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NEW RESULTS ON THE RATE OF RETURN FOR THE RISK ASSETS HELD INDEFINITELY

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Abstract. *This paper focuses on the related problems of the rate of return for risk assets held indefinitely. Under suitable assumptions, we obtain the theoretical formula of the rate of return for risk assets held indefinitely. Several properties of the acquired theoretical formula are given, and we prove that the theoretical formula obtained can be represented by each-order central moment of the random rate of return. When the random rate of return conforms to even distribution and normal distribution, through the use of the deduced theoretical formula, we acquire the rate of return for risk assets held indefinitely which is directly represented by the distributed parameters. When the short-term rate of return conforms to several familiar probability distributions, we can figure out the corresponding rate of return for risk assets held indefinitely by using numerical integral, we also properly analyze the calculated results.*

Keywords: Risk assets, Short-term rate of return, Long-term rate of return, Average annual rate of return, Rate of return for risk assets held indefinitely

1. **Introduction.** The relationship between short-term rate of return and long-term rate of return is familiar to us, the long-term rate of return is the geometric average of each short-term rate of return and the expected value of short-term rate of return is the arithmetic average of each short-term rate of return. In investment analysis, people are used to measuring the income level with the expected value of short-term rate of return and used to measure the indetermination (the risk) of investment income with the variance of short-term rate of return. However, when investors plan to make a long-term continuous investment, it is blind and fallible for them to estimate the income level and the risk according to the expected value and the variance of short-term rate of return for the risk assets. The short-term rate of return is random, and the rate of return of continuous investment in a certain period is necessarily random too. And there are no corresponding simple theoretical formula that can stand for the relationship between the rate of return and variance of short term and long term.

For example, the expectation of the annual rate of return of a certain asset is 10% and the standard deviation of the rate of return is 0.35, which conforms to the normal distribution. If failing to measure and calculate those data carefully, people will deem that they can obtain approximately 10% of the average annual rate of return through holding this asset over a long term and that only the income level has the great indetermination. The author of this paper once made the random simulation experiment [6] with the Monte Carlo method, and the computing results indicated that the average annual rate of return was only 2.54% that was far below 10% if investors continuously held this asset for thirty years. However, the indetermination of the rate of return lessened a lot. Take another asset for example. The expectation of the annual rate of return is 5% and the standard deviation is 0.35, which conforms to the normal distribution. The computing results indicated that the average annual rate of return was only -2.68% if investors continuously held this asset for twenty years. These results are unexpected because the average annual rate of return should be positive according to the normal judgment. We use the Monte Carlo method to make analog computation to the rate of return model which conforms to the normal distribution and has different expectations and variances, and the results indicate that the average annual rate of return for the risk assets held over a long term has sensitive negative correlation with the variance of the short-term rate of return [7], and the greater the variance of the short-term rate of return is, the more the average annual rate of return decreases compared with the short-term rate of return.

When doing the research on the rate of return for risk assets (stock, enterprise bonds, foreign exchange, etc.), researchers usually make assumption with normal distribution. The method is mainly based on the statistical analysis of the rate of return of these risk assets and the good mathematical properties of normal distribution. In the derivation process of Black-Scholes options pricing formula, the change of stock price follows the geometrical Brown motion model, and the mathematical expression is $\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$, S_t represents stock price, μ is the expected return, σ is the volatility and B_t is the standard Brown motion, $dB_t = \varepsilon\sqrt{dt}$, ε conforms to the standard normal distribution and $\frac{dS_t}{S_t}$ stands for the stock return during the period of time dt in time t. For the stock price that follows the geometrical Brown motion, when the holding period is Δt, the stock return conforms to the normal distribution $N(\mu\Delta t, \sigma\sqrt{\Delta t})$. It can be seen that the geometrical Brown motion model assumes that the expectation of the rate of return of stock $(\mu\Delta t)$ is a linear function of holding period. When the stock price follows the geometrical Brown motion model, no matter how long the holding period Δt is, the rate of return conforms to normal distribution, and the rate of return have nothing to do with the variance of short-term rate of return σ^2, so the average rate of return of stock is determined only by the holding period Δt and the expectation μ of short-term rate of return. Evidently, the properties of the geometrical Brown motion model are inconsistent with the results obtained by using the Monte Carlo method to make analog computation. From this point, when describing the rate of return for different investment period, the geometrical Brown motion model will produce significant deviation.

If investors plan to hold a risk asset for a long term, they should know not only the distribution of short-term rate of return, but also the average annual rate of return and corresponding variance of long-term rate of return. This paper focuses on the research on the relationship between the expectation and variance of short-term rate of return for risk assets and the rate of return for risk assets held for a long term, and especially the theoretical formula and the corresponding properties of the average annual rate of return for risk assets held indefinitely. These problems have not yet been intensively studied in the existing investment principles. Undoubtedly, the average annual rate of return for risk assets held indefinitely has important guiding significance to the investors who hold risk assets for a long term.

In Section 2, we derive the mathematical formula r_∞ of the rate of return for risk assets held indefinitely. In Section 3, we acquire several properties of the formula of the rate of return for risk assets held indefinitely, the relationship between the rate of return r_∞ of risk assets held indefinitely and the short-term rate of return μ of the risk assets are obtained, and r_∞ can be represented by each-order central moment of the random rate of return. In Section 4, for several special probability distributions, we derive some tangible results from the formula of the rate of return for risk assets held indefinitely, that is, using parameters of probability distribution to directly represent r_∞. In Section 5, we make use of some familiar distribution density functions to construct several diverse models of the short-term rate of return. When the short-term rate of return respectively conforms to these distributions, we deduce the numerical computational results of the rate of return for risk assets held indefinitely. The last section is conclusion.

2. **The Formula of the Rate of Return for Risk Assets Held Indefinitely.** Take a certain risk asset K, such as a stock, for example. For the sake of narrative convenience, we call the annual rate of return for the risk asset K, the short-term rate of return. In reality, short term and long term are relative, and the short-term rate of return can also be daily, weekly or monthly rate of return. $x_1, x_2, \ldots, x_n$ is the rate of return for n years running. x_i is a random variable conforms the same distribution density function $p(x)$. Because x_i is the rate of return, it is required that $x_i > -1$, $i = 1, 2, \ldots, n$. The average annual rate of return r_n of the risk asset K held for n years running is as follows:

$$r_n = ((1 + x_1)(1 + x_2) \cdots (1 + x_n))^{1/n} - 1 \tag{1}$$

We hope that, when $n \to \infty$, we can derive the formula of the limiting value. This limiting value is the average annual rate of return for the risk asset K held indefinitely.

Theorem 2.1. *Assume that $p(x)$ is the probability density function of the annual rate of return for the risk asset K. When $x \leq -1$, $p(x) = 0$. The theoretical formula of the average annual rate of return r_∞ of the risk asset K held indefinitely is as follows:*

$$r_\infty = \lim_{N \to \infty} r_N = e^{\int_{-1}^{+\infty} p(x) \ln(1+x) dx} - 1 \tag{2}$$

Proof: Dividing the value interval $(-1, +\infty)$ into m mini-intervals $[z_i, z_i + \Delta z_i]$ and $i = 1, 2, \ldots, m$, we have $z_{i+1} = z_i + \Delta z_i$. Examining the situation in interval $[z_i, z_i + \Delta z_i]$, we make random variable to take sufficient enough values in $(-1, +\infty)$, suppose that there are N values $x_1, x_2, \ldots, x_N$ of short-term rate of return. Then the probability that the values of short-term rate of return fall on $[z_i, z_i + \Delta z_i]$ approximately equals to $p(z_i)\Delta z_i$. Therefore, the number of times n_i that the values of short-term rate of return fall on the $[z_i, z_i + \Delta z_i]$ is around:

$$n_i = N \cdot p(z_i)\Delta z_i$$

In the continued product of the expression (1), all the short-term rates of return that fall on $[z_i, z_i + \Delta z_i]$ are represented by the left endpoint z_i of the interval. Hence, the product of the short-term rate of returns that fall on $[z_i, z_i + \Delta z_i]$ can be approximately represented by the following expression:

$$(1 + z_i)^{n_i} = (1 + z_i)^{N \cdot p(z_i)\Delta z_i}$$

There is:

$$\begin{aligned}
\lim_{N \to \infty} r_N &= \lim_{N \to \infty} \left(\prod_{i=1}^{N} (1 + x_i) \right)^{1/N} - 1 \\
&= \lim_{N \to \infty} \left(\prod_{i=1}^{m} (1 + z_i)^{N \cdot p(z_i)\Delta z_i} \right)^{1/N} - 1
\end{aligned}$$

Denote that

$$S_N = \left(\prod_{i=1}^{m} (1 + z_i)^{N \cdot p(z_i)\Delta z_i} \right)^{1/N}$$

And there is:

$$\begin{aligned}\ln(S_N) &= \frac{1}{N}\sum_{i=1}^{N} N\cdot p(z_i)\Delta z_i \ln(1+z_i)\\ &= \sum_{i=1}^{N} p(z_i)\Delta z_i \ln(1+z_i)\end{aligned} \tag{3}$$

According to the definition of definite integral, the above-mentioned (3) is the expression of integral sum. When m is large enough and the length Δz_i of each mini-interval is small enough, there is:

$$\lim_{N\to\infty} \ln(S_N) = \int_{-1}^{+\infty} p(x)\ln(1+x)dx$$

We can acquire the following expression:

$$r_\infty = \lim_{N\to\infty} r_N = e^{\int_{-1}^{+\infty} p(x)\ln(1+x)dx} - 1$$

As long as the integral $\int_{-1}^{+\infty} p(x)\ln(1+x)dx$ in the above-mentioned expression exists, the above-mentioned expression is the mathematical formula of the average annual rate of return for the risk asset K held indefinitely. The theorem is proved.

Because the function $\ln(1+x)$ is equal to $-\infty$ when $x = -1$, in order to guarantee that the integral $\int_{-1}^{+\infty} p(x)\ln(1+x)dx$ in expression (2) exists, we make an assumption that the following conditions are tenable:

$$\lim_{x\to -1^+} p(x)\ln(1+x) = 0, \qquad \lim_{x\to +\infty} p(x)\ln(1+x) = 0$$

When the analytical solution of the integral $\int_{-1}^{+\infty} p(x)\ln(1+x)dx$ in the expression (2) can be solved, the more concrete mathematical formula of the average annual rate of return for the risk asset K held indefinitely will be obtained. On the contrary, when the analytical solution of the integral $\int_{-1}^{+\infty} p(x)\ln(1+x)dx$ in the expression (2) cannot be solved, we can calculate r_∞ through the use of numerical integral and examine the relationship between r_∞ and the expectation and variance of short-term rate of return.

3. **Several Properties about the Formula of the Rate of Return $\boldsymbol{r_\infty}$.** If the asset K has a fixed short-term rate of return r_0, then it is a risk-free asset. At this time, the probability density function $p(x)$ of annual rate of return is a δ function that satisfies the following expression:

$$\delta(x-r_0) = \begin{cases} 0 & x \neq r_0 \\ +\infty & x = r_0 \end{cases}, \quad \int_{-\infty}^{+\infty} \delta(x-r_0)dx = 1$$

According to the Formula (1), if $x_i = r_0$, $i = 1, 2, \ldots, n$, then there are $r_n = ((1+r_0)^n)^{1/n} - 1 = r_0$ and $r_\infty = r_0$. Here, we will validate the correctness of the Formula (2).

Theorem 3.1. *If asset K is a risk-free asset and the annual rate of return is r_0, then the average annual rate of return for asset K held indefinitely provided by Formula (2) is r_0.*

Proof: In accordance with Formula (2), there is:

$$r_\infty = e^{\int_{-1}^{+\infty} p(x)\ln(1+x)dx} - 1$$

When asset K is risk-free, the probability density function $p(x)$ of the annual rate of return is a δ function, that is, $\delta(x - r_0)$. Then we have:

$$r_\infty = e^{\int_{-1}^{+\infty} \delta(x-r_0)\ln(1+x)dx} - 1 = e^{\ln(1+r_0)} - 1 = r_0$$

The theorem is proved.

As for the common probability density function $p(x)$ that possesses the symmetrically distributed short-term rate of return, we hope to know the relationship between r_∞ and the expectation μ of the annual rate of return. There is the following Theorem 3.2.

Theorem 3.2. *If the annual rate of return ξ of the risk asset K is a random variable which possesses the symmetrically distributed probability density function $p(x)$, the mathematic expectation of ξ is μ, the variance of ξ is $\sigma^2 > 0$, and the n-order central moment of ξ is $\sigma_n = \int\limits_{-\infty}^{+\infty} (x-\mu)^n p(x)dx$, then the average annual rate of return r_∞ of the asset K held indefinitely is inevitably less than the expectation μ of the rate of return of the asset K, and there is:*

$$r_\infty = (1+\mu)e^{-\sum\limits_{n=1}^{\infty}\frac{\sigma_{2n}}{2n(1+\mu)^{2n}}} - 1 \tag{4}$$

Proof: Suppose the symmetric point of the density function $p(x)$ is $x = \mu$, that is, $p(\mu+x) = p(\mu-x)$. Then there is:

$$\begin{aligned} E(\xi) &= \int\limits_{-\infty}^{+\infty} xp(x)dx = \int\limits_{0}^{+\infty} [(\mu+x)p(\mu+x) + (\mu-x)p(\mu-x)]dx \\ &= \int\limits_{0}^{+\infty} 2\mu p(\mu+x)dx = 2\mu \int\limits_{0}^{+\infty} p(\mu+x)dx = \mu \end{aligned}$$

That is to say, the expectation of the annual rate of return ξ of the risk asset K is μ which is the value in the symmetric point of $p(x)$. Besides, there is:

$$\int\limits_{\mu}^{+\infty} p(x)dx = \frac{1}{2}, \quad \int\limits_{-1}^{\mu} p(x)dx = \frac{1}{2}$$

According to Formula (2), the mathematic formula of the average annual rate of return r_∞ of the asset K held indefinitely is as follows:

$$r_\infty = e^{\int_{-1}^{+\infty} p(x)\ln(1+x)dx} - 1$$

Consider the following integral:

$$\begin{aligned} \int\limits_{-1}^{+\infty} p(x)\ln(1+x)dx &= \int\limits_{-1}^{\mu} p(x)\ln(1+x)dx + \int\limits_{\mu}^{+\infty} p(x)\ln(1+x)dx \\ &= \int\limits_{0}^{\mu+1} p(\mu-x)\ln(1+\mu-x)dx + \int\limits_{0}^{+\infty} p(\mu+x)\ln(1+\mu+x)dx \end{aligned}$$

Because of the symmetry, there must be $p(\mu+x) = 0$ when $x > 1+\mu$. Therefore, the above-mentioned integral can be combined, and there is:

$$\begin{aligned} \int\limits_{-1}^{+\infty} p(x)\ln(1+x)dx &= \int\limits_{0}^{\mu+1} [p(\mu-x)\ln(1+\mu-x) + p(\mu+x)\ln(1+\mu+x)]dx \\ &= \int\limits_{0}^{\mu+1} p(\mu+x)[\ln(1+\mu-x) + \ln(1+\mu+x)]dx \end{aligned}$$

Disposing the function $\ln(1+\mu-x)+\ln(1+\mu+x)$ mentioned in the above integral, we get:

$$\begin{aligned}\ln(1+\mu-x)+\ln(1+\mu+x) &= \ln\left((1+\mu)\left(1-\frac{x}{1+\mu}\right)\right)+\ln\left((1+\mu)\left(1+\frac{x}{1+\mu}\right)\right)\\ &= 2\ln(1+\mu)+\ln\left(1-\frac{x}{1+\mu}\right)+\ln\left(1+\frac{x}{1+\mu}\right)\end{aligned}$$

On the basis of the theorem's conditions, we have $\left|\frac{x}{1+\mu}\right|<1$. When $|z|<1$, the following Taylor's expansion is tenable:

$$\ln(1-z)=-\sum_{n=1}^{\infty}\frac{1}{n}z^n$$

We have:

$$\begin{aligned}&2\ln(1+\mu)+\ln\left(1-\frac{x}{1+\mu}\right)+\ln\left(1+\frac{x}{1+\mu}\right)\\ &=2\ln(1+\mu)-\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x}{1+\mu}\right)^n-\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{-x}{1+\mu}\right)^n\\ &=2\ln(1+\mu)-2\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{x}{1+\mu}\right)^{2n}\end{aligned}$$

So we have:

$$\begin{aligned}&\int_0^{\mu+1}p(\mu+x)[\ln(1+\mu-x)+\ln(1+\mu+x)]dx\\ &=\int_0^{\mu+1}p(\mu+x)\left(2\ln(1+\mu)-2\sum_{n=1}^{\infty}\frac{1}{2n}\left(\frac{x}{1+\mu}\right)^{2n}\right)dx\\ &=2\ln(1+\mu)\int_0^{\mu+1}p(\mu+x)dx-2\sum_{n=1}^{\infty}\frac{1}{2n(1+\mu)^{2n}}\int_0^{\mu+1}p(\mu+x)x^{2n}dx\\ &=\ln(1+\mu)-\sum_{n=1}^{\infty}\frac{\sigma_{2n}}{2n(1+\mu)^{2n}}\end{aligned}$$

We can obtained the following formula:

$$\begin{aligned}r_\infty &= e^{\int_{-1}^{+\infty}p(x)\ln(1+x)dx}-1=e^{\ln(1+\mu)-\sum\limits_{n=1}^{\infty}\frac{\sigma_{2n}}{2n(1+\mu)^{2n}}}-1\\ &=(1+\mu)e^{-\sum\limits_{n=1}^{\infty}\frac{\sigma_{2n}}{2n(1+\mu)^{2n}}}-1\end{aligned}$$

We obtain the expression (4). And the above-mentioned expression can be transformed into the following forms:

$$\mu-r_\infty=(1+\mu)\left(1-e^{-\sum\limits_{n=1}^{\infty}\frac{\sigma_{2n}}{2n(1+\mu)^{2n}}}\right)$$

For the common and continuously distributed random rate of return, there is $\sigma_{2n} > 0$. So we have:

$$0 < 1 - e^{-\sum\limits_{n=1}^{\infty} \frac{\sigma_{2n}}{2n(1+\mu)^{2n}}} < 1$$

From the expression of $\mu - r_\infty$, we can get $0 < \mu - r_\infty < 1 + \mu$ and $r_\infty < \mu$. And the theorem is proved.

Due to $\sigma_{2n} > 0$, there is:

$$r_\infty = (1+\mu)e^{-\sum\limits_{n=1}^{\infty} \frac{\sigma_{2n}}{2n(1+\mu)^{2n}}} - 1 < (1+\mu)e^{-\frac{1}{2(1+\mu)^2}\sigma^2} - 1$$

From the above expression, we can obtained the following formula:

$$\mu - r_\infty > (1+\mu)\left(1 - e^{-\frac{1}{2(1+\mu)^2}\sigma^2}\right) \tag{5}$$

The conclusion of Theorem 3.2 indicates that for the asset whose short-term rate of return possesses the symmetrical probability density distribution, if it is held for a long term, the average annual rate of return must be less than the expectation of the annual rate of return. And from the expression (5), it can be easily derived that the greater the variance of the short-term rate of return is, the greater the difference $\mu - r_\infty$. If failing to thoroughly probe, people will easily believe that holding the risk asset for a long term can gain the average annual rate of return that is approximately equal to the short-term rate of return. For numerous financial assets, such as stock, bonds and foreign exchange, their short-term rates of return approximately conform to the symmetrical normal distribution. Hence, it can be asserted that the average annual rate of return obtained by holding these financial assets for a long term are inevitably less than the expectation of the annual rate of return. And when the variance of the short-term rate of return is relatively great (which means that the risk is great), the average annual rate of return obtained by holding these financial assets for a long term will decrease sharply.

As for the common probability density function $p(x)$ without symmetry, the following Theorem 3.3 is tenable.

Theorem 3.3. *If the annual rate of return ξ of the risk asset K is a random variable that possesses a common probability density function $p(x)$, the mathematic expectation of ξ is μ and the variance of ξ is $\sigma^2 > 0$. Suppose $\lim\limits_{x\to -1^+} p(x)\ln(1+x) = 0$, when $x \le -1$ then $p(x) = 0$, the variance σ^2 of ξ is greater than zero. Then there is:*

$$r_\infty = e^{\int_{-1}^{+\infty} p(x)\ln(1+x)dx} - 1 \le \int\limits_{-1}^{+\infty} xp(x)dx = \mu \tag{6}$$

Proof: For $a_i \ge 0$, $i = 1, 2, \ldots, n$, the following well-known inequality is tenable:

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{a_1 + a_2 + \ldots + a_n}{n} \tag{7}$$

In this inequality, the necessary and sufficient condition under which the sign of equality is tenable is that a_i takes the same value. For the sequence of the annual rate of return $x_1, x_2, \ldots, x_n$, the average value of the annual rate of return is $\frac{x_1+x_2+\ldots+x_n}{n}$ and the average annual rate of return of n-year continuous investment is $((1+x_1)(1+x_2)\cdots(1+x_n))^{1/n} - 1$. And there is:

$$r_\infty = \lim_{n\to\infty} ((1+x_1)(1+x_2)\cdots(1+x_n))^{1/n} - 1, \quad \mu = \lim_{n\to\infty} \frac{x_1 + x_2 + \ldots + x_n}{n}$$

Denote $a_1 = 1+x_1$, $a_2 = 1+x_2$, $\ldots$, $a_n = 1+x_n$. Because x_i is the annual rate of return, there must be $1+x_i > 0$, $i = 1, 2, \ldots, n$. Because $\sigma^2 > 0$, so a_i, $i = 1, 2, \ldots, n$ are not the same value. According to the expression (7), there is:

$$((1+x_1)(1+x_2)\cdots(1+x_n))^{1/n} < \frac{(1+x_1)+(1+x_2)+\ldots+(1+x_n)}{n}$$

Expand and simplify the above expression, there is:

$$((1+x_1)(1+x_2)\cdots(1+x_n))^{1/n} - 1 < \frac{x_1+x_2+\ldots+x_n}{n}$$

Solving the limit of both sides of the above expression and relying on the expression (2), we gain:

$$r_\infty = e^{\int_{-1}^{+\infty} p(x)\ln(1+x)dx} - 1 \leq \int_{-1}^{+\infty} xp(x)dx = \mu$$

The theorem is proved.

We guess that when σ^2 is greater than zero, there must be $r_\infty < \mu$. Nonetheless, the proof procedure of Theorem 3.3 indicates that we can only assert $r_\infty \leq \mu$.

Theorem 3.4. *If the annual rate of return ξ of the risk asset K is a random variable which possesses a common probability density function $p(x)$, suppose that when $\xi \leq -1$ or $\xi \geq 2\mu+1$ then there is $p(x) = 0$, the mathematic expectation of ξ is μ, the variance of ξ is $\sigma^2 > 0$, and the n-order central moment of ξ is $\sigma_n = \int\limits_{-\infty}^{+\infty} (x-\mu)^n p(x)dx$, then the formula of the average annual rate of return r_∞ of the asset K held indefinitely is as follows:*

$$r_\infty = (1+\mu)e^{-\sum\limits_{n=1}^{\infty}(-1)^n \frac{\sigma_n}{n(1+\mu)^n}} - 1 \tag{8}$$

Proof: On the basis of the expression (2), the mathematical formula of the average annual rate of return r_∞ of the asset K held indefinitely is as follows:

$$r_\infty = e^{\int_{-1}^{+\infty} p(x)\ln(1+x)dx} - 1$$

For the above expression, we make the variable transform $x = z+\mu$ and then we can get:

$$r_\infty = e^{\int_{-1}^{+\infty} p(x)\ln(1+x)dx} - 1 = e^{\int_{-1-\mu}^{1+\mu} p(z+\mu)\ln(1+z+\mu)dz} - 1$$

For the integrand, we process it in the following way:

$$\ln(1+\mu+z) = \ln\left((1+\mu)\left(1+\frac{z}{1+\mu}\right)\right) = \ln(1+\mu) + \ln\left(1+\frac{z}{1+\mu}\right)$$

According to the conditions of the theorem, there is $\left|\frac{z}{1+\mu}\right| < 1$. When $|z| < 1$, the following Taylor's expansion is tenable:

$$\ln(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$$

We have:

$$\ln(1+\mu) + \ln\left(1+\frac{z}{1+\mu}\right) = \ln(1+\mu) - \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left(\frac{z}{1+\mu}\right)^n$$

So we have:

$$\begin{aligned}
&\int_{-\mu-1}^{\mu+1} p(\mu+z)\ln(1+\mu+z)dz \\
&= \int_{-\mu-1}^{\mu+1} p(\mu+z)\left(\ln(1+\mu)-\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{z}{1+\mu}\right)^n\right)dz \\
&= \ln(1+\mu)\int_{-\mu-1}^{\mu+1} p(\mu+z)dz-\sum_{n=1}^{\infty}\frac{(-1)^n}{n(1+\mu)^n}\int_{-\mu-1}^{\mu+1} p(\mu+z)z^n dx \\
&= \ln(1+\mu)-\sum_{n=1}^{\infty}(-1)^n\frac{\sigma_n}{n(1+\mu)^n}
\end{aligned}$$

Therefore, there is:

$$\begin{aligned}
r_\infty &= e^{\int_{-1}^{+\infty} p(x)\ln(1+x)dx}-1 = e^{\ln(1+\mu)-\sum_{n=1}^{\infty}(-1)^n\frac{\sigma_n}{n(1+\mu)^n}}-1 \\
&= (1+\mu)e^{-\sum_{n=1}^{\infty}(-1)^n\frac{\sigma_n}{n(1+\mu)^n}}-1
\end{aligned}$$

The theorem is proved.

If there exist simple computational formulas for n-order central moment σ_n of the random rate of return with probability density function $p(x)$, then it is convenient to calculate r_∞ and $r_\infty-\mu$ through the use of expression (8). In this way, the decreasing amplitude of the rate of return can be examined.

Theorem 3.5. *If the annual rate of return ξ of the risk asset K is a random variable which possesses a common probability density function $p(x)$, suppose that when $\xi \le -1$ or $\xi \ge 2\mu+1$ then there is $p(x)=0$, the mathematic expectation of ξ is μ, the variance of ξ is $\sigma^2>0$, and the n-order central moment of ξ is $\sigma_n=\int_{-\infty}^{+\infty}(x-\mu)^n p(x)dx$, then assume that the following condition tenable for all $n\ge 2$:*

$$\frac{abs(\sigma_n)}{n(1+\mu)^n} > \frac{abs(\sigma_{n+1})}{(n+1)(1+\mu)^{n+1}}$$

Then there must be $r_\infty<\mu$.

Proof: In terms of the conclusion of Theorem 3.4, we have:

$$r_\infty = (1+\mu)e^{-\sum_{n=1}^{\infty}(-1)^n\frac{\sigma_n}{n(1+\mu)^n}}-1$$

It is easy to prove that $\sigma_1=\int_{-\infty}^{+\infty}(x-\mu)p(x)dx=0$, and then we have:

$$r_\infty = (1+\mu)e^{-\sum_{k=1}^{\infty}\left(\frac{\sigma_{2k}}{2k(1+\mu)^{2k}}-\frac{\sigma_{2k+1}}{(2k+1)(1+\mu)^{2k+1}}\right)}-1$$

According to the conditions the theorem provides and $\sigma_{2k}>0$, there is:

$$\gamma=\sum_{k=1}^{\infty}\left(\frac{\sigma_{2k}}{2k(1+\mu)^{2k}}-\frac{\sigma_{2k+1}}{(2k+1)(1+\mu)^{2k+1}}\right)>0$$

Therefore, there is $e^{-\gamma} < 1$, and we have:

$$r_\infty = (1+\mu)e^{-\gamma} - 1 < (1+\mu) - 1 = \mu$$

The theorem is proved.

4. **Some Conclusions Derived from the Formula of the Rate of Return r_∞.** We will examine a simple case that the annual rate of return ξ of the risk asset K conforms to the equally distributed situation. ξ is equally distributed in the interval $[a, b]$ and $a > -1$. The density function of ξ is $p(x)$. And there is:

$$p(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & x < a \text{ or } x > b \end{cases}$$

The mathematical expectation of ξ is $\frac{a+b}{2}$. In accordance with Formula (2), the mathematical formula of the average annual rate of return r_∞ of the risk asset K held indefinitely is as follows:

$$r_\infty = e^{\int_{-1}^{+\infty} p(x)\ln(1+x)dx} - 1 = e^{\frac{1}{b-a}\int_a^b \ln(1+x)dx} - 1$$

Then we have the following Theorem 4.1.

Theorem 4.1. *If the annual rate of return ξ of the risk asset K conforms to the even distribution in the interval $[a, b]$ and $a > -1$, then the average annual rate of return for the risk asset K held indefinitely is inevitably less than $\frac{a+b}{2}$, and its mathematical expression is as follows:*

$$r_\infty = \frac{1}{e}\left[\frac{(1+b)^{(1+b)}}{(1+a)^{(1+a)}}\right]^{\frac{1}{b-a}} - 1 \tag{9}$$

Under some conditions, there is $r_\infty < 0$, when $\frac{a+b}{2} > 0$.

Proof: Because the evenly distributed function is symmetric about the point $\frac{a+b}{2}$, according to Theorem 3.2, the average annual rate of return for the risk asset K held indefinitely is inevitably less than the mathematical expectation $\mu = \frac{a+b}{2}$ of the annual rate of return. Examining the integral:

$$r_\infty = e^{\frac{1}{b-a}\int_a^b \ln(1+x)dx} - 1$$

This integral can be solved, because of $\int \ln(x)dx = x\ln(x) - x + C$. We have:

$$\begin{aligned} r_\infty &= e^{\frac{1}{b-a}\int_{a+1}^{b+1}\ln(x)dx} - 1 = e^{\frac{1}{b-a}(x\ln(x)-x)\big|_{a+1}^{b+1}} - 1 \\ &= e^{\frac{1}{b-a}\left(\ln\frac{(1+b)^{(1+b)}}{(1+a)^{(1+a)}} - (b-a)\right)} - 1 = e^{\ln\left[\frac{(1+b)^{(1+b)}}{(1+a)^{(1+a)}}\right]^{\frac{1}{b-a}} - 1} - 1 \\ &= \frac{1}{e}\left[\frac{(1+b)^{(1+b)}}{(1+a)^{(1+a)}}\right]^{\frac{1}{b-a}} - 1 \end{aligned}$$

Taking $a = -0.8$, $b = 0.9$ and $\mu = \frac{a+b}{2} = 0.05 > 0$, that is, the mathematical expectation of the annual rate of return is 5%, and substituting the values of a and b into the expression (9), we can gain:

$$r_\infty = \frac{1}{e}\left(\frac{1.9^{1.9}}{0.2^{0.2}}\right)^{\frac{1}{1.7}} - 1 = -8.91\%$$

As for the asset K whose annual rate of return is evenly distributed in $[-0.8, 0.9]$, the average value of its annual rate of return is 5%. However, the average annual rate of return for the risk asset K held indefinitely is -8.91%. The theorem is proved.

Theorem 4.2. *If the annual rate of return ξ of the risk asset K conforms to the normal distribution $N(\mu,\sigma^2)$ in which $\mu>0$ and $\mu-3\sigma>-1$, then the mathematical formula of the average annual rate of return for the risk asset K held indefinitely is as follows:*

$$r_\infty=(1+\mu)e^{-\sum\limits_{n=1}^{\infty}\frac{(2n)!}{2n\cdot n!}\left(\frac{\sigma}{\sqrt{2}(1+\mu)}\right)^{2n}}-1 \tag{10}$$

Proof: The moment generating function of normal distribution is $M(s)$:

$$M(s)=e^{\frac{\sigma^2s^2}{2}+\mu s}$$

We can obtain the expression of each-order central moment:

$$\sigma_{2n}=\frac{(2n)!}{n!}\left(\frac{\sigma^2}{2}\right)^n,\quad \sigma_{2n+1}=0,\quad n=1,2,3,\dots$$

Substituting the above expression into the expression (8) and simplifying it, we get the expression (10). The theorem is proved.

Because of $\int\limits_{-1}^{+\infty}p(x)dx=1$, we simplify the expression (6) and we have:

$$e^{\int_{-1}^{+\infty}p(x)\ln(1+x)dx}\le\int\limits_{-1}^{+\infty}(x+1)p(x)dx$$

Take the logarithm of both sides of the above expression, and the expression (6) can be transformed into the following form:

$$\int_{-1}^{+\infty}p(x)\ln(1+x)dx\le\ln\left(\int\limits_{-1}^{+\infty}(x+1)p(x)dx\right) \tag{11}$$

For any $p(x)\ge 0$, if there is $\int\limits_{-1}^{+\infty}p(x)dx=1$ and the integral $\int\limits_{-1}^{+\infty}p(x)\ln(1+x)dx$ exists, then the inequality (11) is tenable. Apart from conveying the relationship between the long-term and the short-term rate of return, the inequality (11) should be able to realize its theoretical value or application value in other fields.

5. **Numerical Results of the Risk Asset Whose Short-Term Rate of Return Takes Different Distributions.** The value of $\mu-r_\infty$ is significant in investment. As Theorem 3.2 and Theorem 3.3 indicate, the average annual rate of return for the risk asset held indefinitely is inevitably less than the average value (expectation) of the annual rate of return for risk assets. For the common probability distribution, as Theorem 3.4 shows, it is impossible to ascertain the value of $\mu-r_\infty$ if we merely know the expectation and variance of the random rate of return ξ. To ascertain the value of $\mu-r_\infty$, we need calculate each-order central moment. Form the qualitative viewpoint, the greater the variance of the annual rate of return for risk assets is, the more the average annual rate of return for the risk asset held indefinitely decrease, compared with the expectation of the annual rate of return. The variance of the annual rate of return is a major factor that determines $\mu-r_\infty$. For common probability distributions, it is difficult to put the analytical expressions (4) and (8) of $\mu-r_\infty$ into use. And in this section, through numerical integration, we use the integral expression ascertained by the expression (2) to examine the quantitative relations between the average annual rate of return r_∞ of the risk asset held indefinitely and the expectation μ and the variance σ^2 of the annual rate of return for the risk asset.

TABLE 1. The computational results of the average annual rate of return r_∞ when the annual rate of return for risk assets conforms to several common-used distributions

Probability model and probability density function	Parameter selection	The expectation μ and standard deviation σ of the annual rate of return	The average annual rate of return r_∞ of the asset K held indefinitely (%)	The difference of the rate of returns $\mu - r_\infty$ (%)
Normal distribution model: $\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $x \geq -0.99$ Expectation $= \mu$ variance $= \sigma^2$	$\mu = 0.05$ $\sigma = 0.025$	$\mu = 0.05$ $\sigma = 0.025$	4.85	0.15
	$\mu = 0.05$ $\sigma = 0.25$	$\mu = 0.05$ $\sigma = 0.25$	1.74	3.26
	$\mu = 0.05$ $\sigma = 0.45$	$\mu = 0.05$ $\sigma = 0.45$	-5.35	10.35
	$\mu = 0.25$ $\sigma = 0.45$	$\mu = 0.25$ $\sigma = 0.45$	14.49	10.51
Even distribution model: $\begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & x < a \text{ or } x > b \end{cases}$ Expectation $= \frac{a+b}{2}$, variance $= \frac{(b-a)^2}{12}$	$a = 0.0067$ $b = 0.0933$	$\mu = 0.05$ $\sigma = 0.025$	4.85	0.15
	$a = -0.383$ $b = 0.483$	$\mu = 0.05$ $\sigma = 0.25$	1.88	3.12
	$a = -0.7294$ $b = 0.8294$	$\mu = 0.05$ $\sigma = 0.45$	-6.43	11.43
	$a = -0.5294$ $b = 1.0294$	$\mu = 0.25$ $\sigma = 0.45$	14.89	10.11
Right-shift t distribution model: $\alpha\frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})}\left(1+\frac{(\alpha x-\beta)^2}{n}\right)^{-\frac{n+1}{2}}$ $x \geq -0.99,\ n = 4$ Expectation $= \frac{\beta}{\alpha}$, variance $= \frac{n}{(n-2)\alpha^2}$	$\alpha = 56.57$ $\beta = 2.83$	$\mu = 0.05$ $\sigma = 0.025$	4.85	0.15
	$\alpha = 5.657$ $\beta = 0.283$	$\mu = 0.05$ $\sigma = 0.25$	2.16	2.84
	$\alpha = 3.143$ $\beta = 0.157$	$\mu = 0.05$ $\sigma = 0.45$	-1.67	6.67
	$\alpha = 3.143$ $\beta = 0.786$	$\mu = 0.25$ $\sigma = 0.45$	16.87	8.13
Left-shift χ^2 distribution model: $\frac{\alpha}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}(\alpha x+\beta)^{\frac{n}{2}-1}e^{-\frac{\alpha x+\beta}{2}}$ $x \geq -\beta/\alpha$ and $x \geq -0.99$ Expectation $= \frac{(n-\beta)}{\alpha}$, variance $= \frac{2n}{\alpha^2}, n = 8$	$\alpha = 160.0$ $\beta = 0.0$	$\mu = 0.05$ $\sigma = 0.025$	4.85	0.15
	$\alpha = 16.0$ $\beta = 7.2$	$\mu = 0.05$ $\sigma = 0.25$	2.24	2.76
	$\alpha = 8.889$ $\beta = 7.556$	$\mu = 0.05$ $\sigma = 0.45$	-4.14	9.14
	$\alpha = 8.889$ $\beta = 5.778$	$\mu = 0.25$ $\sigma = 0.45$	16.15	8.85
Left-shift Fdistribution model: $\alpha\frac{(\alpha x+\beta)^{\frac{m}{2}-1}}{B(\frac{m}{2},\frac{n}{2})}\left(\frac{m}{n}\right)^{\frac{m}{2}}\left(1+\frac{m(\alpha x+\beta)}{n}\right)^{-\frac{m+n}{2}}$ $x \geq -\beta/\alpha$ and $-\beta/\alpha \geq -0.99$ Expectation $= \frac{(\frac{n}{n-2}-\beta)}{\alpha}$ variance $= \frac{2n^2(m+n-2)}{m(n-2)^2(n-4)\alpha^2}$ $n = 10,\ m = 10$	$\alpha = 38.729$ $\beta = -0.686$	$\mu = 0.05$ $\sigma = 0.025$	4.85	0.15
	$\alpha = 3.873$ $\beta = 1.056$	$\mu = 0.05$ $\sigma = 0.25$	2.73	2.27
	$\alpha = 2.152$ $\beta = 1.142$	$\mu = 0.05$ $\sigma = 0.45$	-1.58	6.58
	$\alpha = 2.152$ $\beta = 0.712$	$\mu = 0.25$ $\sigma = 0.45$	17.66	7.34
Left-shift Rayleigh distribution model: $\frac{(x+\beta)}{\alpha^2}e^{-\frac{(x+\beta)^2}{2\alpha^2}}$ $x \geq -\beta$ and $\beta \leq 0.99$ Expectation $= \sqrt{\frac{\pi}{2}}\alpha - \beta$ variance $= \frac{4-\pi}{2}\alpha^2$	$\alpha = 0.0382$ $\beta = -0.0022$	$\mu = 0.05$ $\sigma = 0.025$	4.86	0.14
	$\alpha = 0.382$ $\beta = 0.428$	$\mu = 0.05$ $\sigma = 0.25$	2.19	2.81
	$\alpha = 0.687$ $\beta = 0.811$	$\mu = 0.05$ $\sigma = 0.45$	-4.97	9.97
	$\alpha = 0.687$ $\beta = 0.611$	$\mu = 0.25$ $\sigma = 0.45$	15.71	9.29
Left-shift lognormal distribution model: $\frac{1}{\alpha(x+\beta)\sqrt{2\pi}}e^{-\frac{\ln^2(x+\beta)}{2\alpha^2}}$ $x \geq -\beta$ and $\beta \leq 0.99$ Expectation $= e^{\frac{\alpha^2}{2}} - \beta$ variance $= e^{\alpha^2}(e^{\alpha^2}-1)$	$\alpha = 0.155$ $\beta = 0.962$	$\mu = 0.05$ $\sigma = 0.025$	3.77	1.23
	$\alpha = 0.434$ $\beta = 1.049$	$\mu = 0.05$ $\sigma = 0.25$	-5.57	10.57
	$\alpha = 0.539$ $\beta = 1.106$	$\mu = 0.05$ $\sigma = 0.45$	-13.44	18.44
	$\alpha = 0.539$ $\beta = 0.906$	$\mu = 0.25$ $\sigma = 0.45$	10.17	14.83

We choose the common probability density models, including the normal distribution model, the even distribution model, the t distribution model, the χ^2 distribution model, the F distribution model, the Rayleigh distribution model and the lognormal distribution model. The expectation μ of the short-term rate of return ξ of the risk asset that we usually examine is positive number and the rate of return ξ takes its value in $(-1, +\infty)$. At the same time, it is generally required that $\mu \in (0.05, 2.0)$ and the standard deviation $\sigma \in (0, 0.5)$. Therefore, in order to ensure that the rate of return lies in the reasonable value interval, some distributions shift the average value left, some shift the average value right, and some make the probability density function thin. However, all of the distributions require that the value of the random variable ξ is greater than -0.99, or that when $\xi < -0.99$, the value of the density function is 0. In this way, all the selected distribution models conform to the normal value range of the short-term rate of return for risk assets. Table 1 is the computational results of the average annual rate of return r_∞ when the annual rate of return for risk assets conforms to several common-used distributions. The computational formula is the integral expression (2).

From the computational results of Table 1, we know that when $\sigma = 0.025$ (lower risk), the average annual rate of return r_∞ of the asset held indefinitely decreases slightly; when $\sigma = 0.25$, r_∞ decreases by around $2\% \sim 3\%$; when $\sigma = 0.45$ (higher risk), r_∞ decreases by about 10%, that is, from $\mu = 5\%$ down to around $r_\infty = -5\%$ or from $\mu = 25\%$ down to about $r_\infty = 15\%$. Compared with normal distribution, the decreasing amplitude of r_∞ of the t distribution whose degree of freedom is $n = 4$ and the F distribution $(n = 10, m = 10)$ is slightly small; the decreasing amplitude of the r_∞ of lognormal distribution is the largest.

Marking r_f as the annual rate of return of the risk-free asset and μ as the expectation of the annual rate of return for the risk asset, we have reasons to believe that it is unreasonable to hold this risk asset for a long term when $r_\infty < r_f$. When $r_\infty < 0$, holding this risk asset for a long term will lead to catastrophic results, regardless of the fact that we can have $\mu > 0$ at this time.

6. **Conclusion.** In the investment analysis, people are used to use short-term rate of return to measure the level of rate of return, and using the variance of the short-term rate of return to measure the indetermination (risk) of investment income. However, when investors plan to do a long-term continuous investment, it is blind and fallible to estimate the return level and the risk of the long-term investment on the basis of the expectation and the variance of the short-term rate of return. In this paper, we study the relevant problems about the rate of return for the risk asset held indefinitely and obtain the theoretical formula under appropriate assumptions. The mathematical form of the formula is represented by the integral of the product of the probability density function of the random rate of return and a logarithmic function. Meanwhile, this paper offers some properties of the obtained theoretical formula, the mathematical formula of the average annual rate of return of the asset held indefinitely can be represented by each-order central moment of the random rate of return. And when the short-term rate of return for the risk asset conforms to some familiar probability density functions, we make use of numerical integral method to compute the corresponding average annual rate of return for the risk asset. As the computational results indicate, with the increase of the variance of the short-term rate of return, the level of the rate of return for the risk asset held indefinitely will decline more and more sharply, compared with the expectation of the short-term rate of return.

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REFERENCES

[1] F. Black and M. Scholes, The pricing of option and corporate liabilities, *Journal of Political Economy*, vol.6, no.81, pp.637-654, 1973.

[2] M. F. Osborne, Brownian motion in the stock market, *Operations Research*, no.7, pp.145-173, 1959.

[3] E. F. Fama, The behavior of stock-market prices, *Journal of Business*, vol.38, no.1, pp.34-72, 1965.

[4] S. Kathari and J. Shanken, Stock return variation and expected dividend, *Journal of Financial Economics*, pp.177-210, 1992.

[5] J. Feng and F. Wang, A research on return distribution function of Chinese stock-market, *Chinese Journal of Management Science*, vol.11, no.1, pp.14-21, 2003.

[6] X. Wang and L. Xu, Study on the long-term rate of return when holding risk asset, *International Journal of Innovative Computing, Information and Control*, vol.6, no.7, pp.3275-3288, 2010.

[7] X. Wang, L. Xu and Q. Fang, The critical curve of the long-term rate of return when holding risk assets, *International Journal of Innovative Computing, Information and Control*, vol.6, no.10, pp.4579-4592, 2010.

[8] B. M. Friedman and V. V. Roley, Models of long-term interest rate determination, *The Journal of Portfolio Management*, vol.6, no.3, pp.35-45, 1980.

[9] J. A. Wachter, A consumption-based model of the term structure of interest rates, *Journal of Financial Economics*, vol.79, no.2, pp.365-399, 2006.

[10] Z. Da and M. Warachka, The disparity between long-term and short-term forecasted earnings growth, *Journal of Financial Economics*, vol.100, no.2, pp.424-442, 2011.

[11] M. Lettau and J. A. Wachter, The term structures of equity and interest rates, *Journal of Financial Economics*, vol.101, no.1, pp.90-113, 2011.