International Journal of Innovative
Computing, Information and Control
Volume **10**, Number **5**, October **2014**

ICIC International ©2014 ISSN 1349-4198
pp. 1601–1623

QUANTUM THEORY OF FUNDAMENTAL NETWORK (PATH INTEGRAL EXPRESSION CIRCUITS AND NETWORK'S QUANTIZATION)

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Received May 2013; revised February 2014

Abstract. *We proposed new basic theory and calculation methods for quantum bifurcation, quantum circuits, and neural computer based on path integrals of quantum theory. The problems of classical bifurcation were easily led to Schrödinger equation by considering Nelson's stochastic quantization method. Japanese Amida lottery was a kind of classical bifurcation models because of no interference between each path of lottery. And so we showed how to quantize electric circuits, Amida lottery and complex neural network by path integrals. The bifurcation points of Amida lottery corresponded to diffraction point of polariton in quantum theory. We constructed the method of quantization of basic circuits as AND, OR and NOT. Moreover, we assumed that we could regard classical switches as scattering potentials (switch's operators). Those were quantization concepts, and those quantized circuits with switch operators corresponded to q-AND, q-NOT, and q-OR circuits. The Proca equation of polariton approached to the quaternary Schrödinger equation when the motion of polariton was much slower than light velocity. The kernel K(b,a), which was propagator and an expression of the time development of system, was related to an eigenfunction of Schrödinger equation. We found that the neuro-synaptic junctions were regarded as a kind of switch's potential, whose concepts led to quantization of neural networks by using path integrals.*

Keywords: Polariton, Quantum circuits, Quantum lottery, Path integral, Interference, Hamiltonian, Quaternary Schrödinger equation, Superposition

1. **Introduction.** The models of neurons, their networks and conducting mechanism are not only important bases of biological brain's functions, but also they have been producing many algorithms and their concepts of soft computing as neuro-fuzzy controls, and mechanical learning models in many engineering and information branches [1-4]. And those models have been based on an independence of each axon of neuron. We have been hypothesized that there was not an electromagnetic interference between axon's membranes of neurons. Therefore, a lot of physiological books say that, each neuron holds independence of each other, and there are not electromagnetic interactions between axons and synapses, because the neurons are governed the law of "all or nothing", and those electromagnetic effects are much small since neurons are covered with lipid nonconductor's membranes. And action potentials traveling on the axon and the neural processes (polarization-depolarization-repolarization processes), have been believed not to affect another axon and an ionic current for a long while [1,21]. They say that each neuron is independent and there is not the interference between each axon of neurons.

However, in previous paper, we have proposed the other theory and engineering models. Each neuron has a lot of interferences caused by polarization of the membrane, leak currents, and ionic currents (Na^+, K^+). Neurons have many ionic channels, their currents, and polarizations, whose phenomena generate electromagnetic interactions on our brain as we can detect its field by SQUID. Thus, each neuron gives rise to a holistic macro electromagnetic field, and that electromagnetic field governs each neuron [1].

And we would like to show you another evidence of neural interference. Prof. Arvanitaki discovered the phenomena of ephapse, which was interference between two neural axons [1-4]. When he stimulated one neural axon and generated action potentials, that signal affected another neuron, despite of defection of direct connections between two neurons. He is said to be the first researcher who made up an artificial neuron. So phenomenon, discovered by him, was named as ephapse. His experiments showed that each neuron had directly neural interferences based on the electromagnetic interactions.

As medical examples of neural interferences, we know pathological states of neuralgia and causalgia. However, we would like to assume that our normal brains always actively utilize those electromagnetic interactions so as to make up our holistic and harmonic neural system. And we show the basic equations for those electromagnetic interactions of between each of neuron.

This paper says that those possible forms are the quaternary Schrödinger equations or Proca equation. Moreover, an agency for those electromagnetic interactions is polariton, which is a kind of massive photon. The polariton is the quantized polarization wave on dielectric (cell membrane) and it has the spin-value of one (spin 1). From the standpoint of the mesoscopic science, all electromagnetic interactions should be described as elementary processes based on the interactions of massless or massive photons (polariton), because macro electromagnetic phenomena can be reduced to an approximation of quantum electromagnetic dynamics (Q.E.D.). In the previous paper, we referred to the necessity of polariton, and showed the quantization's process for macro electromagnetic phenomena of neurons [1]. The relativistic quantized electromagnetic field of neurons is governed by the Proca equation. And we show that the Proca equation can be reduced to the quaternary Schrödinger equation of polariton, since a propagating velocity of the polariton (quantized polarization waves) on neurons was so much slower than that of light in vacuum [1].

In this paper, we discuss physical foundation of quantum interferences between artificial neurons based on our previous papers and the Arvanitaki's classical ephapse [1-4]. We attempt to give the descriptions for the polariton's motions on neural axon by using both path integrals and quaternary Schrödinger equation of polariton. We would like to make up the calculating toolbox for polariton's motion, and to show applications for Amida lottery, bifurcations, circuits, scattering problems and for network systems. In order to describe the polariton's theory (quaternary Schrödinger equation), we think the Feynman's path integral is suitable for the neural conductions and of neuron's interferences. We can automatically introduce quantum effects of polaritons to the network systems, and its expression is much similar to classical mechanical Lagrangian. Moreover, we know that the description of path integral is perfectly equivalent to that of Schrödinger equation [9-11,14,15].

In Section 2, we mention that a bifurcation's problem of decision tree and multi-step slit are related to Markov process. Therefore, according to probability's theory, those processes can be expressed as the generalized stochastic equation, i.e., it is Ito equation. Applying Nelson's method, we can reduce that stochastic equation to Schrödinger equation of the wave function ϕ [5], whose process is called the stochastic quantization. On the other hand, Proca equation approximately becomes the quaternary Schrödinger equation

of electromagnetic potential (ϕ, $\mathbf{A}$) in the case of the slow polariton's movement [1,9,10]. And the quaternary Schrödinger equation approaches to the ordinary Schrödinger equation (one component's equation), if a change of the magnetic field is so small (the vector potential $\mathbf{A}$ is constant, or $\delta\mathbf{A} \approx 0$) [1,11,13]. Thus, the polariton's motion can be approximately expressed by Schrödinger equation of scalar potential ϕ, and that ϕ is related to the bifurcation problems of classical mechanics, information theory and the stochastic equation [14-16]. After we explained the principle of Feynman path integrals in Subsection 2.2 and calculated an action S for free polariton and for a harmonic oscillator, we apply those path integrals to the descriptions for the Amida lottery and a slit in Section 3. They are examples of quantum bifurcation's problems of polariton. In Section 4, we discuss some quantum descriptions for simple circuits (for example, AND-, NOT-, OR-circuit and their complex ones) and switches by using path integral. Then we know the path integral is one of the powerful tools so as to describe the quantum networks and circuits [16-20]. Section 5 is mentioned to a perturbation method of Schrödinger equation. Then, we express that our description of neural network based on path integrals automatically leads to perturbation series. Then we mention that the switches of network and circuits are regarded as synaptic junctions or scattering potential of polariton. Section 6 is shared into summary of mathematical tools that we obtain in this paper by using path integral's descriptions. We show that the Schrödinger equation is perfectly equal to path integrals. In Section 7, we conclude the paper.

Main purpose in this paper is to give the ways that we can express the quantum networks containing much interference. We would like to describe the quantization tools for neural networks, Amida lottery, quantum circuits and many complex diagrams. In our neural networks, the polariton conveys physical information, and polariton is quantized particle of the action potentials (impulse) of neurons [11-13]. Thus, our description's method and its development mean the quantum theory of network, bifurcation and circuits. For example, one of great mathematicians, R. Penrose said that our brain cell had many micro-turbines, which worked as conductors causing superposition of wave functions. He thought that those wave functions made reduction to only one wave function when we determined something for various problems [21]. We do not intend to discuss whether his theory is true or not from biological standing points. And we would like to pick up his concepts that our brain utilizes quantum effect and that the brain belongs to a kind of quantum circuit. We have an idea that quantum interferences are playing important roles for our thinking processes.

Therefore, we propose the idea of a quantum circuit and new theory for quantum computers of neural computations in this paper. And we would like to show those quantization-methods of the bifurcation, Amida lottery and decision trees, which contain some fundamental ideas for quantum interferences and the reductions of wave functions.

2. **Uncertainty and Superposition.**

First we would like to discuss a classical bifurcation that contains fundamental problems. The bifurcation is related to both probability and stochastic equations, and its theme leads to Schrödinger equation through Nelson's method, (stochastic quantization) [5,22].

2.1. **Classical bifurcations and Nelson's method.**

There is much difference between classical bifurcation and quantum bifurcation. The former is related to classical probability whose value is always the positive and real number. However, the latter takes complex number, whose function is called probability amplitude.

And the probability amplitude can be connected with a solution of Schrödinger equation. The classical probability cannot automatically expressed interference by superposition principle. However, the probability amplitude essentially contains much interference between each bifurcated branch. And the interference, which arises from superposition principle, plays a lot of important roles in our quantum neural theory.

In this section, we would like to show that problems of decision tree can be regarded as a kind of Brownian motion (Markov process), and then we should notice that Brownian motion is governed with Ito equation (general stochastic equation). And according to Nelson's method (stochastic quantization), the Ito equation reaches Schrödinger equation. Thus, the problems of the decision tree can be rewritten into Schrödinger equation of complex function $\chi(X,t)$ by both Fokker-Planck equation and Chapman equation.

At first, we show that small particles (for example electrons or photons) are flowing on the branches of bifurcation-diagram (a kind of decision tree) (Figure 1). We assume that the particles on the diagram diverge for each branch at an equivalent probability, 50%.

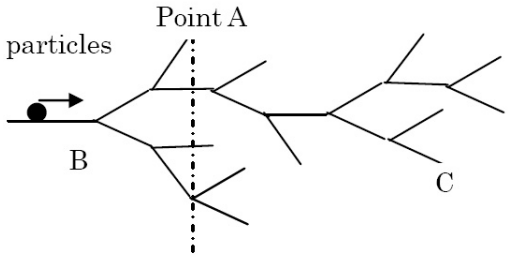
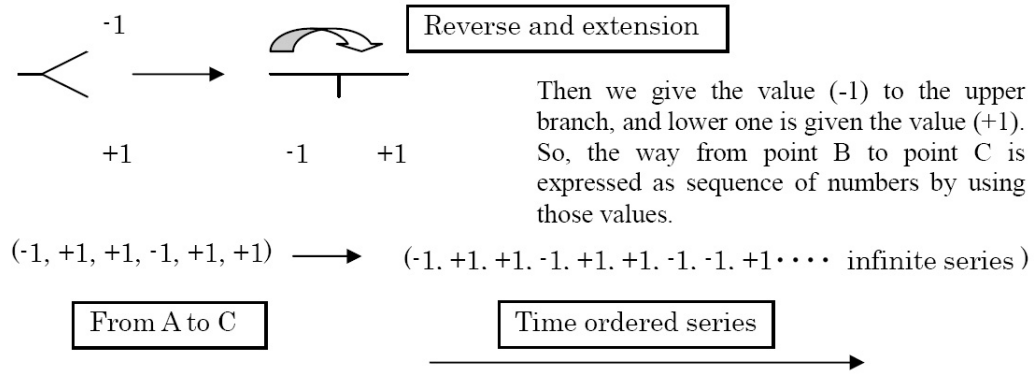


FIGURE 1. Multi-step bifurcation's problem

When we attempt to deform the branch lines of bifurcation diagram Figure 1, then the diagram becomes a following feature: that bifurcation diagram can be represented as the random walk's problem.



If we concretely can show the path (from B to C), we obtain a sequence of numbers: the sequence is $(-1,+1,+1,-1,+1,+1)$. If we hypothesize that bifurcations of the diagram make an infinite series, the above finite bifurcation becomes an infinite random walk's problem. So we notice that the infinite sequence is much similar to Markov process or Brownian motion in one dimension. Thus that Brownian motion truly is expressed by stochastic differential equation [5,22].

We would like to start from a generalized stochastic equation, which is called, Ito equation,

$$dX(t) = b(X(t),t)dt + A(t)dw(t), \tag{1}$$

and then the $dw(t)$ has following characteristics of Brownian motion. (deviation $A(t)$: diffusion coefficient, and an average b: drift coefficient).

$$\langle dw\rangle^2 = \langle w(t+\Delta t) - w(t)\rangle^2 = \beta\,\Delta t \tag{2-1}$$

$$\langle dw\rangle = \langle w(t+\Delta t) - w(t)\rangle = 0 \tag{2-2}$$

According to Nelson's stochastic quantization method with stochastic variable $X(t)$, the trace of a particle is divided into two parts. The one is an anterior average derivative, and another is posterior average derivative. Those terms are defined as

$$Df \equiv \lim_{\Delta t\downarrow 0}\left\langle \frac{f(t+\Delta t)-f(t)}{\Delta t}\;\middle|\; f(s) \text{ is fixed for } s\le t\right\rangle \quad \text{anterior average derivative} \tag{3-1}$$

$$D_*f \equiv \lim_{\Delta t\downarrow 0}\left\langle \frac{f(t)-f(t-\Delta t)}{\Delta t}\;\middle|\; f(s) \text{ is fixed for } s\ge t\right\rangle \quad \text{posterior average derivative} \tag{3-2}$$

and both average velocities for Brownian motion are calculated as

$$DX(t) = b(X(t),t), \quad D_*X(t) = b_*(X(t),t). \tag{4}$$

An acceleration $a(t)$ of Brownian particle was defined by Nelson method, and the $a(t)$,

$$a(t) = \frac{1}{2}\left(D_*D + DD_*\right)X(t), \tag{5}$$

is obtained by performing above derivative for Equation (4). We introduce two new variables, u and v: those are

$$v = \frac{1}{2}\left(b + b_*\right), \quad u = \frac{1}{2}\left(b - b_*\right). \tag{6}$$

Thus, the acceleration $a(t)$ becomes

$$a(t) = -A^2(t)\frac{\beta}{2}\frac{\partial^2 u}{\partial X} + \frac{1}{2}\frac{\partial}{\partial X}\left(v^2-u^2\right) + \frac{\partial v}{\partial t} = -\frac{1}{M}\frac{\partial V}{\partial X}. \tag{7}$$

The symbol M means Brownian particle's mass (we think polariton's mass), and the V is potential energy. Equation (7) corresponds to Newtonian equation of motion for Brownian particle, and it is said to be mechanical condition. Applying the anterior derivative to Chapman equation, we can define an operator (A_Tf) of Equation (8) [5]. ($\rho(X,t_0|Y,t)$: probability that the particle which existed in an initial condition (X,t_0) reaches the point Y at time t) [5]. The operator (A_Tf) is expressed as

$$\left(A_Tf\right)(X) = \lim_{\Delta t\downarrow 0}\int dY\cdot f(Y)\frac{\rho(X,t_0|\,Y,t+\Delta t) - \rho(X,t_0|\,Y,t)}{\Delta t}. \tag{8}$$

Then we can obtain another expression of anterior derivative,

$$\left(A_Tf\right)(X) = Df(X) = b(X,t)\frac{\partial f}{\partial X} + A^2\frac{\beta}{2}\frac{\partial^2 f}{\partial X^2}. \tag{9}$$

We multiply $\rho(X_0,t_0|X,t)$ to Equation (9), and we practice an integration: we finally have

$$\frac{\partial\rho}{\partial t} = -\frac{\partial}{\partial X}\left(b\rho\right) + \frac{\beta}{2}\frac{\partial^2}{\partial X^2}\left(A^2(t)\rho\right). \tag{10}$$

That is Fokker-Planck equation [5]. For the b_*, we have a similar equation:

$$-\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial X}\left(b_*\rho\right) + \frac{\beta}{2}\frac{\partial^2}{\partial X^2}\left(A^2(t)\rho\right). \tag{11}$$

We add up both equations, Equation (10) and Equation (11), whose equations assimilate with one equation that represents a condition of motion for polariton:

$$\frac{\partial u}{\partial t} = -A^2 \frac{\beta}{2} \frac{\partial^2 v}{\partial X^2} - \frac{\partial}{\partial X}(uv). \tag{12}$$

To unify both condition of mechanics and that of motion, we would like to introduce a complex variable, $\chi(X,t) \equiv u + iv$. We transfer two variables u, v into a single variable χ,

$$\chi \equiv A^2 \beta \frac{\partial}{\partial X} \ln \Psi \tag{13-1}$$

$$\Psi(X,t) \equiv \phi(X,t) \exp\left(-\frac{i}{A^2\beta} \int \eta(t) dt\right), \tag{13-2}$$

and the transcription into single equation is achieved as

$$i\frac{\partial}{\partial t}\phi(X,t) = \left[-\frac{A^2\beta}{2}\frac{\partial^2}{\partial X^2} + \frac{1}{A^2\beta\, M} V(X)\right] \phi(X,t). \tag{14}$$

If we take $A^2\beta \to \hbar/M$, we find Equation (14) to be the common Schrödinger equation. Therefore, the probability density $\rho(X,t)$ is given as

$$\rho(X,t) = |\, \phi(X,t)|^2, \tag{15}$$

by a complex probability amplitude ϕ. If we take $t \to it$, Equation (15) is reduced to Feynman-Kac equation. However, there is difference between Schrödinger equation and Feynman-Kac equation. The Feynman-Kac equation has always real number's solution. On the other hand, the Schrödinger equation almost takes complex number's solution. Thus, the Feynman-Kac equation can describe only classical bifurcation and its probability. However, the Schrödinger equation, whose solution is permitted to have the complex number (probability amplitude), is truly suitable for descriptions of interferences between each quantum state. We should notice that the complex number is an essential characteristic for quantum theory, and that the real number is a character of classical bifurcation problem. And the classical bifurcation's problem is always reduced to Weiner process (Brownian motion) and Markov process. So the classical bifurcation is quantized through the Nelson's method [5]. Thus, the classical stochastic problem can be translated into quantum one by introducing the complex variable and the probability amplitude.

We would like to discuss effects of superposition of the probability amplitude, and we mention those of the sensitive limitation caused by uncertainties. If all paths of Figure 1 are governed by uncertainty principle, we find the quantum fluctuations and interferences to exist between each bifurcation's branches. And the fluctuations of position Δx should satisfy the following relation, which is uncertainty:

$$\Delta x \geq \hbar/(\Delta p). \tag{16}$$

Therefore, a path less than the range Δx, is directly governed by effects of quantum mechanics.

We can easily explain the difference between quantum bifurcation and classical one. If particles obey to single-step's bifurcation, a total state vector is written as the superposition and linear combination of all base state vectors. Let us consider two state's model, i.e., those quantum states are ϕ_1 and ϕ_2. If there are those states within uncertainty's range Δx, then a total state ϕ is the summation of the two states:

$$\phi = a\phi_1 + b\phi_2. \tag{17}$$

Thus the total probability density of the above state is expressed as

$$|\phi|^2 = |a|^2 |\phi_1|^2 + |b|^2 |\phi_2|^2 + a^* b \phi_1^* \phi_2 + a b^* \phi_1 \phi_2^*. \tag{18}$$

We notice that the first and second terms of Equation (18) correspond to classical probability densities. The third and fourth terms, which are expression of quantum effects, mean quantum interference's terms. Uncertainty principle tells us that we cannot detect them as the different two states, if their states do not keep away more than the fluctuations' range Δx from each state (Figure 2). As uncertainty of momentum Δp gradually goes to the large value, it is more difficult for us to observe an aspect of bifurcation of particles. Therefore, it will be more clear the difference of both the classical probability and the quantum one.

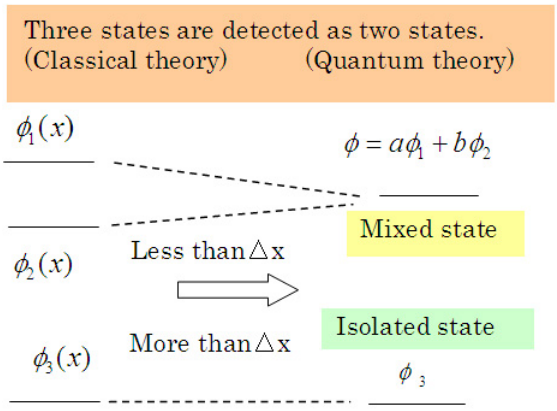


FIGURE 2. Uncertainties and sensitive limitation

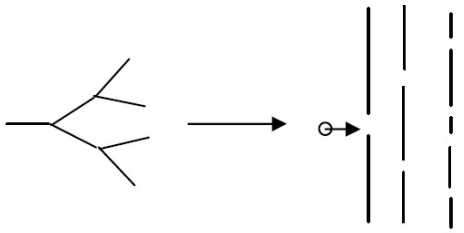


FIGURE 3. Multi-step bifurcations

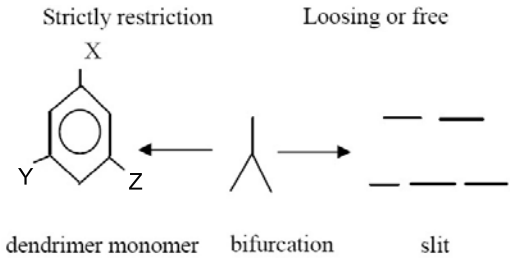


FIGURE 4. Various restricted conditions

(Explanation of Figure 3)

To fix particles on the nano-scale conductors (wires), an external force or some potential is impressed on the particles. If it were not for those restricting conditions, many of particles deviate from their paths or conductors, and then they behave as free particles. And we can apply an example of conductors to the model of slits. So we look upon the bifurcation diagrams as multi-step slits when those restricted conditions are going to weaken.

(Explanation of Figure 4)

When the range of uncertainty Δx is nearly equal to sizes of atoms ($\mathring{A}$), those processes approach to molecular wires of dendrimer monomers.

2.2. **Description of Feynman path integrals.** We would like to mention the principle of Feynman path integrals, and intend to apply its method to the motion of free polariton. Subsequently, we describe the scattering problem or the diffraction of the polariton, by its integrals in order to obtain mathematical tools. At first, we consider a particle whose generalized Lagragian has the following form,

$$L = a(t)t^2 + b(t)\dot{x}x + c(t)x^2 + d(t)\dot{x} + e(t)x + f(t). \tag{19}$$

An action S of its motion is given by the time's integral of the Lagragian between two fixed points, i.e., starting point a and ending point b. We determine the Feynman's kernel $K(b,a)$ that is defined as

$$K(b,a) = \int_a^b \exp\left[\frac{i}{\hbar}S\right] Dx(t), \quad \because S \equiv \int_a^b L(\dot{x},x,t)\,dt, \tag{20}$$

(a: starting point of path, b: ending point of path). Here if we attempt to define a quantum action $S[x(t)]$ in an interval $[a,b]$, then the quantum variable $x(t)$ should be divided into two parts. Thus, its variable $x(t)$ is composed of classical path term $x_c(t)$ and quantum fluctuation $\delta(t)$, and so we have a relation, $x(t) = x_C(t) + \delta(t)$. And the integral $(Dx(t))$ should be performed over all paths in the interval $[a,b]$. Then the action $S[x(t)]$ becomes

$$S[x(t)] = S[x_C(t) + \delta(t)] = \int_{ta}^{tb} dt[a(t)(x_C^2 + 2x_C\dot{\delta} + \dot{\delta}^2) + \cdots]. \tag{21}$$

If it were not for all δ terms, then Equation (21) equals just the classical mechanical action S_c. Notice that S_c contains the only classical variable $x_c(t)$. On the other hand, the quantum action $S[x(t)]$ is composed of two parts. They are the classical action S_c and the second quantum fluctuation's term in Equation (22),

$$S[x(t)] = S_C[x_C] + \int_{ta}^{tb} dt[a\dot{\delta}^2 + b\delta\dot{\delta} + c\dot{\delta}^2]. \tag{22}$$

Thus, the kernel $K(b,a)$, which is calculated in $[a,b]$, can be written as

$$K(b,a) = \int_a^b e^{(i/\hbar)Sc[b,a]} \cdot \exp\left[\frac{i}{\hbar}\int_{ta}^{tb}[a\dot{\delta}^2 + b\delta\dot{\delta} + c\delta^2]dt\right] Dx(t). \tag{23}$$

We would like to give an explicit $S[x(t)]$ and kernel of free particle. So the classical action S_c are described as

$$L = m\frac{\dot{x}^2}{2} \text{ and } S_C[b,a] = \frac{m}{2}\frac{(x_b - x_a)}{t_b - t_a}. \tag{24}$$

Thus, the kernel of Equation (24) of the free particle is given as

$$K(b,a) = \left[\frac{2\pi i\hbar(t_b - t_a)}{m}\right]^{-1/2} \exp\left[\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right]. \tag{25-1}$$

Finally the existence probability of free polaritons at point b, $P(b)dx$, becomes

$$P(b)dx = \frac{m}{2\pi\hbar(t_b - t_a)}dx \propto |K(b,a)|^2. \tag{25-2}$$

Moreover, the wave function of Schrödinger equation $\Psi(b)$ is expressed by the kernel $K(b,a)$, and then we have a simple relation,

$$\psi(b) = \int_{-\infty}^{\infty} K(b,a)\psi(a)\,dx_a, \quad b \equiv (x_b,t_b),\ a \equiv (x_a,t_a). \tag{26}$$

The quantum-polarized waves, which are composed of many photons (there are massive photons), are considered as assembles of harmonic oscillators. The Lagrangian of harmonic oscillator, which means vibration of polariton's field, is given as

$$L = \frac{m\dot{x}^2}{2} - \frac{m\omega^2}{2}x^2. \tag{27}$$

Then the kernel is calculated by the same method as the free particle:

$$K = F(T) \cdot \exp\left[\frac{i}{\hbar} S_C\right], \tag{28}$$

$$S_C = \frac{im\omega}{2\hbar \sin \omega T}\left[(x_a^2 + x_b^2)\cos \omega T - 2x_a x_b\right], \tag{29}$$

$$F(T) = \left(\frac{m\omega}{2\pi i\hbar \sin \omega T}\right)^{1/2}. \tag{30}$$

We give some comments on the calculation of path integral. The all paths (branches) of particle are divided into N divisions so as to obtain the kernel of the propagating particle from point a to point b. The kernel means that we find out a particle at an initial point a, and then it goes to the point a to point x_1. Then it goes ahead from x_1 to x_2. Finally the particle from x_{N-1} arrives at an endpoint b. Therefore, the final kernel $K(a, b)$ is given by multi integrals and product of infinitesimal kernels $K(i+1, i)$, $i = a, 1, 2, b$.

$$K(b, a) = \int \cdots \int dx_1 \cdots dx_{N-1} K(b, N-1) \cdots K(i+1, i) \cdots K(1, a). \tag{31}$$

When the particles to go ahead from (x_i, t) to $(x_{i+1}, t+\varepsilon)$ during an infinitesimal time interval ε, an explicit expression of Equation (31) is

$$K(i+1, i) \equiv \langle i+1 \mid i \rangle = \exp\left[\frac{i\varepsilon}{\hbar} L\left(\frac{x_{i+1} - x_i}{\varepsilon}, \frac{x_{i+1} + x_i}{2}, \frac{t_{i+1} + t_i}{2}\right)\right], \quad L : Lagragian. \tag{32}$$

The second term of kernel $K(i+1, i)$ corresponds to an expression of an inner product using Dirac bra vector $\langle i+1|$ and ket vector $|i\rangle$. Moreover, we should notice that the inner product $\langle B|A\rangle$ contains a time development operator, U-hat,

$$\begin{aligned} &\langle B \mid A \rangle \equiv \langle x_B | \hat{U}(t_B, t_A) | x_A \rangle, \quad \hat{U}(t_B, t_A) \equiv \exp\left\{-i\hat{H}(t_B - t_A)/\hbar\right\}, \\ &B = (x_B, t_B), \quad A = (x_A, t_A). \end{aligned} \tag{33}$$

And the above H-hat is Hamiltonian of Schrödinger equation. The motion of particle from point a to point b reduces to the Dirac bra & ket vector description,

$$\begin{aligned} K(b, a) &= \int \cdots \int dx_1 \cdots dx_{N-1} \langle b \mid N-1 \rangle \cdots \langle i+1 \mid i \rangle \cdots \langle 1 \mid a \rangle \\ &\equiv \int \cdots \int dx_1 \cdots dx_{N-1} \phi[x(t)] \\ &\equiv \lim_{\varepsilon \to 0} \int \cdots \int dx_1 \cdots dx_{N-1} \prod_{i=1}^{N-1} \langle i+1 \mid i \rangle. \end{aligned} \tag{34}$$

Equation (34) mentions to take inter product between the $(i+1)$-th bra and the (i)-th ket vectors and to perform integration over all variables x_i.

3. **Description of Quantum Bifurcation and Diffraction.** We would like to discuss a relationship between path integral and bifurcation diagram in this section. And we apply the path integral to descriptions of polariton's motion on a slit and on Amida lottery. The path integral is another expression of quantum mechanics, and it is perfectly equivalent of Schrödinger equation. According to path integral, the probability $P(a, b)$ is proportion to the absolute square of kernel $K(b, a)$, i.e., $P(b, a) \propto |K(b, a)|^2$. Therefore, the final amplitude $K(b, a)$ is the sum of contribution of each path $\phi[x(t)]$,

$$K(b, a) = \sum\nolimits_{over\ all\ path} \phi\,[x(t)]. \tag{35}$$

The weight of each path is proportional to an exponential of the action S:

$$\phi\,[x(t)] = const. \times \exp\left(\frac{i}{\hbar} S[x(t)]\right). \tag{36-1}$$

At first we consider a bifurcation diagram of a single-step slit (Figure 5).

$$S[x(t)] = \int L(x, \dot{x}, t)\, dt = \int (T - V) dt. \tag{36-2}$$

A particle goes through a hole $G1$ of slit A, and then it experiences the bifurcation by slit B. Finally this particle reaches from the point A to the point B (Figure 5). As shown in Equation (34), the path is written as

$$\phi\,[x(t)] = \lim_{\varepsilon \to 0} \prod_{i=0}^{N-1} K(i+1, i). \tag{37}$$

We would like to show one example of diffraction in the point $x + \alpha_c$ at time T. When a free particle goes ahead from the point x to $x + \alpha_c$, it is diffracted in the point $x + \alpha_c$ by a slit. After that diffraction, the particle arrives at a point (x_2, t_2) on the screen. The probability amplitude $\phi[x(t)]$ with the diffraction becomes

$$\phi[x(t)] = \int_{-b}^{b} d\alpha \, \langle x_2, t_2 \mid x_1 + \alpha, T \rangle \, \langle x_1 + \alpha, T \mid x_1, t_1 \rangle \,, \tag{38}$$

at the point $(x + \alpha_c, T)$. Note that the range of that integral is limited by an interval $[-b, b]$, which is a size of the hole of slit (not infinite). If we assume a Gaussian slit of the width $2b$ whose shape is described by $\exp[-\alpha^2/2b]$, then we can perform integration of Equation (38). The result of probability amplitude is given by Equation (39), since the particle goes through either hole G2-a or G2-b: that result is shown as

$$\begin{aligned} \phi[x(t)] =& \sqrt{\frac{m}{2\pi i\hbar}} \left[T\tau \left(\frac{1}{T} + \frac{1}{\tau} + \frac{i\hbar}{b^2 m} \right) \right]^{-1/2} \\ & \times \exp \left[\frac{im}{2\hbar} \left(\frac{x^2}{\tau} + \frac{x_1^2}{T} \right) - \frac{(im/\hbar)^2}{4(im/2\hbar)^2} \cdot W \right] \end{aligned} \tag{39}$$

where W means $W = \frac{-x/\tau + x_1/T}{1/\tau + 1/T + (i\hbar/b^2 m)}$, $\tau = t_2 - T$, $x = x_2 - x_1$.

Finally, the total wave function of Figure 5 becomes a summation of both paths, $G1 \to G2a \to B$ and $G1 \to G2b \to B$. If there is not an interaction at point $G1$ on slit A and a particle (polariton) freely goes through the slit $G1$, the particle obeys complete condition at point $G1$,

$$\int dx_{G1} \, |G1\rangle \, \langle G1| = 1. \tag{40}$$

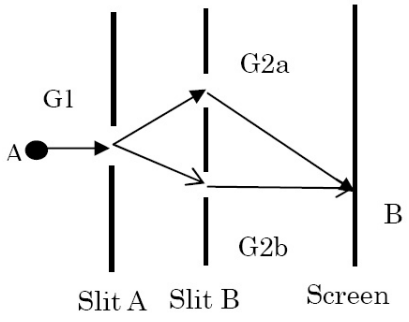


FIGURE 5. Interference of a single-step slit

However, if both gates $G2a$ and $G2b$ are simultaneously open, the probability amplitude of kernel is calculated by Equation (34). Thus the total result of amplitude is given as

$$\begin{aligned}\phi_{all}(x) = &\iint dx_{G1}dx_{G2a}\,\langle B \mid G2a\rangle\,\langle G2a \mid G1\rangle\,\langle G1 \mid A\rangle \\ &+ \iint dx_{G1}dx_{G2b}\,\langle B \mid G2b\rangle\,\langle G2b \mid G1\rangle\,\langle G1 \mid A\rangle\,.\end{aligned} \tag{41}$$

Then $\langle G_1 \mid A\rangle \equiv \langle G_1|\,\hat{U}(t_{G1}, t_0)\,|A_0(t_0)\rangle$, $\hat{U}(t-t_0) = \exp(-i\hat{H}(t-t_0)/\hbar)$.

Notice that those ket vectors $|\,\rangle$ in Equation (41) are not a constant vector, but they contain the time development factors which are related to Hamiltonian of Schrödinger equation. If a particle has no interaction with all slits, then Equation (41) simply reduces to free particle's (free polariton) expression from the point A to the B,

$$\begin{aligned}&\phi_{free}(x_{B,A}) = \langle B(t) \mid A(t_0)\rangle = K(B,A) = \prod_A^B \langle j+1 \mid j\rangle \\ &\text{from point } A \to \text{point } B\end{aligned} \tag{42}$$

If a single slit is set in the point c and the particle is diffracted at that point c $(A < c < B)$, then a trace of particle has following expression:

$$\begin{aligned}\phi_{dif}(x_{B,A}) &= \int \langle B \mid c\rangle\,\langle c \mid A\rangle\,dx_C = \int K(B,c)K(c,A)dx_C \\ &= \int \prod_C^B \langle j+1 \mid j\rangle \cdot \prod_A^C \langle k+1 \mid k\rangle dx_C = A \to (c) \to B.\end{aligned} \tag{43}$$

An Amida lottery is discussed as an example of complex bifurcations and that lottery is a kind of multi-slit (Figure 6). Therefore, Japanese Amida lottery is commonly regarded as one of the examples of classical probability problems. To translate the classical lottery into quantum one, we apply the path integral for classical Amida lottery and introduce quantum interferences into classical Amida lottery. Therefore, those processes are a kind of quantization of Amida lottery. As represented in Figure 6, the photon is diffracted at those following points, $\{G11, G12, G21, G22, G23, G24, G25, G31, G32, G33\}$. This quantum Amida lottery has a lot of paths so as to go ahead from area A to area B, because of sum for all possible paths.

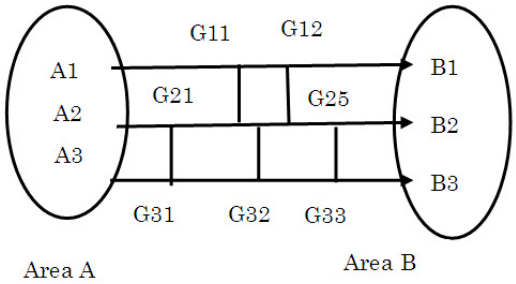


FIGURE 6. Quantum Amida lottery circuit

For example, the amplitude of the paths from area A to $B1$ is given by sum of three paths, i.e., $A1 \to B1$, $A2 \to B1$, and $A3 \to B1$. On the other hand, the classical Amida lottery teaches us no way from point $A1$ to $B1$, because the point $A1$ can only lead to $B2$: $\{A1 \to G11 \to G22 \to G23 \to G32 \to G33 \to G25 \to B2\}$. However, quantum Amida lottery has the other paths to point $B1$ from $A1$: there are two paths, $A1 \to G11 \to G12 \to B1$, $A1 \to G11 \to G22 \to G23 \to G24 \to G12 \to B1$. We can show three paths to $B1$ from point $A2$: $A2 \to G21 \to G22 \to G23 \to G24 \to G12 \to B1$,

$A2 \to G21 \to G22 \to G11 \to G12 \to B1$, $A2 \to G21 \to G31 \to G32 \to G23 \to G24 \to G12 \to B1$. For the path from $A3$ to $B1$: $A3 \to G31 \to G32 \to G23 \to G24 \to G12 \to B1$, $A3 \to G31 \to G21 \to G22 \to G11 \to G12 \to B2$. All of those paths give us a total probability amplitude to $B1$ from the area A, $\phi[B1, A]$, whose complex paths cause many quantum interferences. Here the total amplitude $\phi[B1, A]$ can be written down by linear combination of those three amplitudes. Therefore, they are superposition of those paths,

$$\phi\left[B1, A\right] = C_{A11}\phi\left[B1, A1\right] + C_{A21}\phi\left[B1, A2\right] + C_{A31}\phi\left[B1, A3\right]. \tag{44}$$

An each term of right side of Equation (44) is given by path integrals. The $\phi[A1 \to B1]$ is

$$\begin{aligned}\phi[B1, A1] = &\int dx_{G11}dx_{G12}\left\langle B1 \mid G12\right\rangle\left\langle G12 \mid G11\right\rangle\left\langle G11 \mid A1\right\rangle + \int dx_{G24}dx_{G12}dx_{G11} \\ &\cdot dx_{G22}dx_{G23}\left\langle B1 \mid G12\right\rangle\left\langle G12 \mid G24\right\rangle\left\langle G24 \mid G23\right\rangle\left\langle G23 \mid G22\right\rangle\left\langle G22 \mid G11\right\rangle\left\langle G11 \mid A1\right\rangle.\end{aligned} \tag{45}$$

For $\phi[B1, A2]$, we obtain the relation:

$$\begin{aligned}\phi\left[B1, A2\right] = &\int dx_{G21}dx_{G22}dx_{23}dx_{24}dx_{12}\left\langle B1 \mid G12\right\rangle\left\langle G12 \mid G24\right\rangle\left\langle G24 \mid G23\right\rangle \\ &\cdot\left\langle G23 \mid G22\right\rangle\left\langle G22 \mid G21\right\rangle\left\langle G21 \mid A2\right\rangle + \int dx_{G21}dx_{G22}dx_{G11}dx_{G12}\left\langle B1 \mid G12\right\rangle\left\langle G12 \mid G11\right\rangle \\ &\cdot\left\langle G11 \mid G22\right\rangle\left\langle G22 \mid G21\right\rangle\left\langle G21 \mid A2\right\rangle + \int dx_{G21}dx_{G24}dx_{G23}dx_{G32}dx_{31}dx_{G12}\left\langle B1 \mid G12\right\rangle \\ &\cdot\left\langle G12 \mid G24\right\rangle\left\langle G24 \mid G23\right\rangle\left\langle G23 \mid G32\right\rangle\left\langle G32 \mid G31\right\rangle\left\langle G31 \mid G21\right\rangle\left\langle G21 \mid A2\right\rangle.\end{aligned} \tag{46}$$

And $\phi[A3 \to B1]$ becomes an expression:

$$\begin{aligned}\phi\left[B1, A3\right] = &\int dx_{G12}dx_{G24}dx_{G23}dx_{G32}dx_{G31}\left\langle B1 \mid G12\right\rangle\left\langle G12 \mid G24\right\rangle\left\langle G24 \mid G23\right\rangle \\ &\cdot\left\langle G23 \mid G32\right\rangle\left\langle G32 \mid G31\right\rangle\left\langle G31 \mid A3\right\rangle + \int dx_{G12}dx_{G11}dx_{G22}dx_{G21}dx_{G31}\left\langle B1 \mid G12\right\rangle \\ &\cdot\left\langle G12 \mid G11\right\rangle\left\langle G11 \mid G22\right\rangle\left\langle G22 \mid G21\right\rangle\left\langle G21 \mid G31\right\rangle\left\langle G31 \mid A3\right\rangle.\end{aligned} \tag{47}$$

We apply the same method to the other paths and full total path, i.e., $\phi[B2, A]$ and $\phi[B3, A]$. So, their descriptions are described as

$$\phi\left[B2, A\right] = C_{A12}\phi\left[B2, A1\right] + C_{A22}\phi\left[B2, A2\right] + C_{A32}\phi\left[B2, A3\right]. \tag{48}$$

$$\phi[B3, A] = C_{A13}\phi[B3, A1] + C_{A23}\phi\left[B3, A2\right] + C_{A33}\phi\left[B3, A3\right]. \tag{49}$$

Finally, the total probability amplitude from area A to area B, $\phi[B, A]$, is a summation of those paths. Its expression,

$$\phi\left[B, A\right] = C_{A1}\phi\left[B1, A\right] + C_{A2}\phi[B2, A] + C_{A3}\phi\left[B3, A\right], \tag{50}$$

is given by substituting above equations, Equation (44), Equation (48) and Equation (49) into Equation (50). To observe a part of interferences, we calculate a probability density of $\phi[A \to B1]$ by Equation (44).

$$\begin{aligned}\rho[B1, A] = &\left|\phi\left[B1, A\right]\right|^2 = \left|C_{A11}\phi\left[B1, A1\right]\right|^2 + \left|C_{A21}\phi\left[B1, A2\right]\right|^2 + \left|C_{A31}\phi\left[B1, A3\right]\right|^2 \\ &+ \left\{C^*_{A11}C_{A21}\phi\left[B1, A1\right]^*\phi\left[B1, A2\right] + C^*_{A21}C_{A31}\phi\left[B1, A2\right]^*\phi\left[B1, A3\right]\right. \\ &\left. + C^*_{A31}C_{A11}\phi\left[B1, A3\right]^*\phi\left[B1, A1\right]\right\} + \{\text{counter terms}\}.\end{aligned} \tag{51}$$

Clearly notice that quantum interferences contain those terms $\{C^*_{A11}C_{A21}\phi[B1, A1]^*\phi[B1, A2] + \cdots\} + \{\text{counter terms}\}$ in Equation (44) and Equation (51). In quantum system, we can find also many interferences in following three terms, $|C_{A11}\phi[A1 \to B1]|^2$, $|C_{A21}\phi[A2 \to B1]|^2$, $|C_{A31}\phi[A3 \to B1]|^2$. Because, for example $\phi[A1 \to B1]$, its path is composed of

the combination of many small paths, as $[A1 \to G11 \to G12 \to B1]$ and $[A1 \to G11 \to G22 \to G23 \to G24 \to G12 \to B1]$. The above those many terms, which vanish in the classical bifurcation problems, represent essential quantum effects and interferences.

Really the classical probability has only one term, $|C_{A21}\phi[B1, A2]|^2$, and there is not any interferences of probability (probability amplitude). So normalization condition in that Amida lottery is Equation (52),

$$\int \phi^*[B, A] \cdot \phi[B, A] dx^1 \cdots dx^k = 1. \tag{52}$$

And its transitional amplitude from state $\phi[B1, A]$ to state $\phi[B2, A]$ is defined by

$$\langle \phi[B2, A] | \, \phi[B1, A] \rangle \equiv \int \phi^*[B2, A] \cdot \phi[B1, A] dx^1 \cdots dx^k, \tag{53}$$

in Equation (53). After all, that above transitional probability density becomes

$$P([B2, A] | \, [B1, A]) dx^1 \cdots dx^k = |\langle \phi[B2, A] | \, \phi[B1, A] \rangle|^2 . \tag{54}$$

We can finally obtain the frameworks of quantum bifurcations and interferences by path integral. This section is discussed problems of the diffraction and bifurcations of both the slit and the Amida lottery. We refer to scattering problems of polariton by various potentials in the following section.

4. **Switch Operators and Circuit.** This section is referred to switch operator, which corresponds to potential (scattering potential) of quantum system. And if we assume switches of circuits and networks as scattering potentials, we can easily express classical circuits (NOT, AND, OR) as quantum ones by path integral.

The particle as photon or polariton goes ahead to point B from point A. And that particle is not diffracted at point c but it is scattered by switch (potential) S at point c. This process is described by the bra and ket expression, and then kernel $K(B, A)$ becomes

$$\phi_C[B, A] = K_C(B.A) = \langle B | \, \hat{S}_C \, | A \rangle = \int \langle B | \, c \rangle \, S(c) \, \langle c \, | A \rangle \, dx_C = A \to \boxed{S} \to B. \tag{55}$$

Notice difference between Equation (55) and Equation (43). Equation (55) includes the scattering process by switch potential S at point c, and on the other hand, Equation (43) means the diffraction process at point c. Moreover, Equation (42) simply expresses a free particle having no diffraction process and no scattering potential. Therefore, we show typical three classical circuits which are called as AND-circuit, OR-circuit and NOT-circuit (Figures 7-9).

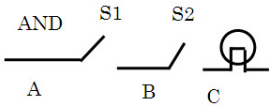


FIGURE 7. AND-circuit

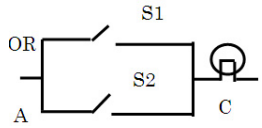


FIGURE 8. OR-circuit

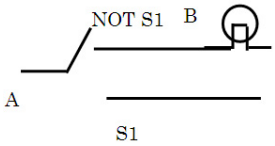


FIGURE 9. NOT-circuit

To obtain quantum description, we apply both rules of Equation (54) and Equation (43) to those circuits.

The AND-circuit can change into quantum one, q-AND, whose schema is simply drawn: the particle goes ahead from point A to scattering center S_1, and then it goes to point B. And after scattered by potential S_2, it arrives at final destination, point C.

$$[A\rightarrow \ (S_1) \ \rightarrow B \ \rightarrow \ (S_2) \ \rightarrow C\,]$$

So we can obtain following expressions of quantum circuit by Figure 7:

$$\phi_{AND}\,[B,A] = \int K_2(c,b)K_1(b,a)\,dx_B = \int \langle c|\,\hat{S}_2\,|b\rangle\,\langle b|\,\hat{S}_1\,|a\rangle\,dx_B. \tag{56-1}$$

$$\because K_2(c,b) = \int \langle c|\ \beta\rangle\,S_2(\beta)\,\langle\beta|\ b\rangle\ dx_\beta = \int K(c,\beta)S_2(\beta)K(\beta,a)\,dx_\beta; \text{for } S_2 \tag{56-2}$$

$$\because K_1(b,a) = \int \langle b|\ \alpha\rangle\,S_1(\alpha)\,\langle\alpha|\ a\rangle\ dx_\alpha = \int K(b,\alpha)S_1(\alpha)K(\alpha,a)\,dx_\alpha;\ \text{for } S_1 \tag{56-3}$$

Above three equations do not correspond to the expressions of classical AND but they are quantum AND-circuit. We would like to call q-AND. The rule of path integral says that an amplitude of different paths works as the additive, and so we can perform superposition of each path (linear combination). Therefore, we apply that rule to classical OR-circuit, which has two parallel switches. Therefore, we can define quantum NOT-circuit,

$$\begin{aligned}\phi_{OR}\,[C,A] &= \langle c|\,S_1\,|a\rangle + \langle c|\,S_2\,|a\rangle \\ &= \int \langle c\mid\alpha\rangle S_1(\alpha)\,\langle\alpha\mid a\rangle\,dx_\alpha + \int \langle c\mid\beta\rangle S_2(\beta)\,\langle\beta\mid a\rangle\,dx_\beta.\end{aligned} \tag{57}$$

The OR diagram becomes $[A\rightarrow\ (\,(S_1)\ \text{OR}\ (S_2)\,)\ \rightarrow C]$. The NOT-circuit is described by a following relation,

$$\phi_{NOT}\,[B,A] = \langle b|\,1-\hat{S}_1\,|a\rangle = \int \langle b\mid\alpha\rangle\,\langle\alpha\mid a\rangle - \langle b\mid\alpha\rangle\,S_1(\alpha)\,\langle\alpha\mid a\rangle\,d\alpha, \tag{58}$$

whose diagram is $[A\rightarrow (1\text{-}S_1)\ \rightarrow C]$. If three logical gates are combined with each other, for example, NOT, AND & OR circuits, we can make up various quantum circuits and calculate their probability amplitude. Those switch's operators are quite different from common classical switches. Because the classical switches are always expressed by c-number, quantum switches take q-number. Those three circuits belong to quantum circuits. An example of combined circuits is showed in diagrams of Figure 10.

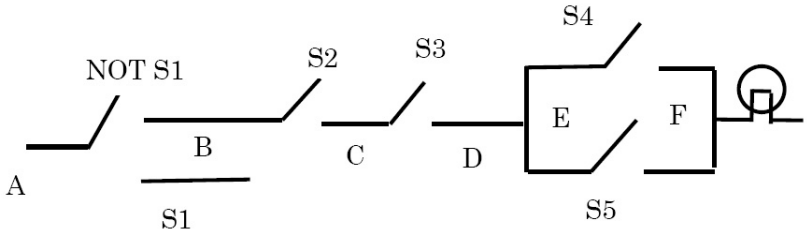


FIGURE 10. Complex circuit (NOT, AND & OR Circuit ϕ_{NAO})

Mathematical representation of above figure is given by using multiple integral:

$$\begin{aligned}\phi_{NAO}[F,A] = &\int\cdots\int dx_\delta dx_e dx_d dx_\gamma dx_C dx_\beta dx_b dx_\alpha [K(f,\delta)S_4(\delta)K(\delta,e)K(e,d)\\ &\times K(d,\gamma)S_3(\gamma)K(\gamma,c))K(c,\beta)S_2(\beta)K(\beta,b)K(b,\alpha)(1-S_1(\alpha))K(\alpha,a)]\\ &+\int\cdots\int dx_\varepsilon dx_e dx_d dx_\gamma dx_C dx_\beta dx_b dx_\alpha [K(f,\varepsilon)S_5(\varepsilon)K(\varepsilon,e)K(e,d)K(d,\gamma)S_3(\gamma)\\ &\times K(\gamma,c))K(c,\beta)S_2(\beta)K(\beta,b)K(b,\alpha)(1-S_1(\alpha))K(\alpha,a)].\end{aligned} \tag{59}$$

Here if propagators (kernels) cause diffractions at points B, C, D, E, then we should perform integration over the slit width. On the other hand, if switch operator S is regarded as a kind of scattering potential, then the range of integral becomes over an infinite range. According to quantum mechanics, physical amount should be described by function of differential operator and time as Hamiltonian: switch operator should be described as

$$\hat{S}_j \equiv \hat{S}_j(\hat{x},\hat{p},t) = S_j(x,-i\hbar\nabla,t). \tag{60}$$

We, as you know, can freely make up an arbitrary circuit by combining those three gates., i.e., those elements are q-AND, q-NOT, and q-OR.

We would like to generalize those quantum gates to m number switch's functions $F_j(S_1,S_2,\cdots,S_N)$, $j=1$ to m, whose variables are composed of N number's switch operators, where F_j means the arbitrary operator's function of N number's switch. Notice that each switch S is an operator and so each F_j is composed of various switch operators. Thus, we can obtain a generalized description of q-AND switches in this case:

$$\phi_{AND}[C,A] = \int dx_{b1}\cdots dx_{bm}\langle c\mid b_m\rangle\langle b_m\mid \hat{F}_m\Big|\, b_m\rangle\cdots\langle b_1\mid \hat{F}_1(\hat{S}_1\cdots\hat{S}_N)\Big|\, a\rangle. \tag{61}$$

We can regard that a circuit has N number's scattering potentials when there are N number switches in its circuit. The rule of switch operator function $F_j(S)$ is generalized as

$$\hat{F}_j \to \int d\chi\cdot|\chi\rangle F_j(S_1(\chi),\cdots)\langle\chi| = \int d\chi\cdot K(\ \ ,\chi)F_j(S_1(\chi),\cdots)K(\chi,\ \ ). \tag{62}$$

When N number's switches are connected in parallel, we have a generalized q-OR

$$\begin{aligned}\phi_{OR}[B,A] &= \sum\nolimits_j^m \langle b\mid \hat{F}_j(\hat{S}_1,\cdots\hat{S}_N)\mid a\rangle\\ &= \sum\nolimits_j^m \int d\chi\langle b\mid\chi\rangle F_j(S_1(\chi),\cdots S_N(\chi))\langle\chi\mid a\rangle\\ &= \sum\nolimits_j^m \int d\chi K(b,\chi)F_j(S_1(\chi),\cdots S_N(\chi))K(\chi,a).\end{aligned} \tag{63}$$

Moreover, we give an expression of a multiple q-NOT,

$$\phi_{AND}[C,A] = \int dx_{b1}\cdots dx_{bm}\langle c\mid b_m\rangle\langle b_m\mid \hat{F}_m(\hat{\bar{S}}_1\cdots\hat{\bar{S}}_N)\Big|\, b_m\rangle\cdots\langle b_1\mid \hat{F}_1(\hat{\bar{S}}_1\cdots\hat{\bar{S}}_N)\Big|\, a\rangle.$$
$$\because\ \hat{\bar{S}}_j = 1-\hat{S}_j \quad j=1,\cdots,m \tag{64}$$

Thus the logical switch can be represented by using kernels $K(B,A)$, and so we need perform an integration at each switch points (scattering potential S). And those procedure and consideration naturally lead us to similarity of perturbation methods.

5. **Application for Perturbation and Scattering of Switch Potentials.** Exactly speaking the massive photon (polariton) is governed by Proca equation. We can reduce Proca equation to quaternary Schrödinger equation [1]. We can apply quaternary Schrödinger equation to many biological problems since the motion of polariton on neurons is much slower than the velocity of light. The quaternary Schrödinger equation has been described as

$$\begin{aligned} i\hbar\frac{\partial\phi^0}{\partial t} &= \left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\right]\phi^0 \\ i\hbar\frac{\partial\phi^a}{\partial t} &= \left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}\right]\phi^a \qquad (65) \\ \because\ \phi^\mu(x) &= (\phi^0(\mathbf{x},t), \phi^a(\mathbf{x},t)), \quad a = 1,2,3. \end{aligned}$$

$$A^\mu(\mathbf{x},t) = \phi^\mu(\mathbf{x},t)\cdot\exp\left(-\frac{i}{\hbar}mc^2t\right), \qquad (66)$$

by using quaternary vector potential, A^μ or ϕ^μ. Therefore, the quaternary potential A^μ represents an total electromagnetic field of polariton (massive photon). On the other hand, the ϕ^μ means kinetic parts of the total field A^μ, and the exponential function of Equation (66) represents longitudinal element of polariton. So the ϕ^0 is scalar potential, and each ϕ^a $(a = 1,2,3)$ is called vector potential of polariton without rest mass. The rest mass limits the range of an existence of polariton. Moreover, we can reduce the quaternary Schrödinger equation to one component (scalar potential ϕ^0) of Schrödinger equation [1]. If a change of vector potential $\mathbf{A}$ is so slow or so small, the following derivative of vector potential $\mathbf{A}$ is nearly equal to zero.

$$\begin{aligned} \mathbf{B}(x,t) &= rot\mathbf{A}(x,t) \\ \mathbf{E}(x,t) &= -grad\mathbf{A}^0 - \frac{1}{c}\frac{\partial\mathbf{A}(x,t)}{\partial t} \end{aligned} \qquad (67)$$

From Equations (65)-(67), the kinetic part of polariton obeys Schrödinger equation of ϕ^0. Then the residual terms become only an electric field as shown in Equation (68), and quaternary Schrödinger equation has only one component ϕ^0 of potariton's vector potential.

$$\mathbf{E}(x,t) \approx -grad\phi^0. \qquad (68)$$

Considering from Equations (61)-(64), we regard various switch operator's function as a kind of potential. So we add up those switch operators to the potential term of Hamiltonian, and finally we have the following form,

$$i\hbar\frac{\partial\phi^0}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}(\alpha,\beta\cdots;t) + \hat{F}(\hat{S}_1,\cdots,\hat{S}_N)\right]\phi^0. \qquad (69)$$

Applying the ordinary perturbation method to Equation (69), the lowest perturbation's expression with potential term V is given. Comparing the results of Equations (55)-(58) with perturbation method of quantum mechanics, we can find easily that those expressions of Equations (55)-(58) are much similar to the first order and the second order term of perturbation method. Thus, the second lowest amplitude of perturbation is described as kernel's expression. As the q-AND circuit has two switch potential terms $S_1(\alpha)$ and

$S_2(\beta)$, the expression of perturbation is given in Equation (70) and Figure 11.

$$\begin{aligned}\phi_{2nd}[C,B,A] &= \left(-\frac{i}{\hbar}\right)^2 \int dx_\beta dx_\alpha \langle c \mid \beta\rangle S_2(\beta) \langle \beta \mid \alpha\rangle S_1(\alpha) \langle \alpha \mid a\rangle \\ &= \left(-\frac{i}{\hbar}\right)^2 \iint dx_\beta dx_\alpha K(c,\beta) S_2(\beta) K(\beta,\alpha) S_1(\alpha) K(\alpha,a).\end{aligned} \tag{70}$$

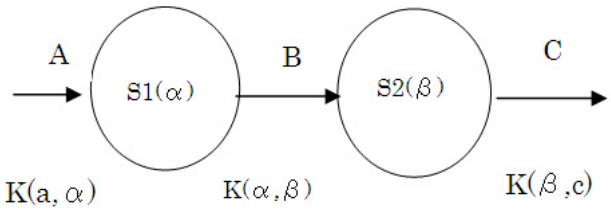


FIGURE 11. Perturbation for second order expansion and q-AND circuit

We take same procedure for q-OR and q-NOT circuits in order to make up perturbation method of propagation for a particle, polariton. According to the diagrams (Figures 12 and 13), the q-OR circuit corresponds to the first ordered perturbation of two potentials connected in parallel. Equation (55) is similar to the first ordered process of perturbation. We know that the scattering process at point C is given as

$$\begin{aligned}\phi_C[B,A] = K_C(B.A) &= \int K(b,c)S(c)K(c,a)dx_C = \int \langle b \mid c\rangle S(c) \langle c \mid a\rangle \, dx_C. \\ &= A \rightarrow \boxed{S_C} \rightarrow B\end{aligned} \tag{71}$$

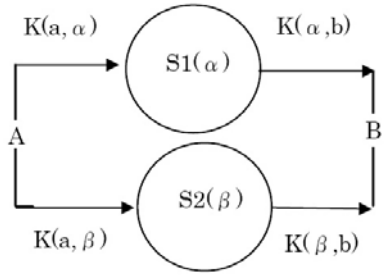


FIGURE 12. Perturbation of q-OR circuit

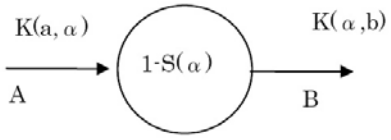


FIGURE 13. Perturbation q-NOT

Applying Equation (71) for both circuits, q-OR and q-NOT, we can easily address the first order expressions of perturbation.

Figure 12 shows perturbation for first order of two parallel potentials, and we should notice that the point A or B is not diffraction's center but port of wave function or propagator. The first perturbation of q-OR becomes, in Figure 13,

$$\begin{aligned}\phi_{1st}[B,A] &= \left(-\frac{i}{\hbar}\right)\left[\langle c \mid S_1(\alpha) \mid a\rangle + \langle c \mid S_2(\beta) \mid a\rangle\right] \\ &= \left(-\frac{i}{\hbar}\right)\int dx_\alpha K(c,\alpha)S_1(\alpha)K(\alpha,a) \; + \left(-\frac{i}{\hbar}\right)\int dx_\beta K(c,\beta)S_2(\beta)K(\beta,a).\end{aligned} \tag{72}$$

And we can know the first ordered amplitude of the switch operator q-NOT,

$$\phi_{1st}[B,A] = \left(-\frac{i}{\hbar}\right)\langle c|\,(1-S(\alpha))|\,a\rangle = \left(-\frac{i}{\hbar}\right)\int dx_\alpha K(c,\alpha)(1-S(\alpha))K(\alpha,a). \quad (73)$$

The q-NOT circuit contains only one scattering center, which is a potential (1-S). Therefore, the q-NOT has the first order perturbation as well as the q-OR circuit. According to perturbation method, we find that the q-AND is the second ordered switch system and that both q-NOT and q-OR mean the first ordered switch system. Iterating those procedures, we can easily obtain the higher ordered perturbation expansions. That perturbation series is given as

$$\begin{aligned} K_T[B,A] = K(B,A) &+ \left(\frac{-i}{\hbar}\right)\int d\alpha\cdot K(B,\alpha)S_1(\alpha)K(\alpha,A) \\ &+ \left(\frac{-i}{\hbar}\right)^2\int d\alpha\, d\beta\cdot K(B,\beta)\times S_2(\beta)K(\beta,\alpha)S_1(\alpha)K(\alpha,A) \\ &+ \cdots\left(\frac{-i}{\hbar}\right)^m\int\cdots\int d\alpha\, d\beta\cdots K(B,L)S_L(L)\cdots K(\alpha,a)\cdots . \end{aligned} \quad (74)$$

Thus, the complex form of kernel, which is propagator or Green's function $K_T[B,A]$, expresses the higher multiple interactions or multi-scattering processes. We find that the perturbation of $K_T[B,A]$ becomes an infinite series of set of $[K(y+1,y)S(y)K(y,y-1)]$.

We would like to apply those rules to constructing a neural network system. The synapses of Figure 14 are looked upon as switch's operators or scattering potentials. Therefore, we can rewrite Figure 11 as shown in Figure 14.

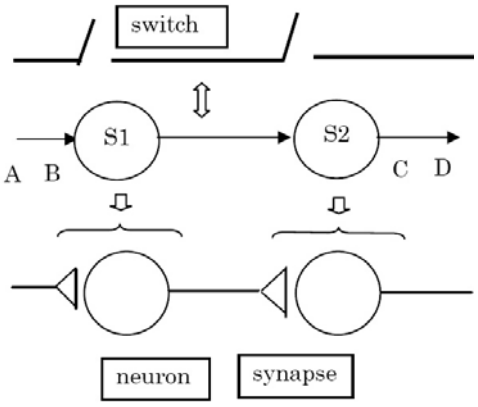


FIGURE 14. Similarity of models

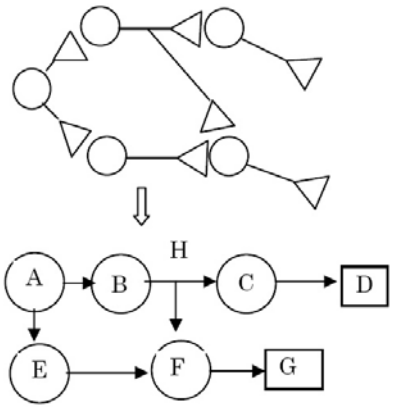


FIGURE 15. Quantum neural network

Figure 14 shows the similarity of the three models, and we can describe the propagation of polariton (quantized polarization wave) from one neuron to another neuron through synaptic junction (synapse). If those above neuron-synapse model does not have any diffractions of polaritons at any points and synaptic junctions are expressed as some potentials, the neuron-synapse model enables us to calculate each propagator and total kernel $K_T[D,A]$, (Fugure 14). That total propagator of polariton is directly given by following expressions. Here is a kernel of Figure 14.

$$K_T[D,A] = \int d\eta\, d\xi\, dB\, dC\; K(D,.C)K(C,\xi)S_2(\xi)K(\xi,\eta)S_1(\eta)K(\eta,B)K(B.A). \quad (75)$$

$$K[b,a] = \left[\frac{2\pi i\hbar(t_b-t_a)}{m}\right]^{-1/2}\exp\left[\frac{im(x_b-x_a)^2}{2\hbar(t_b-t_a)}\right] = \langle b\mid a\rangle\,. \quad (76)$$

Here, $K[b,a]$ means that a free particle goes to point b from point a. The structure of both switch operators S1 and S2 is expressed at functions of each coordinate point (x,y). If we do not have any diffraction's points in both intervals $[A,\eta]$ and $[\xi,D]$ and the particles are perfectly propagating freely, then both integrals dB and dC become equal to 1. Thus, we can remove the integrals of dB and dC from Equation (75). If a neural network is composed of some neurons as shown in Figure 15, then the probability amplitude can be calculated by above calculation procedure. For example, probability amplitude of neuron D is given as

$$\phi_A[D] = \int dCdHdBdA \cdot K(D,C)S_C(C)K(C,H)K(H,B)S_B(B)K(B.A)S_A(A)f(A). \tag{77}$$

The function $f(A)$ of Equation (77) means an arbitrary wave function. And a free particle occurs the diffraction at point H and is scattered both points B and C (Figure 15). For neuron G, we obtain the probability amplitude (propagator):

$$\begin{aligned}\phi_A[G] = &\int dFdHdBdA \cdot K(G,F)S_F(F)K(F,H)K(H,B)S_B(B)K(B.A)S_A(A)f(A) \\ &+ \int dFdEdA \cdot K(G,F)S_F(F)K(F,E)S_E(E)K(E,A)S_A(A)f(A).\end{aligned} \tag{78}$$

Therefore, those two wave functions, $\phi_A[D]$ and $\phi_A[G]$ show that an initial wave function $f(A)$ will arrive at two endpoints D and G, after $f(A)$ was divided into two waves at point A. The $f(A)$ is scattered at many points, A,B,F,E and diffracted at points H,E, by some potentials.

Japanese Amida lottery, which is bifurcation's problem, has many diffraction points as multi-slit. However, Amida lottery does not have any switch's potentials S. On the other hand, quantum circuits and neural networks include both switch's potentials and diffraction's points in their systems.

6. **Summary.** We would like to construct mathematical tools for quantized circuits, neural network and Amida lottery so as to translate classical pictures into quantum ones. Notice that the kernel $K[b,a]$ and inner-product $\langle B|A\rangle$ are not ordinary wave functions but they describe the time development of propagation satisfying Schrödinger equation. They truly express the propagating motion of a particle from point (A,t_A) to point (B,t_B). Thus, an expression of path integral corresponds to dynamics of particle as well as Nowtonian second law of motion. So we would like to summarize important descriptions of the particle's propagation (motion of polariton). If path integral is applied to classical neural networks, then their networks are directly quantized and come to contain various quantum effects in their systems, i.e., for example, tunnel effects, fluctuations and interferences.

1. Free propagation of particle: point A $\to$ point B.

$$K[B,A] = \langle B \mid A\rangle = K(b,a) = \left[\frac{2\pi i\hbar(t_b - t_a)}{m}\right]^{-1/2} \exp\left[\frac{im(x_b - x_a)^2}{2\hbar(t_b - t_a)}\right]. \tag{79}$$

2. Dividing into two parts: two paths are $A \to B$ and $B \to C$. Particle is free propagation. B: relay point or diffraction point.

$$K_B[C,A] = \langle C \mid A\rangle_B = \int dB \cdot K(C,B)K(B,A) = \int dB \langle C \mid B\rangle \langle B \mid A\rangle. \tag{80}$$

3. Diffraction at point B, slit width δ: $A \to (B) \to C$.

$$K[C,A] = \langle C \mid A\rangle = \int dB \cdot K(C,B)K(B,A) = \int_0^\delta dB \langle C \mid B\rangle \langle B \mid A\rangle. \tag{81}$$

4. Various switch's potentials, for example, synaptic junction and scattering potentials for particles, electromagnetic potentials: $A \rightarrow \textcircled{B} \rightarrow C$.

$$\begin{aligned} K_B[C,A] &= \langle C|\ A\rangle_B = \langle C|\,\hat{S}_B\,|A\rangle \\ &= \int dB K(C,B)S(B)K(B,A) = \int dB\cdot\langle C|\ B\rangle\, S(B)\,\langle B|\ A\rangle\,. \end{aligned} \tag{82}$$

5. General switch's potentials: $A \rightarrow \textcircled{f:B} \rightarrow C$, $\textcircled{f:B} = f(S_1(B), S_2(B), \cdots, S_n(B))$.

$$\begin{aligned} K_B[C,A] &= \langle C|\ A\rangle_B = \langle C|\,\hat{F}_B\,|A\rangle \\ &= \int dB\cdot K(C,B)F(S_1(B), S_2(B)\cdots S_n(B))K(B,A) \\ &= \int dB\cdot\langle C\mid B\rangle F(S_1(B), S_2(B)\cdots S_n(B)\,\langle B\mid A\rangle\,. \end{aligned} \tag{83}$$

6. Abbreviation for line and interaction points: $A \rightarrow B \rightarrow \textcircled{C} \rightarrow D \rightarrow E$, then free particle at both points B and D.

$$\begin{aligned} K_C[E,A] &= \langle E|\ A\rangle_C = \langle E|\,\hat{S}_C\,|A\rangle \\ &= \int dB dC dD\cdot K(E,D)K(D,C)S(C))K(C,B)K(B,A) \\ &= \int dC\cdot K(E,C)S(C)K(C,A). \end{aligned} \tag{84}$$

7. Abbreviation for line, interaction and diffraction: $A \rightarrow (B) \rightarrow \textcircled{C} \rightarrow D \rightarrow E$, B: slit or diffraction points. C is scattering point. Notice that we cannot abbreviate dB integral.

$$\begin{aligned} K_C[E,A] &= \langle E|\ A\rangle_C = \langle E|\,\hat{S}_C\,|A\rangle \\ &= \int dB dC dD\cdot K(E,D)K(D,C)S(C))K(C,B)K(B,A) \\ &= \int dB dC\cdot K(E,C)S(C)K(C,B)K(B,A). \end{aligned} \tag{85}$$

8. Propagation and time-development: initial wave function $\phi(A) \rightarrow$ final state B, $\phi_A[B]$.

$$\phi_A[B] = \int dA\cdot K(B,A)\ \phi(A). \tag{86-1}$$

$$K[B,A] = \langle B\mid A\rangle \equiv \langle B;t_B|\ A;t_A\rangle = \langle B|\,\hat{U}(t_B,t_A)\,|A\rangle \tag{86-2}$$

$$\hat{U}(t_B,t_A) = \exp\left(\frac{-i}{\hbar}\hat{H}(t_B-t_A)\right).$$

9. Relationship between an eigenfunction of Schrödinger equation and its propagator.

$$K[B,A] \equiv \sum\nolimits_K \psi^{\mu *}(A)\psi^{\mu}(B)\exp[-i\omega_K^{\mu}(t_B-t_A)]. \tag{87}$$

We would like to show an above relationship between the kernel $K[B,A]$ and eigenfunction of Schrödinger equation. The wave function of S chrödinger equation, whose solution is A^{μ} or ϕ^{μ} (or static approximation of polariton ϕ^0), can be related to the kernel $K[B,A]$.

The general solution of time dependent quaternary Schrödinger equation is represented as

$$\phi^{\mu}[x] = \sum\nolimits_J C_J^{\mu}\psi_J^{\mu}(x)\exp(-i\omega_J^{\mu}t). \tag{88}$$

And the quaternary wave function Ψ_J^{μ}, which is an eigenfunction of stationary state, satisfies Equation (65) or Equation (69). The Ψ_J^{μ} obeys the quaternary Schrödinger equation: that is

$$\begin{aligned}&\left[-\frac{\hbar^2}{2m}\nabla^2 + \hat{V}(\alpha,\beta\cdots;t) + \hat{F}(\hat{S}_1,\cdots,\hat{S}_N)\right]\psi_J^{\mu}(x) = E_J^{\mu}\psi_J^{\mu}(x)\\ &\because\ E = \hbar\omega_J^{\mu}.\end{aligned} \tag{89}$$

The wave function at point (A, t_A) is written as

$$\begin{aligned}&\phi^{\mu}[A;t_A] = \sum\nolimits_J C_J^{\mu}\psi_J^{\mu}(A)\exp(-i\omega_J^{\mu}t_A) \equiv \sum\nolimits_J a_J^{\mu}\psi_J^{\mu}(A)\\ &\therefore\ C_J^{\mu} = a_J^{\mu}\exp\left(i\omega_J^{\mu}t_A\right).\end{aligned} \tag{90}$$

On the other hand, we have a similar expression at point (B, t_B),

$$\phi^{\mu}[B;t_B] = \sum\nolimits_K C_K^{\mu}\psi_K^{\mu}(B)\exp(-i\omega_K^{\mu}t_B) \equiv \sum\nolimits_K a_K^{\mu}\psi_K^{\mu}(B)\exp(-i\omega_K^{\mu}t_B + i\omega_K^{\mu}t_A) \tag{91}$$

where we substituted Equation (88) into C_K^{μ} of Equation (89). The Equation (88) gives us coefficient a_K^{μ}:

$$a_K^{\mu} = \int \psi_K^{\varpi *}(A)\phi^{\mu}[A;t_A]dA. \tag{92}$$

Substituting Equation (90) into (88) and comparing that result with Equation (84), we can obtain an expression of kernel $K[B, A]$.

$$\phi^{\mu}[B;t_B] = \sum\nolimits_K \{\psi_K^{\mu *}(A)\ \psi_K^{\mu}(B)\exp[-i\omega_K^{\mu}(t_B - t_A)]\}\phi^{\mu}[A;t_A]. \tag{93}$$

$$\therefore\ K[B,A] \equiv \sum\nolimits_K \psi^{\mu *}(A)\psi^{\mu}(B)\exp[-i\omega_K^{\mu}(t_B - t_A)]. \tag{94}$$

7. **Conclusion.** We proposed new basic theory and calculation methods for quantum bifurcation, quantum circuits, and neural computer based on path integrals of quantum theory.

At first, we showed that a decision tree can be regarded as a kind of Brownian motion (Markov process), and then the motion was governed with Ito equation (general stochastic equation). And according to Nelson's method (stochastic quantization), the Ito equation finally reached Schrödinger equation. Thus, we knew that problems of classical bifurcation were easily led to Schrödinger equation by considering Nelson's stochastic quantization method. The second example was Japanese Amida lottery, which was a kind of classical bifurcation models because of no interference between each path of lottery. However, we introduced a lot of diffraction points to Amida lottery, and we showed the calculating method of quantum amplitude by path integrals. That path integral was a quantization method of Amida lottery, which contained a lot of diffraction points. If we regarded classical bifurcation points as diffraction points and we summed up the probability amplitudes of all possible paths, we could translate the classical bifurcations into quantum interferences and diffraction's problems of networks.

We discussed the method of quantization of basic circuits as AND, OR and NOT. Those classical circuits did not have any quantum effects. For example, there were quantum effects as the superposition and probability interference. In order to perform quantization of those circuits, we adopted the path integral to above three basic circuits. We assumed that we could regard classical switches as scattering potentials (switch's operators). Therefore,

that was quantization concepts, and those quantized circuits with switch operators corresponded to q-AND, q-NOT, and q-OR circuits. Moreover, we succeeded to show the calculation's methods of complex quantum circuits and neural networks by path integrals. The switch of each circuit was looked upon as switch's potential of Hamiltonian. Thus, Hamiltonian operator H could be described as

$$H = (\text{kinetic energy } T) + (\text{potential energy } V) + (\text{Switch's potential } F(S_1, S_2, S_N)).$$

The Hamiltonian was connected to quaternary Schrödinger equation since the wave function was related to the motion of polariton as massive photon. Exactly speaking, the motion of polariton should be prescribed by Proca equation of relativistic kinematics. However, the Proca equation approached to the quaternary Schrödinger equation when the motion of polariton was much slower than light velocity.

The kernel $K(b, a)$, which was propagator and an expression of the time development of system, was related to an eigenfunction of Schrödinger equation. And we found that the q-OR was similar to the first ordered perturbation of two potential scattering problems. The q-AND was shown to have similarity to the second ordered perturbation of single particle. It is important to notice that the wave function $\phi(x, t)$ was an expression of a situation of wave in the point x at time t, and its expression was static. The kernel $K[B, A]$, however, truly represented the motion of the particle from point (A, t_A) to point (B, t_B), and so its expression was dynamical. Finally, we found that the neuro-synaptic junctions were regarded as a kind of switch's potential, whose concepts led to quantization of neural networks by using path integrals.

We think that quantum interference plays an essential role among many neural networks in our brain. The normal neuron actively utilizes various interferences so as to adjust each neural function through leak polaritons from neural axons and synaptic junctions.

Acknowledgment. The authors also gratefully acknowledge the helpful comments and suggestions of the reviewers, which have improved the presentation.

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