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APPLICATIONS OF BIPOLAR FUZZY THEORY TO HEMIRINGS

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Abstract. *In our real life, bipolar fuzzy theory is a core feature to be considered: positive information represents what is possible or preferred, while negative information represents what is forbidden or surely false. In this paper, we provide a general algebraic framework for handling bipolar information by combining the theory of bipolar fuzzy sets with hemirings. First, we present the concepts of bipolar fuzzy h-ideals and normal bipolar fuzzy h-ideals. Meanwhile, some illustrative examples are given to show the rationality of the definitions introduced in the present paper. Second, the characterizations of bipolar fuzzy h-ideals are investigated by means of positive t-cut, negative s-cut, homomorphism and equivalence relation. Third, we give the range of the values of the non-constant maximal element of all normal bipolar fuzzy h-ideals.*
Keywords: Bipolar fuzzy h-ideal, Hemiring, Image (resp., inverse image), Equivalence relation, Normal bipolar fuzzy h-ideal

1. **Introduction.** It is well-known that a subject may feel at the same time a positive response and a negative one for the same characteristic of an object in daily life. For instance, for a house, being close to downtown is both good (it is convenient) and bad (it is noisy). Therefore, a recent trend in contemporary information processing focuses on bipolar information, both from a knowledge representation point of view, and from a processing and reasoning one. Bipolarity is important to distinguish between (i) positive information, which represents what is guaranteed to be possible, for example because it has already been observed or experienced, and (ii) negative information, which represents what is impossible or forbidden, or surely false [14, 15]. This domain has recently invoked many interesting research topics in database query [11], psychology [7], image processing [5], multicriteria decision making [10], argumentation [4], human reasoning [9], etc.

Fuzzy set theory, originally proposed by Zadeh [32], has provided a mathematical tool for dealing with complex or uncertain behaviors. However, the membership degrees of fuzzy sets are limited on the interval $[0, 1]$ previously, so there is a great difficulty to express the difference of the irrelevant elements of the contrary elements. To overcome the defect, Lee [26] introduced bipolar fuzzy sets, which is applied to solving real-life problems, especially describing the bipolarity mentioned above because bipolar fuzzy sets constitute an appropriate knowledge representation framework.

Semirings, which provide a common generalization of rings and distributive lattices, was introduced by Vandiver [29] in 1934. Since then, they have extensive applications in several fields, such as automata theory, formal languages, optimization theory, graph theory, theory of discrete event dynamical systems, generalized fuzzy computation, coding theory, analysis of computer programs and other branches of applied mathematics (see [3, 23]). Hemirings, as semirings with commutative addition and zero element, have also been proved to be an important algebraic tool in theoretical computer science [16]. It is well known that an important computational technique for problems (such as CSPs) is sequential variable elimination (bucket elimination). This calls for the structure to be rich enough to allow the definition of an internal operator " + " that not only provides an upper bound of its operands (and thus admits 1 as absorbing element and 0 as a neutral element) but is also assumed to be associative, commutative and idempotent. Unsurprisingly, the kind of structure needed is a hemiring, but of a more general form than the hemirings usually used in constraint programming. On the other hand, hemirings are usually as basic elements to describe the bipolarity of decision strategy. For instance, in [6], net predisposition is generalized to hemiring-valued constraints through use of (i) two hemirings, one, L^+, for representing positive degrees of preference, and the other, L^-, for representing negative degrees of preference, equipped with their respective multiplication $\otimes^+$ and $\otimes^-$ and (ii) an operator $\otimes$ defined with $L^+ \cup L^-$ for combining positive and negative elements. Based on the aforementioned analysis, one of our main purposes is to provide algebraic structures for computation technique in sequential variable elimination and the representation of bipolarity. To this goal, by combing the characteristics which can express the difference of the irrelevant elements from the contrary elements of bipolar fuzzy sets with the operations of hemiring, we introduce the notions of bipolar fuzzy h-ideals and normal bipolar fuzzy h-ideals and discuss some fundamental properties. This works can also provide languages and tools for dealing with the problems in computer science and information science by applying the structure ideas. For example, we obtain a completely distributive lattice by means of defining the partial ordering of bipolar fuzzy h-ideals, which will provide a theoretical basis for the reduction of fuzzy information systems [8].

Ideals of hemirings, as a kind of special hemiring, play a crucial role in the algebraic structure theories since many properties of hemirings are characterized by ideals. However, in general, ideals in hemirings do not coincide with the ideals in rings. Observing this problem, Henriksen [17] introduced k-ideas of semirings, which is a class of more restricted ideals in semirings. After that, another more restricted ideals, h-ideals of hemirings, was considered by Iizuka [18]. Subsequently, La Torre [24] studied thoroughly the properties of the h-ideals and k-ideals of hemirings. Then with the development of fuzzy sets, studies about all kinds of h-ideals and k-ideals have been attracting researchers' widespread interest, and related results emerged in increasing numbers (see [2, 12, 13, 19, 21, 27, 28, 30, 31, 33, 34]). Recently, based on the results of bipolar fuzzy sets, more and more researchers have devoted themselves to studying some bipolar fuzzy sub-algebraic structures. For example, Lee [25] discussed bipolar fuzzy ideals of BCK/BCI-algebras. Akram et al. [1] characterized bipolar fuzzy subalgebras of K-algebras. More recently, Kang and Kang [22] developed bipolar Ω-fuzzy sub-semigroups in semigroups. Motivated by this, we investigated the properties of bipolar fuzzy h-ideals of hemirings. We hope that the results can broaden application fields of hemirings. For example, in view of the applications of hemiring theory in theoretical computer science, with bipolar fuzzy h-ideals of hemirings, one can give approximate classification for elements of hemirings.

The rest of this paper is organized as follows. In Section 2, some preliminaries are introduced. In Section 3, we present the concepts of bipolar fuzzy h-ideals and discuss related properties by means of positive t-cut, negative s-cut. In Section 4, we characterize the properties of the image and inverse image of bipolar fuzzy h-ideals by homomorphism of hemirings. In Section 5, the characterizations of the maps from the set of bipolar fuzzy h-ideals to the set of h-ideals are investigated through the equivalence relation. We characterize normal bipolar fuzzy h-ideal of hemiring in Section 6. Finally, we give the conclusions in Section 7.

2. **Preliminaries.** In this section, we review some definitions regarding hemirings [34] and bipolar fuzzy sets [26].

Suppose that $(S,+)$ and $(S,\cdot)$ are two semigroups, then the algebraic system $(S,+,\cdot)$ is called a semiring, in which the two algebraic structures are connected by the distributive laws: $a\cdot(b+c)=a\cdot b+a\cdot c$ and $(b+c)\cdot a=b\cdot a+c\cdot a$ for all $a,b,c\in S$.

The zero element of a semiring $(S,+,\cdot)$ is an element $0\in S$ satisfying $0\cdot x=x\cdot 0=0$ and $0+x=x+0=x$ for all $x\in S$. A semiring with zero and a commutative semigroup $(S,+)$ is called a hemiring.

A non-empty subset I of a hemiring S is called a left (resp., right) ideal of S if I is closed with respect to addition and $SI\subseteq I$ (resp., $IS\subseteq I$). I is called an ideal of S if it is both a left and a right ideal of S.

A left (resp., right) ideal of a hemiring S is called a left (resp., right) h-ideal if any $x,z\in S$, any $a,b\in A$ and $x+a+z=b+z$ implies $x\in A$.

A mapping f from a hemiring S to a semiring T is said to be a homomorphism if for all $x,y\in S$, $f(x+y)=f(x)+f(y)$ and $f(xy)=f(x)f(y)$.

Let S be the universe of discourse. A bipolar-valued fuzzy set A in S is an object with the form $A=\{(x,\mu^P(x),\mu^N(x))|x\in S\}$, where μ^P is a positive map from S into $[0,1]$, and μ^N is a negative map from S into $[-1,0]$. For the sake of simplicity, we shall use the symbol $A=(\mu_A^P,\mu_A^N)$ expressing the bipolar-valued fuzzy set $A=\{(x,\mu^P(x),\mu^N(x))|x\in S\}$. $\mu_A^P(x)$ which expresses the satisfaction degree of an element $x\in S$ about some property is often called a positive membership degree. $\mu_A^N(x)$ which expresses the satisfaction degree of $x\in S$ about some implicit counter-property is often called a negative membership degree. In the rest of this paper, we will use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

Throughout this paper, we only give the proof of results about left cases because the proof of results about right cases can be conducted by similar methods. In order to facilitate discussion, S and T are hemirings unless otherwise specified.

3. **Bipolar Fuzzy *h*-Ideals.** In this section, applying bipolar fuzzy sets theory to semirings, we introduce the notion of bipolar fuzzy left h-ideals of semirings and discuss their properties.

Definition 3.1. *A bipolar fuzzy set $A=(\mu_A^P,\mu_A^N)$ of S is called a bipolar fuzzy left (resp., right) h-ideal of S provide that for all $x,y,z,a,b\in S$:*
(1) $\mu_A^P(x+y)\geq\min\{\mu_A^P(x),\mu_A^P(y)\}$ and $\mu_A^N(x+y)\leq\max\{\mu_A^N(x),\mu_A^N(y)\}$,
(2) $\mu_A^P(xy)\geq\mu_A^P(y)$ (resp., $\mu_A^P(xy)\geq\mu_A^P(x)$) and $\mu_A^N(xy)\leq\mu_A^N(y)$ (resp., $\mu_A^N(xy)\leq\mu_A^N(x)$),
(3) $x+a+z=b+z\Rightarrow\mu_A^P(x)\geq\min\{\mu_A^P(a),\mu_A^P(b)\}$ and $\mu_A^N(x)\leq\max\{\mu_A^N(a),\mu_A^N(b)\}$.

A bipolar fuzzy set which is a bipolar fuzzy left and right h-ideal of S is called a bipolar fuzzy h-ideal of S. In this paper, the collection of all bipolar fuzzy h-ideals of S is denoted by BFhI(S) in short. We note that a bipolar fuzzy set $A=(\mu_A^P,\mu_A^N)$ of S is a bipolar fuzzy h-ideal of S if and only if it satisfies $(1),(3)$ and the following

$$(2')\ \mu_A^P(xy)\geq\max\{\mu_A^P(x),\mu_A^P(y)\}\text{ and }\mu_A^N(xy)\leq\min\{\mu_A^N(x),\mu_A^N(y)\}.$$

Example 3.1. *Let $S=\{0,1,2,3\}$ be a set with the addition operation $(+)$ and the multiplication $(\cdot)$ as follows:*

$+$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	1
3	0	1	1	1

Then S is a hemiring. Define a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ as follows:

	0	1	2	3
μ_A^P	0.5	0.5	0.5	0.3
μ_A^N	–0.6	–0.6	–0.6	–0.2

By routine calculations, we know that $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy h-ideal of S.

The theory of completely distributive lattice provides a base ground for the reduction of fuzzy information systems, fuzzy automaton, mathematical morphology, etc. Next, we will construct a completely distributive lattice that provides bipolar fuzzy h-ideals of hemirings.

Remark 3.1. *Let $A = (\mu_A^P, \mu_A^N) \in BFhI(S)$, $B = (\mu_B^P, \mu_B^N) \in BFhI(S)$. If $\mu_A^P(x) \leq \mu_B^P(x)$ and $\mu_A^N(x) \geq \mu_B^N(x)$ for all $x \in S$, then we write $A \leq B$. Let $A, B \in BFhI(S)$, we define $A \vee B = (\mu_A^P(x) \vee \mu_B^P(x), \mu_A^N(x) \wedge \mu_B^N(x))$ and $A \wedge B = (\mu_A^P(x) \wedge \mu_B^P(x), \mu_A^N(x) \vee \mu_B^N(x))$. It is easy to verify that $(BFhI(S), \leq, \wedge, \vee)$ is a completely distributive lattice, which has the least and greatest elements, $\mathbf{0} = (\mathbf{0}^P, \mathbf{0}^N)$ and $\mathbf{1} = (\mathbf{1}^P, \mathbf{1}^N)$, respectively, where $\mathbf{0}^P(x) = \mathbf{0}^N(x) = 0$, $\mathbf{1}^P(x) = 1$, $\mathbf{1}^N(x) = -1$ for all $x \in S$.*

An interesting consequence of bipolar fuzzy h-ideals of hemirings is the following.

Proposition 3.1. *Let A be a non-empty subset of S. A bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ is defined by*

$$\mu_A^P(x) = \begin{cases} m_1, & \text{if } x \in A, \\ m_2, & \text{otherwise} \end{cases} \quad \text{and} \quad \mu_A^N(x) = \begin{cases} n_1, & \text{if } x \in A, \\ n_2, & \text{otherwise}, \end{cases}$$

where $0 \leq m_2 \leq m_1 \leq 1$, $-1 \leq n_1 \leq n_2 \leq 0$, is a bipolar fuzzy left (resp., right) h-ideal of S if and only if A is a left (resp., right) h-ideal of S.

The research about the relationships of fuzzy subalgebras and crisp subalgebras by cut sets is usual but important, as it is a tie which can connect abstract algebraic structures and fuzzy ones. However, now we encounter a significant challenge that the traditional cut sets are not suitable for the framework of bipolar fuzzy h-ideals of hemirings because the characterization of bipolarity. As a consequence, we defined positive t-cut and negative s-cut.

Definition 3.2. *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set of S and $(s,t) \in [-1,0] \times [0,1]$. We define*

$$A_t^P = \{x \in S | \mu_A^P(x) \geq t\} \text{ and } A_s^N = \{x \in S | \mu_A^N(x) \leq s\},$$

and call them positive $t-cut$ and negative $s-cut$ of $A = (\mu_A^P, \mu_A^N)$, respectively. For every $k \in [0,1]$, the set $A_k^P \cap A_{-k}^N$ is called the $k-cut$ of $A = (\mu_A^P, \mu_A^N)$.

From Definition 3.2, we can easily obtained the relation of bipolar fuzzy h-ideals and h-ideals of hemirings.

Theorem 3.1. *Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set of S, then $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy left (resp., right) h-ideal of S if and only if the followings hold:*
(i) For all $t \in [0,1]$, $A_t^P \neq \emptyset \Rightarrow A_t^P$ is a left (resp., right) h-ideal of S.
(ii) For all $s \in [-1,0]$, $A_s^N \neq \emptyset \Rightarrow A_s^N$ is a left (resp., right) h-ideal of S.

Proof: Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy left h-ideal of S and $t \in [0,1]$ with $A_t^P \neq \emptyset$. Then $\mu_A^P(x) \geq t$, $\mu_A^P(y) \geq t$ for all $x, y \in A_t^P$, $s \in S$. It implies that

$$\mu_A^P(x+y) \geq \min\{\mu_A^P(x), \mu_A^P(y)\} \geq t \quad \text{and} \quad \mu_A^P(sy) \geq \mu_A^P(y) \geq t,$$

i.e., $x+y, sy \in \mu_A^P$. Moreover, let $x, z \in S$, $a, b \in A_t^P$ with $x+a+z = b+z$. Then $\mu_A^P(x) \geq \min\{\mu_A^P(a), \mu_A^P(b)\} \geq t$. This means $x \in A_t^P$. Hence, μ_A^P is a left h-ideal of S. Analogously, we can prove (ii).

Conversely, assume (i) and (ii) are all valid. For any $x \in S$, if $\mu_A^P(x) = t$, $\mu_A^N(x) = s$, then $x \in A_t^P \cap A_s^N$. Thus A_t^P and A_s^N are non-empty. Suppose that $A = (\mu_A^P, \mu_A^N)$ is not a bipolar fuzzy h-ideal of S, then there exists $x, z, a, b \in S$ such that $x + a + z = b + z$,

$$\mu_A^P(x) < t < \min\{\mu_A^P(a), \mu_A^P(b)\} \quad \text{and} \quad \mu_A^N(x) > s > \max\{\mu_A^N(a), \mu_A^N(b)\}.$$

Therefore, $a, b \in A_t^P$ but $x \notin A_t^P$ and $a, b \in A_s^N$ but $x \notin A_s^N$. This is a contradiction. Therefore, $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy left h-ideal of S.

As immediate consequences of Theorem 3.1, we have the following:

Corollary 3.1. *If $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy h-ideal of S, then the $k - cut$ of $A = (\mu_A^P, \mu_A^N)$ is a bipolar h-ideal of S for all $k \in [0,1]$.*

For the sake of simplicity, we denote $S^{(t,s)}$ for the set $\{x \in S | \mu_A^P(x) \geq t\} \cap \{x \in S | \ \mu_A^N(x) \leq s\}$, where $A = (\mu_A^P, \mu_A^N)$.

Corollary 3.2. *If $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy left (resp., right) h-ideal of S, then $S^{(t,s)}$ is a left (resp., right) h-ideal of S for all $(t,s) \in [0,1] \times [-1,0]$. In particular, the non-empty $k - cut$ of $A = (\mu_A^P, \mu_A^N)$ is an h-ideal of S for all $k \in [0,1]$.*

Theorem 3.2. *Assume that $A = (\mu_A^P, \mu_A^N) \in BFhI(S)$ and $\mu_A^P(x) + \mu_A^N(x) \geq 0$ for all $x \in S$, then $A_k^P \cup A_{-k}^N$ is a left (resp., right) h-ideal of S for all $k \in [0,1]$.*

Proof: Let $k \in [0,1]$, evidently, $A_k^P \neq \emptyset$, $A_{-k}^N \neq \emptyset$ and they are all left h-ideals of S from Theorem 3.1. Let $x_1, x_2 \in A_k^P \cup A_{-k}^N$, $x, z \in S$ with $x + x_1 + z = x_2 + z$. To complete the proof, we just need to consider the following four cases:
(i) $x_1 \in A_k^P, x_2 \in A_k^P$, (ii) $x_1 \in A_k^P, x_2 \in A_{-k}^N$, (iii) $x_1 \in A_{-k}^N, x_2 \in A_k^P$, (iv) $x_1 \in A_{-k}^N, x_2 \in A_{-k}^N$. Case (i) implies $\mu_A^P(x_1) \geq k$, $\mu_A^P(x_2) \geq k$. Since $A = (\mu_A^P, \mu_A^N) \in BFhI(S)$, we can obtain

$$\mu_A^P(x_1 + x_2) \geq \min\{\mu_A^P(x_1), \mu_A^P(x_2)\} \geq k, \quad \mu_A^P(xx_1) \geq \mu_A^P(x_1) \geq k$$

and

$$\mu_A^P(x) > \min\{\mu_A^P(x_1), \mu_A^P(x_2)\} \geq k.$$

Then $x_1 + x_2, xx_1, x \in A_k^P \subseteq A_k^P \cup A_{-k}^N$. The proof of case (iv) is similar to case (i). For case (ii), we can easily acquire $\mu_A^P(x_1) \geq k$, $\mu_A^N(x_2) \leq -k$. Since $\mu_A^P(x_2) + \mu_A^N(x_2) \geq 0$, $\mu_A^P(x_2) \geq -\mu_A^N(x_2) \geq k$, we have $\mu_A^P(x_1 + x_2) \geq \min\{\mu_A^P(x_1), \mu_A^P(x_2)\} \geq \min\{\mu_A^P(x_1), -\mu_A^N(x_2)\} \geq k$, $\mu_A^P(xx_1) \geq \mu_A^P(x_1) \geq k$ and $\mu_A^P(x) \geq \min\{\mu_A^P(x_1), \mu_A^P(x_2)\} \geq \min\{\mu_A^P(x_1), -\mu_A^N(x_2)\} \geq k$. Then $x_1 + x_2, xx_1, x \in A_k^P \subseteq A_k^P \cup A_{-k}^N$. The proof of case (iii) is similar to (ii). Hence, $A_k^P \cup A_{-k}^N$ is a left h-ideal of S.

4. **The Image and Inverse Image of Bipolar Fuzzy *h*-Ideals.**

In this section, we discuss the properties of the image and inverse image of bipolar fuzzy h-ideals by homomorphism of hemirings.

Definition 4.1. *Let $\varphi : S \to T$ be a homomorphism of hemirings, and $B = (\mu_B^P, \mu_B^N)$ be a bipolar fuzzy set of T. Then the inverse image of B, $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B^P), \varphi^{-1}(\mu_B^N))$, is the bipolar fuzzy set of S given by $\varphi^{-1}(\mu_B^P)(x) = \mu_B^P(\varphi(x))$, $\varphi^{-1}(\mu_B^N)(x) = \mu_B^N(\varphi(x))$ for* *all $x \in S$. Conversely, let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set of S. The image of A, $\varphi(A) = (\varphi(\mu_A^P), \varphi(\mu_A^N))$, is a bipolar fuzzy set of T defined by*

$$\varphi(\mu_A^P)(y) = \begin{cases} \bigvee\limits_{z\in\varphi^{-1}(y)} \mu_A^P(z), & \text{if } \varphi^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\varphi(\mu_A^N)(y) = \begin{cases} \bigwedge\limits_{z\in\varphi^{-1}(y)} \mu_A^N(z), & \text{if } \varphi^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise} \end{cases}$$

for all $y \in T$, where $\varphi^{-1}(y) = \{x \in S | \varphi(x) = y\}$.

Theorem 4.1. *Let $\varphi : S \to T$ be a homomorphism of hemirings, and $B = (\mu_B^P, \mu_B^N)$ be a bipolar fuzzy left (resp., right) h-ideal of T, then the inverse image $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B^P), \varphi^{-1}(\mu_B^N))$ is a bipolar fuzzy left (resp., right) h-ideal of S.*

Proof: Suppose that $B = (\mu_B^P, \mu_B^N)$ is a bipolar fuzzy left h-ideal of T and φ is a homomorphism of hemirings from S to T. Then for all $x, y \in S$, we have

$$\begin{aligned} \varphi^{-1}(\mu_B^P)(x+y) &= \mu_B^P(\varphi(x+y)) = \mu_B^P(\varphi(x)+\varphi(y)) \\ &\geq \min\{\mu_B^P(\varphi(x)), \mu_B^P(\varphi(y))\} \\ &= \min\{\varphi^{-1}(\mu_B^P)(x), \varphi^{-1}(\mu_B^P)(y)\} \end{aligned}$$

and

$$\begin{aligned} \varphi^{-1}(\mu_B^N)(x+y) &= \mu_B^N(\varphi(x+y)) = \mu_B^N(\varphi(x)+\varphi(y)) \\ &\leq \max\{\mu_B^N(\varphi(x)), \mu_B^N(\varphi(y))\} \\ &= \max\{\varphi^{-1}(\mu_B^N)(x), \varphi^{-1}(\mu_B^N)(y)\}. \end{aligned}$$

Thus, (1) is valid. By the same way, we can show that (2) is hold. Moreover, let $x, z, a, b \in S$ with $x + a + z = b + z$. We can acquire $\varphi(x) + \varphi(a) + \varphi(z) = \varphi(b) + \varphi(z)$ and

$$\begin{aligned} \varphi^{-1}(\mu_B^P)(x) &= \mu_B^P(\varphi(x)) \\ &\geq \min\{\mu_B^P(\varphi(a), \mu_B^P(\varphi(b))\} \\ &= \min\{\varphi^{-1}(\mu_B^P)(a), \varphi^{-1}(\mu_B^P)(b)\}. \end{aligned}$$

Analogously, we have

$$\varphi^{-1}(\mu_B^N)(x) \leq \max\{\varphi^{-1}(\mu_B^N)(a), \varphi^{-1}(\mu_B^N)(b)\}.$$

Hence, $\varphi^{-1}(B) = (\varphi^{-1}(\mu_B^P), \varphi^{-1}(\mu_B^N))$ is a bipolar fuzzy left h-ideal of S.

Theorem 4.2. *Assume that $\varphi : S \to T$ be an epimorphism of hemirings. If $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy left (resp., right) h-ideal of S, then the image $\varphi(A) = (\varphi(\mu_A^P), \varphi(\mu_A^N))$ is a bipolar fuzzy left (resp., right) h-ideal of T.*

Proof: Since φ is an epimorphism, by Theorem 3.1, it is sufficient to show that $\varphi(A)_t^P$ and $\varphi(A)_s^N$ are h-ideals of T for all $(t, s) \in [0, 1] \times [-1, 0]$ satisfying $\varphi(A)_t^P \neq \emptyset, \varphi(A)_s^N \neq \emptyset$. Let $t \in [0, 1]$ and $\varphi(A)_t^P \neq \emptyset$. Then for all $y_1, y_2 \in \varphi(A)_t^P$, we can obtain

$$\varphi(\mu_A^P)(y_1) = \bigvee_{x\in\varphi^{-1}(y_1)} \mu_A^P(x) \geq t \quad \text{and} \quad \varphi(\mu_A^P)(y_2) = \bigvee_{x\in\varphi^{-1}(y_2)} \mu_A^P(x) \geq t.$$

This means that there exist $x_1 \in \varphi^{-1}(y_1)$, $x_2 \in \varphi^{-1}(y_2)$ such that $\mu_A^P(x_1) \geq t$, $\mu_A^P(x_2) \geq t$. Then

$$\varphi(\mu_A^P)(y_1 + y_2) = \bigvee_{x\in\varphi^{-1}(y_1+y_2)} \mu_A^P(x) \geq \mu_A^P(x_1 + x_2) \geq \min\{\mu_A^P(x_1), \mu_A^P(x_2)\} \geq t.$$

Therefore, $y_1 + y_2 \in \varphi(A)_t^P$.

For all $y_0 \in \varphi(A)_t^P$, we have $\varphi(\mu_A^P)(y_0) = \bigvee_{x\in\varphi^{-1}(y_0)} \mu_A^P(x) \geq t$, which implies that there exists $x_0 \in \varphi^{-1}(y_0)$ such that $\mu_A^P(x_0) \geq t$. For each $y \in T$, since φ is an epimorphism and $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy left h-ideal of S, there exists $x \in S$ such that $\varphi(x) = y, \mu_A^P(xx_0) \geq \mu_A^P(x_0)$. Then

$$\varphi(\mu_A^P)(yy_0) = \bigvee_{x\in\varphi^{-1}(yy_0)} \mu_A^P(x) \geq \mu_A^P(xx_0) \geq \mu_A^P(x_0) \geq t.$$

Thus, $yy_0 \in \varphi(A)_t^P$.

Moreover, let any $y, z \in T$ and any $m, n \in \varphi(A)_t^P$ such that $y + m + z = n + z$. Then we can acquire

$$\varphi(\mu_A^P)(m) = \bigvee_{x\in\varphi^{-1}(m)} \mu_A^P(x) \geq t \text{ and } \varphi(\mu_A^P)(n) = \bigvee_{x\in\varphi^{-1}(n)} \mu_A^P(x) \geq t.$$

Since φ is an epimorphism, $\varphi^{-1}(y + m + z) = \varphi^{-1}(n + z)$, i.e., $\varphi^{-1}(y) + \varphi^{-1}(m) + \varphi^{-1}(z) = \varphi^{-1}(n) + \varphi^{-1}(z)$. So there must be $x_0 \in \varphi^{-1}(y)$, $m_0 \in \varphi^{-1}(m)$, $n_0 \in \varphi^{-1}(n)$ and $z_0 \in \varphi^{-1}(z)$ such that $x_0 + m_0 + z_0 = n_0 + z_0$ and $\mu_A^P(m_0) \geq t$, $\mu_A^P(n_0) \geq t$. Due to $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy left h-ideal of S, we have $\mu_A^P(x_0) \geq \min\{\mu_A^P(m_0), \mu_A^P(n_0)\}$. Then

$$\varphi(\mu_A^P)(y) = \bigvee_{x\in\varphi^{-1}(y)} \mu_A^P(x) \geq \mu_A^P(x_0) \geq \min\{\mu_A^P(m_0), \mu_A^P(n_0)\} \geq t.$$

Thus $y \in \varphi(A)_t^P$. This means that $\varphi(A)_t^P$ is a left h-ideal of T. Analogously, we can prove that $\varphi(A)_s^N$ is a left h-ideal of T. This completes the proof.

In the theorem mentioned above, if $\varphi : S \to T$ is not an epimorphism, then $\varphi(A) = (\varphi(\mu_A^P), \varphi(\mu_A^N))$ may not be a bipolar fuzzy left (resp., right) h-ideal of S, as is shown in the following example.

Example 4.1. *Let T be the hemiring which is shown in Example 3.1. If $S = \{0, x, y\}$ is a hemiring with the following addition and multiplication:*

$+$	0	x	y
0	0	x	y
x	x	0	y
y	y	y	0

$\cdot$	0	x	y
0	0	0	0
x	0	0	0
y	0	0	y

Define $\varphi : S \to T$ as $\varphi(0) = \varphi(x) = \varphi(y) = 1$. It is easy to verify that φ is a homomorphism from S to T. Choose a bipolar fuzzy h-ideal $A = (\mu_A^P, \mu_A^N)$ as follows:

	0	x	y
μ_A^P	0.7	0.7	0.3
μ_A^N	-0.4	-0.4	-0.2

By Definition 4.1, $\varphi(\mu_A^P) = (0_0, 1_{0.7}, 2_0, 3_0)$, $\varphi(\mu_A^N) = (0_0, 1_{-0.4}, 2_0, 3_0)$, i.e., $\varphi(\mu_A^P)(0) = \varphi(\mu_A^P)(2) = \varphi(\mu_A^P)(3) = 0$, $\varphi(\mu_A^P)(1) = 0.7$, $\varphi(\mu_A^N)(0) = \varphi(\mu_A^N)(2) = \varphi(\mu_A^N)(3) = 0$, $\varphi(\mu_A^N)(1) = -0.4$. Then $\varphi(A) = (\varphi(\mu_A^P), \varphi(\mu_A^N))$ is not a bipolar fuzzy left h-ideal of T.

5. **Equivalence Relations on Bipolar Fuzzy Ideals.** In this section, we continue to discuss the relation between bipolar fuzzy h-ideals and h-ideals of hemirings by another means: equivalence relations.

For any $(t, s) \in [0, 1] \times [-1, 0]$, define two binary relations P^t and N^s on $BFhI(S)$ as follows:

$$(A, B) \in P^t \Leftrightarrow A_t^P = B_t^P \text{ and } (A, B) \in N^s \Leftrightarrow A_s^N = B_s^N,$$

for all $A = (\mu_A^P, \mu_A^N) \in BFhI(S)$, $B = (\mu_B^P, \mu_B^N) \in BFhI(S)$. It is easy to know P^t and N^s are equivalence relations on $BFhI(S)$.

For any $A = (\mu_A^P, \mu_A^N) \in BFhI(S)$, we use $[A]_{P^t}$ (resp., $[A]_{N^s}$) expressing the equivalence class of a modular P^t (resp., N^s). For all $A = (\mu_A^P, \mu_A^N) \in BFhI(S)$, the family of $[A]_{P^t}$ (resp., $[A]_{N^s}$) is denoted by $BFhI(S)/P^t$ (resp., $BFhI(S)/N^s$). That is, $BFhI(S)/P^t = \{[A]_{P^t} | A = (\mu_A^P, \mu_A^N) \in BFhI(S)\}$ (resp., $BFhI(S)/N^s = \{[A]_{N^s} | A = (\mu_A^P, \mu_A^N) \in BFhI(S)\}$). Let $I(S)$ be the family of all h-ideals of S. Define maps

$$f_t : BFhI(S) \to I(S) \cup \{\emptyset\}, \qquad A \to A_t^P,$$

$$g_s : BFhI(S) \to I(S) \cup \{\emptyset\}, \qquad A \to A_s^N,$$

for all $A = (\mu_A^P, \mu_A^N) \in BFhI(S)$. Then f_t and g_s are clearly well-defined.

In the light of the definition of equivalence relations on $BFhI(S)$, we can obtain the following properties.

Theorem 5.1. *For any $(t, s) \in (0, 1) \times (-1, 0)$, the maps f_t and g_s are surjective.*

Proof: Clearly, a bipolar fuzzy set $\mathbf{0} = (\mathbf{0}^P, \mathbf{0}^N)$ is a bipolar fuzzy h-ideal of S, where $\mathbf{0}^P(x) = \mathbf{0}^N(x) = 0$ for all $x \in S$. Then we have

$$f_t(\mathbf{0}) = \mathbf{0}_t^P = \{x \in S | \mathbf{0}^P(x) \geq t\} = \emptyset \text{ and } g_s(\mathbf{0}) = \mathbf{0}_s^N = \{x \in S | \mathbf{0}^N(x) \leq s\} = \emptyset.$$

For any non-empty B in $I(S)$, consider a bipolar fuzzy set $B_\sim = (\mu_{B_\sim}^P, \mu_{B_\sim}^N)$ in S, where

$$\mu_{B_\sim}^P : S \to [0, 1], \qquad \mu_{B_\sim}^P(x) = \begin{cases} 1, & \text{if } x \in B, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\mu_{B_\sim}^N : S \to [-1, 0], \qquad \mu_{B_\sim}^N(x) = \begin{cases} -1, & \text{if } x \in B, \\ 0, & \text{otherwise} \end{cases}$$

By Proposition 3.1, $B_\sim = (\mu_{B_\sim}^P, \mu_{B_\sim}^N) \in BFhI(S)$. According to the definition, we have

$$f_t(B_\sim) = B_{\sim t}^P = \{x \in S | \mu_{B_\sim}^P(x) \geq t\} = \{x \in S | \mu_{B_\sim}^P(x) = 1\} = B$$

and

$$g_s(B_\sim) = B_{\sim s}^N = \{x \in S | \mu_{B_\sim}^N(x) \leq s\} = \{x \in S | \mu_{B_\sim}^N(x) = -1\} = B.$$

Therefore, f_t and g_s are all surjective.

Now a natural question arises here: are there any relationships between the quotient sets and the set of h-ideals of S. In the following, we will concentrate on giving the answers.

Theorem 5.2. *The quotient sets $BFhI(S)/P^t$ and $BFhI(S)/N^s$ are equipotent to $I(S) \cup \emptyset$ for all $(t, s) \in (0, 1) \times (-1, 0)$.*

Proof: For all $(t, s) \in (0, 1) \times (-1, 0)$ and $A = (\mu_A^P, \mu_A^N) \in BFhI(S)$ let

$$f_t^* : BFhI(S)/P^t \to I(S) \cup \{\emptyset\}, \qquad [A]_{P^t} \to f_t(A)$$

and

$$g_s^* : BFhI(S)/N^s \to I(S) \cup \{\emptyset\}, \qquad [A]_{N^s} \to g_s(A),$$

respectively. For every $A, B \in BFhI(S)$, if $\mu_A^P = \mu_B^P$ and $\mu_A^N = \mu_B^N$, then $(A, B) \in P^t$ and $(A, B) \in N^s$, which means $[A]_{P^t} = [B]_{P^t}$ and $[A]_{N^s} = [B]_{N^s}$. Thus f_t and g_s are all injective.

For any non-empty B in $I(S)$, consider the bipolar fuzzy h-ideal $B_\sim = (\mu_{B_\sim}^P, \mu_{B_\sim}^N)$ which is given in the proof of Theorem 5.1, then we have

$$f_t^*([B_\sim]_{P^t}) = f_t(B_\sim) = B_{\sim t}^P = B \text{ and } g_s^*([B_\sim]_{N^s}) = g_s(B_\sim) = B_{\sim s}^N = B.$$

For the bipolar fuzzy ideal $\mathbf{0} = (\mathbf{0}^P, \mathbf{0}^N)$ of S, we have

$$f_t^*([\mathbf{0}]_{P^t}) = f_t(\mathbf{0}) = \mathbf{0}_t^P = \{x \in S | \mathbf{0}^P(x) \geq t\} = \emptyset$$

and

$$g_s^*([\mathbf{0}]_{N^s}) = g_s(\mathbf{0}) = \mathbf{0}_s^N = \{x \in S | \mathbf{0}^N(x) \leq s\} = \emptyset.$$

Hence, f_t and g_s are surjective. This completes the proof.

For any $0 < k < 1$, we define another relation S^k on $BFhI(S)$ as follows:

$$(A, B) \in S^k \Leftrightarrow A_k = B_k,$$

where $A_k = A_k^P \cap A_{-k}^N$. Then the relation S^k is also an equivalence relation on $BFhI(S)$.

Theorem 5.3. *Let $0 < k < 1$, then the map $\varphi_k : BFhI(S) \to I(S) \cup \emptyset$ defined by $\varphi_k(A) = A_k$ is surjective.*

Proof: Let $0 < k < 1$, we have $\varphi_k(\mathbf{0}) = \mathbf{0}_k^P \cap \mathbf{0}_{-k}^N = \emptyset$. For any non-empty B in $BFhI(S)$, considering a bipolar fuzzy h-ideal $B_\sim = (\mu_{B_\sim}^P, \mu_{B_\sim}^N)$ which is given in the proof of Theorem 5.1, we can obtain

$$\varphi_k(B_\sim) = B_{\sim k} = B_{\sim k}^P \cap B_{\sim(-k)}^N = \{x \in S | \mu_{B_\sim}^P(x) \geq k\} \cap \{x \in S | \mu_{B_\sim}^N(x) \leq (-k)\} = B.$$

Therefore, φ_k is surjective.

Theorem 5.4. *Let $0 < k < 1$, then the quotient set $BFhI(S)/S^k$ is equipotent to $I(S) \cup \emptyset$.*

Proof: Suppose that $0 < k < 1$ and $\varphi_k^* : BFhI(S)/S^k \to I(S) \cup \emptyset$ is a map defined by $\varphi_k^*([A]_{S^k}) = \varphi_k(A)$ for all $[A]_{S^k} \in BFhI(S)/S^k$. Let $\varphi_k^*([A]_{S^k}) = \varphi_k^*([B]_{S^k})$ for every $[A]_{S^k}, [B]_{S^k} \in BFhI(S)/S^k$, then $\varphi_k(A) = \varphi_k(B)$, i.e., $A_k = B_k$. It implies that $(A, B) \in S^k$. Thus, $[A]_{S^k} = [B]_{S^k}$ and φ_k^* is injective.

Moreover, for any non-empty B in $I(S)$, consider the bipolar fuzzy h-ideal $B_\sim = (\mu_{B_\sim}^P, \mu_{B_\sim}^N)$ which is given in the proof of Theorem 5.1, then we have

$$\varphi_k^*([B_\sim]_{S^k}) = \varphi_k(B_\sim) = B_{\sim k} = \{x \in S | \mu_{B_\sim}^P(x) \geq k\} \cap \{x \in S | \mu_{B_\sim}^N(x) \leq (-k)\} = B.$$

On the other hand, $\varphi_k^*([\mathbf{0}]_{S^k}) = \varphi_k(\mathbf{0}) = \mathbf{0}_k^P \cap \mathbf{0}_{-k}^N = \emptyset$. Therefore, φ_k^* is surjective. This completes the proof.

Remark 5.1. *By equivalent relation, we just study the relations of bipolar fuzzy h-ideals of the same hemirings. For the connections of bipolar fuzzy h-ideals of different hemirings, we have not known. To overcome the difficulties, we may conceive fuzzy congruences of bipolar fuzzy h-ideals on hemirings in our future work.*

6. Normal Bipolar Fuzzy *h*-Ideals.

In this section, we introduce and characterize normal bipolar fuzzy h-ideals of hemirings.

By Definition 3.1, it is clear that a bipolar fuzzy set $A = (\mu_A^P, \mu_A^N)$ is an bipolar fuzzy h-ideals of S providing that $\mu_A^P(x) = 1$ and $\mu_A^N(x) = -1$ for all $x \in S$. However, as a general rule, $\mu_A^P(x) = 1$ and $\mu_A^N(x) = -1$ may not always hold. Therefore, it is necessary for us to define the following definition.

Definition 6.1. *A bipolar fuzzy h-ideal $A = (\mu_A^P, \mu_A^N)$ of S is said to be normal if there exists an element $x \in S$ such that $A(x) = (1, -1)$, i.e., $\mu_A^P(x) = 1$ and $\mu_A^N(x) = -1$.*

Example 6.1. *Consider $S = \{0, 1, 2, 3\}$ which is described in Example 3.1. Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set of S defined by*

	0	1	2	3
μ_A^P	1	1	1	0.6
μ_A^N	−1	−1	−1	−0.5

Clearly, $A = (\mu_A^P, \mu_A^N)$ *is a normal bipolar fuzzy h-ideal of* S.

Definition 6.2. *An element* $x_0 \in S$ *is called extremal for a bipolar fuzzy set* $A = (\mu_A^P, \mu_A^N)$ *if* $\mu_A^P(x_0) \geq \mu_A^P(x)$ *and* $\mu_A^N(x_0) \leq \mu_A^N(x)$ *for all* $x \in S$.

From the above definitions, we can easily derived the following properties:

Proposition 6.1. *A bipolar fuzzy set* $A = (\mu_A^P, \mu_A^N)$ *of* S *is a normal bipolar fuzzy h-ideal if and only if* $A(x) = (1, -1)$ *for its all extremal elements.*

Theorem 6.1. *If* x_0 *is an extremal element of a bipolar fuzzy left (resp., right) h-ideal* $A = (\mu_A^P, \mu_A^N)$, *then a bipolar fuzzy set* $\bar{A} = (\bar{\mu}_A^P, \bar{\mu}_A^N)$ *defined by* $\bar{\mu}_A^P(x) = \mu_A^P(x)+1-\mu_A^P(x_0)$ *and* $\bar{\mu}_A^N(x) = \mu_A^N(x) - 1 - \mu_A^N(x_0)$ *for all* $x \in S$ *is a normal bipolar fuzzy left (resp., right) h-ideal of* S *containing* A.

Proof: First, we claim that $\bar{A}$ is normal. In fact, since $\bar{\mu}_A^P(x) = \mu_A^P(x) + 1 - \mu_A^P(x_0)$, $\bar{\mu}_A^N(x) = \mu_A^N(x) - 1 - \mu_A^N(x_0)$ and x_0 is an extremal element of A, we have $\bar{\mu}_A^P(x_0) = 1$, $\bar{\mu}_A^N(x_0) = -1$, $\bar{\mu}_A^P(x) \in [0, 1]$ and $\bar{\mu}_A^N(x) \in [-1, 0]$ for all $x \in S$. Thus $\bar{A} = (\bar{\mu}_A^P, \bar{\mu}_A^N)$ is normal.

Next we show $\bar{A}$ a bipolar fuzzy h-ideal of S. For all $x, y \in S$, we have

$$\begin{aligned}\bar{\mu}_A^P(x+y) &= \mu_A^P(x+y) + 1 - \mu_A^P(x_0)\\ &\geq \min\{\mu_A^P(x), \mu_A^P(y)\} + 1 - \mu_A^P(x_0)\\ &= \min\{\mu_A^P(x) + 1 - \mu_A^P(x_0), \mu_A^P(y) + 1 - \mu_A^P(x_0)\}\\ &= \min\{\bar{\mu}_A^P(x), \bar{\mu}_A^P(y)\}\end{aligned}$$

and

$$\begin{aligned}\bar{\mu}_A^N(x+y) &= \mu_A^N(x+y) - 1 - \mu_A^N(x_0)\\ &\leq \max\{\mu_A^N(x), \mu_A^N(y)\} - 1 - \mu_A^N(x_0)\\ &= \max\{\mu_A^N(x) - 1 - \mu_A^N(x_0), \mu_A^N(y) - 1 - \mu_A^N(x_0)\}\\ &= \max\{\bar{\mu}_A^N(x), \bar{\mu}_A^N(y)\}.\end{aligned}$$

Thus, (1) is valid. Similarly, we can prove that (2) holds. Moreover, let any $x, z, a, b \in S$ such that $x + a + z = b + z$, we have

$$\begin{aligned}\bar{\mu}_A^P(x) &= \mu_A^P(x) + 1 - \mu_A^P(x_0)\\ &\geq \min\{\mu_A^P(a), \mu_A^P(b)\} + 1 - \mu_A^P(x_0)\\ &= \min\{\mu_A^P(a) + 1 - \mu_A^P(x_0), \mu_A^P(b) + 1 - \mu_A^P(x_0)\}\\ &= \min\{\bar{\mu}_A^P(a), \bar{\mu}_A^P(b)\}.\end{aligned}$$

Analogously, we have $\bar{\mu}_A^N(x) \leq \max\{\bar{\mu}_A^N(a), \bar{\mu}_A^N(b)\}$. Thus $\bar{A} = (\bar{\mu}_A^P, \bar{\mu}_A^N)$ is a normal bipolar fuzzy left h-ideal of S. Clearly, $A \subseteq \bar{A}$.

Corollary 6.1. *From the definition of* $\bar{A}$ *in Theorem 6.1, we get* $\bar{\bar{A}} = \bar{A}$ *for all* $A \in BFhI(S)$. *In particular, if* A *is normal, then* $\bar{A} = A$.

Let $\mathcal{N}(S)$ be the set of all normal bipolar fuzzy h-ideals of S. Note that $\mathcal{N}(S)$ is a poset under the set inclusion. Now we arrive at one of our main theorems.

Theorem 6.2. *A non-constant maximal element of* $(\mathcal{N}(S), \subseteq)$ *only takes a value among* $(0, 0), (1, -1)$ *and* $(1, 0)$.

Proof: Assume that $A = (\mu_A^P, \mu_A^N) \in \mathcal{N}(S)$ is a non-constant maximal element of $(\mathcal{N}(S), \subseteq)$. Then $\mu_A^P(x_0) = 1$, $\mu_A^N(x_0) = -1$ for some $x_0 \in S$. Let $x \in S$ such that $\mu_A^P(x) \neq 1$. Then $\mu_A^P(x) = 0$. Otherwise, there exists $m \in S$ such that $0 < \mu_A^P(m) < 1$. On the other hand, let $A_m = (\alpha_A^P, \alpha_A^N)$ be a bipolar fuzzy set of S defined by $\alpha_A^P(x) = \frac{1}{2}(\mu_A^P(x) + \mu_A^P(m))$, $\alpha_A^N(x) = \frac{1}{2}(\mu_A^N(x) + \mu_A^N(m))$ for all $x \in S$. Then we can easily to verify that A_m is well defined. So for all $x \in S$, we have

$$\alpha_A^P(x_0) = \frac{1}{2}(\mu_A^P(x_0) + \mu_A^P(m)) \geq \frac{1}{2}(\mu_A^P(x) + \mu_A^P(m)) = \alpha_A^P(x)$$

and

$$\alpha_A^N(x_0) = \frac{1}{2}(\mu_A^N(x_0) + \mu_A^N(m)) \leq \frac{1}{2}(\mu_A^N(x) + \mu_A^N(m)) = \alpha_A^N(x).$$

Further, for all $x, y \in S$, we have

$$\begin{aligned}\alpha_A^P(x + y) &= \frac{1}{2}(\mu_A^P(x + y) + \mu_A^P(m))\\ &\geq \frac{1}{2}(\min\{\mu_A^P(x), \mu_A^P(y)\} + \mu_A^P(m))\\ &= \min\left\{\frac{1}{2}(\mu_A^P(x) + \mu_A^P(m)), \frac{1}{2}(\mu_A^P(y) + \mu_A^P(m))\right\}\\ &= \min\{\alpha_A^P(x), \alpha_A^P(y)\}\end{aligned}$$

and

$$\begin{aligned}\alpha_A^P(xy) &= \frac{1}{2}(\mu_A^P(xy) + \mu_A^P(m))\\ &\geq \frac{1}{2}(\mu_A^P(y) + \mu_A^P(m)) = \alpha_A^P(y).\end{aligned}$$

By the same argument, we can prove

$$\alpha_A^N(x + y) \leq \max\{\alpha_A^N(x), \alpha_A^N(y)\} \text{ and } \alpha_A^N(xy) \leq \min\{\alpha_A^N(x), \alpha_A^N(y)\}.$$

Moreover, let $x, z, a, b \in S$ with $x + a + z = b + z$, then we have

$$\begin{aligned}\alpha_A^P(x) &= \frac{1}{2}(\mu_A^P(x) + \mu_A^P(m))\\ &\geq \frac{1}{2}(\min\{\mu_A^P(a), \mu_A^P(b)\} + \mu_A^P(m))\\ &= \min\left\{\frac{1}{2}(\mu_A^P(a) + \mu_A^P(m)), \frac{1}{2}(\mu_A^P(b) + \mu_A^P(m))\right\}\\ &= \min\{\alpha_A^P(a), \alpha_A^P(b)\}.\end{aligned}$$

Analogously, $\alpha_A^N(x) \leq \max\{\alpha_A^P(a), \alpha_A^P(b)\}$. This means $A_m \in BFhI(S)$ and A_m has the same extremal elements as A.

By Theorem 6.1, a bipolar fuzzy set $\bar{A}_m = (\bar{\alpha}_A^P, \bar{\alpha}_A^N)$ belongs to $\mathcal{N}(S)$, where

$$\bar{\alpha}_A^P(x) = \alpha_A^P(x) + 1 - \alpha_A^P(x_0) = \frac{1}{2}(1 + \mu_A^P(x))$$

and

$$\bar{\alpha}_A^N(x) = \alpha_A^N(x) - 1 - \alpha_A^N(x_0) = \frac{1}{2}(\mu_A^N(x) - 1).$$

Apparently, $A \subseteq \bar{A}_m$. Since $\bar{\alpha}_A^P(x) = \frac{1}{2}(1 + \mu_A^P(x)) > \mu_A^P(x)$, A is a proper subset of $\bar{A}_m$. Thus $\bar{\alpha}_A^P(m) = \frac{1}{2}\{1 + \mu_A^P(m)\} < 1 = \bar{\alpha}_A^P(x_0)$. This means that $\bar{A}_m$ is non-constant and A is not a maximal element of $\mathcal{N}(S)$. This is a contradiction. Thus μ_A^P only takes two possible values, 0 and 1.

Likewise, we can also prove that μ_A^N just takes a value among 0 and -1. This implies that all the possible values of A are $(0,0)$, $(0,-1)$, $(1,-1)$ and $(1,0)$. Further, if A takes a value from above four values, then
$S^{(0,0)} = \{x \in S|\mu_A^P(x) \geq 0\} \cap \{x \in S|\mu_A^N(x) \leq 0\} = S$,
$S^{(0,-1)} = \{x \in S|\mu_A^P(x) \geq 0\} \cap \{x \in S|\mu_A^N(x) \leq -1\} = \{x \in S|\mu_A^N(x) = -1\}$,
$S^{(1,-1)} = \{x \in S|\mu_A^P(x) \geq 1\} \cap \{x \in S|\mu_A^N(x) \leq -1\} = \{x \in S|\mu_A^P(x) = 1, \mu_A^N(x) = -1\}$,
$S^{(1,0)} = \{x \in S|\mu_A^P(x) \geq 1\} \cap \{x \in S|\mu_A^N(x) \leq 0\} = \{x \in S|\mu_A^P(x) = 1\}$
are all non-empty h-ideals by Corollary 3.2 satisfying

$$(i)\ S^{(1,-1)} \subseteq S^{(0,-1)} \subseteq S^{(0,0)} \ \text{ and } \ (ii)\ S^{(1,-1)} \subseteq S^{(1,0)} \subseteq S^{(0,0)}.$$

For case (i), according to Proposition 3.1, a bipolar fuzzy set $B = (\mu_B^P, \mu_B^N)$, defined by

$$\mu_B^P(x) = \begin{cases} 1, & \text{if } x \in S^{(0,-1)}, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \mu_B^N(x) = \begin{cases} -1, & \text{if } x \in S^{(0,-1)}, \\ 0, & \text{otherwise} \end{cases}$$

is a bipolar fuzzy h-ideal of S. Moreover, it is normal obviously. Now, for all $x \in S^{(0,-1)}$, we have $\mu_B^P(x) = 1 \geq \mu_A^P(x)$ and $\mu_B^N(x) = -1 = \mu_A^N(x)$, that is $A \subseteq B$. For all $x \in S^{(0,0)} - S^{(0,-1)}$, we have $\mu_B^P(x) = 0 = \mu_B^N(x)$. Since μ_A^P only takes two possible values 0 and 1, if $\mu_A^P(x) = 0$, then $\mu_A^P(x) = \mu_B^P(x) = 0$, and $\mu_A^N(x) \leq 0 = \mu_B^N(x)$, hence $B \subseteq A$. Otherwise, if $\mu_A^P(x) = 1$, then $\mu_A^P(x) \geq \mu_B^P(x)$, and $\mu_A^N(x) \leq 0 = \mu_B^N(x)$, hence $B \subseteq A$. In addition, for all $x \in S^{(0,-1)} - S^{(1,-1)}$, we have $\mu_A^P(x) = 0 < 1 = \mu_B^P(x)$ and $\mu_A^N(x) = -1 = \mu_B^N(x)$. That is, $A \subset B$, which contradicts the fact that A is a non-constant maximal element of $(\mathcal{N}(S), \subseteq)$. Therefore, $A \neq (0,-1)$. For case (ii), we can show $A \neq (0,-1)$ similarly. Hence, A only takes the possible values $(0,0)$, $(1,-1)$ and $(1,0)$.

Remark 6.1. *A non-constant bipolar fuzzy h-ideal A of S is called a maximal element of S when $\bar{A}$ defined in Theorem 6.1 is a maximal element of the poset $(\mathcal{N}(S), \subseteq)$. As a consequence of above theorem we obtain the following propositions.*

Proposition 6.2. *A maximal bipolar fuzzy h-ideal of S is normal and takes a value among $(0,0)$, $(1,-1)$ and $(1,0)$.*

Proof: Let $A = (\mu_A^P, \mu_A^N)$ be a maximal bipolar fuzzy h-ideal of S. Then $\bar{A}$ is a maximal element of the poset $(\mathcal{N}(S), \subseteq)$. By Theorem 6.2, $\bar{A}$ only takes three possible values $(0,0)$, $(1,-1)$ and $(1,0)$. In addition, $A \subseteq \bar{A}$ is from Theorem 6.1. Hence, A also takes a value among $(0,0)$, $(1,-1)$ and $(1,0)$. Next, we show A is normal. From the proof of Theorem 6.2, $\bar{\mu}_A^P(x)$ only takes the two possible values 0 and 1. Since $\bar{\mu}_A^P(x) = \mu_A^P(x) + 1 - \mu_A^P(x_0)$, $\bar{\mu}_A^P(x) = 1$ if and only if $\mu_A^P(x) = \mu_A^P(x_0)$, $\bar{\mu}_A^P(x) = 0$ if and only if $\mu_A^P(x) = \mu_A^P(x_0) - 1$, where x_0 is an extremal element of A. From $A \subseteq \bar{A}$ we get $\mu_A^P(x) \leq \bar{\mu}_A^P(x)$ for all $x \in S$. Thus, $\bar{\mu}_A^P(x) = 0$ implies $\mu_A^P(x) = 0$. Consequently, $\mu_A^P(x_0) = 1$. Similarly, from $\bar{\mu}_A^N(x) = \mu_A^N(x) - 1 - \mu_A^N(x_0)$, we have $\mu_A^N(x_0) = -1$. Therefore, A is normal.

Proposition 6.3. *Let $A = (\mu_A^P, \mu_A^N)$ be a maximal bipolar fuzzy h-ideal of S, then $S^{(1,-1)}$ is a maximal h-ideal of S.*

Proof: According to Corollary 3.2, $S^{(1,-1)}$ is an h-ideal of S, so we just need to show it maximal. Let $T = S^{(1,-1)} = \{x \in S|\mu_A^P(x) = 1, \mu_A^N(x) = -1\}$. From Theorem 6.2, $\mu_A^P(x)$ only takes value among 0 and 1, thus $T \neq S$. If M is an h-ideal of S containing T, then $\mu_T^P \subseteq \mu_M^P$. Since $\mu_A^P = \mu_T^P$ and $\mu_A^P(x)$ only takes value among 0 and 1, μ_M^P also takes the possible values. However, by the assumption, A is a maximal bipolar fuzzy h-ideal of S. Thus $\mu_A^P = \mu_T^P = \mu_M^P$ or $\mu_M^P(x) = 1$ for all $x \in S$. In the last cases $T = S$, which is a contradiction, hence $\mu_A^P = \mu_T^P = \mu_M^P$, i.e., $M = T$. This implies that $S^{(1,-1)}$ is a maximal ideal of S.

Definition 6.3. *A non-empty bipolar fuzzy h-ideal of S is called completely normal if there exists $x \in S$ such that $A(x) = (0, 0)$.*

Let all the completely normal bipolar fuzzy h-ideals of S be denoted by $\mathcal{C}(S)$. Apparently, $\mathcal{C}(S) \subseteq \mathcal{N}(S)$. So we can obtain the following results:

Proposition 6.4. *A non-constant maximal element of $(\mathcal{N}(S), \subseteq)$ is also a maximal element of $(\mathcal{C}(S), \subseteq)$.*

Proposition 6.5. *Every maximal bipolar fuzzy h-ideal of S is completely normal.*

Theorem 6.3. *Let $f : [0,1] \to [0,1]$ and $g : [-1,0] \to [-1,0]$ be two increasing functions and $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy set of S. Then $A_{(f,g)} = (\mu_{A_f}^P, \mu_{A_g}^N)$ where $\mu_{A_f}^P(x) = f(\mu_A^P(x))$ and $\mu_{A_g}^N(x) = g(\mu_A^N(x))$ for all $x \in S$ is a bipolar fuzzy h-ideal of S if and only if $A = (\mu_A^P, \mu_A^N) \in BFhI(S)$. In particular, if $f(\mu_A^P(0)) = 1$ and $g(\mu_A^N(0)) = -1$, then $A_{(f,g)}$ is normal.*

Proof: Let $A_{(f,g)} = (\mu_{A_f}^P, \mu_{A_g}^N) \in BFhI(S)$, then for all $x, y \in S$ we have

$$\begin{aligned} f(\mu_A^P(x+y)) &= \mu_{A_f}^P(x+y) \\ &\geq \min\{\mu_{A_f}^P(x), \mu_{A_f}^P(y)\} = \min\{f(\mu_A^P(x)), f(\mu_A^P(y))\} \\ &= f(\min\{\mu_A^P(x), \mu_A^P(y)\}). \end{aligned}$$

Since f is increasing, it follows that $\mu_A^P(x+y) \geq \min\{\mu_A^P(x), \mu_A^P(y)\}$. Conversely, if $A = (\mu_A^P, \mu_A^N) \in BFhI(S)$, then for all $x, y \in S$, we have

$$\begin{aligned} \mu_{A_f}^P(x+y) &= f(\mu_A^P(x+y)) \\ &\geq f(\min\{\mu_A^P(x), \mu_A^P(y)\}) = \min\{f(\mu_A^P(x)), f(\mu_A^P(y))\} \\ &= \min\{\mu_{A_f}^P(x), \mu_{A_f}^P(y)\}. \end{aligned}$$

Similarly, we have $\mu_{A_g}^N(x+y) \leq \max\{\mu_{A_g}^N(x), \mu_{A_g}^N(y)\}$ if and only if $\mu_A^N(x+y) \leq \max\{\mu_A^N(x), \mu_A^N(y)\}$. Thus $A_{(f,g)} = (\mu_{A_f}^P, \mu_{A_g}^N)$ satisfies (1) if and only if $A = (\mu_A^P, \mu_A^N)$ satisfies (1). The analogous connection between $A_{(f,g)} = (\mu_{A_f}^P, \mu_{A_g}^N)$ and $A = (\mu_A^P, \mu_A^N)$ can be obtained in the case of axioms (2′) and (3). This completes the proof.

7. **Conclusions.** Bipolarity plays a very important role in many branches of pure and applied mathematics. The combination of bipolar fuzzy set theory and algebraic system have resulted in many interesting research topics, which have been drawing a widespread attention of many mathematical researchers and computer scientists. In this paper, we have applied bipolar fuzzy sets theories to hemirings and have discussed some basic properties on the subject of bipolar fuzzy h-ideals of hemirings, which is, in fact, just a incomplete beginning of the study of the hemiring theory, so it is necessary to carry out more theoretical researches to establish a general framework for the practical application. We believe that the research in this direction can invoke more new topics and can provide more applications in some fields such as mathematical morphology, logic and information science, engineering, medical diagnosis. So our future work will focus on this field, especially the following problems: (1) By employing bipolar fuzzy h-ideals of hemirings, we establish bipolar fuzzy topologies of hemirings and discuss the correspondences between bipolar fuzzy topologies and bipolar fuzzy ideals of hemirings. (2) The study about bipolar fuzzy h-bi-ideals, bipolar fuzzy h-quasi-ideals, bipolar fuzzy h-interior ideals and so on. (3) The study about applications, especially in information sciences and general systems.

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