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STATE FEEDBACK POSITIVE STABILIZATION OF DISCRETE DESCRIPTOR SYSTEMS

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Abstract. *The positive stabilization problem of discrete-time descriptor systems by means of state-feedback control is studied here. Thus, conditions are developed for a state-feedback controller to make the controlled descriptor system simultaneously stable and positive. An illustrative example is provided to illustrate the proposed approach.*

1. **Introduction.** There is growing interest in the control community to study descriptor (also called singular) systems, since they have broad applications due to their natural capability to describe physical systems [11, 21, 23, 29, 32]. Indeed, descriptor models represent systems described by differential and algebraic equations, so the representation of systems provided by descriptor systems is more general than the representations currently used for the analysis and design of control systems. Thus, descriptor systems models are used in several areas of application such as electrical circuits, computer networks, biotechnology and economics [10, 11, 17, 19, 27]. This is mainly because some systems in these areas cannot be precisely modeled by standard state-space systems, due to the presence of algebraic constraints in the models [7]. Several concepts and results obtained for standard state-space systems have already been extended to descriptor systems; for example, we can cite the analysis and synthesis of state feedback, output feedback, pole placement or robust controllers presented in [9, 10, 11, 20, 26, 29, 31]. In addition, many variables in these systems involve quantities that are intrinsically nonnegative, such as concentrations of substances, level of liquids, and number of proteins. Hence, the mathematical model describing these systems must take into account this nonnegativity constraint. This leads to the notion of positive descriptor systems [30]. Although positive systems have been actively researched and many results have been reported, see for example [2, 3, 12, 24] and references therein, the literature of positive descriptor systems is much more limited. In particular, the fundamental issue of characterizing the stability of positive descriptor systems has only been addressed in [1, 30]. In [30], assuming the non necessary condition that the matrix that represents the projector on the set of admissible initial conditions is nonnegative, the stability issue was addressed by using a generalized Perron-Frobenius

type condition, so that a Lyapunov-type stability condition is derived. In [1] the stability of positive descriptor systems was investigated without making unnecessary assumptions and general necessary and sufficient conditions were proposed by means of Linear Programming (LP). However, the crucial issue of stabilization still remains unexplored to date. The aim of this paper is to present the first attempt to tackle this important problem.

Thus, we present in this manuscript a novel approach to address the problem of effectively computing a state feedback law that makes the descriptor system in closed-loop to be positive and stable. This approach provides a numerically reliable computational framework, as illustrated using an example at the end of the paper.

The remainder of the paper is structured as follows. Section 2 gives some preliminary results and the necessary background required to develop the proposed approach. The state-feedback stabilization problem is considered in Section 3. A numerical example is presented in Section 4, and finally some concluding remarks are given.

2. **Preliminaries.** This section presents some basic results which will be needed in the sequel.

The following class of descriptor systems is considered in this paper:

$$Ex_{k+1} = Ax_k + Bu_k, \tag{1}$$

where $x_0 \in \mathbf{R}^n$ and the matrices $E, A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times p}$ are time-invariant.

The problem dealt with in this paper is to find a feedback law $u_k = Kx_k$ for the system (1) such that the resulting closed-loop system is positive (in the sense defined below) and stable, i.e., such that

$$Ex_{k+1} = (A + BK)x_k,\ x_0 \in \mathbf{R}^n_+, \tag{2}$$

is positive and stable.

To this end, let us recall the basic background of this type of systems.

Definition 2.1. *Let $A, E \in \mathbf{R}^{n\times n}$. The pair (E, A) is called regular if* $\det(\lambda E - A) \neq 0$ *for some $\lambda \in \mathbb{C}$.*

Remark 2.1. *If the pair (E, A) is regular, then there exists a solution of (1), see* [7]. *Thus, we will assume regularity in the remaining part of the paper, and for simplicity, the following notation will be used:*

$$\widehat{E} := (\lambda E - A)^{-1}E$$
$$\widehat{A} := (\lambda E - A)^{-1}A$$

Remark 2.2. *The solutions of (1) will be described in terms of the so-called Drazin inverse: for a matrix M this Drazin inverse is denoted M^D. Further details on this inverse can be found for example in* [7].

Next, we define the positivity of the following autonomous descriptor system,

$$Ex_{k+1} = Ax_k,\ x_0 \in \mathbf{R}^n_+. \tag{3}$$

Definition 2.2. *The descriptor system (3) is said to be positive if, for all $k \in \mathbf{R}_+$, we have that $x_k \in \mathbf{R}^n_+$ for any consistent initial condition $x_0 \geq 0$, where the set of consistent initial conditions, is described by*

$$\widehat{E}^D\widehat{E}x_0 \geq 0.$$

Conditions under which system (3) is positive are presented in the following result.

Theorem 2.1. *Consider the system (3) and assume that $\widehat{E}^D\widehat{E} \geq 0$. Then the following statements are equivalent:*

(1) System (1) is positive.
(2) $\widehat{E}^D\widehat{A} \geq 0$.

Proof: The equivalence of 1) and 2) follows from [30, Theorem 3.8].

For the notion of stability we now introduce the following standard definitions.

Definition 2.3. *A scalar* $\lambda \in \mathbb{C}$ *is an eigenvalue of* (E, A) *if* $\det(\lambda E - A) = 0$. *The finite spectral radius* $\rho(E, A)$ *of* (E, A) *is defined as the maximum absolute value among the eigenvalues of* (E, A). *The descriptor system (3) is said to be stable if* $\rho(E, A) < 1$.

Next we recall the necessary and sufficient conditions for the asymptotic stability of the system (3) in the case when the system (3) is positive.

Theorem 2.2. *Let system (3) be positive. Then it is asymptotically stable if and only if there exists a positive definite diagonal matrix* Y *such that*

$$\left(\widehat{E}^D\widehat{A}\right)^T Y \left(\widehat{E}^D\widehat{A}\right) - Y$$

is negative definite or equivalently if and only if $\widehat{E}^D\widehat{A}$ *is a Schur matrix.*

Proof: The first equivalence was first established in [30, Theorem 4.8] and the second one follows from standard Lyapunov arguments.

The following result will be used in order to derive our main results in the next section, see [2, Theorem 2.4].

Theorem 2.3. *If* $\widehat{E}^D\widehat{A}$ *is a nonnegative matrix, then the descriptor system (1) is asymptotically stable if and only if there exists* $\alpha > 0$ *such that*

$$\left(\widehat{E}^D\widehat{A} - I\right)\alpha < 0. \tag{4}$$

3. **Positive Stabilization via Steady State Feedback.** We now propose a new approach for solving the problem of positive stabilization of a given descriptor system, i.e., for finding a steady state feedback law in such a way that the resulting closed-loop system is simultaneously positive and stable.

Making use of the ideas in [1] and Theorems 2.1 and 2.3, we present now a result on the problem of the stabilization of system (4) by using the following class of controlled descriptor systems:

$$\begin{cases} Ex_{k+1} = Ax_k + Bu_k, \\ x_k \geq 0 \end{cases} \tag{5}$$

where $B \in \mathbb{R}^{n\times p}$ and the control signal is calculated from static state feedback $u(k) = Kx(k)$.

Theorem 3.1. *Consider system (5) with* (E, A) *regular. Then, the following statements are equivalent:*

(i) There exists a stabilizing matrix K *such that the state-feedback law* $u(t) = Kx(t)$ *for system (5) makes the closed-loop system positive and stable.*

(ii) There exist $\lambda \in \mathbb{C}$, $\bar{x} = [\bar{x}(1), \ldots, \bar{x}(n)]^T \in \mathbf{R}^n$, $y_1, \ldots, y_n \in \mathbf{R}^p$ *such that the following conditions are satisfied:*

(a) $(\lambda E - A - BK)^{-1}$ *exists;*

(b)

$$\left(\overline{E}^D\overline{A} - I\right)\bar{x} + \overline{E}^D\overline{B}\sum_{i=1}^{n} y_i < 0, \tag{6}$$

$$\bar{x} > 0, \tag{7}$$

$$\left(\overline{E}^D\overline{A}\right)(i,j)\bar{x}(j) + \bar{b}_i y_i \geq 0, \tag{8}$$

for $i, j = 1, \ldots, n$, *where*

$$\overline{E} = (\lambda E - A - BK)^{-1}E, \tag{9}$$

$$\overline{B} = (\lambda E - A - BK)^{-1}B, \tag{10}$$

$$\overline{A} = (\lambda E - A - BK)^{-1}A \tag{11}$$

and $\bar{b}_i$ *are the rows of the matrix* $\overline{E}^D\overline{B}$.

Proof: Clearly, the closed-loop system is described by the formula $Ex_{k+1} = (A + BK)x_k$. Define $\tilde{A} := (\lambda E - A - BK)^{-1}(A + BK) = (\overline{A} + \overline{B}K)$. Thus, in the light of Theorems 2.1 and 2.2, statement (i) is equivalent to the condition

$$\overline{E}^D\overline{A} + \overline{E}^D\overline{B}K = \overline{E}^D\tilde{A} \text{ is nonnegative and Schur},$$

which is equivalent, by Theorem 2.3, to

$$\overline{E}^D\tilde{A} \geq 0 \text{ and } \left(\overline{E}^D\tilde{A} - I\right)\bar{x} < 0, \tag{12}$$

for some $\bar{x} > 0$.

Thus, it is sufficient to show the equivalence between statement (ii) and the existence of a gain matrix K such that conditions (12) are satisfied.

Define $K = [k_i]_{i=1}^n = [\bar{x}(1)^{-1}y_1 \quad \ldots \quad \bar{x}(n)^{-1}y_n] \in \mathbf{R}^{p\times n}$. Since $k_i = \bar{x}(i)^{-1}y_i$ for $i = 1, \ldots, n$, inequality (8) is equivalent to

$$\begin{aligned} \overline{E}^D\overline{A}(i,j) + \bar{b}_i\bar{x}(i)^{-1}y_i &= \overline{E}^D\overline{A}(i,j) + \bar{b}_i k_i \geq 0 \\ &\Updownarrow \\ \left(\overline{E}^D\overline{A} + \overline{E}^D\overline{B}K\right) &= \overline{E}^D\tilde{A} \geq 0, \end{aligned}$$

which amounts to saying that inequality (8) is equivalent to the positivity of the closed-loop system. For stability, we work out Equation (6) to obtain,

$$\begin{aligned} \left(\overline{E}^D\overline{A} - I\right)\bar{x} + \overline{E}^D\overline{B}\sum_{i=1}^n y_i &= \left[\left(\overline{E}^D\overline{A} - I\right) + \overline{E}^D\overline{B}\left(\sum_{i=1}^n y_i\right)(\bar{x}^{-1})\right]\bar{x} \\ &= \left[\left(\overline{E}^D\overline{A} - I\right) + \overline{E}^D\overline{B}K\right]\bar{x} \\ &= \left(\overline{E}^D\tilde{A} - I\right)\bar{x}. \end{aligned}$$

where $\bar{x}^{-1} = \text{diag}\,(\bar{x}_i^{-1})$ and $K\bar{x} = \sum_{i=1}^n y_i$. Hence, by Theorem 2.3, we have that $\bar{x} > 0$ and

$$\begin{aligned} \left(\overline{E}^D\overline{A} - I\right)\bar{x} + \overline{E}^D\overline{B}\sum_{i=1}^n y_i &= \left(\overline{E}^D\left(\overline{A} + \overline{B}K\right) - I\right)\bar{x} \\ &= \left(\overline{E}^D\tilde{A} - I\right)\bar{x} < 0 \end{aligned}$$

if and only if $\overline{E}^D\tilde{A}$ is Schur. This concludes the proof.

Using the last statement of Theorem 3.1 combined with Theorem 2.1, one can obtain the following corollary.

Corollary 3.1. *Let system (5) be given and assume that* $E^D E \geq 0$. *Then, system (5) is positive and stable if there exist* $\bar{x} = [\bar{x}(1), \ldots, \bar{x}(n)]^T \in \mathbf{R}^n$, $y_1, \ldots, y_n \in \mathbf{R}^n$ *such that the following conditions hold:*

(1) $EA + EBK = AE + BKE$;
(2)

$$(EA - I)\bar{x} + EB\sum_{i=1}^{n} y_i < 0, \tag{13}$$

$$\bar{x} > 0, \tag{14}$$

$$EA(i,j)\bar{x}(j) + b_i^\top y_i \geq 0, \tag{15}$$

for $i, j = 1, \ldots, n$, *where* b_i *are the rows of* EB.

Corollary 3.2. *It follows from Proposition 1 in [5] that if, instead of assuming* $\widehat{E}^D\widehat{E} \geq 0$, *we have that* $E^DE \geq 0$ *and* $EA = AE$, *then system (1) is positive if and only if* $E^DA \geq 0$. *This corollary then follows easily using Theorem 3.1.*

Remark 3.1. *Note that the condition* $(E\lambda - A - BK)^{-1}$ *in Theorem 3.1 amounts to saying that* $\overline{E}^D\overline{A} = \overline{A}\,\overline{E}^D$, *or equivalently, that the pair* $(\overline{E}, \overline{A})$ *is regular. In addition, it is not difficult to check that* $E^DE > 0$ *implies that* $\overline{E}^D\overline{E} > 0$.

We must remark that the proposed method, via the conditions of Theorem 3.1 (and Corollaries 3.1 and 3.2), is heuristic due to the non-convexity of the problem. Therefore, it may not always be possible to determine whether the problem admits a feasible solution. The modification of the proposed conditions is currently being researched, in such a way that the conditions will be computationally more effective.

4. **Example.**

A simple numerical example is now dealt with using the proposed approach to illustrate its applicability.

Example 4.1. *Consider the descriptor system in (5) with*

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; A = \begin{pmatrix} 1/2 & 0 \\ 1 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The objective is to design a state-feedback controller that makes the system simultaneously positive and stable (observe that the original system is not positive: thus, we want to impose positivity by feedback). For this, a suitable gain matrix $K = [K_1 \; K_2]$ *can be obtained from Theorem 3.1. First, matrices* $\overline{A}$, $\overline{B}$ *and* $\overline{E}$ *can be evaluated from (9)-(11) by fixing (for instance)* $\lambda = 2$,

$$\overline{A} = (\lambda E - A - BK)^{-1}A = \begin{pmatrix} -1/2(1+K_2) & 0 \\ 1/2(1+K_1) + 3/2 & 3/2 \end{pmatrix},$$

$$\overline{B} = (\lambda E - A - BK)^{-1}B = \frac{-2/3}{1+K_2}\begin{pmatrix} 0 \\ 3/2 \end{pmatrix},$$

$$\overline{E} = (\lambda E - A - BK)^{-1}E = \frac{-2/3}{1+K_2}\begin{pmatrix} -(1+K_2) & 0 \\ 1+K_1 & 0 \end{pmatrix},$$

(where $K_2 <> -1$ *to ensure regularity). To evaluate the conditions in Theorem 3.1 the Drazin inverse of* E *is needed, that can be seen to be*

$$\overline{E}^D = \frac{1}{\lambda_1^2}\overline{E} = \frac{-3}{2(1+K_2)}\begin{pmatrix} -(1+K_2) & 0 \\ 1+K_1 & 0 \end{pmatrix}.$$

The conditions of (6)-(8) of Theorem 3.1 turn out to be the following:

$$\frac{1}{1+K_2}\left[-\frac{1}{2}(1+K_1)\overline{x}_1-(1+K_2)\overline{x}_2\right] < 0, \tag{16}$$

$$\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \end{pmatrix} > 0, \tag{17}$$

$$-\frac{1}{2}(1+K_1)(1+K_2) \geq 0 \tag{18}$$

Any combination of K_1 *and* K_2 *that fulfill conditions (19) would provide positive stabilization; it is easy to see that if* $(1+K_2)<0$ *then* K_1 *just needs to fulfill*

$$1+K_1 \geq 0 \tag{19}$$

From the combinations of values that fulfill these conditions we select, for example, $K=[1\ \ -2]$*. Effectively, the closed-loop system, when* $K=[1\ \ -2]$*, is given by*

$$E=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ A+BK=\begin{pmatrix} 1/2 & 0 \\ 2 & -1 \end{pmatrix},$$

which is clearly a positive and stable system, as it corresponds to the following dynamic equations:

$$x_{1,k+1}=\frac{1}{2}x_{1,k},\ x_{2,k}=2x_{1,k}.$$

5. **Conclusions.** In this paper, we have addressed the problem of finding (for a given descriptor system) a steady state feedback law such that the resulting system is positive and stable. We have presented a necessary and sufficient condition for the solvability of this problem, which was illustrated by way of an example. Although the solvability of the conditions proposed is not direct in a general case, they do provide some ideas for solving the problem. In addition, the proposed approach might be extended to other particular cases, such as the case in which the system matrices are uncertain, but bounded by known matrices; or the case in which the state and the input vectors must satisfy additional constraints.

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