linearizable*".

Before presenting the necessary and sufficient conditions for state (respectively, feedback) linearizability, we need some concepts from differential geometry. To the system $\mathcal{S}$, we associate distributions $\mathcal{D}^1 \subset \mathcal{D}^2 \subset \cdots \subset \mathcal{D}^n$ defined as

$$\mathcal{D}^k \triangleq \Big\{ ad_f^j g_i,\ 0 \le j \le k-1,\ 1 \le i \le p \Big\},\quad k = 1, \cdots, n, \tag{11}$$

where $ad_f^j g_i = [f, ad_f^{j-1} g_i]$ for $j \ge 1$, $ad_f^0 g_i = g_i$ and $[f, g_i] \triangleq \dfrac{\partial g_i}{\partial x} f(x) - \dfrac{\partial f}{\partial x} g_i(x)$.

**Theorem 2.1.** [9, 10] *Consider the system $\mathcal{S}$ defined in* (6).

(*a*) *The system $\mathcal{S}$ is locally state-linearizable if and only if*

(*i*) $dim\big(span\{g(x), ad_f^1 g(x), \cdots, ad_f^{n-1} g(x)\}\big) = n$,

(*ii*) $[ad_f^j g,\ ad_f^r g] = 0$, $0 \le j < r \le n$.

(*b*) *The system $\mathcal{S}$ is locally feedback-linearizable if and only if*

(*i*) $dim\big(span\{g(x), ad_f^1 g(x), \cdots, ad_f^{n-1} g(x)\}\big) = n,$

(*ii*) $\mathcal{D}^{(n-1)}$ *is involutive, meaning that* $[\mathcal{D}^{(n-1)}, \mathcal{D}^{(n-1)}] \subseteq \mathcal{D}^{(n-1)}$.

Although the above conditions are useful to test whether a system is state-linearizable (respectively, feedback-linearizable), the transformation itself is determined through the solution of a system of partial differential Equations (8), which may be difficult, if not impossible, to solve in general. This constitutes the main disadvantage of input-state feedback linearization methods. The second disadvantage is that when the original nonlinear system state equation is linearized, the corresponding output equations may become nonlinear which introduces difficulties in the control design phase.

## 3. A Hybrid Nonlinear Control Methodology.

### 3.1. Derivation of the method.

Being aware of the disadvantages/difficulties or non-applicability of the above mentioned classical feedback linearization methods, we now present an alternative approach for input-affine *u-x-square* nonlinear systems, next extend to a more general class of input-affine nonlinear systems in Section 4. To start with, consider

$$\dot{x} = Kx + f(x) + G(x)u, \tag{12}$$

with

$$K = \begin{bmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & \ddots & \vdots \\ k_{n1} & \cdots & k_{nn} \end{bmatrix}, \quad f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}, \quad G(x) = \begin{bmatrix} g_{11}(x) & \cdots & g_{1n}(x) \\ \vdots & \ddots & \vdots \\ g_{n1}(x) & \cdots & g_{nn}(x) \end{bmatrix},$$

where $k_{ij}$, $i, j = 1, \cdots, n$, are real constants and $f_i(x)$, $g_{ij}(x)$, $i, j = 1, \cdots, n$, are real functions of $x$. The objective is to transform the system (12) in a form where the resulting nonlinearity is "weaker". Consider the input transformation given by

$$\tilde{u} \triangleq G(x)u, \tag{13}$$

and, let $x(t) \in I \triangleq [I_1, I_2, \cdots, I_n]$ for all $t$ means that $x_i(t) \in I_i \triangleq [\underline{x}_i, \overline{x}_i]$, $i = 1, \cdots, n$, for all $t$. In general, $G(x)$ is not invertible for all $x(t) \in I$ but invertibility is required to pass back from $\tilde{u}$ to $u$. We modify $G(x)$ so that the modified input transformation, $\widetilde{G}(x)$, is non-singular for all $x(t) \in I$. To this end, we add-subtract $\ell_i u_i$ ($1 \le i \le n$) to/from the $i$-th subequation in (12):

$$\dot{x}_i = \sum_{j=1}^{n} k_{ij}\, x_j + f_i(x) + \underbrace{\sum_{j=1}^{n} g_{ij}(x)\, u_j + \ell_i\, u_i}_{= \sum_{j=1}^{n} \left(\widetilde{G}_\ell(x)\right)_{ij} u_j} - \ell_i\, u_i. \tag{14}$$

This operation is repeated for several subequations in (12) until $\widetilde{G}_\ell(x)$ is non singular for all $x(t) \in I$. As will be made clear below, the number of add-substract operations drives the number of time-varying parameters in the subsequent LPV control and must then be minimized. Let the new system after the "add-subtract" operations be given as

$$\dot{x} = Kx + f(x) + \underbrace{\left(G(x) + G_{\ell,s}\right)}_{\widetilde{G}_{\ell,s}(x)} u - G_{\ell,s} u, \tag{15}$$

with $G_{\ell,s}$ a diagonal matrix

$$G_{\ell,s} = \begin{bmatrix} s_1\ell_1 & & \\ & \ddots & \\ & & s_n\ell_n \end{bmatrix},$$

where $s_i$, $i = 1, \cdots, n$, are binary variables (0 or 1) and $\ell_i$ are constants. Specifically, $s_i$ denotes whether $\ell_i u_i$ is added-subtracted to/from the $i$-th equation. If $\ell_i u_i$ is added-subtracted, then $s_i = 1$, otherwise $s_i = 0$. The modified input transformation matrix $\widetilde{G}_{\ell,s}(x)$ then writes

$$\widetilde{G}_{\ell,s}(x) = \begin{bmatrix} g_{11}(x) + s_1\ell_1 & g_{12}(x) & \cdots & g_{1n}(x) \\ g_{21}(x) & g_{22}(x) + s_2\ell_2 & \cdots & g_{2n}(x) \\ \vdots & \vdots & \vdots & \vdots \\ g_{n1}(x) & g_{n2}(x) & \cdots & g_{nn}(x) + s_n\ell_n \end{bmatrix}. \tag{16}$$

Next, we formulate the minimization of the number of add-subtract operations as follows. For an element $\mathscr{S}$ of the power set $2^{\mathsf{S}}$ of $\mathsf{S} \triangleq \big\{s = (s_1, s_2, \cdots, s_n)\big\}$, define $\Pi_{\mathscr{S}}$ as the determinant of $\widetilde{G}_{\ell,s}(x)$, i.e.,

$$\Pi_{\mathscr{S}}(x;\ell) = \det\Big(\widetilde{G}_{\ell,s}(x)\Big)\Big|_{s=\mathscr{S}}.$$

To minimize the number of "add-subtract" operations which are required to make $\widetilde{G}_{\ell,s}(x)$ invertible, the binary set $\mathscr{S}$ is chosen as the minimum cardinality (with minimum number of "1"s) element of $\mathsf{S}$ such that

$$\exists \ell \;/\; \Pi_{\mathscr{S}}(x;\ell) \neq 0, \quad \forall x(t) \in I. \tag{17}$$

**Remark 3.1.** *In the determination of $\ell$, in general there are infinitely many solutions. From now on, we will denote the chosen one by $\ell^\star$ and the associated vector $s$ as $s^\star$.*

Assume without loss of generality that solution of (17) results in $s_i^\star = 1$, $1 \le i \le r$ and $s_i^\star = 0$, $r+1 \le i \le n$. Equation (15) then becomes

$$\dot{x} = Kx + f(x) + \underbrace{\big(G(x) + G_{\ell^\star,s^\star}\big)}_{\widetilde{G}_{\ell^\star,s^\star}(x)} u - G_{\ell^\star,s^\star} u \tag{18}$$

with

$$G_{\ell^\star,s^\star} = \mathrm{diag}(\ell_1^\star, \cdots, \ell_r^\star, 0, \cdots, 0).$$

Next, assume that we shift one of the components of the state vector $x$, say $x_k$, $1 \le k \le n$, by $\delta$ and define $\tilde{x}_k = x_k + \delta$ so that $\tilde{x}_k \in [\underline{\tilde{x}}_k, \overline{\tilde{x}}_k]$, where $\underline{\tilde{x}}_k > 0$. After multiplying-dividing each entry of $G_{\ell^\star,s^\star}u$ in (18) by $\tilde{x}_k$, this yields

$$G_{\ell^\star,s^\star} u = N_{\ell^\star}(\rho)\tilde{x}_k,$$

where

$$N_{\ell^\star}(\rho) = \begin{bmatrix} \ell_1^\star\rho_1 \\ \vdots \\ \ell_r^\star\rho_r \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad \rho_j \triangleq \frac{u_j}{\tilde{x}_k}, \quad j = 1, \cdots, r.$$

To generalize the developments, let us introduce $\tilde{x} = x + \delta x$ ($\dot{\tilde{x}} = \dot{x}$) where $\delta x_i = \delta\delta_{ik}$ with $\delta_{ij}$ the Kronecker symbol. Equation (18) may then rewrite as

$$\dot{\tilde{x}} = \underbrace{[K - K_{\ell^\star}(\rho)]}_{A(\rho)}\tilde{x} + f^{\delta x}(\tilde{x}) + \widetilde{G}^{\delta x}_{\ell^\star,s^\star}(\tilde{x})u - K\delta x, \tag{19}$$

where

$$K_{\ell^\star}(\rho) = \begin{bmatrix} 0 & \cdots & \ell_1^\star\rho_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \ell_r^\star\rho_r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ & & k\text{th col.} & & & \end{bmatrix},$$

and

$$f^{\delta x}(\tilde{x}) = f(x + \delta x), \quad \widetilde{G}^{\delta x}_{\ell^\star,s^\star}(\tilde{x}) = \widetilde{G}_{\ell^\star,s^\star}(x + \delta x).$$

Defining the transformation

$$\tilde{u} \triangleq f^{\delta x}(\tilde{x}) + \widetilde{G}^{\delta x}_{\ell^\star,s^\star}(\tilde{x})u - K\delta x, \tag{20}$$

(19) becomes

$$\dot{\tilde{x}} = A(\rho)\tilde{x} + \tilde{u}. \tag{21}$$

**Remark 3.2.** *In the framework of input-state linearization presented in Section 2.2, the transformation* (20) *is an input-state transformation with* $\mathscr{R}(x) = I_n$, $\beta(x) = (\widetilde{G}^{\delta x}_{\ell^\star,s^\star})^{-1}(\tilde{x})$ *and* $\alpha(x) = (\widetilde{G}^{\delta x}_{\ell^\star,s^\star})^{-1}(\tilde{x})\left(K\delta x - f^{\delta x}(\tilde{x})\right)$.

State Equation (21) is a special case of the state equation of a general LPV system (in the sense that only the state matrix $A$ is parameter-dependent)

$$\dot{x} = \mathcal{A}(\rho)x + \mathcal{B}_u(\rho)u,$$

where $\rho \triangleq \begin{pmatrix} \rho_1 & \cdots & \rho_{n_\rho} \end{pmatrix}^T$ denotes the vector of time-varying parameters in the system. When disturbances, as well as measured and controlled outputs, are considered, the general LPV system takes the form

$$\dot{x} = \mathcal{A}(\rho)x + \mathcal{B}_w(\rho)w + \mathcal{B}_u(\rho)u, \tag{22a}$$
$$z = \mathcal{C}_z(\rho)x + \mathcal{D}_{zw}(\rho)w + \mathcal{D}_{zu}(\rho)u, \tag{22b}$$
$$y = \mathcal{C}_y(\rho)x + \mathcal{D}_{yw}(\rho)w + \mathcal{D}_{yu}(\rho)u, \tag{22c}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, $u \in \mathbb{R}^{n_u}$ is the manipulated input, $w \in \mathbb{R}^{n_w}$ is the disturbance input on the system, $z \in \mathbb{R}^{n_z}$ is the controlled output, $y \in \mathbb{R}^{n_y}$ is the measured output and $\rho \in \mathbb{R}^{n_\rho}$ is the time-varying parameter vector. We will control the system (21) using LPV control theory. As mentioned before, there exists a large body of literature on control of such systems. The approaches presented in [12, 13, 14, 15] require gridding of the parameter space, which becomes computationally involved even if the number of parameters is very moderate (say more than three parameters). In the case of medium-to-large scale systems, it is very likely that the number of parameters will be relatively large and hence we will use a method suitable for LPV systems in rational form and which does not require griding. In Section 5, this underlying LPV control theory will be explained very briefly but the interested reader can refer to [8] where this method was proposed and to the references therein for more detailed information.

3.2. **Summary of the approach and remarks.** The core ideas on how to combine feedback linearization and LPV to derive a widely applicable control technique are now briefly summarized. The main implementation steps are as follows:

1. The original system (12) is input added-subtracted and the subtracted input terms are divided-multiplied by a shifted state component. This introduces a new nonlinearity, parameterized by $\rho_i$ which is a function of inputs and the shifted state.
2. The introduction of the scheduling parameters $\rho_i$, $i = 1, \cdots, r$, along with the definition of a new control variable $\tilde{u}$, see (20), reformulates the original system as a LPV system (21). The minimization of the number of add-subtract operations minimizes the number of time-varying parameters $\rho_i$, i.e., we typically want to have $r \ll n$.
3. The control input $\tilde{u}$ will be determined from an LPV control algorithm to be described in Section 5.
4. Using (20), $u$ is finally determined (states are measurable and $\widetilde{G}^{\delta x}_{\ell^\star,s^\star}(\tilde{x})$ is invertible).

4. **Extension of the Method to a More General Class of Input-Affine Nonlinear Systems.** In this section, we extend the method proposed in Section 3 to a more general class of input-affine nonlinear systems. The generalization is based on an initial parametrization of part of the dynamics and subsequent application of the method developed for *u-x-square* systems in Section 3 to the square part of the rest of dynamics. Given the following class of input-affine nonlinear systems

$$\dot{x}_a = f_a(x), \tag{23a}$$

$$\dot{x}_b = K_b x + f_b(x) + G_b(x)u, \tag{23b}$$

$$y = h(x), \tag{23c}$$

with $x = [x_a \ x_b]^T \in \mathbb{R}^n$, $x_b$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $f_b(x) \in \mathbb{R}^m$, $G_b(x) \in \mathbb{R}^{m\times m}$, and $K_b \in \mathbb{R}^{m\times n}$ is a constant matrix. Applying the proposed method of Section 3 to (23b) results in

$$\dot{\tilde{x}}_a = \tilde{f}_a(\tilde{x}), \tag{24a}$$

$$\dot{\tilde{x}}_b = A_b(\tilde{\rho})\tilde{x} + \tilde{u}, \tag{24b}$$

$$y = \tilde{h}(\tilde{x}), \tag{24c}$$

where $\tilde{u} \triangleq f^{\delta x}_b(\tilde{x}) + \widetilde{G}^{\delta x}_{b_{\ell^\star},s^\star}(\tilde{x})u - K_b\delta x$ as before and $\tilde{\rho}(t) \in R^{n_{\tilde{\rho}}}$ is the parameter vector resulting from application of the proposed method. After parameterizing $\tilde{f}_a(\tilde{x})$ (using system states) as $\tilde{f}_a(\tilde{x}) = A_a(\hat{\rho})\tilde{x}$ and $\tilde{h}(\tilde{x})$ as $\tilde{h}(\tilde{x}) = C(\hat{\rho})$, the system (24) can be compactly represented as

$$\dot{\tilde{x}} = \begin{bmatrix} A_a(\hat{\rho}) \\ A_b(\tilde{\rho}) \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ \tilde{u} \end{bmatrix} \triangleq A(\rho)\tilde{x} + \tilde{\tilde{u}}, \tag{25a}$$

$$y = C(\hat{\rho})\tilde{x}, \tag{25b}$$

where $\rho(t) = [\hat{\rho}(t) \ \tilde{\rho}(t)]^T$ and $\tilde{x} = [x_1, \cdots, x_{k-1}, \tilde{x}_k, x_{k+1}, \cdots, x_n]^T$ with $\tilde{x}_k = x_k + \delta$. When the disturbance and controlled output channels are included, the system (25) can be represented as the general LPV form (22). To clarify the extended method by a simple example, consider the following system

$$\begin{aligned} \dot{x}_1 &= 4x_1 + x_2^2, \\ \dot{x}_2 &= -2x_1 + x_1x_2 + 5x_1u_1 + x_2u_2, \\ \dot{x}_3 &= x_1 + x_3 + \sin(x_2) + 2x_1u_1 + u_2, \\ y &= x_2. \end{aligned}$$

Letting $x_a = x_1$, $x_b = [x_2\ x_3]^T$, $\hat{\rho}_1 = x_2$, we have

$$K_b = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad f_b(x) = \begin{pmatrix} x_1 x_2 \\ \sin(x_2) \end{pmatrix}, \quad G_b(x) = \begin{bmatrix} 5x_1 & x_2 \\ 2x_1 & 1 \end{bmatrix}, \quad h(x) = x_2$$

and

$$\begin{aligned} \dot{x}_1 &= 4x_1 + \hat{\rho}_1 x_2, \\ \dot{x}_b &= K_b x + f_b(x) + G_b(x)u. \end{aligned}$$

Next, assume that for a control problem associated with the given system we add/subtract $\ell_1^\star u_1$ to/from the second state equation to make $G_b$ invertible. Next, multiply-divide the subtracted $\ell_1^\star u_1$ in the second state equation by $\tilde{x}_1 = x_1 + \delta$, which is positive in the range of interest. Defining $\tilde{\rho} = \dfrac{u_1}{\tilde{x}_1}$, we have

$$A_b(\tilde{\rho}) = \begin{bmatrix} -2 - \ell_1^\star \tilde{\rho} & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \tilde{u} = \begin{pmatrix} 2\delta + (\tilde{x}_1 - \delta)x_2 \\ -\delta + \sin(x_2) \end{pmatrix} + \begin{bmatrix} 5(\tilde{x}_1 - \delta) + \ell_1^\star & x_2 \\ 2(\tilde{x}_1 - \delta) & 1 \end{bmatrix} u,$$

$C(\hat{\rho}) = [0 \quad 1 \quad 0]$ and

$$\begin{aligned} \dot{\tilde{x}}_1 &= 4\tilde{x}_1 - 4\delta + \hat{\rho}_1 x_2, \\ \dot{\tilde{x}}_b &= A_b(\tilde{\rho})\tilde{x} + \tilde{u}. \end{aligned}$$

Finally, parameterizing $-4\delta$ as $-4\delta = -4\dfrac{\delta}{\tilde{x}_1}\tilde{x}_1 \triangleq -4\hat{\rho}_2\tilde{x}_1$, we have the form (25) with $\tilde{x}_a = \tilde{x}_1$, $\tilde{x}_b = [x_2\ \ x_3]^T$, $\tilde{x} = [\tilde{x}_a\ \ \tilde{x}_b]^T$, $A_a(\hat{\rho}) = [4 - 4\hat{\rho}_2\ \ \hat{\rho}_1]$.

## 5. **Recap of LPV Control Theory.**

We now briefly recall the basics of the LPV approach.

### 5.1. **The LPV control problem.**

Consider system (22) where the dependence on $\rho$ is fractional in each term. We assume $\mathcal{D}_{yu}(\rho) = 0$, i.e., there is no direct effect of control inputs on measured outputs. The parameter vector $\rho$ can be "pulled out" [20] from the system such that (22) becomes

$$\dot{x} = Ax + B_p p + B_w w + B_u u, \tag{26a}$$
$$q = C_q x + D_{qp} p + D_{qw} w + D_{qu} u \quad \text{and} \quad p = \Delta q, \tag{26b}$$
$$z = C_z x + D_{zp} p + D_{zw} w + D_{zu} u, \tag{26c}$$
$$y = C_y x + D_{yp} p + D_{yw} w + D_{yu} u, \tag{26d}$$

where $p \in \mathbb{R}^{n_p}$, $q \in \mathbb{R}^{n_q}$ are internal signals entering and leaving the perturbation block $\Delta \triangleq \operatorname{diag}\left(\rho_1 I_{n_1}, \rho_2 I_{n_2}, \cdots, \rho_{n_\rho} I_{n_\rho}\right)$ for some positive integers $n_1, \cdots, n_\rho$.

The control design problem is to find a controller $\mathscr{K}$ with state space equations

$$\dot{x}_c = A_c x_c + B_{p_c} p_c + B_y y, \tag{27a}$$
$$q_c = C_{q_c} x_c + D_{q_c p_c} p_c + D_{q_c y} y \quad \text{and} \quad p_c = \Delta_c q_c, \tag{27b}$$
$$u = C_u x_c + D_{u p_c} p_c + D_{uy} y, \tag{27c}$$

where $\Delta_c = \Delta$ is the controller scheduling function. The plant with controller is shown in Figure 1, where

$$\mathscr{G} \triangleq \left[\begin{array}{c|ccc} A & B_p & B_w & B_u \\ \hline C_q & D_{qp} & D_{qw} & D_{qu} \\ C_z & D_{zp} & D_{zw} & D_{zu} \\ C_y & D_{yp} & D_{yw} & D_{yu} \end{array}\right], \quad \mathscr{K} \triangleq \left[\begin{array}{c|cc} A_c & B_y & B_{p_c} \\ \hline C_u & D_{uy} & D_{u p_c} \\ C_{q_c} & D_{q_c y} & D_{q_c p_c} \end{array}\right]. \tag{28}$$

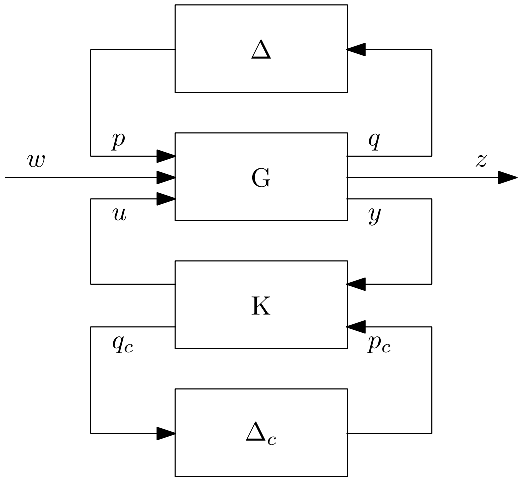



FIGURE 1. Scheduled LTI-plant with scheduled LTI-controller

For all admissible parameter trajectories $\rho(t) \in \mathbb{R}^{n_\rho}$, the closed-loop system is required to be stable and guarantees an $\mathcal{L}_2$-gain from the disturbance $w$ to the controlled output channel $z$ to be less than $\tau$, i.e.,

$$\sup_{w \in \mathcal{L}_2(\mathbb{R}^{nw}), w \neq 0} \frac{||z||_2}{||w||_2} < \tau, \tag{29}$$

where the $\mathcal{L}_2$-norm of a signal $s(t)$ is defined as

$$||s(t)||_2 = \sqrt{\int_{-\infty}^{\infty} s(t)^T s(t) dt}.$$

We normalize the parameter vector $\rho$ to $\bar{\rho}$ such that $|\bar{\rho}_i(t)| \leq 1$ for all $t \geq 0$. For the rest of the paper, we replace $\rho$ with $\bar{\rho}$ and assume that each parameter is bounded in magnitude by 1. With such a parameter vector, we associate the "multiplier" sets

$$\mathcal{P} = \left\{ \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} : Q \prec 0, S = -S^T, R = -Q \right\}, \tag{30}$$

$$\widetilde{\mathcal{P}} = \left\{ \begin{pmatrix} \widetilde{Q} & \widetilde{S} \\ \widetilde{S}^T & \widetilde{R} \end{pmatrix} : \widetilde{R} \succ 0, \widetilde{S} = -\widetilde{S}^T, \widetilde{Q} = -\widetilde{R} \right\}. \tag{31}$$

5.2. **Existence conditions for the required controller.** A theorem by [8] is now given which provides sufficient conditions for the existence of a stabilizing controller of the form (27) that guarantees an upper bound on the $\mathcal{L}_2$-gain from the disturbance to the controlled output. In the statement of the theorem, $\mathcal{M}_\perp$ denotes a basis for the orthogonal complement of the image of $\mathcal{M}$ and the symbol $\star$ stands for $\mathcal{B}$ in $\begin{pmatrix} \mathcal{A} & \star \\ \mathcal{B}^T & \mathcal{C} \end{pmatrix}$ and $\star^T \mathcal{M} \mathcal{B}$. Let

$$F_1 = \left(\begin{array}{c:c:c:c} 0 & 0 & X & 0 \\ \hdashline 0 & P & 0 & 0 \\ \hdashline X & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & -\tau I \end{array}\right), \quad F_2 = \left(\begin{array}{c:c:c:c} 0 & 0 & Y & 0 \\ \hdashline 0 & \widetilde{P} & 0 & 0 \\ \hdashline Y & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \tau I \end{array}\right),$$

$$G_1 = \left(\begin{array}{ccc} A & B_p & B_w \\ \hdashline C_q & D_{qp} & D_{qw} \\ 0 & I & 0 \\ \hdashline I & 0 & 0 \\ \hdashline 0 & 0 & I \end{array}\right), \quad G_2 = \left(\begin{array}{ccc} I & 0 & 0 \\ \hdashline 0 & I & 0 \\ -B_p^T & -D_{qp}^T & -D_{zp}^T \\ \hdashline -A^T & -C_q^T & -C_z^T \\ \hdashline 0 & 0 & I \end{array}\right).$$

**Theorem 5.1.** [8] *A controller of the form* (27) *exists such that the closed-loop system is stable for all admissible parameter trajectories and the* $\mathcal{L}_2$*-gain from* $w$ *to* $z$ *is less than* $\tau$ *if there exist* $P \in \mathcal{P}$, $\widetilde{P} \in \widetilde{\mathcal{P}}$, $X = X^T \in \mathbb{R}^{n_x \times n_x}$ *and* $Y = Y^T \in \mathbb{R}^{n_x \times n_x}$ *such that*

$$\begin{pmatrix} \left(\star\right)_{\perp}^{T}\left(\star\right)^{T} F_1 G_1 \begin{pmatrix} C_y^T \\ D_{yp}^T \\ D_{yw}^T \end{pmatrix}_{\perp} & \star \\ \begin{pmatrix} C_z & D_{zp} & D_{zw} \end{pmatrix} \begin{pmatrix} C_y^T \\ D_{yp}^T \\ D_{yw}^T \end{pmatrix}_{\perp} & -\tau I \end{pmatrix} \prec 0,$$

$$\begin{pmatrix} \left(\star\right)_{\perp}^{T}\left(\star\right)^{T} F_2 G_2 \begin{pmatrix} B_u \\ D_{qu} \\ D_{zu} \end{pmatrix}_{\perp} & \star \\ \begin{pmatrix} B_w^T & D_{qw}^T & D_{zw}^T \end{pmatrix} \begin{pmatrix} B_u \\ D_{qu} \\ D_{zu} \end{pmatrix}_{\perp} & \tau I \end{pmatrix} \succ 0, \tag{32a}$$

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0, \quad \begin{pmatrix} R & I \\ I & \widetilde{R} \end{pmatrix} \succ 0. \tag{32b}$$

The controller is constructed from $X$, $Y$, the multipliers $P$, $\widetilde{P}$ and from the system matrices. The controller construction steps are lengthy and technical and hence are skipped here. The details can be found in [8].

5.3. **Number of scheduling parameters in LPV control methods.** As mentioned in Section 1, there exist many different linear-parameter varying control methods, in which parameter gridding approaches or multiplier-based approaches can be used. In all cases of LPV control algorithms, the computational burden of controller design, $C_{\text{burd}}$, can be expressed as $C_{\text{burd}} \approx \sigma 2^{\kappa N_{\text{par}}}$ with $N_{\text{par}}$ denoting the number of parameters. The constants $\sigma > 0$ and $\kappa > 0$ depend on the LPV control method used. As a result, for all LPV control applications it is always desirable to reduce the number of scheduling parameters from computational point of view. Moreover, the feasibility of the associated LMIs for a given problem and the performance of a synthesized LPV controller may be affected by the number of scheduling parameters because the required stability and performance have to be satisfied for all parameter trajectories. Beyond a certain number of parameters, the LMIs of controller synthesis become infeasible. If not, we would have the chance of controlling all nonlinear systems that can be put in LPV form; which is obviously not the case. The proposed method improves these issues by reducing the number of scheduling parameters.

6. **Contribution of the Paper and Discussion of Pros-Cons.** This paper contributes to the field of nonlinear control by hybridizing two nonlinear control methods: feedback linearization and LPV control.

As will be shown in the case studies in Section 7, there exist many examples where feedback linearization and LPV control are difficult to apply or simply unapplicable. The present hybrid method combining the power of two methods yields a more widely applicable solution. The method was motivated by developing a solution to the control of nonlinear systems encountered in flow control applications as shown for the case study in Example 7.3. However, the applicability of the method is for a more general class of input-affine nonlinear systems as demonstrated by its use for a practical case study in Example 7.4. One of the main powerful sides of the proposed method is its applicability to nonlinear systems where a time-dependent reference tracking can be done under process disturbances owing to the fact that the core of the method depends on LPV control theory. However, as discussed in Section 5.3, LPV method may suffer from a high number of scheduling parameters, a limitation alleviated by the proposed method through cancelation of some or all of the system nonlinearities.

Although the developed method is in principle applicable to the given class of input-affine nonlinear systems, the final control relies on an LPV control algorithm and hence there is no guarantee that the LMIs related to the LPV control of the resulting LPV model (corresponding to a chosen $x_k$) will be feasible. The structure of the LPV model of the system is dependent on which $x_k$ is chosen. It is not known a priori which state should be shifted. A state is selected, the corresponding LPV model is obtained and a feasible solution for LMIs (32) is searched. If no feasible solution is found, another state is shifted and the same procedure is re-applied.

As for the pros, the proposed approach is applicable to systems where feedback linearization is not applicable either because the necessary geometric conditions are not satisfied and/or because it is difficult to find the associated transformations.

As shown in the examples in the case studies in Section 7, it may provide a solution to some problems where the LPV control itself is not working. Further, as discussed in Section 5.3, this method achieves the reduction in the number of scheduling parameters. When applied to dense/large-scale nonlinear systems, the proposed approach provides a way to put the underlying system into LPV form with minimum number of scheduling parameters via reducing nonlinearities through feedback linearization, a crucial property as discussed above. For example, the system (12) could be put into LPV form (21) by defining $\rho_i \triangleq x_i$, $i = 1, \ldots, n$. In this case, there would be $n$ time-varying parameters and, due to the nonlinearity involved, the size of $\Delta$ would be large and make application of LPV control difficult, as demonstrated in the third example in Section 7. In contrast, the presented method is expected to lead to a lower number of parameters in (21) and to a reduced size of the corresponding $\Delta$.

Finally, although we do not know which state to shift nor how much to shift a priori, the possibility of different shifts adds a flexibility to the proposed method.

7. **Case Studies.** In this section, we demonstrate the developed method first on two simple numerical examples, then on a complex benchmark problem in flow control applications and finally on a more practical example. Each case study also presents a different difficulty for feedback linearization or LPV-based control. The first case study is a second order nonlinear system for which neither feedback linearization control nor LPV control is applicable. The second one is a more complex system for which again feedback linearization and LPV control methods do not work. The third case study is an extensively studied benchmark problem from flow control applications [21] and the practical case study involves control of an inverted pendulum on a cart [22]. All these case studies demonstrate the widely and efficient applicability of the proposed solution.

**Example 7.1.**

The system is described by

$$\begin{aligned}
\dot{x}_1 &= -x_1 + 6x_2 + 2x_1x_2u_1, \\
\dot{x}_2 &= 0.1x_1^2 + 8x_2 + 1.5x_2u_2, \\
x_1(0) &= 2, \quad x_2(0) = 3.
\end{aligned} \tag{33}$$

Assume that the measured output is $y = [x_1\ x_2]^T$ and that the objective is the tracking of references $r_1 = 1 + \sin(t)$ and $r_2 = 2 + 2\cos(t)$. Before applying the proposed method, note that if the input-output feedback linearization was considered with $h_i = x_i$, $i = 1, 2$, the $H(x)$ matrix in (3) would be

$$H(x) = \begin{bmatrix} 2x_1x_2 & 0 \\ 0 & 1.5x_2 \end{bmatrix},$$

which becomes singular as $x_1 \to 0$ or $x_2 \to 0$. From the targeted references, we see that $x_1 \to 0$ or $x_2 \to 0$ is possible, hence preventing feedback linearization based on the input-output linearization method to be applied. On the other hand, feedback linearization based on the input-state linearization is clearly difficult since it would require to solve a set of nonlinear partial differential equations. The application of LPV control to this example by selecting $\rho_1 = x_1$ and $\rho_2 = x_2$ does not work (the LMIs are infeasible). Indeed, as $x_2 \to 0$, $\rho_2 \to 0$ and for the zero trajectory of $\rho_2$ the control inputs have no effect on the system.

As a result, we resort to the proposed method. Note that $G(x)$ defined in (13) is the same as $H(x)$ and we hence need addition-subtraction of some input(s) to make it invertible. In case of perfect tracking we would have $x_1(t) \in [0,\ 2]$ and $x_2(t) \in [0,\ 4]$. However, in practice there will be tracking errors and we assume a margin of two units: $x_1(t) \in [-2,\ 4] \triangleq I_1$ and $x_2(t) \in [-2,\ 6] \triangleq I_2$. Adding-subtracting $s_1\ell_1u_1$ to/from the first state equation and $s_2\ell_2u_2$ to/from the second state equation, we want to find $\ell_1$ and $\ell_2$ such that

$$\begin{aligned}
\det\left(\widetilde{G}_{\ell,s}(x)\right) &= (2x_1x_2 + s_1l_1) \times (1.5x_2 + s_2l_2) \\
&\neq 0, \quad \forall x(t) \in I \triangleq [I_1\, I_2].
\end{aligned} \tag{34}$$

A feasible solution to (34) is $\ell_1^\star = 25$ and $\ell_2^\star = 7$ (and hence $s_1^\star = s_2^\star = 1$). Next $x_2$ is shifted by $\delta = 15$ ($\tilde{x}_2 \triangleq x_2 + \delta$) and the subtracted inputs are multiplied-divided by $\tilde{x}_2$ in the equations. Defining $\rho_i \triangleq \dfrac{u_i}{\tilde{x}_2}$, $i = 1, 2$, (33) becomes

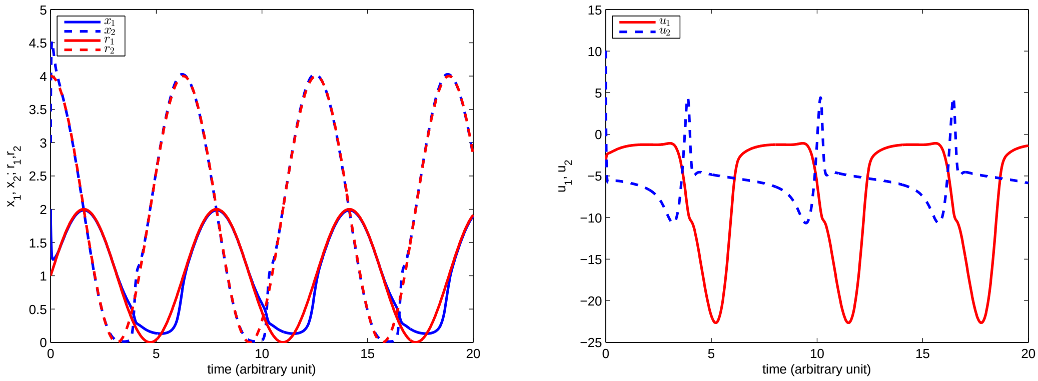
$$\begin{aligned}
\dot{x}_1 &= -x_1 + (6 - \ell_1^\star\rho_1)\tilde{x}_2 - 6\delta + [2x_1(\tilde{x}_2 - \delta) + \ell_1^\star]u_1, \\
\dot{\tilde{x}}_2 &= 0.1x_1^2 + (8 - \ell_2^\star\rho_2)\tilde{x}_2 - 8\delta + [1.5(\tilde{x}_2 - \delta) + \ell_2^\star]u_2, \\
x_1(0) &= 2, \quad \tilde{x}_2(0) = x_2(0) + \delta = 18.
\end{aligned} \tag{35}$$

Now the modified input transformation matrix

$$\widetilde{G}^{\delta x}_{\ell^\star,s^\star}(x_1, \tilde{x}_2) = \begin{bmatrix} 2x_1(\tilde{x}_2 - \delta) + \ell_1^\star & 0 \\ 0 & 1.5(\tilde{x}_2 - \delta) + \ell_2^\star \end{bmatrix}$$

is invertible in the range of interest, which is based on the range of desired references plus some margins. Letting

$$\tilde{u} \triangleq \begin{bmatrix} -6\delta \\ 0.1x_1^2 - 8\delta \end{bmatrix} + \widetilde{G}^{\delta x}_{\ell^\star,s^\star} u, \tag{36}$$




(a) (b)

FIGURE 2. Tracking results (a) and the corresponding control inputs (b) for Example 7.1

(35) becomes

$$\begin{aligned}\dot{x}_1 &= -x_1 + (6 - \ell_1^\star \rho_1)\tilde{x}_2 + \tilde{u}_1,\\ \dot{\tilde{x}}_2 &= (8 - \ell_2^\star \rho_2)\tilde{x}_2 + \tilde{u}_2,\\ x_1(0) &= 2, \quad \tilde{x}_2(0) = 18. \end{aligned} \tag{37}$$

For designing an LPV controller for (37), an initial range for $\rho_{1,2}$ is assumed, which is a posteriori validated from the simulation of the system under the control action. We here assume $\rho_{1,2} \in [-3, 3]$. For physical systems, since most of the time the states and/or control inputs are physical variables, their ranges, and hence the ranges associated with the time-varying parameters defined algebraically from them, are known. The control input to be determined by the LPV controller is $\tilde{u}$. After it is determined, the real control input $u$ is determined from (36). The tracking results and the associated control inputs shown in Figure 2 demonstrate the effectiveness of the resulting control. From the controlled system simulation results we have $\rho_1 \in [-1.27, -0.1]$ and $\rho_2 \in [-0.7, 0.56]$, which are included in the ranges assumed before designing the controller.

**Example 7.2.**

The second example is as follows

$$\begin{aligned}\dot{x}_1 &= x_3 - x_4 + x_1 \sin(x_4) + \sin(0.3x_2)u_1,\\ \dot{x}_2 &= x_1 + 0.5x_4^2 + (x_1 - 1)u_2,\\ \dot{x}_3 &= x_4 - x_3x_1 + u_2 + x_2u_3,\\ \dot{x}_4 &= x_2 + x_1x_2 + u_3 + (x_3 + x_4)u_4,\\ x(0) &= [4.5 \;\; -0.5 \;\; 2.5 \;\; 3]^T. \end{aligned} \tag{38}$$

The measured outputs are $y_1 = x_1$, $y_2 = x_3 + x_4$. The objective is the tracking of references $r_1 = 4$, $r_2 = 6$. For this problem, note that input-output linearization presented in Section 2.1 is not applicable since the system is not *u-y-square*. Although the presented discussion of Section 2.1 can be extended to non-square systems [19], some kind of invertibility of a matrix like $H(x)$ defined previously is unavoidable. When the invertibility condition is not satisfied, the use of Moore-Penrose pseudo-inverse leads to a performance degradation [23]. The difficulty in applying state-input linearization lies in solving a set of partial differential equations and in the fact that the transformed outputs will be nonlinear. The direct application of LPV control with $\rho_1 \triangleq \sin(x_4) \in [-1,\, 1]$, $\rho_2 \triangleq \sin(0.3x_2) \in [-1,1]$, $\rho_3 \triangleq x_4 \in [2.5, 3.5]$, $\rho_4 \triangleq x_1 - 1 \in [3,4]$, $\rho_5 \triangleq x_3 \in [2,3]$ and $\rho_6 \triangleq x_2 \in [-1,0]$ does not give a feasible solution. Note that here we tried to choose small parameter bounds based on initial conditions (even though in reality we may need wider limits) to help the synthesis of a feasible direct LPV controller but it failed. Further, in addition to the above parametrization, many others were tried but a feasible direct LPV controller was not obtained. As a result, we resort to the proposed method.

Assuming a margin of two units for tracking errors, we have: $x_1(t) \in [2,6] \triangleq I_1$ and $x_3(t) + x_4(t) \in [4,8] \triangleq I_3$. The non-modified input transformation matrix $G(x)$ for this example is

$$G(x) = \begin{bmatrix} \sin(0.3x_2) & 0 & 0 & 0 \\ 0 & x_1 - 1 & 0 & 0 \\ 0 & 1 & x_2 & 0 \\ 0 & 0 & 1 & x_3 + x_4 \end{bmatrix} \tag{39}$$

and since it is a lower triangular matrix, its determinant is given as

$$\det\left(G(x)\right) = \sin(0.3x_2) \times (x_1 - 1) \times x_2 \times (x_3 + x_4). \tag{40}$$

From the given tracking references we specified ranges for $x_1(t)$ and $x_3(t) + x_4(t)$. Since we have $\sin(0.3x_2) \in [-1,1]$, the only range needed to ensure a non-zero determinant is that of $x_2(t)$. We assume $x_2(t) \in [-2,12] \triangleq I_2$, which needs to be verified after simulation of the controlled system. To ensure a positive determinant (negative determinant can be considered as well) over the specified ranges, we add-subtract $s_1\ell_1 u_1$ to/from the first state equation and $s_3\ell_3 u_3$ to/from the third state equation to find $\ell_1$ and $\ell_3$ such that

$$\begin{aligned} \det\left(\widetilde{G}_{l,s}(x)\right) &= (\sin(0.3x_2) + s_1 l_1) \times (x_1 - 1) \times (x_2 + s_3 l_3) \\ &\quad \times (x_3 + x_4) \neq 0, \quad \forall x_1(t) \in I_1,\ x_2(t) \in I_2,\ x_3(t) + x_4(t) \in I_3. \end{aligned} \tag{41}$$

The determined solution is $\ell_1^\star = 1.2$ and $\ell_3^\star = 1$ (hence, $s_1^\star = s_3^\star = 1$, $s_2^\star = s_4^\star = 0$). Next, we multiply-divide each subtracted control input by $x_1$ (which is positive from the initial condition and from the fact that $r_1 = 4$) and finally define $\rho_1 \triangleq \dfrac{u_1}{x_1}$ and $\rho_2 \triangleq \dfrac{u_3}{x_1}$. Equation (38) then becomes

$$\begin{aligned} \dot{x}_1 &= -\ell_1^\star \rho_1 x_1 + x_3 - x_4 + x_1 \sin(x_4) + \left[\sin(0.3x_2) + \ell_1^\star\right] u_1, \\ \dot{x}_2 &= x_1 + 0.5x_4^2 + (x_1 - 1)\, u_2, \\ \dot{x}_3 &= -\ell_3^\star \rho_2 x_1 + x_4 - x_3 x_1 + u_2 + (x_2 + \ell_3^\star) u_3, \\ \dot{x}_4 &= x_2 + x_1 x_2 + u_3 + (x_3 + x_4) u_4, \\ x(0) &= [4.5\ -0.5\ \ 2.5\ \ 3]^T. \end{aligned} \tag{42}$$

The modified input transformation matrix is

$$\widetilde{G}^{\delta x}_{\ell^\star, s^\star}(x) = \begin{bmatrix} \sin(0.3x_2) + \ell_1^\star & 0 & 0 & 0 \\ 0 & x_1 - 1 & 0 & 0 \\ 0 & 1 & x_2 + \ell_3^\star & 0 \\ 0 & 0 & 1 & x_3 + x_4 \end{bmatrix}. \tag{43}$$

Letting

$$\tilde{u} \triangleq \begin{bmatrix} x_1 \sin(x_4) \\ 0.5x_4^2 \\ -x_3x_1 \\ x_1x_2 \end{bmatrix} + \widetilde{G}^{\delta x}_{\ell^\star,s^\star}(x)u, \tag{44}$$

(42) becomes

$$\begin{aligned} \dot{x}_1 &= -\ell_1^\star \rho_1 x_1 + x_3 - x_4 + \tilde{u}_1, \\ \dot{x}_2 &= x_1 + \tilde{u}_2, \\ \dot{x}_3 &= -\ell_3^\star \rho_2 x_1 + x_4 + \tilde{u}_3, \\ \dot{x}_4 &= x_2 + \tilde{u}_4, \\ \tilde{x}(0) &= [4.5 \ \ -0.5 \ \ 2.5 \ \ 3]^T. \end{aligned} \tag{45}$$

We here assume $\rho_1 \in [-6, 10]$, $\rho_2 \in [-3, 12]$ and design an LPV controller for (45). The tracking results, the corresponding control inputs and the second state of the system are shown in Figures 3 and 4. From Figure 4(b), note that $x_2(t) \in I_2$. Again, the overall

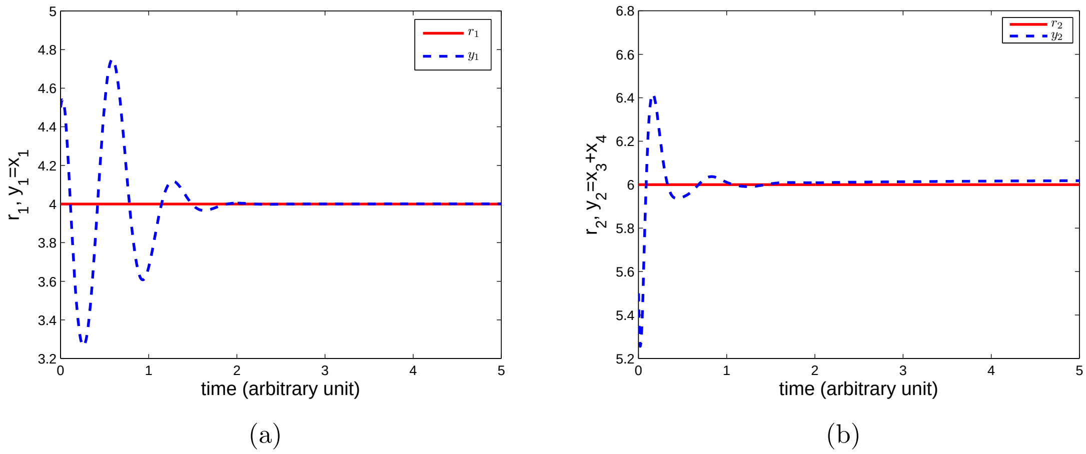



FIGURE 3. References and system response for Example 7.2

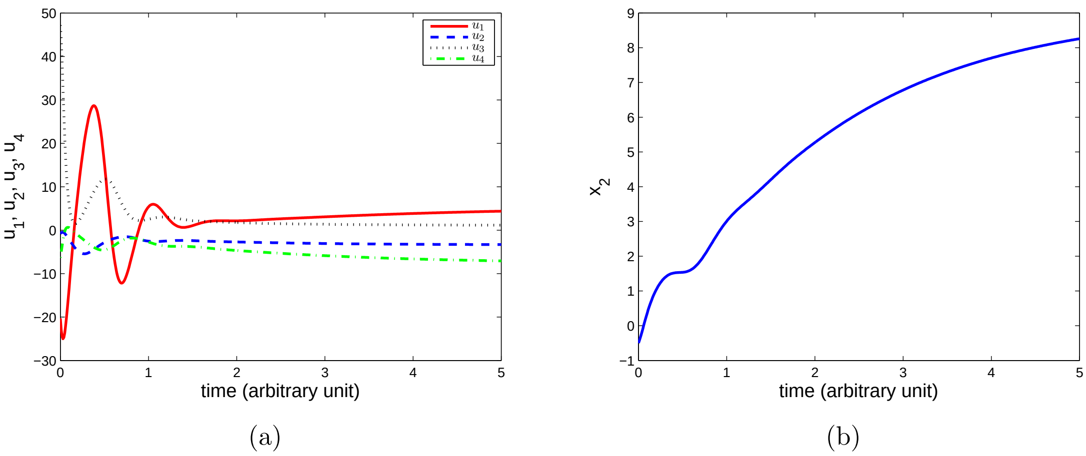



FIGURE 4. Control inputs and $x_2(t)$ for Example 7.2

control methodology presented here is seen to be effective. From the controlled system simulation results we have $\rho_1 \in [-5.9, 7.9]$ and $\rho_2 \in [0.4, 10.5]$, which are included in the ranges assumed before the design of the controller. Note that $x_2$ can be zero and hence $G(x)$ can be singular, which clearly shows the non-applicability of feedback linearization if the input transformation is based on $G(x)$.

**Example 7.3.** *Modified 1-D Burgers' System.*

Consider the following partial differential equation, a modified 1-D Burgers' equation encountered in fluid mechanics:

$$\frac{\partial v}{\partial t} + (v+V)\frac{\partial v}{\partial x} = \nu\frac{\partial^2 v}{\partial x^2} + gv, \tag{46}$$

where $v(x,t) : \mathcal{D} \times \mathcal{T} \to \mathbb{R}$, $\mathcal{D}$, $\mathcal{T} \subseteq \mathbb{R}$, is the variable of interest to be controlled (mainly to follow a predetermined trajectory), $g(x,t) : \mathcal{D} \times \mathcal{T} \to \mathbb{R}$ is the forcing function or the control signal, $\nu > 0$ is the viscosity and $V$ is a constant. To obtain an ODE model of the system amenable to control, we use the classical Proper Orthogonal Decomposition (POD) method [24, 25] for representing $v$ and Fourier functions for describing $g$. Namely, consider the expansions

$$v(x,t) \approx \varphi_0(x) + \sum_{i=1}^{n} a_i(t)\varphi_i(x), \tag{47a}$$

$$g(x,t) \approx \phi_0(x) + \sum_{i=1}^{n} b_i(t)\phi_i(x), \tag{47b}$$

where $a_i(t)$, $b_i(t)$, $i = 1, \ldots, n$, are respectively the temporal coefficients of the orthonormal spatial modes $\varphi_i(x)$, $\phi_i(x)$, $i = 1, \ldots, n$. Here, $\{\varphi_i(x)\}$ are obtained through POD by post-processing the output of the model (46) with a given reference forcing function, chosen as $g(x,t) = g^{ref}(x,t) \triangleq \sin(t) + \sum_{i=1}^{n-1} \frac{3i}{n} \sin\left(\frac{2\pi i}{5}x\right)\sin(t)$. The POD approximation is constructed from 500 snapshots over the time span $t \in [0, 10]$, see [24] for details on the method. Retaining the first $n = 10$ modes already accounts for 99.6% of the total energy of the system (in the $L^2(\mathcal{D} \times \mathcal{T})$-norm) and was hence deemed sufficient for the present control purpose. The forcing function $g(x,t)$ is decomposed over functions $\{\phi_i(x)\}$:

$$\{\phi_i(x)\} \triangleq \left\{1, \sin\left(\frac{2\pi}{L}x\right), \ldots, \sin\left(\frac{2\pi}{L}(n-1)x\right)\right\}, \tag{48}$$

where $L \triangleq 5$ is the length of the spatial interval and $x \in [0, L]$. The functions $\varphi_0(x)$, $\phi_0(x)$ are mean values of $v(x,t)$ and $g(x,t)$ at $x$ over the considered time span. After taking the required temporal and spatial derivatives of the approximation $v(x,t)$ given by (47a), inserting the resulting expressions together with (47b) into (46), multiplying both sides by $\varphi_k(x)$, $k = 1, \cdots, n$ and integrating over space from 0 to $L$, we finally obtain the following system

$$\dot{a}_k = \sum_{i=1}^{n} L_{ik}a_i + \sum_{i,j=1}^{n} Q_{ijk}a_ia_j + \sum_{i,j=1}^{n} B_{ijk}a_ib_j + \sum_{i=1}^{n} C_{ik}b_i + c_k, \quad k = 1, \ldots, n, \tag{49}$$

with $L$, $Q$, $B$ and $C$ time-independent projection tensors. Note that the orthogonal property of $\{\varphi_k(x)\}_1^n$ over the spatial domain with uniform measure was used. As a test case, the temporal coefficients $a_i^{ref}(t)$ of the reference solution with the forcing function as $g(x,t) = g^{ref}(x,t)$, $V = 1$, $n = 10$, computed from (46) are tracked (noted $r_i(t)$ for figure

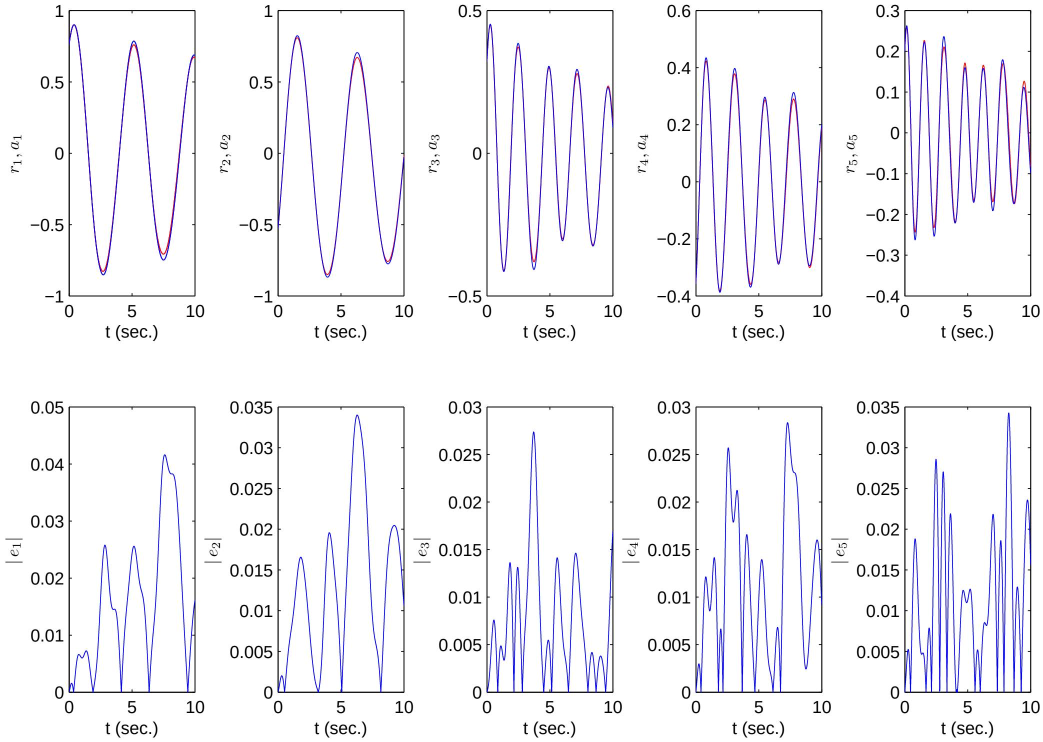



FIGURE 5. LPV-control simulation results (top row) and the corresponding error variations (bottom row): responses of states 1 to 5

readability). To guarantee invertibility of $\widetilde{G}^{\delta x}_{\ell^\star,s^\star}(x)$, we follow the same steps as before and obtain $\ell_1^\star = \ell_2^\star = \cdots = \ell_{10}^\star = 1$ and set $\rho_j \triangleq \dfrac{b_j}{\tilde{a}_{10}}$, $j = 1, \ldots, 10$, where $\tilde{a}_{10} \triangleq a_{10} + 2$.

The tracking results are shown in Figures 5 and 6 together with the corresponding errors $e_i$ over time, $e_i \triangleq r_i - a_i$, $i = 1, \ldots, 10$. In the figures, the solid color signals are the references and the dashed ones are the system responses. The error is seen to remain low, for all states at all time, again demonstrating the efficiency of the present method.

Note that if one tries to directly apply LPV control to (49) by taking $\rho_i \triangleq x_i$, $i = 1, \ldots, 10$, the size of $\Delta$ would be more than $1000 \times 1000$ and LMIs in Theorem 5.1 will not be feasible to obtain an LPV controller.

**Example 7.4.** *Inverted pendulum on a cart.*

The last example considers a practical application of the method introduced in Section 3 to an inverted pendulum mounted on a motor driven cart. This example also demonstrates the use of the method for a non-square $x$-$u$ system (see Section 4) and comparison of the performance of the proposed method with a fuzzy control approach [22]. The system is shown in Figure 7 and the dynamics of the system is given by [22]:

$$\dot{x}_1 = x_2, \tag{50a}$$

$$\dot{x}_2 = \frac{g\sin(x_1) - amlx_2^2\sin(2x_1)/2 - a\cos(x_1)u}{4l/3 - aml\cos^2(x_1)}, \tag{50b}$$

where $x_1$ denotes the angle $\theta$ of the pendulum from the vertical, and $x_2 = \dot{\theta}$ is the angular velocity. $g = 9.8$ m/s$^2$ is the gravity constant, $m$ is the mass of the pendulum, $M$ is the

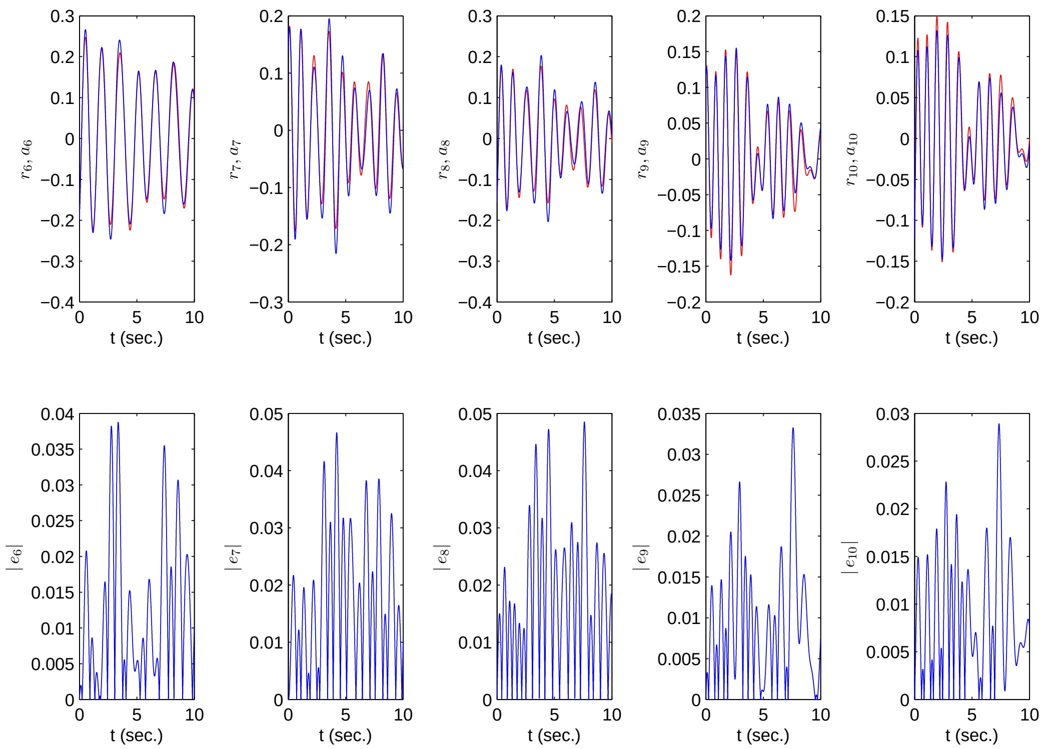


FIGURE 6. LPV-control simulation results (top row) and the corresponding error variations (bottom row): responses of states 6 to 10

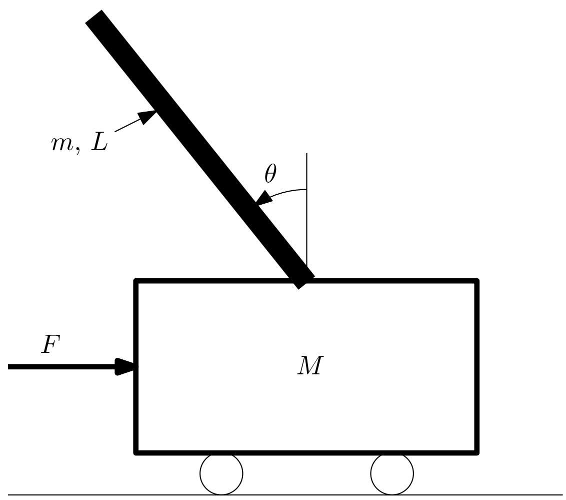
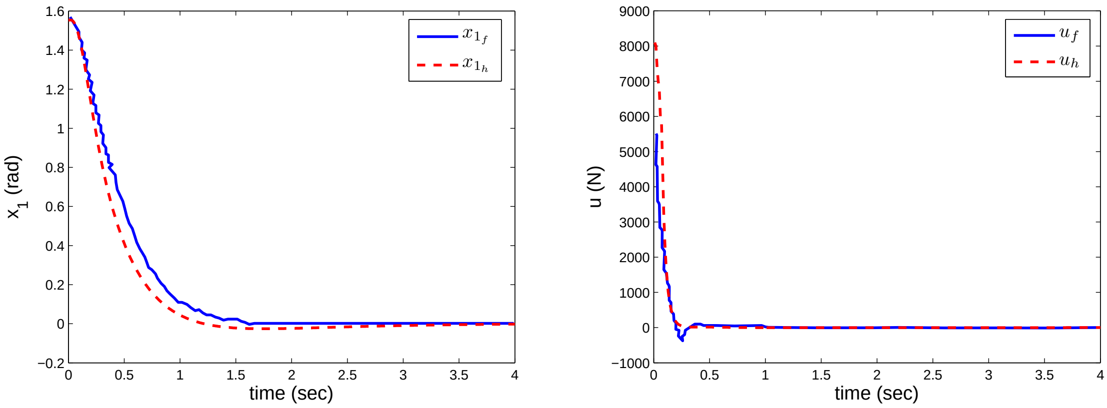



FIGURE 7. Inverted pendulum on a cart

mass of the cart, $a = 1/(m + M)$, $L = 2l$ is the length of the pendulum and $u = F$ is the force applied to the cart. Since the system is not *u-y-square*, input-output feedback linearization is not applicable and the input-state linearization will involve solution of a nonlinear PDE as before.

Letting

$$G(x) = G(x_1, x_2) = \frac{-a\cos(x_1)}{4l/3 - aml\cos^2(x_1)},$$




(a) (b)

FIGURE 8. Comparison of hybrid and the fuzzy control methods for an inverted pendulum on a cart

we see that $G(x)$ will be singular when $x_1 = \pi/2$. As a result, we add-subtract $\ell u$ to/from (50b) to obtain

$$\widetilde{G}_\ell(x) = \widetilde{G}_\ell(x_1, x_2) = \frac{-a\cos(x_1)}{4l/3 - aml\cos^2(x_1)} + \ell.$$

Shifting $x_1$ by $\delta = 2$, multiplying-dividing the subtracted $\ell u$ by $\tilde{x}_1 = x_1 + \delta$ and letting $\rho = \frac{u}{\tilde{x}_1}$, (50) becomes

$$\dot{\tilde{x}}_1 = x_2, \tag{51a}$$

$$\dot{x}_2 = -\ell\rho\tilde{x}_1 + \tilde{u}, \tag{51b}$$

where

$$\tilde{u} = \frac{g\sin(\tilde{x}_1 - \delta) - amlx_2^2\sin\left(2\left(\tilde{x}_1 - \delta\right)\right)/2}{4l/3 - aml\cos^2(\tilde{x}_1 - \delta)} + \widetilde{G}_\ell(\tilde{x}_1 - \delta, x_2)u.$$

We consider the stabilization of the system with the initial condition $x(0) = [\pi/2, 0]^T$. For the proposed method with the given objective, we set $\ell^\star = 0.15\,10^{-3}$. Note that a small positive $\ell$ value is enough to make $\widetilde{G}_\ell(x)$ invertible for the given objective. We prefer a small $\ell$ value since we observe (through a simulation study) that the associated control input, and hence $\rho(t)$, is large for the considered system. An unnecessary parameter interval for $\rho(t)$ will degrade the performance of the designed controller. The physical system parameter values are taken from [22] as $m = 2\,\text{kg}$, $M = 8\,\text{kg}$ and $l = 0.5\,\text{m}$. The parameter range for $\rho(t)$ was taken to be $[-100, 6000]$. The results of the hybrid method are compared in Figure 8 with the fuzzy control method of [22]. The subscript "$h$" is used for the proposed hybrid control approach and "$f$" is used for the fuzzy control approach. The proposed method is seen to be faster than the fuzzy method in stabilizing the system. Next, we try to demonstrate the tracking performance of the hybrid algorithm for the inverted pendulum on a cart by tracking $r = \cos(t)$. The result is shown in Figure 9. Note that for this case we cannot apply the fuzzy control method of [22] since it is for stabilization. In general, nonlinear tracking control is significantly more difficult than nonlinear stabilization for an arbitrary tracking of a reference in the reachable set of the system. As seen from Figure 9, in contrast, the present approach can be applied both for stabilization and tracking thanks to the underlying LPV control theory.

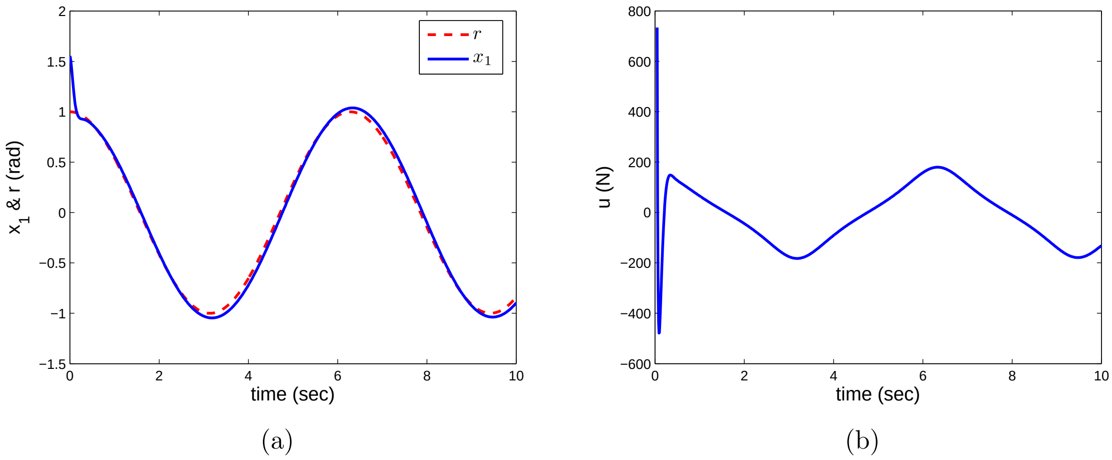



FIGURE 9. Tracking performance of hybrid method for an inverted pendulum on a cart

8. **Conclusion.** In spite of an extensive research over the last decades, nonlinear system control remains a challenging field for general nonlinear systems. The combination of some already existing approaches often achieves good results or provides a solution to problems which cannot be solved by the individual approaches. In this paper, we have presented such a method for the control of a class of input-affine nonlinear systems by combining the feedback linearization and the linear parameter-varying control methods. The main idea behind the present hybrid method is to use the linearizing power of feedback linearization to alleviate the limitation of LPV control and hence increase its applicability. The resulting method is most suitable for cases where feedback linearization is not applicable or a linear-parameter varying control cannot be used or as an alternative hybrid control method for the given class of input-affine nonlinear systems. The proposed hybrid method provides an LPV modeling of the system with a minimum number of time-varying parameters by reducing the system nonlinearities via feedback linearization made possible through input add-subtract operations. The role of the added inputs is to make the input transformation matrix invertible in the range of interest. The subtracted inputs are divided-multiplied by a shifted state and these introduced nonlinearities are defined as scheduling parameters, functions of both inputs and the shifted state. A greedy-like algorithm is used to minimize the number of input add-subtractions, leading to a small number of scheduling parameters. The resulting model is finally put into an LPV form and controlled using LPV-based control theory. Numerical examples of various complexity and exhibiting different control difficulties were considered to illustrate the power of the method. They are all effectively handled by the proposed method. As seen from the examples, the proposed method can be used both for stabilization and time-varying reference tracking.

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