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MATHEMATICAL MODEL FOR RECTANGULAR BEAMS OF VARIABLE CROSS SECTION OF SYMMETRICAL LINEAR SHAPE FOR UNIFORMLY DISTRIBUTED LOAD

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Abstract. *This paper presents a mathematical model for rectangular beams subject to a uniformly distributed load of variable cross section of symmetric linear shape to obtain fixed-end moments, carry-over factors and stiffness factors. The properties of the rectangular cross section of the beam vary along its axis "x", i.e., the width "b" is constant and the height "h" varies along the beam, and this variation is a linear type. The consistent deformation method is used to solve such problems; a method based on the superposition of effects and by means of the Bernoulli-Euler theory obtains the deformations at any point of the beam. Traditional methods to obtain deflections of variable section members are Simpson's rule, or any other technique to perform numerical integration and other authors present tables which are restricted to certain relationships. Besides the effectiveness and accuracy of the developed method, a significant advantage is that the displacement and moments are calculated at any cross section of the beam using the respective integral representations as mathematical formulas.*

Keywords: Fixed-end moments, Carry-over factors, Stiffness factors, Variable rectangular cross section, Linear shape, Consistent deformation method, Bernoulli-Euler theory

1. **Introduction.** One of the major concerns of structural engineering over the past 50 years is to propose elastic methods dependable to satisfactorily model to the variable cross section members, such that it is having certainty in the determination of mechanical elements, strains and displacements that allows to properly design this type of members.

During the last century, between 1950 and 1960 were developed several design aids, as those presented by Guldan [1], and the most popular tables published by Portland Cement Association (PCA) in 1958 "Handbook" [2], where stiffness constants and fixed-end moments of variable section members are presented because the limitations for extensive calculations at that time, in the PCA tables were used several hypotheses to simplify the problem, among the most important pondering the variation of the stiffness (linear or parabolic, according to the case of geometry) in function of main moment of inertia in flexure, considering independent cross section, which was demonstrated as incorrect. Furthermore, the shear deformations and the ratio of length-height of beam are neglected in the definition of stiffness factors; simplifications can lead to significant errors in determining stiffness factors [3].

Elastic formulation of stiffness for members of variable section was evolved over time, and after the publication of the PCA tables, the following works deserve special mention all based on beams theory. Just [4] was the first to propose a rigorous formulation for members of variable cross section of drawer type and "I" based on the classical theory of beams of Bernoulli-Euler for two-dimensional member without including axial deformations. Schreyer [5] proposed a more rigorous theory of beams for members varying linearly, in which the hypotheses generalized by Kirchhoff were introduced to take into account the shear deformations. Medwadowski [6] resolved the problem of flexure in beam of shear nonprismatic using the theory of variational calculus. Brown [7] presented a method which uses approximate interpolation functions consistent with beam elastic theory and the principle of virtual work to define the stiffness matrix of members of variable section.

Matrices of elastic stiffness for two-dimensional and three-dimensional members of variable section based on classical theory of beam by Euler-Bernoulli and flexibilities method taking into account the axial and shear deformations, and the cross section shape are found in Tena-Colunga and Zaldo [8] and in the appendix B [9].

In traditional methods used for the variable cross section members, the deflections are obtained by Simpson's rule or some other techniques to perform numerical integration, and tables presenting some books are limited to certain relationships [10-12].

This paper presents a mathematical model to obtain the fixed-end moments, stiffness factors and carry-over of a beam subject to a uniformly distributed load of variable rectangular cross section taking into account the width "b" being constant and height "h_x" varying of linear shape.

2. Mathematical Model.

2.1. General principles of the straight line.

Figure 1 shows a beam in elevation and also presents its rectangular cross-section taking into account the width "b" being constant and height "h_x" varying of linear shape in three different parts.

The value "h_x" varies with respect to "x"; this gives:

$$h_x = h + y \tag{1}$$

Now, the properties of the straight line are used:

Equation for $0 \leq x \leq a$:

$$h_x = \frac{ah + ay_1 - y_1 x}{a} \tag{2}$$

Equation for $a \leq x \leq L - a$:

$$h_x = h \tag{3}$$

Equation for $L - a \leq x \leq L$:

$$h_x = \frac{ah + y_1 x - y_1 L + ay_1}{a} \tag{4}$$

2.2. Derivation of the equations for uniformly distributed load.

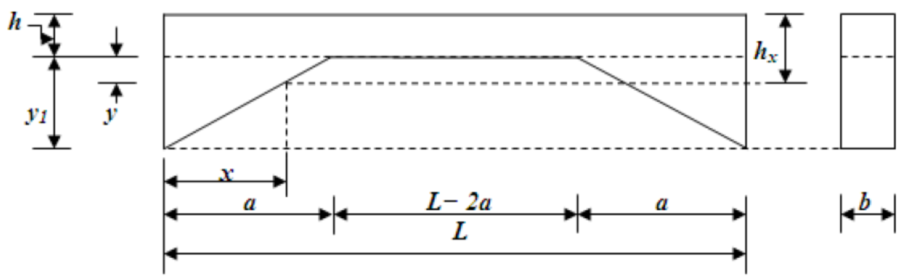

FIGURE 1. Rectangular section varying the height of linear shape

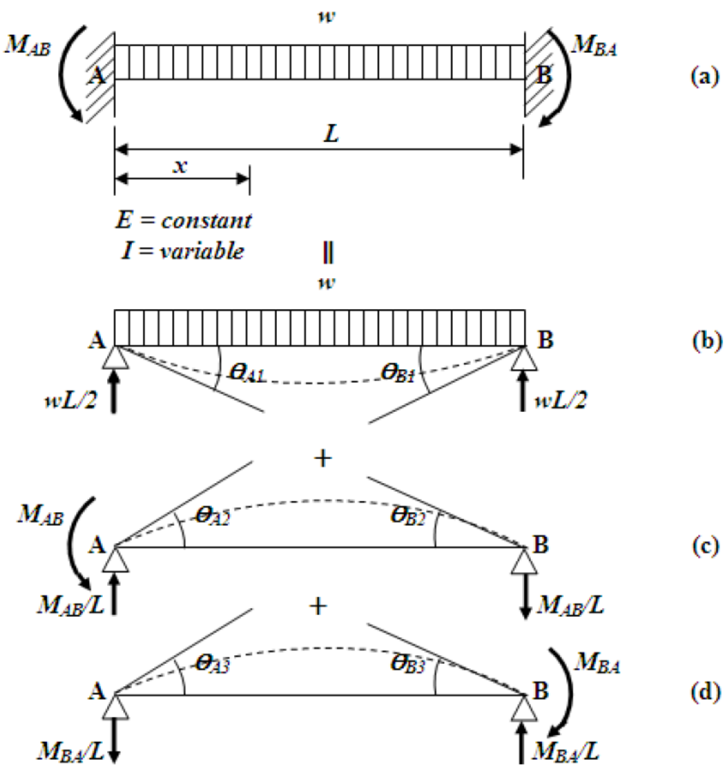

FIGURE 2. Beam fixed at its ends

2.2.1. *Fixed-end moments.* Figure 2(a) shows the beam "AB" subject to a uniformly distributed load and fixed ends. The fixed-end moments are found by the sum of the effects. The moments are considered positive in counterclockwise and the moments are considered negative in clockwise. In Figure 2(b) shows the same beam simply supported at their ends at the load applied to find the rotations "θ_{A1}" and "θ_{B1}". Now, the rotations "θ_{A2}" and "θ_{B2}" are caused by the moment "M_{AB}" applied in the support "A" according to Figure 2(c), and in terms of "θ_{A3}" and "θ_{B3}" are caused by the moment "M_{BA}" applied in the support "B", seen in Figure 2(d) [13,14].

The conditions of geometry are [13-15]:

$$\theta_{A1} + \theta_{A2} + \theta_{A3} = 0 \tag{5}$$

$$\theta_{B1} + \theta_{B2} + \theta_{B3} = 0 \tag{6}$$

The beam of Figure 2(b) is analyzed to find "θ_{A1}" and "θ_{B1}" by Euler-Bernoulli theory to calculate the deflections [16,17]. The equation is:

$$\frac{dy}{dx} = \int \frac{M_z}{EI_z} dx \tag{7}$$

where $dy/dx = \theta_z$ is the total rotation around the axis "z", E is the modulus of elasticity of the material, M_z is the moment around the axis "z", I_z is the moment of inertia around the axis "z".

The moment at any point of the beam on axis "x" is [18]:

$$M_z = \frac{w(x^2 - Lx)}{2} \tag{8}$$

The moment of inertia for a rectangular member is:

$$I_z = \frac{bh_x^3}{12} \tag{9}$$

Equations (2)-(4) for the three different segments are substituted into Equation (9); it is presented:

Equation for $0 \leq x \leq a$:

$$I_z = \frac{b}{12}\left[\frac{ah + ay_1 - y_1 x}{a}\right]^3 \tag{10}$$

Equation for $a \leq x \leq L - a$:

$$I_z = \frac{bh^3}{12} \tag{11}$$

Equation for $L - a \leq x \leq L$:

$$I_z = \frac{b}{12}\left[\frac{ah + y_1 x - y_1 L + ay_1}{a}\right]^3 \tag{12}$$

Then, the moment of inertia for a member of rectangular section is presented in Equations (10)-(12).

a) For the segment $0 \leq x \leq a$:

Equations (8) and (10) are substituted into Equation (7); it is presented:

$$\frac{dy}{dx} = \frac{6wa^3}{Eb}\int \frac{(x^2 - Lx)}{(ah + ay_1 - y_1 x)^3}dx \tag{13}$$

Equation (13) is evaluated; it is shown:

$$\begin{aligned}\frac{dy}{dx} = \frac{6wa^3}{Eb}\bigg\{ &-\frac{\ln[xy_1 - a(y_1 + h)]}{y_1^3} + \frac{x[y_1(2a - L) + 2ah]}{y_1^2[xy_1 - a(y_1 + h)]^2} \\ &- \frac{a[y_1^2(3a - L) + hy_1(6a - L) + 3ah^2]}{2y_1^3[xy_1 - a(y_1 + h)]^2} + C_1\bigg\}\end{aligned} \tag{14}$$

In Equation (14) is replaced $x = a$, to find the rotation $dy/dx = \theta_{a1}$, where the height "h_x" varies the linear shape; it is:

$$\theta_{a1} = \frac{6wa^3}{Eb}\left\{\frac{-2ah^2\ln(-ah) + y_1^2(a - L) + hy_1(L - 2a) - 3ah^2}{2ah^2 y_1^3} + C_1\right\} \tag{15}$$

b) For the segment $a \leq x \leq L - a$:

Equations (8) and (11) are substituted into Equation (7); it is presented:

$$\frac{dy}{dx} = \frac{6w}{Ebh^3}\int (x^2 - Lx)dx \tag{16}$$

Equation (16) is evaluated; it is shown:

$$\frac{dy}{dx} = \frac{6w}{Ebh^3}\left(\frac{x^3}{3} - \frac{Lx^2}{2} + C_2\right) \tag{17}$$

The boundary conditions are replaced into Equation (17), when $x = L/2$ and $dy/dx = 0$ by symmetry that occurs, we get $C_2 = L^2/12$. Then Equation (17) is showed:

$$\frac{dy}{dx} = \frac{6w}{Ebh^3}\left(\frac{x^3}{3} - \frac{Lx^2}{2} + \frac{L^3}{12}\right) \tag{18}$$

Now, the boundary conditions are replaced into Equation (18), when $x = a$, to find the rotation $dy/dx = \theta_{a2}$; it is:

$$\theta_{a2} = \frac{6w}{Ebh^3}\left(\frac{a^3}{3} - \frac{La^2}{2} + \frac{L^3}{12}\right) \tag{19}$$

The boundary conditions are replaced into Equation (18), when $x = L - a$, to obtain the rotation $dy/dx = \theta_{b2}$; it is:

$$\theta_{b2} = \frac{6w}{Ebh^3}\left[\frac{(L-a)^3}{3} - \frac{L(L-a)^2}{2} + \frac{L^3}{12}\right] \tag{20}$$

Then, Equations (15) and (19) are equalized, because rotations must be equal at the point $x = a$, where the height "h_x" varies linearly to find the constant "C_1", this value is:

$$C_1 = \frac{4a^3 - 6La^2 + L^3}{12a^3h^3} - \left[\frac{-2ah^2\ln(-ah) + y_1^2(a-L) + hy_1(L-2a) - 3ah^2}{2ah^2y_1^3}\right] \tag{21}$$

Equation (21) is substituted into Equation (14) to obtain the rotations anywhere of the segment $0 \le x \le a$:

$$\begin{aligned}\frac{dy}{dx} = \frac{6wa^3}{Eb}\Bigg\{ &- \frac{\ln[xy_1 - a(y_1+h)]}{y_1^3} + \frac{x[y_1(2a-L)+2ah]}{y_1^2[xy_1 - a(y_1+h)]^2} \\ &- \frac{a[y_1^2(3a-L) + hy_1(6a-L) + 3ah^2]}{2y_1^3[xy_1 - a(y_1+h)]^2} + \frac{4a^3 - 6La^2 + L^3}{12a^3h^3} \\ &- \left[\frac{-2ah^2\ln(-ah) + y_1^2(a-L) + hy_1(L-2a) - 3ah^2}{2ah^2y_1^3}\right]\Bigg\}\end{aligned} \tag{22}$$

In Equation (22) is replaced $x = 0$, to find the rotation in support "A"; it is presented:

$$\begin{aligned}\theta_{A1} = \frac{6wa^3}{Eb}\Bigg\{ &\frac{1}{y_1^3}\left[\ln\left(\frac{h}{y_1+h}\right)\right] + \frac{4a^3 - 6a^2L + L^3}{12a^3h^3} \\ &+ \left[\frac{y_1^3(L-a) + hy_1^2L + 3ah^2y_1 + 2ah^3}{2ah^2y_1^2(y_1+h)^2}\right]\Bigg\}\end{aligned} \tag{23}$$

c) For the segment $L - a \le x \le L$:
Equations (8) and (12) are substituted into Equation (7), it is presented:

$$\frac{dy}{dx} = \frac{6wa^3}{Eb}\int \frac{(x^2 - Lx)}{(ah + y_1x - y_1L + ay_1)^3}dx \tag{24}$$

Equation (24) is evaluated; it is shown:

$$\begin{aligned}\frac{dy}{dx} = \frac{6wa^3}{Eb}\Bigg\{ &\frac{\ln[xy_1 + y_1(a-L) + ah]}{y_1^3} + \frac{x[y_1(2a-L) + 2ah]}{y_1^2[xy_1 + y_1(a-L) + ah]^2} \\ &+ \frac{y_1^2(a-L)(3a-2L) + ahy_1(6a-5L) + 3a^2h^2}{2y_1^3[xy_1 + y_1(a-L) + ah]^2} + C_3\Bigg\}\end{aligned} \tag{25}$$

In Equation (25) is replaced $x = L - a$, to find the rotation $dy/dx = \theta_{b3}$, where the height "h_x" varies the linear shape; it is:

$$\theta_{b3} = \frac{6wa^3}{Eb}\left\{\frac{2ah^2\ln(ah) - y_1^2(a-L) - hy_1(L-2a) + 3ah^2}{2ah^2y_1^3} + C_3\right\} \tag{26}$$

Then, Equations (20) and (26) are equalized, because rotations must be equal at the point $x = L - a$, where the height "h_x" varies linearly to find the constant "C_3", this value is:

$$C_3 = \left\{\frac{4(L-a)^3 - 6L(L-a)^2 + L^3}{12a^3h^3} - \frac{2ah^2\ln(-ah) - y_1^2(a-L) - hy_1(L-2a) + 3ah^2}{2ah^2y_1^3}\right\} \tag{27}$$

Equation (27) is substituted into Equation (25) to obtain the rotations anywhere of the segment $L - a \leq x \leq L$:

$$\begin{aligned}\frac{dy}{dx} = \frac{6wa^3}{Eb}\Bigg\{ &\left[\frac{\ln[xy_1 + y_1(a-L) + ah]}{y_1^3}\right] + \left[\frac{x[y_1(2a-L) + 2ah]}{y_1^2[xy_1 + y_1(a-L) + ah]^2}\right] \\ &+ \left[\frac{y_1^2(a-L)(3a-2L) + ahy_1(6a-5L) + 3a^2h^2}{2y_1^3[xy_1 + y_1(a-L) + ah]^2}\right] \\ &+ \left[\frac{4(L-a)^3 - 6L(L-a)^2 + L^3}{12a^3h^3}\right] \\ &- \left[\frac{2ah^2\ln(-ah) - y_1^2(a-L) - hy_1(L-2a) + 3ah^2}{2ah^2y_1^3}\right]\Bigg\}\end{aligned} \tag{28}$$

In Equation (28) is replaced $x = L$, to find the rotation in support "B"; it is presented:

$$\begin{aligned}\theta_{B1} = \frac{6wa^3}{Eb}\Bigg\{ &-\frac{1}{y_1^3}\left[\ln\left(\frac{h}{y_1+h}\right)\right] - \left[\frac{4a^3 - 6a^2L + L^3}{12a^3h^3}\right] \\ &- \left[\frac{y_1^3(L-a) + hy_1^2L + 3ah^2y_1 + 2ah^3}{2ah^2y_1^2(y_1+h)^2}\right]\Bigg\}\end{aligned} \tag{29}$$

Now, the beam of Figure 2(c) is analyzed to find "θ_{A2}" and "θ_{B2}" in function of "M_{AB}" [16,17]:

The moment at any point of the beam on axis "x" is [18]:

$$M_z = \frac{M_{AB}(L-x)}{L} \tag{30}$$

a) For the segment $0 \leq x \leq a$:

Equations (10) and (30) are substituted into Equation (7); it is presented:

$$\frac{dy}{dx} = \frac{12M_{AB}a^3}{EbL}\int \frac{(L-x)}{(ah + ay_1 - y_1x)^3}dx \tag{31}$$

Equation (31) is evaluated; it is shown:

$$\frac{dy}{dx} = \frac{12M_{AB}a^3}{EbL}\left\{-\frac{2xy_1 - y_1(a+L) - ah}{2y_1^2[xy_1 - a(y_1+h)]^2} + C_1\right\} \tag{32}$$

In Equation (32) is replaced $x = a$, to find the rotation $dy/dx = \theta_{a1}$, where the height "h_x" varies the linear shape; it is:

$$\theta_{a1} = \frac{12M_{AB}a^3}{EbL}\left\{\frac{ah - y_1(a-L)}{2a^2h^2y_1^2} + C_1\right\} \tag{33}$$

Equation (32) is integrated to obtain the displacements, because there are not known conditions for rotations; this is as follows:

$$y = \frac{12M_{AB}a^3}{EbL}\int\left\{-\frac{2xy_1 - y_1(a+L) - ah}{2y_1^2[xy_1 - a(y_1+h)]^2} + C_1\right\}dx \tag{34}$$

Equation (34) is evaluated; it is shown:

$$y = \frac{12M_{AB}a^3}{EbL}\left\{-\frac{\ln[xy_1 - a(y_1+h)]}{y_1^3} + \frac{y_1(L-a) - ah}{2y_1^3[a(y_1+h) - xy_1]} + C_1x + C_2\right\} \tag{35}$$

The boundary conditions are replaced into Equation (35), when $x = 0$ and $y = 0$ to find the constant "C_2", this value is:

$$C_2 = \left\{\frac{2a(y_1+h)\ln[-a(y_1+h)] + y_1(a-L) + ah}{2ay_1^3(y_1+h)}\right\} \tag{36}$$

Equation (36) is substituted into Equation (35):

$$y = \frac{12M_{AB}a^3}{EbL}\left\{ -\frac{\ln[xy_1 - a(y_1+h)]}{y_1^3} + \frac{y_1(L-a)-ah}{2y_1^3[a(y_1+h)-xy_1]} + C_1x \right. \\ \left. + \frac{\ln[-a(y_1+h)]}{y_1^3} + \frac{y_1(a-L)+ah}{2ay_1^3(y_1+h)} \right\} \tag{37}$$

In Equation (37) is replaced $x = a$, to find the displacement $y = y_{a1}$, where the height "h_x" varies the linear shape; it is:

$$y_{a1} = \frac{12M_{AB}a^3}{EbL}\left\{ \frac{1}{y_1^3}\ln\left(\frac{y_1+h}{h}\right) - \frac{y_1[y_1(a-L)+ah]}{2ahy_1^3(y_1+h)} + C_1a \right\} \tag{38}$$

b) For the segment $a \le x \le L-a$:

Equations (11) and (30) are substituted into Equation (7); it is presented:

$$\frac{dy}{dx} = \frac{12M_{AB}}{Ebh^3L}\int (L-x)dx \tag{39}$$

Equation (39) is evaluated; it is shown:

$$\frac{dy}{dx} = \frac{12M_{AB}}{Ebh^3L}\left(Lx - \frac{x^2}{2} + C_3\right) \tag{40}$$

In Equation (40) is replaced $x = a$, to find the rotation $dy/dx = \theta_{a2}$, where the height "h_x" varies the linear shape; it is:

$$\theta_{a2} = \frac{12M_{AB}}{Ebh^3L}\left(La - \frac{a^2}{2} + C_3\right) \tag{41}$$

In Equation (40) is replaced $x = L-a$, to find the rotation $dy/dx = \theta_{b2}$, where the height "h_x" varies the linear shape; it is:

$$\theta_{b2} = \frac{12M_{AB}}{Ebh^3L}\left[L(L-a) - \frac{(L-a)^2}{2} + C_3\right] \tag{42}$$

Equation (40) is integrated to obtain the displacements, because there are not known conditions for rotations; this is as follows:

$$y = \frac{12M_{AB}}{Ebh^3L}\int \left(Lx - \frac{x^2}{2} + C_3\right)dx \tag{43}$$

Equation (43) is evaluated; it is shown:

$$y = \frac{M_{AB}}{Ebh^3L}\left(\frac{Lx^2}{2} - \frac{x^3}{6} + C_3x + C_4\right) \tag{44}$$

In Equation (44) is replaced $x = a$, to find the displacement $y = y_{a2}$, where the height "h_x" varies the linear shape; it is:

$$y_{a2} = \frac{12M_{AB}}{Ebh^3L}\left(\frac{La^2}{2} - \frac{a^3}{6} + C_3a + C_4\right) \tag{45}$$

In Equation (44) is replaced $x = L-a$, to find the displacement $y = y_{b2}$, where the height "h_x" varies the linear shape; it is:

$$y_{b2} = \frac{12M_{AB}}{Ebh^3L}\left[\frac{L(L-a)^2}{2} - \frac{(L-a)^3}{6} + C_3(L-a) + C_4\right] \tag{46}$$

c) For the segment $L-a \le x \le L$:

Equations (12) and (30) are substituted into Equation (7); it is presented:

$$\frac{dy}{dx} = \frac{12M_{AB}a^3}{EbL}\int \frac{(L-x)}{(ah+y_1x-y_1L+ay_1)^3}dx \tag{47}$$

Equation (47) is evaluated; it is shown:

$$\frac{dy}{dx} = \frac{12M_{AB}a^3}{EbL}\left\{\frac{2xy_1+y_1(a-2L)+ah}{2y_1^2[xy_1+y_1(a-L)+ah]^2}+C_5\right\} \tag{48}$$

In Equation (48) is replaced $x = L-a$, to find the rotation $dy/dx = \theta_{b3}$, where the height "h_x" varies the linear shape; it is:

$$\theta_{b3} = \frac{12M_{AB}a^3}{EbL}\left(\frac{h-y_1}{2ah^2y_1^2}+C_5\right) \tag{49}$$

Equation (48) is integrated to obtain the displacements, because there are not known conditions for rotations; this is as follows:

$$y = \frac{12M_{AB}a^3}{EbL}\int\left\{\frac{2xy_1+y_1(a-2L)+ah}{2y_1^2[xy_1+y_1(a-L)+ah]^2}+C_5\right\}dx \tag{50}$$

Equation (50) is evaluated; it is shown:

$$y = \frac{12M_{AB}a^3}{EbL}\left\{\frac{\ln[xy_1+y_1(a-L)+ah]}{y_1^3}+\frac{a(y_1+h)}{2y_1^3[xy_1+y_1(a-L)+ah]}+C_5x+C_6\right\} \tag{51}$$

The boundary conditions are replaced into Equation (51), when $x = L$ and $y = 0$ to find the constant "C_6" in function of "C_5", this value is:

$$C_6 = -\left\{\frac{2\ln[a(y_1+h)]+1}{2y_1^3}\right\}-C_5L \tag{52}$$

Equation (52) is substituted into Equation (51):

$$\begin{aligned} y = \frac{12M_{AB}a^3}{EbL}&\left\{\frac{1}{y_1^3}\ln\left[\frac{xy_1+y_1(a-L)+ah}{a(y_1+h)}\right]\right. \\ &\left.+\frac{a(y_1+h)}{2y_1^3[xy_1+y_1(a-L)+ah]}-\frac{1}{2y_1^3}-C_5(L-x)\right\} \end{aligned} \tag{53}$$

In Equation (53) is replaced $x = L-a$, to find the displacement $y = y_{b3}$, where the height "h_x" varies the linear shape; it is:

$$y_{b3} = \frac{12M_{AB}a^3}{EbL}\left[-\frac{1}{y_1^3}\ln\left(\frac{y_1+h}{h}\right)+\frac{1}{2hy_1^2}-C_5a\right] \tag{54}$$

Then, Equations (33) and (41) are equalized, because rotations must be equal at the point $x = a$, where the height "h_x" varies linearly to find the constant "C_3" in function of "C_1", this value is:

$$C_3 = a^3h^3\left\{\frac{ah-y_1(a-L)}{2a^2h^2y_1^2}+C_1\right\}-La+\frac{a^2}{2} \tag{55}$$

Also, Equations (38) and (45) are equalized, because the displacements must be equal at the point $x = a$, and subsequently Equation (55) is replaced to find the constant "C_4", this value is:

$$C_4 = \frac{a^3h^3}{y_1^3}\ln\left(\frac{y_1+h}{h}\right)-\left\{\frac{a^3h^2y_1-a^3hy_1^2+a^2hy_1^2L}{2y_1^2(y_1+h)}\right\}+\frac{La^2}{2}-\frac{a^3}{3} \tag{56}$$

Next, Equations (42) and (49) are equalized, because rotations must be equal at the point $x = L-a$, where the height "h_x" varies linearly to find the constant "C_3" in function of "C_5", this value is:

$$C_3 = \frac{a^2h(h-y_1)}{2y_1^2} + C_5a^3h^3 - L(L-a) + \frac{(L-a)^2}{2} \tag{57}$$

Also, Equations (46) and (54) are equalized, because the displacements must be equal at the point $x = L - a$, and subsequently Equations (56) and (57) are replaced to find the constant "C_5", this value is:

$$\begin{aligned} C_5 = \Bigg\{ &-\frac{2}{y_1^3L}\ln\left(\frac{y_1+h}{h}\right) - \left[\frac{hL - y_1L + ay_1 - 2ah}{2ah^2y_1^2L}\right] \\ &+ \left[\frac{ahy_1 - ay_1^2 + y_1^2L}{2ah^2y_1^2L(y_1+h)}\right] + \frac{(L-a)^2}{2a^3h^3} - \frac{(L-a)^3}{3a^3h^3L} - \frac{a^2}{2a^3h^3} + \frac{a^3}{3a^3h^3L}\Bigg\} \end{aligned} \tag{58}$$

Equation (58) is substituted into Equation (57) to find the constant "C_3":

$$\begin{aligned} C_3 = \Bigg\{ &-\frac{2a^3h^3}{y_1^3L}\ln\left(\frac{y_1+h}{h}\right) + \frac{a^3h^2y_1 - a^3hy_1^2 + a^2hy_1^2L}{2y_1^2L(y_1+h)} \\ &- \frac{a^3hy_1 - 2a^3h^2}{2y_1^2L} + \frac{a^2(2a-3L)}{6L} - \frac{(L^3-a^3)}{3L}\Bigg\} \end{aligned} \tag{59}$$

Then, Equation (59) is substituted into Equation (55) to obtain the value of "C_1":

$$\begin{aligned} C_1 = \Bigg\{ &-\frac{2}{y_1^3L}\ln\left(\frac{y_1+h}{h}\right) + \frac{ah - ay_1 + y_1L}{2ah^2y_1L(y_1+h)} \\ &- \frac{y_1 - 2h}{2h^2y_1^2L} - \frac{ah - ay_1 + y_1L}{2a^2h^2y_1^2} + \frac{(a-L)^3}{3a^3h^3L} + \frac{a^3}{3a^3h^3L}\Bigg\} \end{aligned} \tag{60}$$

Equation (60) is substituted into Equation (32) to obtain the rotations anywhere of the segment $0 \le x \le a$:

$$\begin{aligned} \frac{dy}{dx} = \frac{12M_{AB}a^3}{EbL}\Bigg\{ &-\frac{2xy_1 - y_1(a+L) - ah}{2y_1^2[xy_1 - a(y_1+h)]^2} - \frac{2}{y_1^3L}\ln\left(\frac{y_1+h}{h}\right) \\ &+ \left[\frac{ah - ay_1 + y_1L}{2ah^2y_1L(y_1+h)}\right] - \left[\frac{y_1-2h}{2h^2y_1^2L}\right] - \left[\frac{ah - ay_1 + y_1L}{2a^2h^2y_1^2}\right] + \frac{(a-L)^3}{3a^3h^3L} + \frac{a^3}{3a^3h^3L}\Bigg\} \end{aligned} \tag{61}$$

In Equation (61) is replaced $x = 0$, to find the rotation in support "A"; it is presented:

$$\begin{aligned} \theta_{A2} = \frac{12M_{AB}a^3}{EbL}\Bigg\{ &-\frac{2}{y_1^3L}\ln\left(\frac{y_1+h}{h}\right) + \left[\frac{y_1(a+L) + ah}{2a^2y_1^2(y_1+h)^2}\right] + \left[\frac{ah - ay_1 + y_1L}{2ah^2y_1L(y_1+h)}\right] \\ &- \left[\frac{y_1-2h}{2h^2y_1^2L}\right] - \left[\frac{ah - ay_1 + y_1L}{2a^2h^2y_1^2}\right] + \frac{(a-L)^3}{3a^3h^3L} + \frac{1}{3h^3L}\Bigg\} \end{aligned} \tag{62}$$

Equation (58) is substituted into Equation (48) to obtain the rotations anywhere of the segment $L - a \le x \le L$:

$$\begin{aligned} \frac{dy}{dx} = \frac{12M_{AB}a^3}{EbL}\Bigg\{ &\frac{2xy_1 + y_1(a-2L) + ah}{2y_1^2[xy_1 + y_1(a-L) + ah]^2} - \frac{2}{y_1^3L}\ln\left(\frac{y_1+h}{h}\right) \\ &+ \left[\frac{a^3h^2y_1 - a^3hy_1^2 + a^2hy_1^2L}{2a^3h^3y_1^2L(y_1+h)}\right] - \left[\frac{a^2h^2L - a^2hy_1L + a^3hy_1 - 2a^3h^2}{2a^3h^3y_1^2L}\right] \\ &+ \frac{(L-a)^2}{2a^3h^3} - \frac{(L-a)^3}{3a^3h^3L} - \frac{a^2}{2a^3h^3} + \frac{a^3}{3a^3h^3L}\Bigg\} \end{aligned} \tag{63}$$

In Equation (63) is replaced $x = L$, to find the rotation in support "B"; it is presented:

$$\begin{aligned}\theta_{B2} = \frac{12M_{AB}a^3}{EbL}\Bigg\{ &-\frac{2}{y_1^3 L}\ln\left(\frac{y_1+h}{h}\right) + \left[\frac{h^2L + ahy_1 - ay_1^2 + y_1^2L}{2ah^2y_1^2L(y_1+h)}\right] \\ &- \left[\frac{hL - y_1L + ay_1 - 2ah}{2ah^2y_1^2L}\right] + \frac{(L-a)^2}{2a^3h^3} - \frac{(L-a)^3}{3a^3h^3L} - \frac{a^2}{2a^3h^3} + \frac{a^3}{3a^3h^3L}\Bigg\}\end{aligned} \tag{64}$$

Subsequently, the member of Figure 2(d) is analyzed to find "θ_{A3}" and "θ_{B3}" in function of "M_{BA}" [16,17]:

The moment at any point of the beam on axis "x" is [18]:

$$M_z = \frac{M_{BA}(x)}{L} \tag{65}$$

a) For the segment $0 \le x \le a$:

Equations (10) and (65) are substituted into Equation (7); it is presented:

$$\frac{dy}{dx} = \frac{12M_{BA}a^3}{EbL}\int \frac{(x)}{(ah + ay_1 - y_1x)^3}dx \tag{66}$$

Equation (66) is evaluated; it is shown:

$$\frac{dy}{dx} = \frac{12M_{BA}a^3}{EbL}\left\{\frac{2xy_1 - a(y_1+h)}{2y_1^2[xy_1 - a(y_1+h)]^2} + C_1\right\} \tag{67}$$

In Equation (67) is replaced $x = a$, to find the rotation $dy/dx = \theta_{a1}$, where the height "h_x" varies the linear shape; it is:

$$\theta_{a1} = \frac{12M_{BA}a^3}{EbL}\left[\frac{(y_1 - h)}{2ah^2y_1^2} + C_1\right] \tag{68}$$

Equation (67) is integrated to obtain the displacements, because there are not known conditions for rotations, this is as follows:

$$y = \frac{12M_{BA}a^3}{EbL}\int\left\{\frac{2xy_1 - a(y_1+h)}{2y_1^2[xy_1 - a(y_1+h)]^2} + C_1\right\}dx \tag{69}$$

Equation (69) is evaluated; it is shown:

$$y = \frac{12M_{BA}a^3}{EbL}\left\{\frac{\ln[xy_1 - a(y_1+h)]}{y_1^3} - \frac{a(y_1+h)}{2y_1^3[xy_1 - a(y_1+h)]} + C_1x + C_2\right\} \tag{70}$$

The boundary conditions are replaced into Equation (70), when $x = 0$ and $y = 0$ to find the constant "C_2"; it is presented:

$$C_2 = -\left\{\frac{2\ln[-a(y_1+h)] + 1}{2y_1^3}\right\} \tag{71}$$

Equation (71) is substituted into Equation (70):

$$y = \frac{12M_{BA}a^3}{EbL}\left\{\frac{\ln[xy_1 - a(y_1+h)]}{y_1^3} - \frac{a(y_1+h)}{2y_1^3[xy_1 - a(y_1+h)]} + C_1x - \frac{2\ln[-a(y_1+h)] + 1}{2y_1^3}\right\} \tag{72}$$

In Equation (72) is replaced $x = a$, to find the displacement $y = y_{a1}$, where the height "h_x" varies the linear shape; it is:

$$y_{a1} = \frac{12M_{BA}a^3}{EbL}\left[-\frac{1}{y_1^3}\ln\left(\frac{y_1+h}{h}\right) + \frac{1}{2hy_1^2} + C_1a\right] \tag{73}$$

b) For the segment $a \le x \le L - a$:

Equations (11) and (65) are substituted into Equation (7); it is presented:

$$\frac{dy}{dx} = \frac{12M_{BA}}{Ebh^3L}\int (x)dx \tag{74}$$

Equation (74) is evaluated; it is shown:

$$\frac{dy}{dx} = \frac{12M_{BA}}{Ebh^3L}\left(\frac{x^2}{2} + C_3\right) \tag{75}$$

In Equation (75) is replaced $x = a$, to find the rotation $dy/dx = \theta_{a2}$; it is:

$$\theta_{a2} = \frac{12M_{BA}}{Ebh^3L}\left(\frac{a^2}{2} + C_3\right) \tag{76}$$

In Equation (75) is replaced $x = L - a$, to find the rotation $dy/dx = \theta_{b2}$; it is:

$$\theta_{b2} = \frac{12M_{BA}}{Ebh^3L}\left[\frac{(L-a)^2}{2} + C_3\right] \tag{77}$$

Equation (75) is integrated to obtain the displacements, because there are not known conditions for rotations; this is as follows:

$$y = \frac{12M_{BA}}{Ebh^3L}\int \left(\frac{x^2}{2} + C_3\right) dx \tag{78}$$

Equation (78) is evaluated; it is shown:

$$y = \frac{M_{BA}}{Ebh^3L}\left(\frac{x^3}{6} + C_3x + C_4\right) \tag{79}$$

In Equation (79) is replaced $x = a$, to find the displacement $y = y_{a2}$; it is:

$$y_{a2} = \frac{12M_{BA}}{Ebh^3L}\left(\frac{a^3}{6} + C_3a + C_4\right) \tag{80}$$

In Equation (79) is replaced $x = L - a$, to find the displacement $y = y_{b2}$; it is:

$$y_{b2} = \frac{12M_{BA}}{Ebh^3L}\left[\frac{(L-a)^3}{6} + C_3(L-a) + C_4\right] \tag{81}$$

c) For the segment $L - a \leq x \leq L$:
Equations (12) and (65) are substituted into Equation (7); it is presented:

$$\frac{dy}{dx} = \frac{12M_{BA}a^3}{EbL}\int \frac{(x)}{(ah + y_1x - y_1L + ay_1)^3}dx \tag{82}$$

Equation (82) is evaluated; it is shown:

$$\frac{dy}{dx} = \frac{12M_{BA}a^3}{EbL}\left\{-\frac{2xy_1 + y_1(a-L) + ah}{2y_1^2[xy_1 + y_1(a-L) + ah]^2} + C_5\right\} \tag{83}$$

In Equation (83) is replaced $x = L - a$, to find the rotation $dy/dx = \theta_{b3}$; it is:

$$\theta_{b3} = \frac{12M_{BA}a^3}{EbL}\left[\frac{y_1(a-L) - ah}{2a^2h^2y_1^2} + C_5\right] \tag{84}$$

Equation (83) is integrated to obtain the displacements, because there are not known conditions for rotations, this is as follows:

$$y = \frac{12M_{BA}a^3}{EbL}\int \left\{-\frac{2xy_1 + y_1(a-L) + ah}{2y_1^2[xy_1 + y_1(a-L) + ah]^2} + C_5\right\} dx \tag{85}$$

Equation (85) is evaluated, it is shown:

$$y = \frac{12M_{BA}a^3}{EbL}\left\{-\frac{\ln[xy_1 + y_1(a-L) + ah]}{y_1^3} - \frac{y_1(a-L)+ah}{2y_1^3[xy_1 + y_1(a-L)+ah]} + C_5x + C_6\right\} \tag{86}$$

The boundary conditions are replaced into Equation (86), when $x = L$ and $y = 0$ to find the constant "C_6" in function of "C_5", this value is:

$$C_6 = \left\{\frac{\ln[a(y_1+h)]}{y_1^3} + \frac{y_1(a-L)+ah}{2ay_1^3(y_1+h)}\right\} - C_5L \tag{87}$$

Equation (87) is substituted into Equation (86):

$$\begin{aligned} y = \frac{12M_{BA}a^3}{EbL}\Bigg\{ & -\frac{\ln[xy_1 + y_1(a-L)+ah]}{y_1^3} - \frac{y_1(a-L)+ah}{2y_1^3[xy_1+y_1(a-L)+ah]} \\ & + \frac{\ln[a(y_1+h)]}{y_1^3} + \frac{y_1(a-L)+ah}{2ay_1^3(y_1+h)} - C_5(L-x)\Bigg\} \end{aligned} \tag{88}$$

In Equation (88) is replaced $x = L - a$, to find the displacement $y = y_{b3}$; it is:

$$y_{b3} = \frac{12M_{BA}a^3}{EbL}\left[\frac{1}{y_1^3}\ln\left(\frac{y_1+h}{h}\right) - \frac{y_1[y_1(a-L)+ah]}{2ahy_1^3(y_1+h)} - C_5a\right] \tag{89}$$

Then, Equations (68) and (76) are equalized, because rotations must be equal at the point $x = a$, where the height "h_x" varies linearly to find the constant "C_3" in function of "C_1", this value is:

$$C_3 = a^3h^3\left[\frac{(y_1-h)}{2ah^2y_1^2} + C_1\right] - \frac{a^2}{2} \tag{90}$$

Also, Equations (73) and (80) are equalized, because the displacements must be equal at the point $x = a$, and subsequently Equation (90) is replaced to find the constant "C_4", this value is:

$$C_4 = -\frac{a^3h^3}{y_1^3}\ln\left(\frac{y_1+h}{h}\right) + \left[\frac{2a^3h^4 - a^3h^3y_1}{2h^2y_1^2}\right] + \frac{a^3}{3} \tag{91}$$

Next, Equations (77) and (84) are equalized, because rotations must be equal at the point $x = L-a$, where the height "h_x" varies linearly to find the constant "C_3" in function of "C_5", this value is:

$$C_3 = \frac{ahy_1(a-L) - a^2h^2}{2y_1^2} + C_5a^3h^3 - \frac{(L-a)^2}{2} \tag{92}$$

Also, Equations (81) and (89) are equalized, because the displacements must be equal at the point $x = L - a$, and subsequently Equations (91) and (92) are replaced to find the constant "C_5", this value is:

$$\begin{aligned} C_5 = \Bigg\{ & \frac{ahy_1^2L^2 - 2a^2hy_1^2L + a^3hy_1^2 + ah^2y_1L^2 + a^2h^3L - a^3h^2y_1}{2a^3h^3y_1^2L(y_1+h)} \\ & + \frac{2}{y_1^3L}\ln\left(\frac{y_1+h}{h}\right) - \frac{2a^3h^2 - a^3hy_1}{2a^3h^3y_1^2L} + \frac{(L-a)^3 - a^3}{3a^3h^3L}\Bigg\} \end{aligned} \tag{93}$$

Equation (93) is substituted into Equation (92) to find the constant "C_3":

$$\begin{aligned} C_3 = \Bigg\{ & \frac{2a^3h^3}{y_1^3L}\ln\left(\frac{y_1+h}{h}\right) - \frac{2a^3h^4 - a^3h^3y_1}{2h^2y_1^2L} \\ & + \frac{-a^2hy_1^2L + a^3hy_1^2 - a^3h^2y_1}{2y_1^2L(y_1+h)} - \frac{(L^3 - 3a^2L + 4a^3)}{6L}\Bigg\} \end{aligned} \tag{94}$$

Then, Equation (94) is substituted into Equation (90) to obtain the value of "C_1":

$$\begin{aligned} C_1 = \Bigg\{ & \frac{2}{y_1^3 L} \ln\left(\frac{y_1+h}{h}\right) - \frac{2a^3h^2 - a^3hy_1 + a^2hy_1L - a^2h^2L}{2a^3h^3y_1^2L} \\ & + \frac{-a^2hy_1^2L + a^3hy_1^2 - a^3h^2y_1}{2a^3h^3y_1^2L(y_1+h)} - \frac{(L^3 - 6a^2L + 4a^3)}{6a^3h^3L} \Bigg\} \end{aligned} \tag{95}$$

Equation (95) is substituted into Equation (67) to obtain the rotations anywhere of the segment $0 \le x \le a$:

$$\begin{aligned} \frac{dy}{dx} = \frac{12M_{BA}a^3}{EbL} \Bigg\{ & \frac{2xy_1 - a(y_1+h)}{2y_1^2[xy_1 - a(y_1+h)]^2} - \frac{2a^3h^2 - a^3hy_1 + a^2hy_1L - a^2h^2L}{2a^3h^3y_1^2L} \\ & + \frac{-a^2hy_1^2L + a^3hy_1^2 - a^3h^2y_1}{2a^3h^3y_1^2L(y_1+h)} - \frac{(L^3 - 6a^2L + 4a^3)}{6a^3h^3L} + \frac{2}{y_1^3L} \ln\left(\frac{y_1+h}{h}\right) \Bigg\} \end{aligned} \tag{96}$$

In Equation (96) is replaced $x = 0$, to find the rotation in support "A"; it is presented:

$$\begin{aligned} \theta_{A3} = \frac{12M_{BA}a^3}{EbL} \Bigg\{ & \frac{2}{y_1^3L} \ln\left(\frac{y_1+h}{h}\right) - \frac{2a^3h^2 - a^3hy_1 + a^2hy_1L - a^2h^2L}{2a^3h^3y_1^2L} \\ & + \frac{-a^2hy_1^2L + a^3hy_1^2 - a^3h^2y_1 - a^2h^3L}{2a^3h^3y_1^2L(y_1+h)} - \frac{(L^3 - 6a^2L + 4a^3)}{6a^3h^3L} \Bigg\} \end{aligned} \tag{97}$$

Equation (93) is substituted into Equation (83) to obtain the rotations anywhere of the segment $L - a \le x \le L$:

$$\begin{aligned} \frac{dy}{dx} = \frac{12M_{BA}a^3}{EbL} \Bigg\{ & -\frac{2xy_1 + y_1(a-L) + ah}{2y_1^2[xy_1 + y_1(a-L) + ah)]^2} + \frac{2}{y_1^3L} \ln\left(\frac{y_1+h}{h}\right) - \frac{2a^3h^2 - a^3hy_1}{2a^3h^3y_1^2L} \\ & + \frac{ahy_1^2L^2 - 2a^2hy_1^2L + a^3hy_1^2 + ah^2y_1L^2 + a^2h^3L - a^3h^2y_1}{2a^3h^3y_1^2L(y_1+h)} + \frac{(L-a)^3 - a^3)}{3a^3h^3L} \Bigg\} \end{aligned} \tag{98}$$

In Equation (98) is replaced $x = L$, to find the rotation in support "B"; it is presented:

$$\begin{aligned} \theta_{B3} = \frac{12M_{BA}a^3}{EbL} \Bigg\{ & \frac{2}{y_1^3L} \ln\left(\frac{y_1+h}{h}\right) - \frac{y_1L + ay_1 + ah}{2a^2y_1^2(y_1+h)^2} - \frac{2a^3h^2 - a^3hy_1}{2a^3h^3y_1^2L} \\ & + \frac{ahy_1^2L^2 - 2a^2hy_1^2L + a^3hy_1^2 + ah^2y_1L^2 + a^2h^3L - a^3h^2y_1}{2a^3h^3y_1^2L(y_1+h)} + \frac{(L-a)^3 - a^3)}{3a^3h^3L} \Bigg\} \end{aligned} \tag{99}$$

Equations (23), (62) and (97) are substituted of the support "A" into Equation (5):

$$\begin{aligned} & \frac{6wa^3}{Eb} \left\{ \frac{1}{y_1^3} \ln\left(\frac{h}{y_1+h}\right) + \frac{4a^3 - 6a^2L + L^3}{12a^3h^3} + \frac{y_1^3(L-a) + hy_1^2L + 3ah^2y_1 + 2ah^3}{2ah^2y_1^2(y_1+h)^2} \right\} \\ & + \frac{12M_{AB}a^3}{EbL} \Bigg\{ -\frac{2}{y_1^3L} \ln\left(\frac{y_1+h}{h}\right) + \frac{y_1(a+L) + ah}{2a^2y_1^2(y_1+h)^2} - \frac{y_1 - 2h}{2h^2y_1^2L} + \frac{ah - ay_1 + y_1L}{2ah^2y_1L(y_1+h)} \\ & - \frac{ah - ay_1 + y_1L}{2a^2h^2y_1^2} + \frac{(a-L)^3}{3a^3h^3L} + \frac{1}{3h^3L} \Bigg\} + \frac{12M_{BA}a^3}{EbL} \Bigg\{ \frac{2}{y_1^3L} \ln\left(\frac{y_1+h}{h}\right) \\ & - \frac{2a^3h^2 - a^3hy_1 + a^2hy_1L - a^2h^2L}{2a^3h^3y_1^2L} + \frac{-a^2hy_1^2L + a^3hy_1^2 - a^3h^2y_1 - a^2h^3L}{2a^3h^3y_1^2L(y_1+h)} \\ & - \frac{(L^3 - 6a^2L + 4a^3)}{6a^3h^3L} \Bigg\} = 0 \end{aligned} \tag{100}$$

Equations (29), (64) and (99) are substituted of the support "B" into Equation (6):

$$\begin{aligned}&\frac{6wa^3}{Eb}\left\{-\frac{1}{y_1^3}\ln\left(\frac{h}{y_1+h}\right)-\frac{4a^3-6a^2L+L^3}{12a^3h^3}-\frac{y_1^3(L-a)+hy_1^2L+3ah^2y_1+2ah^3}{2ah^2y_1^2(y_1+h)^2}\right\}\\&+\frac{12M_{AB}a^3}{EbL}\left\{-\frac{2}{y_1^3L}\ln\left(\frac{y_1+h}{h}\right)+\frac{h^2L+ahy_1-ay_1^2+y_1^2L}{2ah^2y_1^2L(y_1+h)}-\frac{hL-y_1L+ay_1-2ah}{2ah^2y_1^2L}\right.\\&\left.+\frac{(L-a)^2}{2a^3h^3}-\frac{(L-a)^3}{3a^3h^3L}-\frac{a^2}{2a^3h^3}+\frac{a^3}{3a^3h^3L}\right\}+\frac{12M_{BA}a^3}{EbL}\left\{\frac{2}{y_1^3L}\ln\left(\frac{y_1+h}{h}\right)\right.\\&-\frac{y_1L+ay_1+ah}{2a^2y_1^2(y_1+h)^2}-\frac{2a^3h^2-a^3hy_1}{2a^3h^3y_1^2L}\\&\left.+\frac{ahy_1^2L^2-2a^2hy_1^2L+a^3hy_1^2+ah^2y_1L^2+a^2h^3L-a^3h^2y_1}{2a^3h^3y_1^2L(y_1+h)}+\frac{(L-a)^3-a^3}{3a^3h^3L}\right\}=0\end{aligned} \tag{101}$$

Equations (100) and (101) are developed to find "M_{AB}" and "M_{BA}"; it is as follows:

$$\begin{aligned}M_{AB}=&\,w(y_1+h)\left\{12a^3h^3(y_1+h)\ln\left(\frac{y_1+h}{h}\right)-y_1[y_1^3(4a^3-6a^2L+L^3)\right.\\&\left.+hy_1^2(L^3-2a^3)+6a^3h^2y_1+12a^3h^3]\right\}\\&/12y_1^3[y_1^2(2a-L)+hy_1(3a-2L)-h^2L]\end{aligned} \tag{102}$$

$$\begin{aligned}M_{BA}=&\,w(y_1+h)\left\{12a^3h^3(y_1+h)\ln\left(\frac{y_1+h}{h}\right)-y_1[y_1^3(4a^3-6a^2L+L^3)\right.\\&\left.+hy_1^2(L^3-2a^3)+6a^3h^2y_1+12a^3h^3]\right\}\\&/12y_1^3[y_1^2(2a-L)+hy_1(3a-2L)-h^2L]\end{aligned} \tag{103}$$

Then, value of "M_{AB}" and "M_{BA}" are equal, this is a logical situation, since the member is symmetrical.

2.2.2. *Factor of carry-over and stiffness.* In order to develop the method to obtain the factor of carry-over and stiffness, it will be helpful to consider the following problem: If a clockwise moment of "M_{AB}" is applied at the simple support of a straight member of variable cross section simply supported at one end and fixed at the other, find the rotation "θ_A" at the simple support and the moment "M_{BA}" at the fixed end, as shown in Figure 3.

The additional end moments, "M_{AB}" and "M_{BA}", should be such as to cause rotations of "θ_A" and "θ_B", respectively. If "θ_{A2}" and "θ_{B2}" are the end rotations caused by "M_{AB}", according to Figure 3(b) and "θ_{A3}" and "θ_{B3}" by "M_{BA}", these are observed in Figure 3(c).

The conditions of geometry required are [13]:

$$\theta_A=\theta_{A2}-\theta_{A3} \tag{104}$$

$$0=\theta_{B2}-\theta_{B3} \tag{105}$$

The beam of Figure 3(b) is analyzed to find "θ_{A2}" and "θ_{B2}" in function of "M_{AB}" are shown in Equations (62) and (64).

The beam of Figure 3(c) is analyzed to find "θ_{A3}" and "θ_{B3}" in function of "M_{BA}" of the same way; these were obtained by Equations (97) and (99).

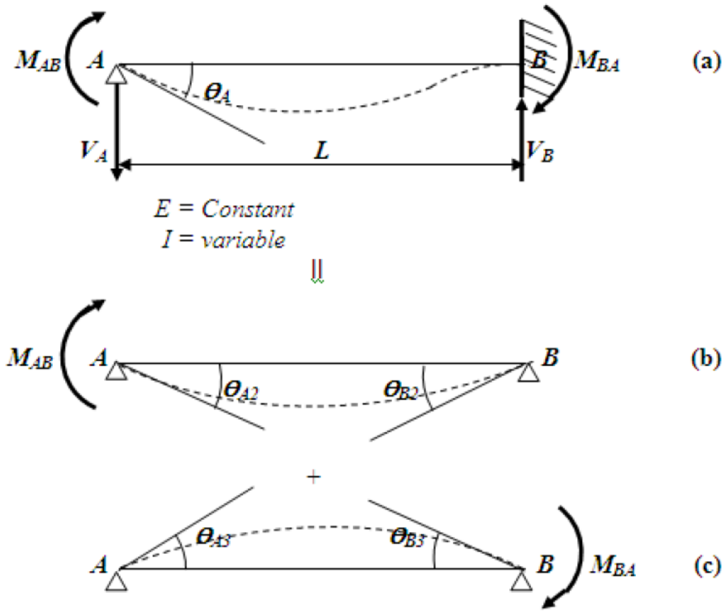

FIGURE 3. Beam simply supported at one end and fixed at the other

Now, Equations (64) and (99) are substituted into Equation (105):

$$\begin{aligned}&\frac{12M_{AB}a^3}{EbL}\left\{-\frac{2}{y_1^3L}\ln\left(\frac{y_1+h}{h}\right)+\frac{h^2L+ahy_1-ay_1^2+y_1^2L}{2ah^2y_1^2L(y_1+h)}\right.\\&\left.-\frac{hL-y_1L+ay_1-2ah}{2ah^2y_1^2L}+\frac{(L-a)^2}{2a^3h^3}-\frac{(L-a)^3}{3a^3h^3L}-\frac{a^2}{2a^3h^3}+\frac{a^3}{3a^3h^3L}\right\}\\&-\frac{12M_{BA}a^3}{EbL}\left\{\frac{2}{y_1^3L}\ln\left(\frac{y_1+h}{h}\right)-\frac{y_1L+ay_1+ah}{2a^2y_1^2(y_1+h)^2}-\frac{2a^3h^2-a^3hy_1}{2a^3h^3y_1^2L}\right.\\&\left.+\frac{ahy_1^2L^2-2a^2hy_1^2L+a^3hy_1^2+ah^2y_1L^2+a^2h^3L-a^3h^2y_1}{2a^3h^3y_1^2L(y_1+h)}+\frac{(L-a)^3-a^3)}{3a^3h^3L}\right\}=0\end{aligned} \tag{106}$$

Equation (106) is used to obtain "M_{BA}" in function of "M_{AB}":

$$\begin{aligned}M_{BA}=\Bigg[\Bigg\{&y_1(y_1+h)[y_1^3(4a^3-6a^2L+L^3)+hy_1^2(L^3-2a^3)+6a^3h^2y_1+6a^3h^3]\\&-12a^3h^3(y_1+h)^2\ln\left(\frac{y_1+h}{h}\right)\Bigg\}\Bigg/\Bigg\{12a^3h^3(y_1+h)^2\left[\ln\left(\frac{y_1+h}{h}\right)\right]\\&-y_1\left[2y_1^4(2a^3-3a^2L+3aL^2-L^3)+hy_1^3(2a^3-6a^2L+9aL^2-4L^3)\right.\\&\left.+2h^2y_1^2(2a^3-L^3)+12a^3h^3y_1+6a^3h^4\right]\Bigg\}\Bigg]M_{AB}\end{aligned} \tag{107}$$

Therefore, the factor of carry-over of "A" to "B" is the ratio of the moment induced at point "B" due to the moment applied at point "A"; this is the moment coefficient "M_{AB}" expressed in Equation (107). The factor of carry-over of "B" to "A" is equal, since the member is symmetrical.

Now, Equations (62) and (97) are substituted into Equation (104):

$$\begin{aligned}\theta_A = {} & \frac{12M_{AB}a^3}{EbL}\bigg\{-\frac{2}{y_1^3L}\ln\left(\frac{y_1+h}{h}\right)+\frac{y_1(a+L)+ah}{2a^2y_1^2(y_1+h)^2}+\frac{ah-ay_1+y_1L}{2ah^2y_1L(y_1+h)} \\ & -\frac{y_1-2h}{2h^2y_1^2L}-\frac{ah-ay_1+y_1L}{2a^2h^2y_1^2}+\frac{(a-L)^3}{3a^3h^3L}+\frac{1}{3h^3L}\bigg\}-\frac{12M_{BA}a^3}{EbL} \\ & \bigg\{\frac{2}{y_1^3L}\ln\left(\frac{y_1+h}{h}\right)-\frac{2a^3h^2-a^3hy_1+a^2hy_1L-a^2h^2L}{2a^3h^3y_1^2L} \\ & +\frac{-a^2hy_1^2L+a^3hy_1^2-a^3h^2y_1-a^2h^3L}{2a^3h^3y_1^2L(y_1+h)}-\frac{(L^3-6a^2L+4a^3)}{6a^3h^3L}\bigg\}\end{aligned} \tag{108}$$

Then, Equation (107) is replaced into Equation (108):

$$\begin{aligned}\frac{EbL}{12a^3}\theta_A = {} & M_{AB}\bigg\{-\frac{2}{y_1^3L}\ln\left(\frac{y_1+h}{h}\right)+\frac{y_1(a+L)+ah}{2a^2y_1^2(y_1+h)^2}+\frac{ah-ay_1+y_1L}{2ah^2y_1L(y_1+h)} \\ & -\frac{y_1-2h}{2h^2y_1^2L}-\frac{ah-ay_1+y_1L}{2a^2h^2y_1^2}+\frac{(a-L)^3}{3a^3h^3L}+\frac{1}{3h^3L}\bigg\} \\ & -\bigg\{\bigg[\bigg\{y_1(y_1+h)[y_1^3(4a^3-6a^2L+L^3)+hy_1^2(L^3-2a^3)+6a^3h^2y_1+6a^3h^3] \\ & -12a^3h^3(y_1+h)^2\ln\left(\frac{y_1+h}{h}\right)\bigg\}\bigg/\bigg\{12a^3h^3(y_1+h)^2\ln\left(\frac{y_1+h}{h}\right) \\ & -y_1[2y_1^4(2a^3-3a^2L+3aL^2-L^3)+hy_1^3(2a^3-6a^2L+9aL^2-4L^3) \\ & +2h^2y_1^2(2a^3-L^3)+12a^3h^3y_1+6a^3h^4]\bigg\}\bigg]M_{AB}\bigg\}\bigg\{\frac{2}{y_1^3L}\ln\left(\frac{y_1+h}{h}\right) \\ & -\frac{2a^3h^2-a^3hy_1+a^2hy_1L-a^2h^2L}{2a^3h^3y_1^2L} \\ & +\frac{-a^2hy_1^2L+a^3hy_1^2-a^3h^2y_1-a^2h^3L}{2a^3h^3y_1^2L(y_1+h)}-\frac{(L^3-6a^2L+4a^3)}{6a^3h^3L}\bigg\}\end{aligned} \tag{109}$$

Equation (109) is used to obtain "M_{AB}" in function of "θ_A":

$$\begin{aligned}M_{AB} = {} & Ebh^3(y_1+h)^2\bigg\{y_1[2y_1^4(2a^3-3a^2L+3aL^2-L^3) \\ & +hy_1^3(2a^3-6a^2L+9aL^2-4L^3)+2h^2y_1^2(2a^3-L^3) \\ & +12a^3h^3y_1+6a^3h^4]-12a^3h^3(y_1+h)^2\ln\left(\frac{y_1+h}{h}\right)\bigg\} \\ & \bigg/\bigg[6[y_1^2(2a-L)+hy_1(3a-2L)-h^2L]\bigg\{24a^3h^3(y_1+h)^2\ln\left(\frac{y_1+h}{h}\right) \\ & -y_1[y_1^4(8a^3-12a^2L+6aL^2-L^3)+hy_1^3(4a^3-12a^2L+9aL^2-2L^3) \\ & +h^2y_1^2(8a^3-L^3)+24a^3h^3y_1+12a^3h^4]\bigg\}\bigg]\theta_A\end{aligned} \tag{110}$$

Therefore, the stiffness factor is the moment applied at the point "A" due to the rotation induced of 1 radian in the support "A"; this is the coefficient of the rotation "θ_A" expressed in Equation (110). The stiffness factor of the moment applied at the point "B" due to the rotation induced of 1 radian in the support "B", also is expressed in Equation (110), since the member is symmetrical.

3. **Application.** In the following example are determined the internal moments at supports of the beam shown in Figure 4.

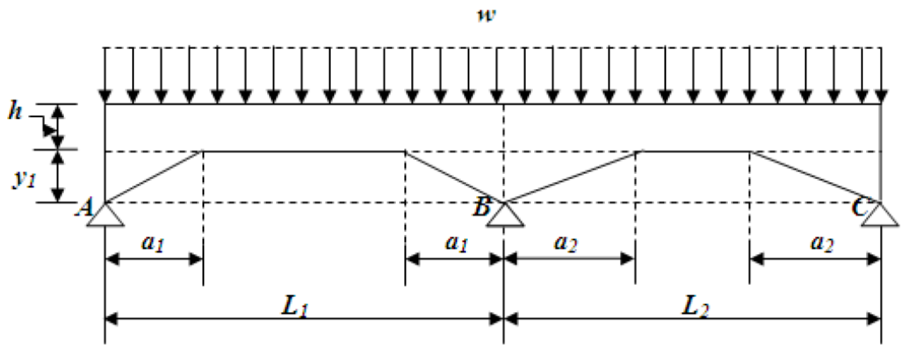

FIGURE 4. Continuous beam with straight haunches simply supported

Data of the continuous beam are:
$w = 3000$kg/m
$b = 30$cm
$h = 30$cm
$y_1 = 30$cm
$a_1 = 2.50$m
$L_1 = 10.00$m
$a_2 = 3.50$m
$L_2 = 10.00$m
Span AB
Using Equations (102) and (103) to obtain moments:

$$M_{AB} = 30501.27\text{kg-m} \quad M_{BA} = 30501.27\text{kg-m}$$

And taking into account the direction of the moments, are as follows:

$$M_{AB} = +30501.27\text{kg-m} \quad M_{BA} = -30501.27\text{kg-m}$$

Substituting into Equation (107) to find the carry-over factor of two ends of the beam:

$$C_{AB} = 0.6233 \quad C_{BA} = 0.6233$$

Now, Equation (110) is used to obtain the stiffness factor of two ends of the beam:

$$K_{AB} = 0.000521\text{E} \quad K_{BA} = 0.000521\text{E}$$

Span BC
Using Equations (102) and (103) to obtain moments:

$$M_{BC} = 31316.70\text{kg-m} \quad M_{CB} = 31316.70\text{kg-m}$$

And taking into account the direction of the moments, are as follows:

$$M_{BC} = +31316.70\text{kg-m} \quad M_{CB} = -31316.70\text{kg-m}$$

Substituting into Equation (107) to find the carry-over factor of two ends of the beam:

$$C_{BC} = 0.5182 \quad C_{CB} = 0.5182$$

Now, Equation (110) is used to obtain the stiffness factor of two ends of the beam:

$$K_{BC} = 0.000498\text{E} \quad K_{CB} = 0.000498\text{E}$$

Using the foregoing values for stiffness factor, the distribution factors are computed and entered in Table 1. The moment distribution follows the same procedure outlined by Hardy Cross method. The results in kg-m are shown on the last line of the table.

Table 1. Moment distribution method

Joint		A	B		C
Member		AB	BA	BC	CB
Stiffness factor		0.000521E	0.000521E	0.000498E	0.000498E
Distribution factor		1.0000	0.5113	0.4887	1.0000
Carry-over factor		0.6233	0.6233	0.5182	0.5182
Fixed-end moments		+30501.27	−30501.27	+31316.70	−31316.70
Cycle 1	FEM	+30501.27	−30501.27	+31316.70	−31316.70
	Balance	−30501.27	−416.93	−398.50	+31316.70
Cycle 2	CO	−259.87	−19011.44	+16228.31	−206.50
	Balance	+259.87	+1423.01	+1360.12	+206.50
Cycle 3	CO	+886.96	+161.98	+107.01	+704.81
	Balance	−886.96	−137.53	−131.46	−704.81
Cycle 4	CO	−85.72	−552.84	−365.23	−68.12
	Balance	+85.72	+469.41	+448.66	+68.12
Cycle 5	CO	+292.58	+53.43	+35.30	+232.50
	Balance	−292.58	−45.37	−43.36	−232.50
Cycle 6	CO	−28.28	−182.37	−120.48	−22.47
	Balance	+28.28	+154.85	+148.00	+22.47
Cycle 7	CO	+96.52	+17.63	+11.64	+76.69
	Balance	−96.52	−14.97	−14.30	−76.69
Cycle 8	CO	−9.33	−60.16	−39.74	−7.41
	Balance	+9.33	+51.08	+48.82	+7.41
Cycle 9	CO	+31.84	+5.82	+3.84	+25.30
	Balance	−31.84	−4.94	−4.72	−25.30
Cycle 10	CO	−3.07	−19.85	−13.11	−2.45
	Balance	+3.07	+16.85	+16.11	+2.45
Cycle 11	CO	+10.50	+1.91	+1.27	+8.35
	Balance	−10.50	−1.63	−1.55	−8.35
Cycle 12	CO	−1.02	−6.54	−4.33	−0.80
	Balance	+1.02	+5.56	+5.31	+0.80
Cycle 13	CO	+3.47	+0.64	+0.41	+2.75
	Balance	−3.47	−0.54	−0.51	−2.75
Cycle 14	CO	−0.34	−2.16	−1.43	−0.26
	Balance	+0.34	+1.84	+1.75	+0.26
Cycle 15	CO	+1.15	+0.21	+0.13	+0.91
	Balance	−1.15	−0.17	−0.17	−0.91
Cycle 16	CO	−0.11	−0.72	−0.47	−0.09
	Balance	+0.11	+0.61	+0.58	+0.09
Total moments		0	−48594.6	+48594.6	0

4. **Results.** Table 1 shows the application of the mathematical model developed in this paper. Moment distribution method or Hardy Cross method was used to develop this example to present the application of the carry-over factor, stiffness factor and fixed-end moments.

A way to validate the proposed model is as follows: in Equations (102) and (103) is substituted "$a = 0L$" to obtain the fixed-end moments "$M_{AB} = wL^2/12$" and "$M_{BA} = wL^2/12$", in Equation (107) is substituted "$a = 0L$" to find the carry-over factor "$C_{AB} =$

$C_{BA} = 0.5$" and to obtain the stiffness factor is substituted "$a = 0L$" into Equation (110) and this is "$K_{AB} = K_{BA} = Ebh^3/3L = 4EI/L$". The values presented above correspond to a constant cross section.

Therefore, the model proposed in this paper is valid and is not limited to certain dimensions or proportions as shown in some books.

5. **Conclusions.** This paper developed a mathematical model for the fixed-end moments, carry-over factors and stiffness for a uniformly distributed load. The properties of the rectangular cross section of the beam vary along its axis, i.e., the width "b" is constant and the height "h" varies along the beam, and this variation is a linear type.

The mathematical technique presented in this research is very adequate for the fixed-end moments, rotations, factors of carry-over and stiffness for beams of variable rectangular cross section subject to a uniformly distributed load, since the results are accurate, and it presents the mathematical expression.

The significant application of fixed-end moments, rotations and displacements is in the matrix methods of structural analysis to obtain the moments acting and the stiffness of a member. The factor of carry-over is used in the moment distribution method or Hardy Cross method.

Traditional methods were used for variable section members, the deflections are obtained by Simpson's rule, or any other techniques to perform numerical integration, and tables showing some authors are restricted to certain relationships. Besides the efficiency and accuracy of the method developed in this research, a significant advantage is that the rotations, displacements and moments are obtained in any cross section of the beam using the respective integral representations in mathematical expression corresponding.

The mathematical model developed in this paper applies only for rectangular beams subject to a uniformly distributed load of variable cross section of symmetric linear shape. The suggestions for future research are shown as follows: 1) when the member presented another type of cross section, by example variable cross section of drawer type, "T" and "I"; 2) when the member has another type of configuration, by example parabolic type, circular and elliptic; 3) when the member is subject to another type of load, by example concentrated load and triangularly distributed load.

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