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NONLINEAR FEEDBACK CONTROL BASED ON POSITIVE INVARIANCE FOR A NUTRIENT REMOVAL BIOLOGICAL PLANT

Hicham El Bahja and Pastora Vega Cruz

Automatic and Computer Science
Faculty of Science
University of Salamanca
Plaza de los Cados s/n, Salamanca 37008, Spain
{ hicham; pvega }@usal.es

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Abstract. *Activated sludge wastewater treatment processes are difficult to be controlled because of their complex and nonlinear behavior; however, the control of the dissolved oxygen and nitrogen levels in the reactors plays an important role in the operation of the facility. For this reason a new approach is proposed in this work. The paper describes the theory and applications of a nonlinear control technique, i.e., the so-called nonlinear feedback control. The control law consists of a linear feedback part which is designed using the positive invariance concept technique and a nonlinear feedback part without any switching elements. The control approach structure is combined with a state space estimation algorithm, for the on-line reconstruction of unmeasured biological states of the bioprocess. The efficiency of both the control and estimation is demonstrated via computer simulations.*
Keywords: Observation, Nonlinear control, Wastewater plant, Feedback control

1. **Introduction.** Wastewater treatment plants are large non-linear systems subject to significant perturbations in flow and load, together with variation in the composition of the incoming wastewater. Nevertheless, these plants have to be operated continuously, meeting stricter and stricter regulations. The tight effluent requirements defined by the European Union a decade ago (European Directive 91/271 "Urban wastewater") become effective in 2005 and are likely to increase both operational costs and economic penalties to upgrade existing wastewater treatment plants in order to comply with the future effluent standards. Many control strategies have been proposed in the literature but their evaluation and comparison, either practical or based on simulation are difficult. This is partly due to the variability of the influent, to the complexity of the biological and biochemical phenomena and to the large range of time constants (from a few minutes to several days) but also to the lack of standard evaluation criteria (among other things, due to region specific effluent requirements and cost levels). In addition, the microorganisms that are involved in the process and their adaptive behavior coupled with nonlinear dynamics of the system make the WWTP to be really challenging from the control point of view [1-3].

A number of works focused in the control of the WWTPs are found in the literature. The proposed control strategies differ in objectives and methods. The most important control variable is the dissolved oxygen concentration (DO) in the aerobic reactors; it is regulated by the aeration system. An appropriate level of DO guaranties a satisfactory nitrification and ensures adequate stirring in the tank; however, the energy requirements to control DO (aeration energy) are high. Therefore, most of the control strategies focus on DO control [7]. However, another typical control variable is the nitrate concentration in the anoxic zone and it is regulated with the internal recycle flow [8]. The control methods include simple control [9]; addition of a feedforward action based on the measurement of the influent flow rate [10] and feedforward compensation [11]; linearized and optimal control [12,13], nonlinear control [14-17], fuzzy control [18,19], optimal control [20] and supervisory control [21]. Model Predictive Control (MPC), in particular, has been a research topic for WWTP since the mid of 1990's. There are many developments of the classical MPC such as robust MPC, adaptive MPC and non linear MPC [22-26]. Furthermore, works combining estimation to the control for monitoring such process can also be found [27].

On the other hand, the state space representation is frequently used to form multivariable approach to linear control system synthesis and design. These control schemes are based on the assumption that the system state vector is available for feedback control purposes. In some applications, this assumption is not satisfied because it is either impossible or inappropriate, in practical situations, to measure all elements of system state. To retain many useful properties of the linear state feedback control, it is necessary to overcome the problem of the incomplete state information. The state observation problem is based on the construction of an auxiliary dynamical system, known as the state observer, driven by the inputs and outputs of the original system [4]. The reconstructed state vector is then substituted for the inaccessible one in the usual linear state feedback. Furthermore, as pointed out above in many practical situations, linear systems are subjected to state and (or) input constraints. Such constraints are generally associated with physical limitations in process variables. The respect of these constraints can be accomplished by designing suitable feedback control laws. In many cases, this can be done by constructing positively invariant domains inside the set of the constraints [5]. Other important applications were derived from this concept. In particular, one of them consists in using a large set of initial states while the constraints on the control vector are respected [6].

The objective of this work is to design and to apply the non linear feedback control based on the positive invariance concept techniques to a WWTP. This controller consists of a linear feedback law (computed from the invariance positive) and a nonlinear feedback law without any switching elements. The linear feedback part is designed to yield a closed-loop system with a small damping ratio for a quick response, while at the same time does not exceed the actuator limits for the desired command input levels. The nonlinear feedback law is used to increase the damping ratio of the closed-loop system as the system output approaches the target reference to reduce the overshoot caused by the linear part. The obtained model combines the problems of non availability of the state to measure with the limitations of some variables. The control is achieved by an observer based controller that can take into account constraints on the control and on the error. The model is worked out to meet all design required conditions. The efficiency of the controller is showed via simulations with the real plant.

The remainder of the paper is organized as follows. The modeling of the continuous wastewater treatment plants is detailed in Section 2. In fact, the modeling of the aerated basin, the anoxic basin and the settler are depicted. The control of the process is presented in Section 3. The simulation results are then described in Section 4. Finally, Section 5 ends the paper with concluding remarks.

Notations:

- For two vectors $x, y \in \mathbb{R}^n, x \leq y$ (respectively, $x \prec y$) means $x_i \leq y_i$ (respectively, $x_i \prec y_i$), $i = 1, \ldots, n$.
- $x_i^+ = sup(x_i, 0)$, $x_i^- = sup(-x_i, 0)$.

- For $A \in \mathbb{R}^{n\times n}, \sigma(A)$ denotes its spectrum and
$\tilde{A} = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}, \quad A_1(i,j) = \begin{cases} a_{ij}, & i = j \\ a_{ij}^+, & i \neq j \end{cases}$ and $A_2(i,j) = \begin{cases} 0, & i = j \\ a_{ij}^-, & i \neq j \end{cases}$
- $\mathbb{I}$ and $int\mathbb{R}_+^n$ are respectively the identity matrix of dimension n and the interior of $\mathbb{R}_+^n$.

2. **Process Modeling.** A typical, conventional activated sludge plant for the removal of carbonaceous and nitrogen materials consists of an anoxic basin followed by an aerated one, which is aerated by a submerged air bubble system or mechanical agitation at its surface and a settler (Figure 1). In the presence of dissolved oxygen, wastewater, that is mixed with the returned activated sludge, is biodegraded in the reactor. Treated affluent is separated from the sludge in the settler. A portion of the activated sludge is wasted while a large fraction is returned to anoxic reactor to maintain the appropriate substrate-to-biomass ratio.

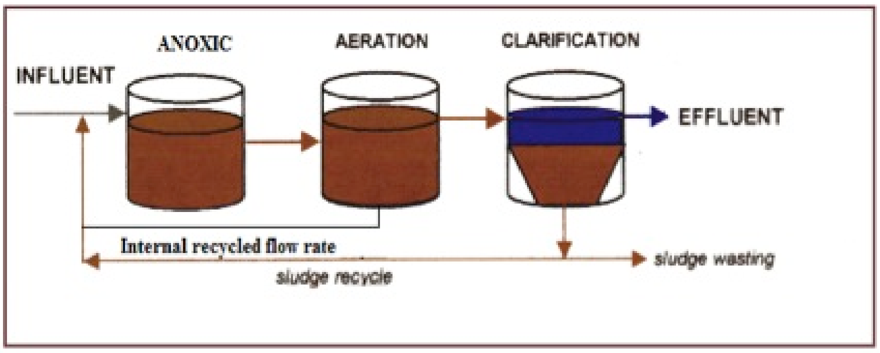

Figure 1. WWTP

In this study, we consider six basic components present in the wastewater: autotrophic bacteria X_A, heterotrophic bacteria X_H, readily biodegradable carbonaceous substrates S_S, nitrogen substrates S_{NH}, S_{NO} and dissolved oxygen S_O, where X_A, X_H, S_S, S_{NH}, S_{NO}, and S_O represent the concentrations of these elements. In the formulation of the model, the following assumptions are considered: the physical properties of fluid are constant; there is no concentration gradient across the vessel; substrates and dissolved oxygen are considered as a rate-limiting with a bi-substrate Monod-type Kinetic and finally no bio-reaction takes place in the settler which is perfect.

Based on the above description and assumptions, we can formulate the full set of ordinary differential equations (mass balance equations), making up the IWA AS Model NO.1 [28,29].

2.1. **Modeling of the aerated basin.**

$$\dot{X}_{A,nit}(t) = (1 + r_1 + r_2)\, D_{nit}\,(X_{A,denit} - X_{A,nit}) + (\mu_{A,nit} - b_A)\, X_{A,nit} \tag{1}$$

$$\dot{X}_{H,nit}(t) = (1 + r_1 + r_2)\, D_{nit}\,(X_{H,denit} - X_{H,nit}) + (\mu_{H,nit} - b_H)\, X_{H,nit} \tag{2}$$

$$\dot{S}_{S,nit}(t) = (1 + r_1 + r_2)\, D_{nit}\,(S_{S,denit} - S_{S,nit}) + (\mu_{H,nit} + \mu_{Ha,nit})\, X_{H,nit}/Y_H \tag{3}$$

$$\begin{aligned}\dot{S}_{NH,nit}(t) = {} & (1 + r_1 + r_2)\, D_{nit}\,(S_{NH,denit} - S_{NH,nit}) + (i_{xb} + 1/Y_A)\,\mu_{A,nit} X_{A,nit} \\ & - (\mu_{H,nit} + \mu_{Ha,nit})\, i_{xb} X_{H,nit}\end{aligned} \tag{4}$$

$$\dot{S}_{NO,nit}(t) = (1 + r_1 + r_2)\, D_{nit}\, (s_{NO,denit} - s_{NO,nit}) + \mu_{A,nit} \frac{X_{A,nit}}{Y_A} - \frac{1 - Y_H}{2.86 Y_H} \mu_{Ha,nit} X_{H,nit} \quad (5)$$

$$\dot{S}_{O,nit}(t) = (1 + r_1 + r_2)\, D_{nit}\, (S_{O,denit} - S_{O,nit}) + a_0 Q_{air}(C_S - S_{O,nit}) - \frac{4.57 - Y_A}{Y_A} \mu_{A,nit} X_{A,nit} - \frac{1 - Y_H}{Y_H} \mu_{Ha,nit} X_{H,nit} \quad (6)$$

where $\mu_{A,nit}$ and $\mu_{H,nit}$ are the growth rates of autotrophs and heterotrophs in aerobic conditions and $\mu_{Ha,nit}$ is the growth rate of heterotrophs in anoxic conditions. Defined by:

$$\mu_{A,nit} = \mu_{\max,A} \frac{S_{NH,nit}}{\left(K_{NH,A} + S_{NH,nit}\right)} \frac{S_{O,nit}}{(K_{O,A} + S_{O,nit})}$$
$$\mu_{H,nit} = \mu_{\max,H} \frac{S_{S,nit}}{(K_S + S_{S,nit})} \frac{S_{NH,nit}}{\left(K_{NH,H} + S_{NH,nit}\right)} \times \frac{S_{O,nit}}{(K_{O,H} + S_{O,nit})}$$
$$\mu_{Ha,nit} = \mu_{\max,H} \frac{S_{S,nit}}{(K_S + S_{S,nit})} \frac{S_{NH,nit}}{\left(K_{NH,H} + S_{NH,nit}\right)} \times \frac{K_{O,H}}{(K_{O,H} + S_{O,nit})} \frac{S_{NO,nit}}{(K_{NO} + S_{NO,nit})} . \eta_{NO}$$

2.2. **Modeling of the anoxic basin.**

$$\dot{X}_{A,denit}(t) = D_{denit}(X_{A,in} + r_1 X_{A,nit}) + \alpha . r_2 D_{denit} \times X_{rec} - (1 + r_1 + r_2)\, D_{denit} X_{A,denit} + (\mu_{A,denit} - b_A) X_{A,denit} \quad (7)$$

$$\dot{X}_{H,denit}(t) = D_{denit}(X_{H,in} + r_1 X_{H,nit}) - (1 + r_1 + r_2) \times D_{denit} X_{H,denit} + (1 - \alpha) r_2 D_{denit} X_{rec} + (\mu_{H,denit} - b_H) X_{H,denit} \quad (8)$$

$$\dot{S}_{S,denit}(t) = -(\mu_{H,denit + \mu_{Ha,denit}}) \frac{X_{H,denit}}{Y_H} - (1 + r_1 + r_2) \times D_{denit} S_{S,denit} + D_{denit}(S_{S,in} - r_1 S_{S,nit}) \quad (9)$$

$$\dot{S}_{NH,denit}(t) = D_{denit}(S_{NH,in} + r_1 S_{NH,nit}) - (1 + r_1 + r_2) \times D_{denit} S_{NH,denit} - (i_{xb} + 1/Y_A)\, \mu_{A,denit} \times X_{A,denit} - (\mu_{H,denit} + mu_{Ha,denit}) \times i_{xb} X_{H,denit} \quad (10)$$

$$\dot{S}_{NO,denit}(t) = D_{denit}(S_{NO,in} + r_1 S_{NO,nit}) - (1 + r_1 + r_2) \times D_{denit} S_{NO,denit} + \frac{\mu_{A,denit} X_{A,denit}}{Y_A} - \frac{1 - Y_H}{2.86 Y_H} \mu_{Ha,denit} X_{H,denit} \quad (11)$$

where

$$\mu_{A,denit} = \mu_{\max,A} \frac{S_{NH,denit}}{\left(K_{NH,A} + S_{NH,denit}\right)}$$
$$\mu_{H,denit} = \mu_{\max,H} \frac{S_{S,denit}}{(K_S + S_{S,denit})} \frac{S_{NH,denit}}{\left(K_{NH,H} + S_{NH,denit}\right)}$$
$$\mu_{Ha,denit} = \mu_{\max,H} \frac{S_{S,denit}}{(K_S + S_{S,denit})} \frac{S_{NH,denit}}{\left(K_{NH,H} + S_{NH,denit}\right)} \times \frac{S_{NO,denit}}{(K_{NO} + S_{NO,denit})} \eta_{NO}$$

2.3. **Modeling of the settler.** A mass balance on the settler leads to the following equation:

$$\dot{X}_{rec} = (1 + r_2)D_{dec}(X_{A,nit} + X_{H,nit}) - (r_2 + w)D_{dec}X_{rec} \tag{12}$$

where r_1, r_2 and w represent respectively, the ratio of the internal recycled flow Q_{r1} to influent flow Q_{in}, the ratio of the recycled flow Q_{r2} to the influent flow and the ratio of waste flow Q_w to influent flow, and C_S is the maximum dissolved oxygen concentration. D_{nit}, D_{denit} and D_{dec} are the dilution rates in respectively, nitrification, denitrification, denitrification basins and settler tank; X_{rec} is the concentration of the recycled biomass. The other variables and parameters of the system Equations (1)-(12) are defined in Tables 1 and 2.

3. **The Control Problem.** Before computing and applying the nonlinear feedback control, the linearization and the decomposition of the nonlinear system are needed. Also the reconstruction of the unmeasurable states has been achieved by the Luenberger observer.

Table 1. Process characteristics

Variables	Value	Description
V_{nit}	$1333m^3$	Volume of nitrification basin
V_{dnit}	$1000m^3$	Volume of denitrification basin
V_{dec}	$6000m^3$	Volume of settler
Q_{in}	$18446m^3/j$	Influant flow rate
Q_w	$385m^3/j$	Waste flow rate
$X_{A,in}$	$0mg/l$	Autotrophs in the influent
$X_{H,in}$	$30mg/l$	Heterotrophs in the influent
$S_{NH,in}$	$30mg/l$	Ammonium in the influent
$S_{NO,in}$	$2mg/l$	Nitrate in the influent
$S_{O,in}$	$0mg/l$	Oxygen in the influent

Table 2. Kinetic parameters and stoechiometric coefficient characteristics

Variables	Value	Description
Y_A	0.24	Yield of autotroph mass
Y_H	0.67	Yield of heterotroph mass
i_{xb}	0.086	
K_S	$20mg/l$	Affinity constant
$K_{NH,A}$	$1mg/l$	Affinity constant
$K_{NH,H}$	$0.05mg/l$	Affinity constant
K_{NO}	$0.5mg/l$	Affinity constant
$K_{O,A}$	$0.4mg/l$	Affinity constant
$K_{O,H}$	$0.2mg/l$	Affinity constant
$\mu_{A\max}$	$0.8l/j$	Maximum specific growth rate
$\mu_{H\max}$	$0.6l/j$	Maximum specific growth rate
b_A	$0.2l/j$	Decay coefficient of autotrophs
b_H	$0.68l/j$	Decay coefficient of heterotrophs
η_{NO}	$0.8l/j$	Correction factor for anoxic growth
α	0.5	

3.1. **Linearization.** Biological Wastewater treatment models can be generally represented as follows:

$$\begin{cases} \frac{dx}{dt(t)} = f(u(t), x(t)) \\ y(t) = g(u(t), x(t)) \end{cases} \tag{13}$$

where $x(t)$ is a vector of variables reflecting the systems state, called state variables, $u(t)$ is a vector of input variables, and $y(t)$ is a vector of outputs or measured variables. The linear model is formed by numerically evaluating the change in function values (f and g) resulting from small changes in model variables (x, u) at any particular operating point:

$$A = \frac{\partial f}{\partial x}\Big|_{OP}; \quad B = \frac{\partial f}{\partial u}\Big|_{OP} \tag{14}$$

$$C = \frac{\partial g}{\partial x}\Big|_{OP}; \quad D = \frac{\partial g}{\partial u}\Big|_{OP} \tag{15}$$

where all of the above partial derivatives are evaluated at the chosen operating point and will, therefore, change depending on the operating point. This results in the well known linear "state-space" model format:

$$\begin{cases} \frac{dx}{dt} = Ax + Bu \\ y(t) = Cx + Du \end{cases} \tag{16}$$

To obtain a model in the state space, the state vector is considered as

$$\begin{aligned} X(t) = [&X_{A,nit}(t) \quad X_{H,nit}(t) \quad S_{S,nit}(t) \quad S_{NH,nit}(t) \quad S_{NO,nit}(t) \quad S_{O,nit}(t) \quad X_{A,denit}(t) \\ &X_{H,denit}(t) \quad S_{S,denit}(t) \quad S_{NH,denit}(t) \quad S_{NO,denit}(t) \quad X_{rec}(t)]^T \end{aligned} \tag{17}$$

Further, to complete the model, the following input and output vectors are used:

$$Y(t) = [S_{NH,nit}(t) \quad S_{NO,nit}(t) \quad S_{O,nit}(t)]^T \tag{18}$$

$$U(t) = [Q_{r1} \quad Q_{r2} \quad Q_{air}]^T \tag{19}$$

Linearizing the system around the equilibrium point computed from the nonlinear equations leads to the new variables (x, u, y) that are now deviation variables. That is, they are deviations from the point the model is linearized about, not their original absolute values. The equilibrium point is given by

$$\bar{x}(t) = [69.6 \quad 623 \quad 13.5 \quad 3.2 \quad 10.4 \quad 2.4 \quad 68.9 \quad 624.6 \quad 20.9 \quad 8.9 \quad 5.3 \quad 1356.8]^T \tag{20}$$

The constraints on the control are given by the following limitations:

$$\begin{cases} -\bar{Q}_{r1} \le Q_{r1} \le 4\bar{Q}_{r1} \\ -\bar{Q}_{r2} \le Q_{r2} \le \bar{Q}_{r2} \\ -\bar{Q}_{air} \le Q_{air} \le 2\bar{Q}_{air} \end{cases} \tag{21}$$

with $\bar{Q}_{r1} = 2300m^3/j$, $\bar{Q}_{r2} = 18446m^3/j$ and $\bar{Q}_{air} = 100m^3/j$.

3.2. **Decomposition.** In order to apply the concept of positive-invariance, the studied system must be controllable and observable. However, our system does not completely satisfy the two later conditions. For this reason, we used the decomposition process that allows us to extract only the controllable and observable part.

Any representation in the state space can be transformed into the equivalent form using the transformation $Z = T_o x$ [5]:

$$\begin{cases} \dot{Z} = \bar{A}Z + \bar{B}u \\ y(t) = \bar{C}Z \end{cases} \tag{22}$$

with

$$\bar{A} = \begin{pmatrix} A_{no} & A_{12} \\ 0 & A_o \end{pmatrix}; \quad \bar{B} = \begin{pmatrix} B_{no} \\ B_o \end{pmatrix}$$
$$\bar{C} = \begin{pmatrix} 0 & C_o \end{pmatrix}; \quad Z = \begin{pmatrix} Z_{no} \\ Z_o \end{pmatrix}$$

Therefore, we obtain the following system of equations:

$$\begin{cases} \dot{Z}_{no} = A_{no}Z_{no} + A_{12}Z_o + B_{no}u \\ \dot{Z}_o = A_oZ_o + B_ou \\ y = C_oZ_o \end{cases} \tag{23}$$

where A_o and C_o are constant matrices of appropriate dimension and the pair (A_o, B_o) is controllable. It is assumed that A_o possesses at last $(n-m)$ stable eigenvalues. The control u is constrained in the set Ω defined as follows:

$$\Omega = \{u \in \mathbb{R}^m \ -u_{\min} \leq u \leq u_{\max}. \quad u_{\min}, u_{\max} \in int\mathbb{R}_+^m\} \tag{24}$$

using a state feedback control:

$$u(t) = sat(Fx(t)), \quad F \in \mathbb{R}^{n\times n} \tag{25}$$

where the saturation function is as follows:

$$sat(Fx(t)) = \begin{cases} u_{\max} & \text{if } Fx \geq u_{\max} \\ u & \text{if } -u_{\min} \prec Fx \prec u_{\max} \\ -u_{\min} & \text{if } Fx < -u_{\min} \end{cases}$$

leads to a domain of linear behavior for the closed loop system that is given by:

$$\mathbb{D}(F, u_{\min}, u_{\max}) = \{x \in \mathbb{R}^n, \quad -u_{\min} \leq Fx \leq u_{\max}\} \tag{26}$$

and the closed loop system in this case

$$\dot{x}(t) = (A_o + B_oF)x(t) \tag{27}$$

Hence, if the domain (27) is positively invariant, in the sense of the definition given below, one guarantees the respect of the control constraints for all $t \geq 0$.

Definition 3.1. *A subset $\mathbb{D}$ of $\mathbb{R}^m$ is said to be positively invariant with respect to system (28) if the condition $x(t_0) \in \mathbb{D}$ implies that $x(t) \in \mathbb{D}$, $\forall t \geq t_0$.*

3.3. **Luenberger observer.** Since the objective of the use of an observer is to reconstruct the unavailable states, the presence of p linear combinations of the state in the output suggests that the remaining $n-p$ linear combinations may be reconstructed by an observer of order no greater than $n-p$. Such an observer is called a minimal-order observer [5]. Thus, we wish to generate the remaining state combinations as follows:

$$z(.) = TZ_o(.) \tag{28}$$

where matrix T is chosen in such a way that the matrix $\begin{pmatrix} C_o \\ T \end{pmatrix}$ is invertible. Using this linear combination, with the matrix T of dimension $(n-p, n)$, the estimated state is obtained from:

$$\hat{Z}_o = \begin{pmatrix} C_o \\ T \end{pmatrix}^{-1} \begin{pmatrix} y(.) \\ z(.) \end{pmatrix} = \begin{pmatrix} V_o & P_o \end{pmatrix} \begin{pmatrix} y(.) \\ z(.) \end{pmatrix} \tag{29}$$

Furthermore, the amount $TZ_o(.)$ can be measured which leads us to generate $z(.)$ from an auxiliary dynamical system as follows:

$$\dot{z}(.) = D_oz(.) + E_oy(.) + G_ou(.) \tag{30}$$

where $z(.)$ is the state of the observer dynamics. Note here that the matrices V_o, C_o, T, P_o, satisfy

$$V_oC_o + P_oT = \mathbb{I}. \tag{31}$$

The control problem with constraints via an observer of minimal order may be stated in the following way

$$u(.) = sat\left(F\hat{Z}_o(.)\right) \tag{32}$$

We choose the state feedback F and matrices D_o, E_o and G_o are chosen so that the asymptotic stability and the constraints on the inputs are respected.

The observation error in this case is given by

$$\epsilon(.) = z(.) - TZ_o(.) \tag{33}$$

recalling that the matrices of the observer of minimal order are given by [7]

$$D_o = TA_oP_o, \quad E_o = TA_oV_o, \quad G_o = TB_o \tag{34}$$

which is equivalent to write these matrices satisfying the following relation

$$TA_o - E_oC_o = D_oT \tag{35}$$

where the matrices T and P_o are chosen to ensure asymptotic stability of the matrix D_o, and canceling the observation error asymptotically, indeed [2]:

$$\begin{aligned}\dot{\epsilon}(.) &= \dot{z}(.) - T\dot{Z}_o \\ &= D_oz(.) + E_oy(.) + G_ou(.) - T(A_oZ_o(.) + B_ou(.)) \\ &= D_oz(.) - D_oTZ_o(.) \\ &= D_o\epsilon(.)\end{aligned}$$

For the observation error, we define the field $D(\mathbb{I}, \epsilon_{\max}, \epsilon_{\min})$ that gives us the limits within which we allow change of the error $\epsilon(.)$. The reconstruction error is always given by

$$e(.) = \hat{Z}_o(.) - Z_o(.) \tag{36}$$

and is related to the observation error in the following way:

$$\begin{aligned}e(.) &= V_oy(.) + P_oz(.) - Z_o(.) \\ &= V_oC_oZ_o(.) + P_oz(.) - (V_oC_o + P_oT)Z_o(.) \\ &= P_o(z(.) - TZ_o(.)) \\ &= P_o\epsilon(.)\end{aligned}$$

Lemma 3.1. *The field $D(\mathbb{I}, u_{\max}, u_{\min}) \times D(\mathbb{I}, \epsilon_{\max}, \epsilon_{\min})$ is positively invariant with respect to the system trajectory $\begin{pmatrix} u(.) \\ \epsilon(.) \end{pmatrix}$ if and only if, there exists a matrix $H\_oinR^{m\times m}$ such that:*

$$\begin{cases} H_oF = FA_o + FB_oF \\ \widetilde{M}q_\epsilon \leq 0 \end{cases} \tag{37}$$

where

$$M = \begin{pmatrix} H_o & L_r \\ 0 & D_o \end{pmatrix}; \quad q_\epsilon = \begin{pmatrix} u_{\max} \\ \epsilon_{\max} \\ u_{\min} \\ \epsilon_{\min} \end{pmatrix}; \quad L_r = -FV_oC_oA_oP_o$$

for every pair: $(u(0), \epsilon(0)) \in D(\mathbb{I}, u_{\max}, u_{\min}) \times D(\mathbb{I}, \epsilon_{\max}, \epsilon_{\min})$.

To compute the feedback gain, the inverse procedure is used [5,6]. Hence, matrix H_o satisfies all required conditions such that a solution existing is chosen and the feedback F is obtained as a solution to the equation:

$$FA_o + FB_oF = H_oF \tag{38}$$

Remark 3.1. *Note here that all computation effort is handled off line. Choice of an adequate matrix H_o with all required conditions is studied in [5], solution of Equation (38) is detailed in [6].*

We start by writing the equation for the evolution of the control $u(t)$ always in the case of a linear behavior using previous relationship (28), (31), (32):

$$\begin{aligned}
\dot{u} &= F\dot{\hat{Z}}_o \\
&= FP_o\dot{z}(.) + FV_oC_o\dot{Z}_o(.) \\
&= FP_oTA_o\hat{Z}_o(.) + FB_ou(.) + FV_oC_oA_o(\hat{Z}(.) - e(.)) \\
&= (FA_o + FB_oF)\hat{Z}_o(.) - FV_oC_oA_oe(.) \\
&= H_oF\hat{Z}_o(.) - FV_oC_oA_oP_o\epsilon(.) \\
&= H_ou(.) + L_r\epsilon(.)
\end{aligned}$$

Therefore, the system formed by the control $u(.)$ and the error $\epsilon(.)$, can be expressed as:

$$\begin{pmatrix} \dot{u}(.) \\ \dot{\epsilon}(.) \end{pmatrix} = \begin{pmatrix} H_o & L_r \\ 0 & D_o \end{pmatrix} \begin{pmatrix} u(.) \\ \epsilon(.) \end{pmatrix}$$

3.4. **The nonlinear feedback control.** The objective here is to design a nonlinear feedback control law for the system (16) with the constraints (21) that will cause the output to track a step input rapidly without expressing large overshoot respecting the constraints below. The following assumptions on the system matrices are required:
1) (A_o, B_o) is stabilizable.
2) (A_o, B_o, C_o) is invertible and has no zero at $s = 0$.

In this section, we follow the idea of the work presented in [30] to develop a nonlinear feedback control technique for the case where we have $(n - p)$ states of the plant (16) which are measurable as mentioned before. We have the following step-by-step design procedure.

Step 1: Design a linear feedback law

$$u_L = Fx + Gr \tag{39}$$

where r is a step command input and F is chosen such that 1) $A_o + B_oF$ is an asymptotically stable matrix (see the section before). Furthermore, G is a scalar and is given by

$$G = -[C_o(A_o + B_oF)^{-1}B_o]^{-1} \tag{40}$$

Here, we note that G is well defined because $A_o + B_oF$ is stable, and the triple (A_o, B_o, C_o) is invertible and has no invariant zeros at $s = 0$.

Step 2: Next, we compute

$$H := [1 - F(A_o + B_oF)^{-1}B_o]G \tag{41}$$

and

$$x_e := G_er := -(A_o + B_oF)^{-1}B_oGr \tag{42}$$

Given a positive-definite matrix $W \in \mathbb{R}^{n\times n}$, solve the following Lyapunov equation:

$$(A_o + B_oF)^TP + P(A_o + B_oF) = -W \tag{43}$$

for $P > 0$. Note that such a P exists since $A_o + B_oF$ is asymptotically stable. Then, the nonlinear feedback control law $u_N(t)$ is given by

$$u_N = \rho(r, y)B_o^T P(x - x_e) \tag{44}$$

where ρ is any nonpositive function locally Lipschitz in y, which is used to change the system closed-loop damping ratio as the output approaches the step command input.
Step 3. The linear and nonlinear feedback laws derived in the previous step are now combined to form a composite nonlinear feedback controller

$$\begin{aligned} u &= u_L + u_N \\ &= Fx + Gr + \rho(r, y)B_o^T P(x - x_e) \end{aligned} \tag{45}$$

The following theorem shows that the closed-loop system comprising the given plant in (16) and the nonlinear feedback control law in (44) is asymptotically stable. It also determines the magnitude of r that can be tracked by such a control law without exceeding the control limit.

Theorem 3.1. *Consider the given system in* (16)*, the linear control law of* (25) *and the nonlinear feedback control law of* (44)*. For any $\alpha \in (0, 1)$, let $c_\alpha > 0$ be the largest positive scalar satisfying the following condition:*

$$|Fx| \le u_{\max}(1 - \alpha), \quad \forall x \in X_{alpha} := \{x : x^T Px \le c_\alpha\} \tag{46}$$

Then the linear control law of (25) *is capable of driving the system controller output $y(t)$ to track asymptotically a step command input r, provided that the initial state x_0 and r satisfy*

$$\tilde{x}_0 = (x_0 - x_e) \in X_{alpha}, \quad |Hr| \le \alpha u_{\max} \tag{47}$$

Furthermore, for any nonpositive function $\rho(r, y)$, locally Lipschitz in y, the composite nonlinear feedback law in (44) *is capable of driving the system controller output $y(t)$ to track asymptotically the step command input of amplitude r, provided that the initial state x_0 and r satisfy* (47)*.*

Proof: Let $\tilde{x} = x - x_e$. It is simple to verify that the linear control law of (39) can be rewritten as

$$\begin{aligned} u_L &= F\tilde{x}(t) + [1 - F(A_o + B_oF)^{-1}B_o]Gr \\ &= F\tilde{x}(t) + Hr. \end{aligned}$$

Hence, for all $\tilde{x} \in X_{alpha}$, provided that $|Hr| \le \alpha u_{\max}$, $|F\tilde{x} + Hr| \le u_{\max}$ and the closed-loop system is linear and it is given by

$$\dot{\tilde{x}} = (A_o + B_oF)\tilde{x} + A_o x_e + B_oHr \tag{48}$$

Noting that

$$\begin{aligned} A_o x_e + B_oHr &= B_o[1 - F(A_o + B_oF)^{-1}B_o]Gr \\ &\quad - A_o(A_o + B_oF)^{-1}B_oGr \\ &= [I - B_oF(A_o + B_oF)^{-1}]B_oGr \\ &\quad - A_o(A_o + B_oF)^{-1}B_oGr \\ &= 0 \end{aligned}$$

the closed-loop system in (48) can then be simplified as

$$\dot{\tilde{x}} = (A_o + B_oF)\tilde{x} \tag{49}$$

Similarly, the closed-loop system comprising the given plant in (16) and the nonlinear feedback control (49) can be expressed as

$$\dot{\tilde{x}} = (A_o + B_o F)\tilde{x} + B_o w \tag{50}$$

where

$$w = sat\,(F\tilde{x} + Hr + u_N) - F\tilde{x} - Hr \tag{51}$$

Clearly, for the given x_0 satisfying (47), we have $\tilde{x}_0 = (x_0 - x_e) \in X_\alpha$. We note that (51) is reduced to (50) if $\rho = 0$. Thus, we can prove the results, respectively, under the linear control and the non linear feedback control in one shot.

Next, we define a Lyapunov function $V = \tilde{x}^T P \tilde{x}$, and evaluate the derivation of V along the trajectories of the closed-loop system in (47), i.e.,

$$\begin{aligned} \dot{V} &= \dot{\tilde{x}}^T P \tilde{x}^T P \dot{\tilde{x}} + 2\tilde{x}^T P B_o w \\ &= \tilde{x}^T (A_o + B_o F)^T P \tilde{x} + \tilde{x}^T P (A_o + B_o F)\tilde{x} \\ &\quad + 2\tilde{x}^T P B_o w \\ &= -\tilde{x}^T W \tilde{x} + 2\tilde{x}^T P B_o w \end{aligned} \tag{52}$$

Note that for all

$$\tilde{x} \in X_\alpha = \{\tilde{x} : \tilde{x}^T P \tilde{x} \le c_\alpha\} \Rightarrow |F\tilde{x}| \le u_{\max}(1 - \alpha) \tag{53}$$

We next study the $\dot{V}$ for the different cases of the constraints on the input.

Case 1) If $|F\tilde{x} + Hr + u_N| \le u_{\max}$, then $w = u_N = \rho B^T P \tilde{x}$ thus

$$\dot{V} = -\tilde{x}^T W \tilde{x} + 2\rho \tilde{x}^T P B_o B_o^T P \tilde{x} \le -\tilde{x}^T W \tilde{x} \tag{54}$$

Case 2) If $F\tilde{x} + Hr + u_N \succ u_{\max}$, and by construction $|F\tilde{x} + Hr| \le u_{\max}$, we have

$$0 \prec w = u_{\max} - F\tilde{x} - Hr \prec u_N = \rho B_o^T P \tilde{x} \tag{55}$$

which implies that $\tilde{x}^T P B_o \prec 0$ and hence $\dot{V} = -\tilde{x}^T W \tilde{x} + 2\tilde{x}^T P B_o w \le -\tilde{x}^T W \tilde{x}$.

Case 3) Finally, if $F\tilde{x} + Hr + u_N \le -u_{\min}$, we have

$$\rho B_o^T P \tilde{x} = u_n \prec -u_{\min} - F\tilde{x} - Hr \prec 0 \tag{56}$$

implying $\tilde{x}^T P B_o \succ 0$ and hence $\dot{V} \le -\tilde{x}^T W \tilde{x}$.

In conclusion, we have shown that

$$\dot{V} \le -\tilde{x}^T W \tilde{x} \;\; \tilde{x} \in X_{alpha} \tag{57}$$

which implies that X_{alpha} is an invariant set of the closed-loop system in (50). This, in turn, indicates that, for all initial state x_0 and the step command input of amplitude r that satisfy (51)

$$\lim_{t \to \infty} x(t) = x_e \Rightarrow \lim_{t \to \infty} y(t) = r \tag{58}$$

This completes the proof.

4. **Simulation Results.** The simulation results are obtained using a fourth order Runge-Kutta with the same typical values of process and kinetic parameters defined in Tables 1 and 2 and the controller is tested with two kinds of disturbances:

Thereafter decomposition, the obtained system is represented by (24), and A_o, B_o, C_o are injected in closed loop and coupled with a minimal order Luenberger observer which has the role of estimating the non-measurable states from the measurable ones ($S_{NH,nit}, S_{NO,nit}, S_{O,nit}$). We assume that in our case the control constraints are such as

$$u_{\max} = \begin{pmatrix} 5000 \\ 18466 \\ 110 \end{pmatrix}, \quad u_{\min} = \begin{pmatrix} 2000 \\ 18425 \\ 80 \end{pmatrix}$$

and reconstruction errors limits are such as:

$$\epsilon_{\max} = \begin{pmatrix} 1 & 1 & 0.5 & 1 & 1 & 1 \end{pmatrix}; \quad \epsilon_{\min} = \begin{pmatrix} 0.5 & 0.5 & 0.25 & 0.8250 & 0.5 & 0.5 \end{pmatrix}$$

we choose the matrix H assigning spectrum $\{-170; -55; -51\}$ as follows:

$$H = \begin{pmatrix} -170 & 0 & 0 \\ 0 & -55 & 0 \\ 0 & 0 & -51 \end{pmatrix}$$

Hence, solving Equation (38) leads to:

$$F = \begin{pmatrix} 0.0010 & -0.0044 & 0.0319 & 0.0725 & -0.0791 & -0.0944 & 1.0349 & -0.0000 & 0.4162 \\ 0.0001 & -0.0005 & 0.0000 & 0.0001 & 0.0000 & -0.0002 & 0.0017 & -0.0000 & 0.0007 \\ 0.0148 & -0.0774 & 0.1421 & 0.0036 & -0.1591 & -0.2023 & -0.0352 & -0.0024 & 0.0864 \end{pmatrix}$$

Figures below are devoted to present the evolution of the disturbances, outputs and inputs of the system. In fact, the nonlinear feedback controller, as defined in the sections above, is applied to the WWTP. The linear and non linear feedback controllers are compared by computing the indices IQ, EQ, PE and ISE as summarized in Tables 3 and 4. The output variables evolution, are $S_{NH,nit}$, $S_{NO,nit}$ and the dissolved oxygen concentrations, and their corresponding reference trajectories are 3.2, 10.4 and 2.4, respectively.

As general remarks asymptotic stability is obtained, all constraints are respected and the the amount of all non desired organic matter is reduced in the output to the desired values. Figures (3,5) and (7,9) show the performance and the effectiveness of the regulator. In particular, one can appreciate the ability of the controller to track the desired values of the controlled variables. Hence, in practice, the nonlinear controller is able to reduce

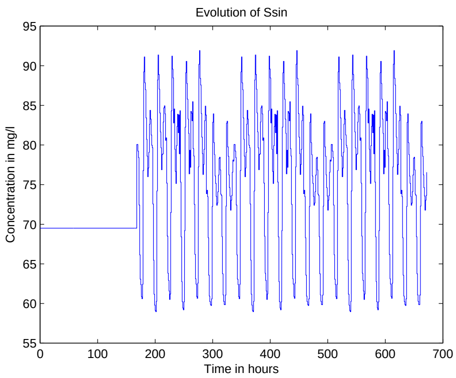

FIGURE 2. Evolution of the disturbance S_{in1}

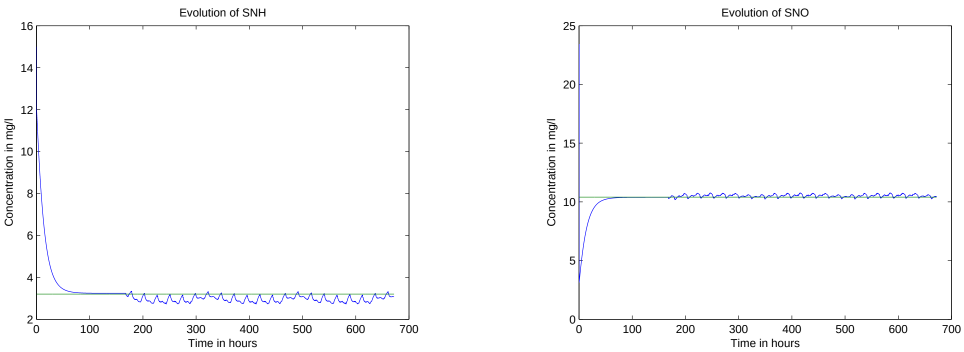

FIGURE 3. Evolutions of the concentrations $S_{NH,nit}$ and $S_{NO,nit}$

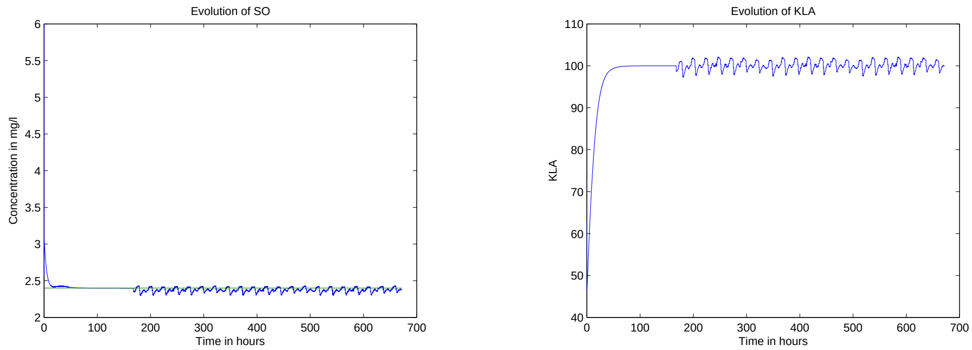

FIGURE 4. Evolutions of the concentration S_O and the dissolved oxygen Q_{air}

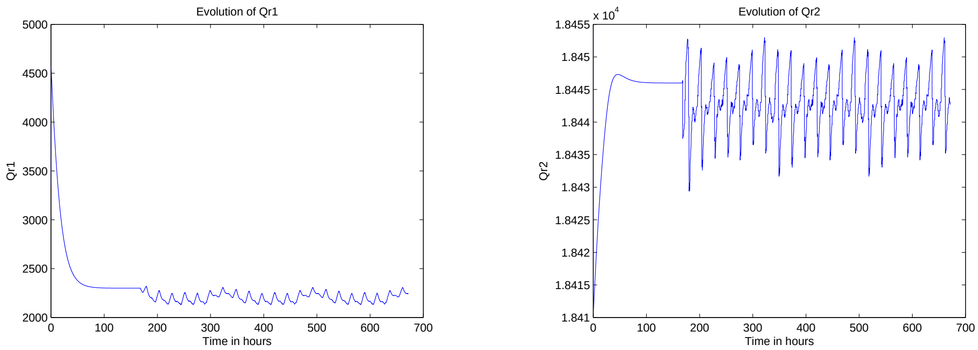

FIGURE 5. Evolutions of the recycled flows Q_{r1} and Q_{r2}

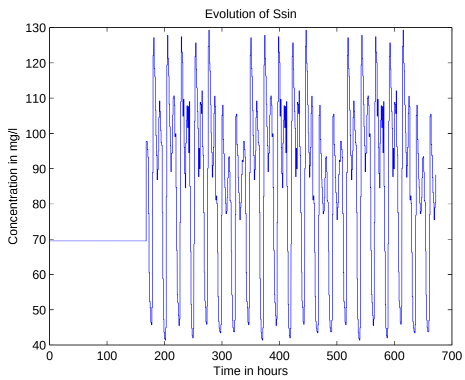

FIGURE 6. Evolution of the disturbance S_{in2}

TABLE 3. Indices of the plant with the first disturbance (Figure 1)

Indices	$u = u_l$	$u = u_l + u_{nl}$	Units
Influent Quality (IQ)	5.4306e+007	5.4306e+007	$gPUd^{-1}$
Effluent Quality (EQ)	8.9048e+007	8.9015e+007	$gPUd^{-1}$
Pumping Energy (PE)	8.5923e+004	8.5923e+004	Whd^{-1}
Aeration Energy (AE)	2.8433e+004	2.8173e+004	$Whd-1$
Integral of the Squared Error (ISE)	10.3677	1.9225	

the EQ, AE and ISE with an important percentage and this is clear from Tables 3 and 4. No change in IQ because the influent is constant.

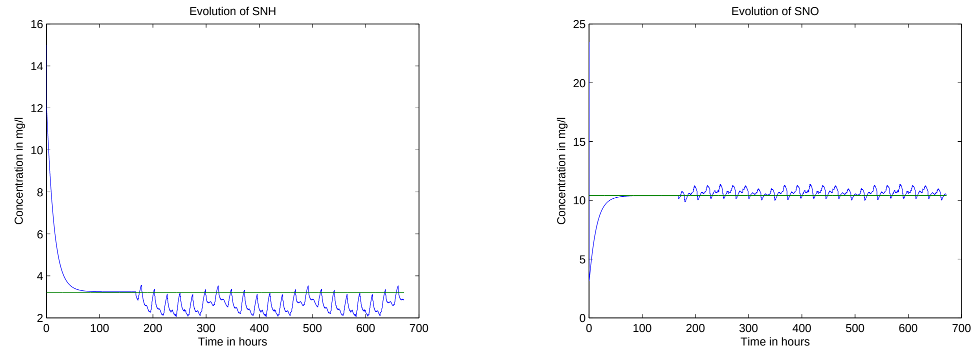

FIGURE 7. Evolutions of the concentrations $S_{NH,nit}$ and $S_{NO,nit}$

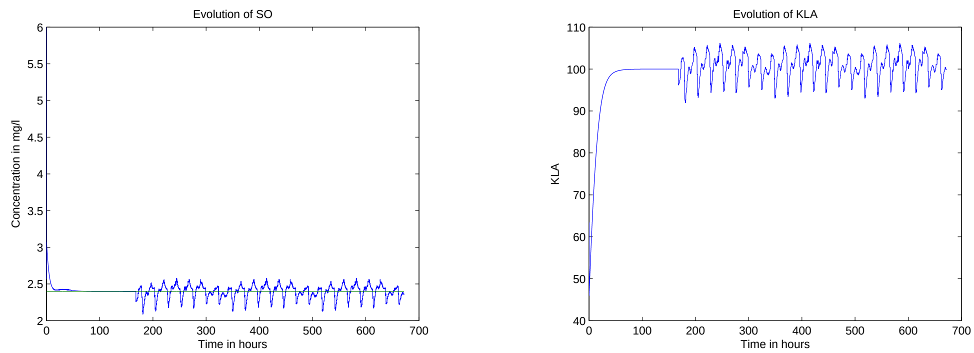

FIGURE 8. Evolutions of the concentration S_O and the dissolved oxygen Q_{air}

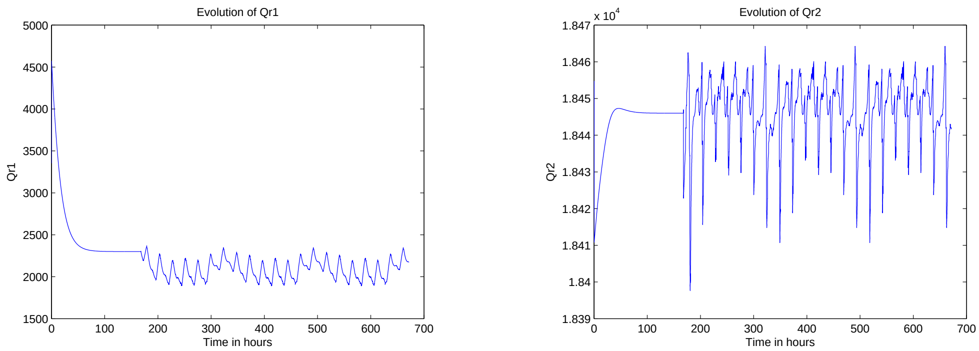

FIGURE 9. Evolutions of the recycled flows Q_{r1} and Q_{r2}

TABLE 4. Indices of the plant with the second disturbance (Figure 5)

Indices	$u = u_l$	$u = u_l + u_{nl}$	Units
Influent Quality (IQ)	5.4678e+007	5.4678e+007	$gPUd^{-1}$
Effluent Quality (EQ)	9.1354e+007	9.1306e+007	$gPUd^{-1}$
Pumping Energy (PE)	8.5923e+004	8.5923e+004	Whd^{-1}
Aeration Energy (AE)	2.8712e+004	2.8313e+004	Whd^{-1}
Integral of the Squared Error (ISE)	15.8029	5.2772	

5. **Conclusion.** In this paper, we introduced the nonlinear feedback control of a non linear system with input constraints. In fact, positive invariance techniques together with minimal order observer (software sensor) are used to control the linearized model of a

WWTP. For this process, modeled as a linear process, some state variables are unavailable to measure and more than that no adequate sensor exists. Hence, the introduction of the observer is of great interest. Further, in general case, linearizing a non linear process leads the variables (control in our case) to be limited within neighborhood of the steady point functioning. The positive invariance techniques that had emerged as very efficient to handle similar problems of constrained control are successfully used to control the nitrogen removal process. The observer based constrained control, as presented above may compete with approaches in easiness, applicability and computing effort.

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Appendix A. The linearization of the system around the equilibrium point computed from the non linear equations gives the following matrices:

$$A = \begin{pmatrix} -29.07 & 0 & 0 & 2.65 & 0 & 2.17 & 29.40 & 0 & 0 & 0 & 0 & 0 \\ 0 & -29.48 & 6.04 & 0.642 & 0 & 4.40 & 0 & 29.40 & 0 & 0 & 0 & 0 \\ 0 & -0.34 & -38.99 & -1.02 & -0.05 & 5.01 & 0 & 0 & 29.40 & 0 & 0 & 0 \\ -2.22 & -0.02 & -0.55 & -40.74 & 0 & 12.23 & 0 & 0 & 0 & 29.40 & 0 & 0 \\ 2.18 & 0 & -0.06 & 11.05 & -29.41 & 9.64 & 0 & 0 & 0 & 0 & 29.40 & 0 \\ -9.45 & 0 & -0.18 & -47.90 & -0.02 & -167.01 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2.30 & 0 & 0 & 0 & 0 & 0 & -38.67 & 0 & 0 & 0.55 & 0 & 1.84 \\ 0 & 2.30 & 0 & 0 & 0 & 0 & 0 & -39.18 & 4.44 & 0.12 & 0 & 16.60 \\ 0 & 0 & 2.30 & 0 & 0 & 0 & 0 & -0.78 & -50.67 & -0.30 & -3.42 & 0 \\ 0 & 0 & 0 & 2.30 & 0 & 0 & -3.06 & -0.04 & -0.66 & -39.33 & -0.19 & 0 \\ 0 & 0 & 0 & 0 & 2.30 & 0 & 2.99 & -0.03 & -0.55 & 1.14 & -39.58 & 0 \\ 6.14 & 6.14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.13 \end{pmatrix}$$

$$B = 10^4 \begin{pmatrix} -0.0011 & -0.0011 & 0 \\ 0.0023 & 0.0023 & 0 \\ 0.0102 & 0.0102 & 0 \\ 0.0079 & 0.0079 & 0 \\ -0.0071 & -0.0071 & 0 \\ -0.0033 & -0.0033 & 0.0008 \\ 0.0014 & 0.1233 & 0 \\ -0.0031 & 1.1003 & 0 \\ -0.0386 & -0.0386 & 0 \\ -0.0105 & -0.0165 & 0 \\ 0.0094 & -0.0097 & 0 \\ 0 & -0.2042 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The similarity matrix T_o is used to compute the matrices A_o, B_o and C_o:

$$T_o = \begin{pmatrix} 0.0001 & 0.9975 & -0.0395 & 0 & 0 & 0 & 0 & -0.0561 & 0.0034 & -0.0001 & 0 & 0.0185 \\ -0.0014 & 0.0589 & 0.0680 & 0 & 0 & 0 & 0.0009 & 0.9943 & -0.0573 & 0.0012 & 0.0003 & -0.0003 \\ 0 & -0.0184 & 0.0007 & 0 & 0 & 0 & 0 & 0.0013 & -0.0001 & 0 & 0 & 0.9998 \\ -0.0200 & 0.0355 & 0.9965 & 0 & 0 & 0 & 0.0064 & -0.0703 & 0.0001 & 0.0172 & 0.0037 & 0 \\ -0.0001 & 0 & 0.0041 & 0 & 0 & 0 & -0.0214 & 0.0572 & 0.9981 & 0.0001 & 0 & 0 \\ 0.0001 & -0.0002 & -0.0064 & 0 & 0 & 0 & 0.9997 & 0.0008 & 0.0214 & -0.0001 & 0 & 0 \\ -0.9929 & -0.0007 & -0.0188 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0842 & 0.0821 & 0 \\ -0.0017 & 0.0005 & 0.0146 & 0 & 0 & 0 & 0 & 0 & 0 & -0.6895 & -0.7242 & 0 \\ 0.1173 & 0.0004 & 0.0122 & 0 & 0 & 0 & 0 & 0 & 0 & -0.7192 & 0.6847 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_o = \begin{pmatrix} -38.85 & 28.99 & -0.01 & 0.18 & 0.01 & -0.02 & -5.10 & 0.05 & 1.01 \\ 2.33 & -50.29 & -0.27 & -0.25 & 2.72 & -2.09 & -0.02 & 0 & 0.01 \\ 0.01 & -0.44 & -38.68 & -2.34 & -0.33 & -0.17 & 0.03 & 0 & -0.01 \\ 0.01 & 0.07 & -28.69 & -29.23 & -0.04 & -1.26 & 2.26 & -0.19 & 2.81 \\ -0.01 & 1.29 & -0.08 & 0.11 & -38.99 & 0.81 & -0.07 & 1.67 & 1.60 \\ 0.04 & 0.28 & 7.71 & -1.26 & -0.51 & -39.77 & -0.32 & -1.57 & 1.35 \\ 0 & 0 & 0 & -9.39 & -0.01 & 1.11 & -167.01 & -0.02 & -47.91 \\ 0 & 0 & 0 & -0.24 & 21.29 & -20.38 & 9.65 & -29.41 & 11.05 \\ 0 & 0 & 0 & 0.25 & 20.27 & 21.41 & 12.24 & -0.01 & -40.75 \end{pmatrix}$$

$$B_o = 10^3 \begin{pmatrix} 0.1040 & -0.6657 & 0 \\ -0.3872 & 0.2181 & 0 \\ 0.0052 & 1.2322 & 0 \\ 0.0252 & 0.0145 & 0 \\ 0.0057 & 0.1856 & 0 \\ 0.1403 & 0.0522 & 0 \\ 0.0332 & 0.0332 & -0.0076 \\ 0.0708 & 0.0708 & 0 \\ -0.0789 & -0.0789 & 0 \end{pmatrix}$$

$$C_o = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

The matrix T is chosen such that only the part $z(t) = TZ_{co}(t)$ is estimated. Further, matrix P is chosen to ensure asymptotic stability of matrix $D = TA_{co}P$. In fact, in this case matrix T and P are given by:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

According to Equation (34), the matrices D, G and E are computed.

$$D = \begin{pmatrix} -38.8580 & 28.9925 & -0.0070 & 0.1784 & 0.0119 & -0.0185 \\ 2.3306 & -50.2968 & -0.2666 & -0.2478 & 2.7231 & -2.0948 \\ 0.0147 & -0.4418 & -38.6780 & -2.3376 & -0.3296 & -0.1756 \\ 0.0119 & 0.0735 & -28.6943 & -29.2308 & -0.0443 & -1.2648 \\ -0.0105 & 1.2947 & -0.0824 & 0.1128 & -38.9935 & 0.8130 \\ 0.0442 & 0.2873 & 7.7115 & -1.2630 & -0.5140 & -39.7722 \end{pmatrix}$$

$$G = 10^3 \begin{pmatrix} 0.1040 & -0.6657 & 0 \\ -0.3872 & 0.2181 & 0 \\ 0.0052 & 1.2322 & 0 \\ 0.0252 & 0.0145 & 0 \\ 0.0057 & 0.1856 & 0 \\ 0.1403 & 0.0522 & 0 \end{pmatrix} \quad E = \begin{pmatrix} -1.0078 & -0.0489 & 5.1034 \\ -0.0042 & -0.0002 & 0.0205 \\ 0.0064 & 0.0003 & -0.0327 \\ -2.8104 & 0.1898 & -2.2577 \\ -1.6047 & -1.6664 & 0.0718 \\ -1.3549 & 1.5741 & 0.3185 \end{pmatrix}$$