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ON THE BIFURCATION BEHAVIOR OF A THREE-SPECIES LOTKA-VOLTERRA FOOD CHAIN MODEL WITH TWO DISCRETE DELAYS

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Abstract. *In this paper, a three-species Lotka-Volterra food chain model with two discrete delays is investigated, where the time delays are regarded as parameters. Its dynamics are studied in terms of local analysis and Hopf bifurcation analysis. By analyzing the associated characteristic transcendental equation, it is found that Hopf bifurcation occurs when these delays pass through a sequence of critical value. Some explicit formulae for determining the stability and the direction of the Hopf bifurcation periodic solutions are derived by using the normal form theory and center manifold theory. Finally, numerical simulations supporting the theoretical analysis are carried out.*
Keywords: Three-species Lotka-Volterra system, Stability, Hopf bifurcation, Discrete delay, Periodic solution

1. **Introduction.** During the past decade, the dynamics of two-species predator-prey models with delays or without delays has become a subject of intense research activity of mathematical fields due to their theoretical and practical significance. It is well known that the interactions between two species have mainly three kinds of fundamental forms: competition, cooperation, and prey-predation in population biology. Many excellent and interesting results on the three fundamental forms predator-prey models have been reported. For example, May [1] first proposed and discussed briefly the delayed predator-prey model

$$\begin{cases} \dot{x}(t) = x(t)[r_1 - a_{11}x(t-\tau) - a_{12}y(t)], \\ \dot{y}(t) = y(t)[-r_2 + a_{21}x(t) - a_{22}y(t)], \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ represent the population densities of prey and predator at time t, respectively; $\tau \geq 0$ is the feedback time delay of the prey to the growth of the species itself; $r_1 > 0$ denotes intrinsic growth rate of the prey and $r_2 > 0$ denotes the death rate of the predator; the parameters a_{ij} $(i, j = 1, 2)$ are all positive constants. In 2005, Song and Wei [2] investigated further the Hopf bifurcation nature of system (1). Yan and Li [3] considered the Hopf bifurcation behavior of the following predator-prey system

$$\begin{cases} \dot{x}(t) = x(t)[r_1 - a_{11}x(t-\tau) - a_{12}y(t)], \\ \dot{y}(t) = y(t)[-r_2 + a_{21}x(t) - a_{22}y(t-\tau)]. \end{cases} \tag{2}$$

Faria [4] dealt with the stability and Hopf bifurcation of the following predator-prey with feedback control and two different discrete delays:

$$\begin{cases} \dot{x}(t) = x(t)[r_1 - a_{11}x(t) - a_{12}y(t-\tau_2)], \\ \dot{y}(t) = y(t)[-r_2 + a_{21}x(t-\tau_1) - a_{22}y(t)]. \end{cases} \tag{3}$$

Yan and Zhang [5] focused on the Hopf bifurcation analysis of the delayed Lokta-Volterra predator-prey system as follows

$$\begin{cases} \dot{x}(t) = x(t)[r_1 - a_{11}x(t-\tau) - a_{12}y(t-\tau)], \\ \dot{y}(t) = y(t)[-r_2 + a_{21}x(t-\tau) - a_{22}y(t-\tau)]. \end{cases} \tag{4}$$

Hu et al. [6] analyzed the stability and Hopf bifurcation of the following predator-prey model with multiple delays

$$\begin{cases} \dot{x}(t) = x(t)[r_1 - a_{11}x(t-\tau) - a_{12}y(t-\tau_1)], \\ \dot{y}(t) = y(t)[-r_2 + a_{21}x(t-\tau_2) - a_{22}y(t-\tau)]. \end{cases} \tag{5}$$

Gao et al. [7] studied the Hopf bifurcation and global stability of the following delayed predator-prey system with stage structure for predator

$$\begin{cases} \dot{x}(t) = x(t)[r_1 - \sum_{j=1}^{m} a_{ij}x(t-\tau_{1j}) - \sum_{j=1}^{m} b_{1j}y(t-\rho_{1j})], \\ \dot{y}(t) = y(t)[r_2 - \sum_{j=1}^{m} a_{2j}x(t-\tau_{2j}) - \sum_{j=1}^{m} b_{2j}y(t-\rho_{2j})], \end{cases} \tag{6}$$

where $i, j = 1, 2, \cdots, m$. Yuan and Zhang [8] investigated the following stability and global Hopf bifurcation in a delayed predator-prey model

$$\begin{cases} \dot{x}(t) = x(t)[r_1 - a_{11}\int_{-\infty}^{t} F(t-s)x(s)ds - a_{12}y(t)], \\ \dot{y}(t) = y(t)[-r_2 + a_{21}x(t) - a_{22}y(t)]. \end{cases} \tag{7}$$

Xu and Shao [9] discussed the bifurcation of a predator-prey model with discrete and distributed time delay which takes the form

$$\begin{cases} \dot{x}(t) = x(t)[r_1 - a_{11}\int_{-\infty}^{t} F(t-s)x(s)ds - a_{12}y(t-\tau)], \\ \dot{y}(t) = y(t)[-r_2 + a_{21}\int_{-\infty}^{t} F(t-s)x(s)ds - a_{22}y(t-\tau)]. \end{cases} \tag{8}$$

Xu et al. [10] considered the bifurcation behavior of the following predator-prey model with two delays

$$\begin{cases} \dot{x}(t) = x(t)[r_1 - a_{11}x(t-\tau_1) - a_{12}y(t-\tau_2)], \\ \dot{y}(t) = y(t)[-r_2 + a_{21}x(t-\tau_1) - a_{22}y(t-\tau_2)]. \end{cases} \tag{9}$$

Zhang [11] investigated the stability and bifurcation periodic solutions of the following Lotka-Volterra competition system with multiple delays

$$\begin{cases} \dot{u}(t) = u(t)[r_1 - a_1u(t-\tau) - b_1v(t-\tau_1)], \\ \dot{v}(t) = v(t)[r_2 - b_2u(t-\tau_2) - a_2v(t-\tau)], \end{cases} \tag{10}$$

where $u(t), v(t)$ represent the population densities of the two competing species at time t. r_i, a_i, b_i are all positive constants. For more work on two-species predator-prey systems, one can see [12-20]. In real natural world, there are maybe more species in some habitat and they can construct a food chain. Therefore, we think that it is more realistic to consider a multiple-species predator-prey system. In 2008, Baek and Lee [21] proposed and studied the following three-species food chain system

$$\begin{cases} \dot{x}_1(t) = x_1(t)[a - bx_1(t) - cx_2(t)], \\ \dot{x}_2(t) = x_2(t)[-d_1 + c_1x_1(t) - e_1x_3(t)], \\ \dot{x}_3(t) = x_3(t)[-d_2 + e_2x_2(t)], \end{cases} \tag{11}$$

where $x_1(t)$, $x_2(t)$ and $x_3(t)$ represent the population densities of the lowest-level prey, mid-level predator and top predator at time t, respectively; the constant $a > 0$ is called the intrinsic growth rate of the prey species; $b > 0$ measures the intraspecific competition of the prey; $c > 0$ and $e_1 > 0$ are the predation rate per capita of the mid-level and top predator, respectively; $c_1 > 0$ and $e_2 > 0$ represent the conversion rates of the low-level prey to the mid-level predator and the mid-level predator to the top one, respectively; d_1 and d_2 denote the death rate of the mid-level and top predator, respectively.

Cui and Yan [22] argued that in model (11), the gestation periods and maturation time of some species have been ignored. In fact, predator species need to take time to have the ability to reproduce and capture food. Therefore, it is often reasonable to incorporate time delays into system in order to consider the gestation periods and maturation time of species. Motivated by the viewpoint, Cui and Yan [22] modified system (11) as follows

$$\begin{cases} \dot{x}_1(t) = x_1(t)[a - bx_1(t) - cx_2(t)], \\ \dot{x}_2(t) = x_2(t)[-d_1 + c_1x_1(t) - e_1x_3(t - \tau_1)], \\ \dot{x}_3(t) = x_3(t)[-d_2 + e_2x_2(t - \tau_2)], \end{cases} \tag{12}$$

where $\tau_1 \geq 0$ denotes the time from birth to having the ability to predate for top predator and τ_2 denotes the maturation time that the mid-level predator can be served as food for the top predator, respectively. By regarding the sum τ of two delay τ_1 and τ_2 as the bifurcation parameter, they investigated the Hopf bifurcation nature of system (12).

It shall be pointed out that the hunting delay of mid-level predator to prey and the delay in mid-level predator maturation in model (11) have been omitted. Motivated by the viewpoint, we modified system (11) as follows:

$$\begin{cases} \dot{x}_1(t) = x_1(t)[a - bx_1(t) - cx_2(t - \tau_1)], \\ \dot{x}_2(t) = x_2(t)[-d_1 + c_1x_1(t - \tau_2) - e_1x_3(t)], \\ \dot{x}_3(t) = x_3(t)[-d_2 + e_2x_2(t)], \end{cases} \tag{13}$$

where $\tau_1 \geq 0$ stands for the hunting delay of mid-level predator to prey and τ_2 denotes the delay in the mid-level predator maturation. In order to establish the main results for model (13), it is necessary to make the following assumption:

$$\text{(H1) } \tau_1 + \tau_2 = \tau.$$

This paper is organized as follows. In Section 2, the stability of the equilibrium and the existence of Hopf bifurcation at the equilibrium are studied. In Section 3, the direction of Hopf bifurcation and the stability and periods of bifurcating periodic solutions on the center manifold are determined. In Section 4, numerical simulations are carried out to illustrate the validity of the main results. Some main conclusions are drawn in Section 5.

2. **Stability of Positive Equilibrium and Local Hopf Bifurcation.**

In this section, we discuss the local asymptotic stability of the positive equilibrium of system (13) and the existence of local Hopf bifurcation near the positive equilibrium.

It is easy to see that system (13) has three boundary equilibria $E_1(0,0,0)$, $E_2(\frac{d_1}{b},0,0)$, $E_3(\frac{d_1}{c_1}, \frac{ac_1-d_1b}{cc_1}, 0)$ and a unique positive equilibrium $E^*(x_1^*, x_2^*, x_3^*)$ provided that the condition

$$\text{(H2) } ae_2c_1 - d_2cc_1 - d_1be_2 > 0$$

holds, where

$$x_1^* = \frac{ae_2 - d_2c}{be_2},\ x_2^* = \frac{d_2}{e_2},\ x_3^* = \frac{ae_2c_1 - d_2cc_1 - d_1be_2}{be_1e_2}.$$

Under the hypothesis (H2), making the change of variables $\bar{x}_i(t) = x_1(t) - x_1^*(i = 1, 2, 3)$ and still denoting $\bar{x}_i(t)$ by $x_i(t)$, then system (13) can be rewritten as the following equivalent form

$$\begin{cases} \dot{x}_1(t) = (x_1(t) + x_1^*)[-bx_1(t) - cx_2(t - \tau_1)], \\ \dot{x}_2(t) = (x_2(t) + x_2^*)[c_1x_1(t - \tau_2) - e_1x_3(t)], \\ \dot{x}_3(t) = (x_3(t) + x_3^*)e_2x_2(t), \end{cases} \tag{14}$$

which leads to

$$\begin{cases} \dot{x}_1(t) = -bx_1^*x_1(t) - cx_1^*x_2(t - \tau_1) - bx_1^2(t) + cx_1(t)x_2(t - \tau_1), \\ \dot{x}_2(t) = -e_1x_2^*x_3(t) + c_1x_2^*x_1(t - \tau_2) + c_1x_1(t - \tau_2) - e_1x_2(t)x_3(t), \\ \dot{x}_3(t) = x_3^*e_2x_2(t) + e_2x_2(t)x_3(t). \end{cases} \tag{15}$$

The linear system of (15) at $E^*(x_1^*, x_2^*, x_3^*)$ takes the form:

$$\begin{cases} \dot{x}_1(t) = -bx_1^*x_1(t) - cx_1^*x_2(t - \tau_1), \\ \dot{x}_2(t) = -e_1x_2^*x_3(t) + c_1x_2^*x_1(t - \tau_2), \\ \dot{x}_3(t) = x_3^*e_2x_2(t). \end{cases} \tag{16}$$

Under the condition (H1), then the associated characteristic equation of (16) is

$$\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + n_1\lambda e^{-\lambda\tau} = 0, \tag{17}$$

where $m_0 = e_1e_2bx_1^*x_2^*x_3^*, m_1 = e_1e_2x_2^*x_3^*, m_2 = bx_1^*, n_1 = -cc_1x_1^*x_2^*$. Clearly, $\lambda = 0$ is not a root of (17). When $\tau = 0$, the characteristic Equation (17) becomes

$$\lambda^3 + m_2\lambda^2 + (m_1 + n_1)\lambda + m_0 = 0. \tag{18}$$

It follows from the Routh-Hurwitz criteria that all roots of (18) have negative real parts if the following condition

$$\text{(H3)} \quad m_2(m_1 + n_1) > m_0$$

holds.

Let $\lambda = i\omega_0$, $\tau = \tau_0$, and substitute this into (17), for the sake of simplicity, denote ω_0 and τ_0 by ω, τ, respectively. Separating the real and imaginary parts, we have

$$n_1\omega \sin \omega\tau = m_2\omega^2 - m_0, \tag{19}$$

$$n_1\omega \cos \omega\tau = \omega^3 - m_1\omega. \tag{20}$$

According to $\sin^2 \omega\tau + \cos^2 \omega\tau = 1$, we obtain

$$(n_1\omega)^2 = (m_2\omega^2 - m_0)^2 + (\omega^3 - m_1\omega)^2 \tag{21}$$

which leads to

$$\omega^6 + r_1\omega^4 + r_2\omega^2 + r_3 = 0, \tag{22}$$

where $r_1 = m_2^2 - 2m_1, r_2 = m_1^2 - 2m_0m_2 - n_1^2, r_3 = m_0^2$.

Let $z = \omega^2$, and then (22) becomes

$$h(z) = z^3 + r_1z^2 + r_2z + r_3 = 0. \tag{23}$$

Since $r_3 \geq 0$, then following Wang et al. [23] and Guo et al. [24], we have the following results on the roots of (23).

Lemma 2.1. [23] *If $r_2 > 0$, then (23) has no positive real parts.*

Lemma 2.2. [24] *Define $\Delta = \frac{4}{27}r_2^3 - \frac{1}{27}r_1^2r_2^2 + \frac{4}{27}r_1^3r_3 - \frac{2}{3}r_1r_2r_3 + r_3^2$. Then the necessary and sufficient conditions that the cubic Equation (23) has one simple positive real root for z are*

(i) either $r_2 \geq 0$ and $r_1^2 > 3r_2$, or $r_2 < 0$; and

(ii) $\Delta \leq 0$; and

(iii) $z^* = \frac{1}{3}\left(-r_1 + \sqrt{r_1^2 - 3r_2}\right)$ and $h(z^*) < 0$.

It follows from Lemma 2.1 that we have the stability result on system (13).

Lemma 2.3. *If (H1), (H2), (H3) and $r_2 > 0$ are satisfied, then system (13) is asymptotically stable for all $\tau \geq 0$.*

From (20), the conditions of Lemma 2.2 imply that there is an unique positive ω_0 satisfying Equation (22), namely, the characteristic Equation (17) has a pair of purely imaginary roots of the form $\pm i\omega_0$. From (20), we get

$$\tau_n = \frac{1}{\omega_0}\left[\arccos\left(\frac{\omega_0^2 - m_1}{n_1}\right) + 2n\pi\right] \quad (n = 0, 1, 2, \cdots). \tag{24}$$

Theorem 2.1. *In addition to (H1)-(H3), if (H4) $(ae_2 - d_2c)^2 > 2be_2d_2(ae_2c_1 - d_2cc_1 - d_1be_2)$ holds, then system (13) undergoes a Hopf bifurcation at the positive equilibrium $E^*(x_1^*, x_2^*, x_3^*)$ when $\tau = \tau_n$, $n = 0, 1, 2, \cdots$; furthermore, $E^*(x_1^*, x_2^*, x_3^*)$ is asymptotically stable if $0 \leq \tau \leq \tau_0$ and unstable if $\tau > \tau_0$.*

Proof: Now we will show that $\frac{d(\text{Re}\lambda)}{d\tau}|_{\tau=\tau_n} > 0$ which implies that there exists at least one eigenvalue with positive real part for $\tau > \tau_n$. Differentiating Equation (17) with respect to τ, we obtain

$$\left[3\lambda^2 + 2m_2\lambda + m_1 + n_1e^{-\lambda\tau} - n_1\lambda\tau e^{-\lambda\tau}\right]\frac{d\lambda}{d\tau} = n_1\lambda^2e^{-\lambda\tau}.$$

Then

$$\begin{aligned}\left[\frac{d\lambda}{d\tau}\right]^{-1} &= \frac{3\lambda^2 + 2m_2\lambda + m_1}{n_1\lambda^2e^{-\lambda\tau}} + \frac{1}{\lambda^2} - \frac{\tau}{\lambda} \\ &= \frac{3\lambda^2 + 2m_2\lambda + m_1}{-\lambda(\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0)} + \frac{1}{\lambda^2} - \frac{\tau}{\lambda}.\end{aligned}$$

Thus

$$\text{sign}\left\{\text{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}\Bigg|_{\lambda=i\omega_0}\right\} = \frac{1}{\omega_0^2}\text{sign}\left\{\frac{2n_1^2\omega_0^6 + (m_2^2n_1^2 - 2m_1n_1^2)\omega_0^4 - n_1^2m_0^2}{[(m_0 - m_2\omega_0^2)^2 + (m_1 - \omega_0^2)^2](n_1\omega_0)^2}\right\}.$$

In the sequel, we rewrite the numerator as follows. Let $\delta = \omega_0^2$, and then

$$g(\delta) = 2n_1^2\delta^3 + (m_2^2n_1^2 - 2m_1n_1^2)\delta^2 - n_1^2m_0^2.$$

It is easy to obtain

$$\frac{dg(\delta)}{d\delta} = 2[3n_1^2\delta^2 + (m_2^2n_1^2 - 2m_1n_1^2)\delta].$$

$\frac{dg(\delta)}{d\delta}$ has two real roots, which take the form

$$\delta_1 = 0,\ \delta_2 = -\frac{m_2^2 - 2m_1}{6} = -\frac{(ae_2 - d_2c)^2 - 2be_2d_2(ae_2c_1 - d_2cc_1 - d_1be_2)}{6be_2^2} < 0.$$

Then we can conclude that $\frac{dg(\delta)}{d\delta}$ monotonously increases in $(\delta_1, +\infty)$ (i.e.,$(0, +\infty)$), that is, $g(\delta)$ monotonously increases in $(0, +\infty)$. Since $g(0) = 0$, we have $g(\delta) > 0$ for all $\delta > 0$. Thus we derive

$$\text{sign}\left\{\text{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_n}\right\} = \text{sign}\left\{\text{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\lambda=i\omega_0}\right\} > 0.$$

Therefore, the transversality condition holds and Hopf bifurcation occurs. Equation (17) has negative real parts, i.e., the equilibrium $E^*(x_1^*, x_2^*, x_3^*)$ is locally stable for $\tau = 0$, while τ_0 is the minimum τ_n at which the real parts of these roots are zero. Thus $E^*(x_1^*, x_2^*, x_3^*)$ is locally asymptotically stable if $\tau \in [0, \tau_0)$ and unstable if $\tau > \tau_0$.

3. **Direction and Stability of the Hopf Bifurcation.** In the previous section, the stability and positive equilibrium $E^*(x_1^*, x_2^*, x_3^*)$ of system (13) and the existence of Hopf bifurcations near the positive equilibrium $E^*(x_1^*, x_2^*, x_3^*)$ are presented. In this section, we shall investigate the properties of Hopf bifurcations obtained in Theorem 2.1 and the stability of bifurcating periodic solutions occurring through Hopf bifurcations by using the normal form theory and the center manifold reduction for the retarded functional differential equations due to Hassard et al. [25]. Throughout this section, we assume that $\tau_1 < \tau_2$.

In Section 2, we know that system (13) undergoes a Hopf bifurcation at the positive equilibrium $E^*(x_1^*, x_2^*, x_3^*)$ when $\tau = \tau_n$, $n = 0, 1, 2, \cdots$. For fixed $n \in \{0, 1, 2, \cdots\}$, there exist τ_{1n} and τ_{2n} such that $\tau_{1n} + \tau_{2n} = \tau_n$. Now we consider the problem above in the phase space $C = C([-\tau_{2n}, 0], R^3)$. Linear part of system (13) at the positive equilibrium $E^*(x_1^*, x_2^*, x_3^*)$ is given by

$$\begin{cases} \dot{x}_1(t) = -bx_1^* x_1(t) - cx_1^* x_2(t - \tau_1), \\ \dot{x}_2(t) = -e_1 x_2^* x_3(t) + c_1 x_2^* x_1(t - \tau_2), \\ \dot{x}_3(t) = x_3^* e_2 x_2(t) \end{cases} \tag{25}$$

and non-linear part is given by

$$f(x_t) = \begin{pmatrix} -bx_1^2(t) + cx_1(t)x_2(t - \tau_1), \\ -e_1 x_2(t) x_3(t) \\ e_2 x_2(t) x_3(t) \end{pmatrix}. \tag{26}$$

Denote

$$C^k[-\tau_{2n}, 0] = \{\varphi | \varphi : [-\tau_{2n}, 0] \to R^3, \text{each component of } \varphi \text{ has } k \text{ order continuous derivative}\}.$$

For convenience, denote $C[-\tau_{2n}, 0]$ by $C^0[-\tau_{2n}, 0]$.

For $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta), \varphi_3(\theta))^T \in C([-\tau_{2n}, 0], R^3)$, define a family of operators

$$L_\mu \varphi = B\varphi(0) + B_1 \varphi(-\tau_{1n} - \mu) + B_2(-\tau_{2n}) \tag{27}$$

and

$$G(\mu, \varphi) = \begin{pmatrix} -b\varphi_1^2(0) + c\varphi_1(0)\varphi_2(-\tau_1) \\ -e_1 \varphi_2(0)\varphi_3(0) \\ e_2 \varphi_2(0)\varphi_3(0) \end{pmatrix}, \tag{28}$$

where

$$B = \begin{pmatrix} -bx_1^* & 0 & 0 \\ 0 & 0 & -e_1 x_2^* \\ 0 & x_3^* e_2 & 0 \end{pmatrix}, \ B_1 = \begin{pmatrix} 0 & -cx_1^* & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ B_2 = \begin{pmatrix} 0 & 0 & 0 \\ c_1 x_1^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

and L_μ is a one-parameter family of bounded linear operators in $C([-\tau_{2n}, 0], R^3) \to R^3$. By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta, \mu)$ in $[-\tau_{2n}, 0] \to R^{3^2}$, such that

$$L_\mu \varphi = \int_{-\tau_{2n}}^0 d\eta(\theta, \mu)\varphi(\theta). \tag{29}$$

In fact, choosing

$$\eta(\theta, \mu) = \begin{cases} B, & \theta = 0, \\ B_1 \delta(\theta + \tau_{1n} + \mu), & \theta \in [-\tau_{1n} - \mu, 0), \\ -B_2 \delta(\theta + \tau_{2n}), & \theta \in [-\tau_{2n}, -\tau_{1n} - \mu), \end{cases} \tag{30}$$

where $\delta(\theta)$ is Dirac function, then (29) is satisfied. For $(\varphi_1, \varphi_2, \varphi_3) \in (C^1[-\tau_{2n}, 0], R^3)$, define

$$A(\mu)\varphi = \begin{cases} \dfrac{d\varphi(\theta)}{d\theta}, & -\tau_{2n} \le \theta < 0, \\ \int_{-\tau_{2n}}^{0} d\eta(s,\mu)\varphi(s), & \theta = 0 \end{cases} \tag{31}$$

and

$$R\varphi = \begin{cases} 0, & -\tau_{2n} \le \theta < 0, \\ f(\mu, \varphi), & \theta = 0. \end{cases} \tag{32}$$

Then (13) is equivalent to the abstract differential equation

$$\dot{x_t} = A(\mu)x_t + R(\mu)x_t, \tag{33}$$

where $x = (x_1, x_2, x_3)^T$, $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau_{2n}, 0]$.

For $\psi \in C([-\tau_{2n}, 0], (R^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, \tau_{2n}], \\ \int_{-\tau_{2n}}^{0} d\eta^T(t,0)\psi(-t), & s = 0. \end{cases} \tag{34}$$

For $\phi \in C([-\tau_{2n}, 0], R^3)$ and $\psi \in C([0, \tau_{2n}], (R^3)^*)$, define the bilinear form

$$\langle \psi, \phi \rangle = \overline{\psi}(0)\phi(0) - \int_{-\tau_{2n}}^{0} \int_{\xi=0}^{\theta} \psi^T(\xi - \theta) d\eta(\theta)\phi(\xi)d\xi, \tag{35}$$

where $\eta(\theta) = \eta(\theta, 0)$. We have the following result on the relation between the operators $A = A(0)$ and A^*.

Lemma 3.1. *$A = A(0)$ and A^* are adjoint operators.*

Proof: Let $\phi \in C^1([-\tau_{2n}, 0], R^3)$ and $\psi \in C^1([0, \tau_{2n}], (R^3)^*)$. It follows from (35) and the definitions of $A = A(0)$ and A^* that

$$\begin{aligned}
\langle \psi(s), A(0)\phi(\theta) \rangle &= \bar{\psi}(0)A(0)\phi(0) - \int_{-\tau_{2n}}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)d\eta(\theta)A(0)\phi(\xi)d\xi \\
&= \bar{\psi}(0)\int_{-\tau_{2n}}^{0} d\eta(\theta)\phi(\theta) - \int_{-\tau_{2n}}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)d\eta(\theta)A(0)\phi(\xi)d\xi \\
&= \bar{\psi}(0)\int_{-\tau_{2n}}^{0} d\eta(\theta)\phi(\theta) - \int_{-\tau_{2n}}^{0} [\bar{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)]_{\xi=0}^{\theta} \\
&\quad + \int_{-\tau_2^*}^{0} \int_{\xi=0}^{\theta} \frac{d\bar{\psi}(\xi - \theta)}{d\xi} d\eta(\theta)\phi(\xi)d\xi \\
&= \int_{-\tau_{2n}}^{0} \bar{\psi}(-\theta)d\eta(\theta)\phi(0) - \int_{-\tau_{2n}}^{0} \int_{\xi=0}^{\theta} \left[-\frac{d\bar{\psi}(\xi - \theta)}{d\xi} \right] d\eta(\theta)\phi(\xi)d\xi \\
&= A * \bar{\psi}(0)\phi(0) - \int_{-\tau_{2n}}^{0} \int_{\xi=0}^{\theta} A^*\bar{\psi}(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi \\
&= \langle A^*\psi(s), \phi(\theta) \rangle.
\end{aligned}$$

This shows that $A = A(0)$ and A^* are adjoint operators and the proof is complete.

By the discussion in Section 2, we know that $\pm i\omega_0$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_0$ and $-i\omega_0$, respectively. We have the following result.

Lemma 3.2. *The vector*

$$q(\theta) = (1, a_1, a_2)^T e^{i\omega_0\theta},\ \ \theta \in [-\tau_{2n}, 0],$$

is the eigenvector of $A(0)$ *corresponding to the eigenvalue* $i\omega_0$*, and*

$$q^*(s) = D(1, a_1^*, a_2^*)e^{i\omega_0 s},\ \ s \in [0, \tau_{2n}],$$

is the eigenvector of A^* *corresponding to the eigenvalue* $-i\omega_0$*, and moreover,* $\langle q^*(s), q(\theta)\rangle = 1$*, where*

$$D = \frac{1}{1 + \bar{a}_1 a_1^* + \bar{a}_2 a_2^* + a_1^* c_1 x_1^* \tau_{2n} + \bar{a}_1 c x_1^* \tau_{1n}}. \tag{36}$$

Proof: Let $q(\theta)$ be the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0$ and $q^*(s)$ be the eigenvector of A^* corresponding to the eigenvalue $-i\omega_0$, namely, $A(0)q(\theta) = i\omega_0 q(\theta)$ and $A^*q(s) = -i\omega_0 q^*(s)$. From the definitions of $A(0)$ and A^*, we have $A(0)q(\theta) = dq(\theta)/d\theta$ and $A^*q(s) = -dq^*(s)/ds$. Thus, $q(\theta) = q(0)e^{i\omega_0\theta}$ and $q^*(s) = q(0)e^{i\omega_0 s}$. In addition,

$$\begin{aligned}\int_{-\tau_{2n}}^{0} d\eta(\theta)q(\theta) &= \begin{pmatrix} -bx_1^* & 0 & 0 \\ 0 & 0 & -e_1x_2^* \\ 0 & x_3^*e_2 & 0 \end{pmatrix} q(0) + \begin{pmatrix} 0 & -cx_1^* & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} q(-\tau_{1n}) \\ &+ \begin{pmatrix} 0 & 0 & 0 \\ c_1x_1^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} q(-\tau_{2n}) = A(0)q(0) = i\omega_0 q(0).\end{aligned} \tag{37}$$

That is

$$\begin{pmatrix} -bx_1^* - a_1cx_1^*e^{-i\omega_0\tau_{1n}} \\ c_1x_1^*e^{-i\omega_0\tau_{2n}} - a_2e_1x_2^* \\ a_1x_3^*e_2 \end{pmatrix} = \begin{pmatrix} i\omega_0 \\ ia_1\omega_0 \\ ia_2\omega_0 \end{pmatrix}. \tag{38}$$

Therefore, we can easily obtain

$$a_1 = -\frac{i\omega_0 + bx_1^*}{cx_1^*e^{-i\omega_0\tau_{1n}}},\ \ a_2 = -\frac{x_3^*e_2(i\omega_0 + bx_1^*)}{i\omega_0 cx_1^*e^{-i\omega_0\tau_{1n}}}.$$

On the other hand,

$$\begin{aligned}\int_{-\tau_{2n}}^{0} q^*(-t)d\eta(t) &= \begin{pmatrix} -bx_1^* & 0 & 0 \\ 0 & 0 & x_3^*e_2 \\ 0 & -e_1x_2^* & 0 \end{pmatrix} q^*(0) + \begin{pmatrix} 0 & 0 & 0 \\ -cx_1^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} q^*(-\tau_{1n}) \\ &+ \begin{pmatrix} 0 & c_1x_1^* & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} q^*(-\tau_{2n}) = A^*q^*(0) = -i\omega_0 q^*(0).\end{aligned} \tag{39}$$

Namely,

$$\begin{pmatrix} i\omega_0 - bx_1^* + c_1x_1^*a_1^*e^{i\omega_0\tau_{2n}} \\ i\omega_0a_1^* + x_3^*e_2 - cx_1^*a_1^*e^{i\omega_0\tau_{1n}} \\ i\omega_0a_2^* - e_1x_2^*a_1^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{40}$$

Therefore, we can easily obtain

$$a_1^* = \frac{bx_1^* - i\omega_0}{c_1x_1^*e^{i\omega_0\tau_{2n}}},\ \ a_2^* = \frac{e_1x_2^*(bx_1^* - i\omega_0)}{i\omega_0c_1x_1^*e^{i\omega_0\tau_{2n}}}.$$

In the sequel, we shall verify that $\langle q^*(s), q(\theta)\rangle = 1$. In fact, from (35), we have

$$\begin{aligned}\langle q^*(s), q(\theta)\rangle &= \bar{D}(1, \bar{a_1^*}, \bar{a_2^*})(1, a_1, a_2)^T \\ &\quad - \int_{-\tau_{2n}}^{0}\int_{\xi=0}^{\theta} \bar{D}(1, \bar{a_1^*}, \bar{a_2^*})e^{-i\omega_0(\xi-\theta)}d\eta(\theta)(1, a_1, a_2)^T e^{i\omega_0\xi}d\xi \\ &= \bar{D}\left[1 + a_1\bar{a_1^*} + a_2\bar{a_2^*} - \int_{-\tau_{2n}}^{0}(1, \bar{a_1^*}, \bar{a_2^*})\theta e^{i\omega_0\theta}d\eta(\theta)(1, a_1, a_2)^T\right] \\ &= \bar{D}\left[1 + a_1\bar{a_1^*} + a_2\bar{a_2^*} + (1, \bar{a_1^*}, \bar{a_2^*})\left(-\tau_{1n}e^{-i\omega_0\tau_{1n}}B_1 - \tau_{2n}e^{-i\omega_0\tau_{1n}}B_2\right)(1, a_1, a_2)^T\right] \\ &= \bar{D}\left[1 + a_1\bar{a_1^*} + a_2\bar{a_2^*} + \bar{a_1^*}c_1x_1^*\tau_{2n} + a_1cx_1^*\tau_{1n}\right] = 1.\end{aligned}$$

Next, we use the same notations as those in Hassard et al. [25], and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let x_t be the solution of Equation (13) when $\mu = 0$.

Define

$$z(t) = \langle q^*, x_t\rangle,\ W(t, \theta) = x_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\} \tag{41}$$

on the center manifold C_0, and we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta), \tag{42}$$

where

$$W(z(t), \bar{z}(t), \theta) = W(z, \bar{z}) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots \tag{43}$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Noting that W is also real if x_t is real, we consider only real solutions. For solutions $x_t \in C_0$ of (13),

$$\begin{aligned}\dot{z}(t) &= \langle q^*(s), \dot{x}_t\rangle = \langle q^*(s), A(0)x_t + R(0)x_t\rangle \\ &= \langle q^*(s), A(0)x_t\rangle + \langle q^*(s), R(0)x_t\rangle \\ &= \langle A^*q^*(s), x_t\rangle + \bar{q}^*(0)R(0)x_t \\ &\quad - \int_{-\tau_2^*}^{0}\int_{\xi=0}^{\theta}\bar{q}^*(\xi-\theta)d\eta(\theta)A(0)R(0)x_t(\xi)d\xi \\ &= \langle i\omega_0 q^*(s), x_t\rangle + \bar{q}^*(0)f(0, x_t(\theta) \overset{\text{def}}{=} i\omega_0 z(t) + \bar{q}^*(0)f_0(z(t), \bar{z}(t)).\end{aligned} \tag{44}$$

That is

$$\dot{z}(t) = i\omega_0 z + g(z, \bar{z}), \tag{45}$$

where

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{46}$$

Hence, we have

$$\begin{aligned}g(z, \bar{z}) &= \bar{q}^*(0)f_0(z, \bar{z}) = f(0, x_t) \\ &= \bar{D}(1, \bar{a}_1^*, \bar{a}_2^*)(f_1(0, x_t), f_2(0, x_t), f_3(0, x_t))^T,\end{aligned} \tag{47}$$

where

$$\begin{aligned}f_1(0, x_t) &= -bx_{1t}^2(0) + cx_{1t}(0)x_{2t}(-\tau_1), \\ f_2(0, x_t) &= -e_1x_{2t}(0)x_{3t}(0), \\ f_3(0, x_t) &= e_2x_{2t}(0)x_{3t}(0).\end{aligned}$$

Noticing that

$$x_t(\theta) = (x_{1t}(\theta), x_{2t}(\theta), x_{3t}(\theta))^T = W(t, \theta) + zq(\theta) + \bar{z}\bar{q}$$

and

$$q(\theta) = (1, a_1, a_2)^T e^{i\omega_0\theta},$$

we have

$$x_{1t}(0) = z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots,$$
$$x_{2t}(0) = a_1 z + \bar{a}_1\bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots,$$
$$x_{3t}(0) = a_2 z + \bar{a}_2\bar{z} + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\bar{z} + W_{02}^{(3)}(0)\frac{\bar{z}^2}{2} + \cdots,$$
$$x_{2t}(-\tau_{1n}) = a_1 e^{-i\omega_0\tau_{1n}}z + \bar{a}_1 e^{i\omega_0\tau_{1n}}\bar{z} + W_{20}^{(2)}(-\tau_{1n})\frac{z^2}{2} + W_{11}^{(2)}(-\tau_{1n})z\bar{z} + W_{02}^{(2)}(-\tau_{1n})\frac{\bar{z}^2}{2} + \cdots.$$

From (46) and (47), we have

$$\begin{aligned}
g(z,\bar{z}) &= \bar{q}^*(0)f_0(z,\bar{z}) = \bar{D}\left[f_1(0,x_t) + \bar{a}_1^* f_2(0,x_t) + \bar{a}_2^* f_3(0,x_t)\right] \\
&= \bar{D}\left[-b + ca_1 e^{-i\omega_0\tau_{1n}} - \bar{a}_1^* a_1 a_2 e_1 + \bar{a}_2^* a_1 a_2 e_2\right] z^2 \\
&\quad + \bar{D}\left[-2b + 2ca_1\mathrm{Re}\{e^{-i\omega_0\tau_{1n}}\} - 2\bar{a}_1^*\mathrm{Re}\{a_1\bar{a}_2\}e_1 + 2\bar{a}_2^*\mathrm{Re}\{a_1\bar{a}_2\}e_2\right] z\bar{z} \\
&\quad + \bar{D}\left[-b + c\bar{a}_1 e^{i\omega_0\tau_{1n}} - \bar{a}_1^*\bar{a}_1\bar{a}_2 e_1 + \bar{a}_2^*\bar{a}_1\bar{a}_2 e_2\right] \bar{z}^2 \\
&\quad + \bar{D}\Bigg[-b\left(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\right) + c\bigg(\frac{1}{2}W_{20}^{(2)}(-\tau_{1n}) + W_{11}^{(2)}(-\tau_{1n}) \\
&\quad + \frac{1}{2}W_{11}^{(1)}(0)a_1 e^{-i\omega_0\tau_{1n}} + \frac{1}{2}W_{20}^{(1)}(0)\bar{a}_1 e^{i\omega_0\tau_{1n}}\bigg) \\
&\quad - \bar{a}_1^* e_1\left(a_1 W_{11}^{(3)}(0) + \frac{1}{2}\bar{a}_1 W_{20}^{(3)}(0) + a_2 W_{11}^{(1)}(0) + \frac{1}{2}\bar{a}_2 W_{20}^{(1)}(0)\right) \\
&\quad + \bar{a}_2^* e_2\left(a_1 W_{11}^{(3)}(0) + \frac{1}{2}\bar{a}_1 W_{20}^{(3)}(0) + a_2 W_{11}^{(1)}(0) + \frac{1}{2}\bar{a}_2 W_{20}^{(1)}(0)\right)\Bigg] z^2\bar{z} + \cdots.
\end{aligned}$$

Then we have

$$\begin{aligned}
g_{20} &= 2\bar{D}\left[-b + ca_1 e^{-i\omega_0\tau_{1n}} - \bar{a}_1^* a_1 a_2 e_1 + \bar{a}_2^* a_1 a_2 e_2\right], \\
g_{11} &= 2\bar{D}\left[-b + ca_1\mathrm{Re}\{e^{-i\omega_0\tau_{1n}}\} - \bar{a}_1^*\mathrm{Re}\{a_1\bar{a}_2\}e_1 + \bar{a}_2^*\mathrm{Re}\{a_1\bar{a}_2\}e_2\right], \\
g_{02} &= 2\bar{D}\left[-b + c\bar{a}_1 e^{i\omega_0\tau_{1n}} - \bar{a}_1^*\bar{a}_1\bar{a}_2 e_1 + \bar{a}_2^*\bar{a}_1\bar{a}_2 e_2\right], \\
g_{21} &= 2\bar{D}\Bigg[-b\left(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\right) + c\bigg(\frac{1}{2}W_{20}^{(2)}(-\tau_{1n}) + W_{11}^{(2)}(-\tau_{1n}) \\
&\quad + \frac{1}{2}W_{11}^{(1)}(0)a_1 e^{-i\omega_0\tau_{1n}} + \frac{1}{2}W_{20}^{(1)}(0)\bar{a}_1 e^{i\omega_0\tau_{1n}}\bigg) \\
&\quad - \bar{a}_1^* e_1\left(a_1 W_{11}^{(3)}(0) + \frac{1}{2}\bar{a}_1 W_{20}^{(3)}(0) + a_2 W_{11}^{(1)}(0) + \frac{1}{2}\bar{a}_2 W_{20}^{(1)}(0)\right) \\
&\quad + \bar{a}_2^* e_2\left(a_1 W_{11}^{(3)}(0) + \frac{1}{2}\bar{a}_1 W_{20}^{(3)}(0) + a_2 W_{11}^{(1)}(0) + \frac{1}{2}\bar{a}_2 W_{20}^{(1)}(0)\right)\Bigg].
\end{aligned}$$

In the sequel, we compute the following values

$$W_{20}^{(1)}(0), W_{11}^{(1)}(0), W_{20}^{(2)}(-\tau_{1n}), W_{11}^{(2)}(-\tau_{1n}), W_{20}^{(3)}(0), W_{11}^{(3)}(0)$$

in g_{21}. From (33) and (34), we get

$$W' = \begin{cases} AW - 2\mathrm{Re}\{\bar{q}^*(0)f_0 q(\theta)\}, & -\tau_{2n} \le \theta < 0, \\ AW - 2\mathrm{Re}\{\bar{q}^*(0)f_0 q(\theta)\} + f_0, & \theta = 0. \end{cases}$$
$$\stackrel{\text{def}}{=} AW + H(z, \bar{z}, \theta), \tag{48}$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots . \tag{49}$$

Comparing the coefficients, we obtain

$$(A - 2i\omega_0)W_{20} = -H_{20}(\theta), \tag{50}$$

$$AW_{11}(\theta) = -H_{11}(\theta). \tag{51}$$

We know that for $\theta \in [-\tau_{2n}, 0)$,

$$H(z, \bar{z}, \theta) = -\bar{q}^*(0)f_0 q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta). \tag{52}$$

Comparing the coefficients of (49) with (52) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tag{53}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{54}$$

From (50), (53) and the definition of A, we get

$$\dot{W}_{20}(\theta) = 2i\omega_0 W_{20}(\theta) + g_{20}q(\theta) + \bar{g_{02}}\bar{q}(\theta). \tag{55}$$

Noting that $q(\theta) = q(0)e^{i\omega_0\theta}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0}q(0)e^{i\omega_0\theta} + \frac{i\bar{g}_{02}}{3\omega_0}\bar{q}(0)e^{-i\omega_0\theta} + E_1 e^{2i\omega_0\theta}, \tag{56}$$

where E_1 is a constant vector. Similarly, from (51), (54) and the definition of A, we have

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta), \tag{57}$$

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0}q(0)e^{i\omega_0\theta} + \frac{i\bar{g}_{11}}{\omega_0}\bar{q}(0)e^{-i\omega_0\theta} + E_2, \tag{58}$$

where E_2 is a constant vector. In what follows, we shall seek appropriate E_1 and E_2 in (56) and (58), respectively. It follows from the definition of A, (53) and (54) that

$$\int_{-\tau_{2n}}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0 W_{20}(0) - H_{20}(0) \tag{59}$$

and

$$\int_{-\tau_{2n}}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{60}$$

where $\eta(\theta) = \eta(0, \theta)$. From (50), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g_{02}}\bar{q}(0) + 2\begin{pmatrix} -b + ca_1 e^{-i\omega_0\tau_{1n}} \\ -a_1 a_2 e_1 \\ a_1 a_2 e_2 \end{pmatrix}. \tag{61}$$

By (51), we have

$$H_{11}(0) = -g_{11}q(0) - \bar{g_{11}}(0)\bar{q}(0) + 2\begin{pmatrix} -b + ca_1\mathrm{Re}\{e^{-i\omega_0\tau_{1n}}\} \\ \mathrm{Re}\{a_1\bar{a}_2\}e_1 \\ \mathrm{Re}\{a_1\bar{a}_2\}e_2 \end{pmatrix}. \tag{62}$$

From (50), (51) and the definition of A, we have

$$\begin{cases} BW_{20}(0) + B_1W_{20}(-\tau_{1n}) + B_2(-\tau_{2n}) = 2i\omega_0 W_{20} - H_{20}(0), \\ BW_{11}(0) + B_1W_{11}(-\tau_{1n}) + B_2(-\tau_{2n}) = -H_{11}(0). \end{cases} \tag{63}$$

Noting that

$$\left(i\omega_0 I - \int_{-\tau_{2n}}^{0} e^{i\omega_0\theta} d\eta(\theta)\right) q(0) = 0, \tag{64}$$

$$\left(-i\omega_0 I - \int_{-\tau_{2n}}^{0} e^{-i\omega_0\theta} d\eta(\theta)\right) \bar{q}(0) = \begin{pmatrix} -b + ca_1 e^{-i\omega_0\tau_{1n}} \\ -a_1a_2e_1 \\ a_1a_2e_2 \end{pmatrix} \tag{65}$$

and substituting (56) and (61) into (59), we have

$$\left(2i\omega_0 I - \int_{-\tau_{2n}}^{0} e^{2i\omega_0\theta} d\eta(\theta)\right) E_1 = 2 \begin{pmatrix} -b + ca_1 e^{-i\omega_0\tau_{1n}} \\ -a_1a_2e_1 \\ a_1a_2e_2 \end{pmatrix}. \tag{66}$$

That is

$$\begin{pmatrix} 2i\omega_0 + bx_1^* & cx_1^* e^{-2i\omega_0\tau_{1n}} & 0 \\ -c_1x_1^* e^{-2i\omega_0\tau_{2n}} & 2i\omega_0 & e_1x_2^* \\ 0 & x_3^*e_2 & 2i\omega_0 \end{pmatrix} \begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \\ E_1^{(3)} \end{pmatrix} = 2 \begin{pmatrix} -b + ca_1 e^{-i\omega_0\tau_{1n}} \\ -a_1a_2e_1 \\ a_1a_2e_2 \end{pmatrix}.$$

Hence

$$E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1},\ E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1},\ E_1^{(3)} = \frac{\Delta_{13}}{\Delta_1}, \tag{67}$$

where

$$\begin{aligned}
\Delta_1 &= \det \begin{pmatrix} 2i\omega_0 + bx_1^* & cx_1^* e^{-2i\omega_0\tau_{1n}} & 0 \\ -c_1x_1^* e^{-2i\omega_0\tau_{2n}} & 2i\omega_0 & e_1x_2^* \\ 0 & x_3^*e_2 & 2i\omega_0 \end{pmatrix} \\
&= cc_1(x_1^*)^2 e^{-2i\omega_0\tau_n} - (2i\omega_0 + bx_1^*)(4\omega_0^2 + e_1e_2x_2^*x_3^*), \\
\Delta_{11} &= 2\det \begin{pmatrix} -b + ca_1 e^{-i\omega_0\tau_{1n}} & cx_1^* e^{-2i\omega_0\tau_{1n}} & 0 \\ -a_1a_2e_1 & 2i\omega_0 & e_1x_2^* \\ a_1a_2e_2 & x_3^*e_2 & 2i\omega_0 \end{pmatrix} \\
&= 2\left[4\omega_0^2(b - ca_1e^{-i\omega_0\tau_{1n}}) + a_1a_2e_1e_2cx_1^*e^{-2i\omega_0\tau_{1n}} \right. \\
&\quad \left. + 2i\omega_0a_1a_2e_1cx_1^*e^{-2i\omega_0\tau_{1n}} + e_1e_2x_2^*x_3^*(b - ca_1e^{-i\omega_0\tau_{1n}})\right], \\
\Delta_{12} &= 2\det \begin{pmatrix} 2i\omega_0 + bx_1^* & -b + ca_1 e^{-i\omega_0\tau_{1n}} & 0 \\ -c_1x_1^* e^{-2i\omega_0\tau_{2n}} & -a_1a_2e_1 & e_1x_2^* \\ 0 & a_1a_2e_2 & 2i\omega_0 \end{pmatrix} \\
&= -2\left[2i\omega_0a_1a_2e_1(2i\omega_0 + bx_1^*) + a_1a_2e_1e_2x_2^*(2i\omega_0 + bx_1^*) \right. \\
&\quad \left. + 2i\omega_0c_1x_1^*(b - ca_1e^{-i\omega_0\tau_{1n}})e^{-2i\omega_0\tau_{2n}}\right], \\
\Delta_{13} &= 2\det \begin{pmatrix} 2i\omega_0 + bx_1^* & cx_1^* e^{-2i\omega_0\tau_{1n}} & -b + ca_1 e^{-i\omega_0\tau_{1n}} \\ -c_1x_1^* e^{-2i\omega_0\tau_{2n}} & 2i\omega_0 & -a_1a_2e_1 \\ 0 & x_3^*e_2 & a_1a_2e_2 \end{pmatrix} \\
&= 2\left[2i\omega_0a_1a_2e_2(2i\omega_0 + bx_1^*) + c_1e_2x_1^*x_3^*e^{-2i\omega_0\tau_{2n}}(b - ca_1e^{-i\omega_0\tau_{1n}}) \right. \\
&\quad \left. + a_1a_2e_2cc_1(x_1^*)^2e^{-2i\omega_0\tau_n} + a_1a_2e_1e_2x_3^*(2i\omega_0 + bx_1^*)\right].
\end{aligned}$$

Similarly, substituting (58) and (62) into (60), we have

$$\left(\int_{-\tau_{2n}}^{0} d\eta(\theta)\right) E_2 = 2\begin{pmatrix} -b + ca_1\mathrm{Re}\{e^{-i\omega_0\tau_{1n}}\} \\ \mathrm{Re}\{a_1\bar{a}_2\}e_1 \\ \mathrm{Re}\{a_1\bar{a}_2\}e_2 \end{pmatrix}. \tag{68}$$

That is

$$\begin{pmatrix} -bx_1^* & cx_1^* & 0 \\ c_1x_1^* & 0 & e_1x_2^* \\ 0 & e_2x_3^* & 0 \end{pmatrix}\begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \end{pmatrix} = 2\begin{pmatrix} b - ca_1\mathrm{Re}\{e^{-i\omega_0\tau_{1n}}\} \\ -\mathrm{Re}\{a_1\bar{a}_2\}e_1 \\ -\mathrm{Re}\{a_1\bar{a}_2\}e_2 \end{pmatrix}. \tag{69}$$

Then

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2},\ E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2},\ E_2^{(3)} = \frac{\Delta_{23}}{\Delta_2}, \tag{70}$$

where

$$\begin{aligned}
\Delta_2 &= \det\begin{pmatrix} -bx_1^* & cx_1^* & 0 \\ c_1x_1^* & 0 & e_1x_2^* \\ 0 & e_2x_3^* & 0 \end{pmatrix} \\
&= -be_1e_2x_1^*x_2^*x_3^*, \\
\Delta_{21} &= 2\det\begin{pmatrix} b - ca_1\mathrm{Re}\{e^{-i\omega_0\tau_{1n}}\} & cx_1^* & 0 \\ -\mathrm{Re}\{a_1\bar{a}_2\}e_1 & 0 & e_1x_2^* \\ -\mathrm{Re}\{a_1\bar{a}_2\}e_2 & e_2x_3^* & 0 \end{pmatrix} \\
&= -2\left[ce_1x_1^*x_2^*\mathrm{Re}\{a_1\bar{a}_2\}e_2 + \left(b - ca_1\mathrm{Re}\{e^{-i\omega_0\tau_{1n}}\}\right)e_1e_2x_2^*x_3^*\right], \\
\Delta_{22} &= 2\det\begin{pmatrix} -bx_1^* & b - ca_1\mathrm{Re}\{e^{-i\omega_0\tau_{1n}}\} & 0 \\ c_1x_1^* & -\mathrm{Re}\{a_1\bar{a}_2\}e_1 & e_1x_2^* \\ 0 & -\mathrm{Re}\{a_1\bar{a}_2\}e_2 & 0 \end{pmatrix} \\
&= -2be_1e_2x_1^*x_2^*\mathrm{Re}\{a_1\bar{a}_2\}, \\
\Delta_{23} &= 2\det\begin{pmatrix} -bx_1^* & cx_1^* & b - ca_1\mathrm{Re}\{e^{-i\omega_0\tau_{1n}}\} \\ c_1x_1^* & 0 & -\mathrm{Re}\{a_1\bar{a}_2\}e_1 \\ 0 & e_2x_3^* & -\mathrm{Re}\{a_1\bar{a}_2\}e_2 \end{pmatrix} \\
&= 2\left[(b - ca_1\mathrm{Re}\{e^{-i\omega_0\tau_{1n}}\})c_1e_2x_1^*x_3^*\right. \\
&\quad \left. + cc_1(x_1^*)^2\mathrm{Re}\{a_1\bar{a}_2\}e_2 + be_2x_1^*x_3^*\mathrm{Re}\{a_1\bar{a}_2\}e_1\right].
\end{aligned}$$

In view of (56), (58), (67) and (70), we can calculate g_{21} and derive the following values:

$$\begin{aligned}
c_1(0) &= \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\
\mu_2 &= -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\tau_0)\}}, \\
\beta_2 &= 2\mathrm{Re}(c_1(0)), \\
T_2 &= -\frac{\mathrm{Im}\{c_1(0)\} + \mu_2\mathrm{Im}\{\lambda'(\tau_0)\}}{\omega_0}.
\end{aligned}$$

These formulae give a description of the Hopf bifurcation periodic solutions of (13) at $\tau = \tau_n$, on the center manifold. From the discussion above, we have the following result.

Theorem 3.1. *For system (13), if (H1)-(H4) hold, the periodic solution is supercritical (subcritical) if* $\mu_2 > 0$ $(\mu_2 < 0)$*; the bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if* $\beta_2 < 0$ $(\beta_2 > 0)$*; the periods of the bifurcating periodic solutions increase (decrease) if* $T_2 > 0$ $(T_2 < 0)$.

Remark 3.1. *The investigation on the stability and Hopf bifurcation behavior of system (13) has important biological significance. By regarding the delays as bifurcation parameter, we can show what time delays have the effect on the dynamical behavior, that is to say, how do the gestation periods and maturation time of species of system (13) influence the stability and Hopf bifurcation behavior? It has practical significance in effective control of biological systems and human services.*

Remark 3.2. *Although many authors [1-10] dealt with the stability and Hopf bifurcation nature of the predator-prey models, they focused on the two-species predator-prey systems. In fact, there are more species in some habitat and they can construct a food chain. Then multiple-species predator-preys are more realistic in real natural world. Thus the investigation on multiple-species predator-preys system (13) has more important practical significance.*

Remark 3.3. *Considering that there are the gestation periods and maturation time of species, Cui and Yan [22] have incorporated time delays into system and investigated the existence and Hopf bifurcation behavior of multiple-species predator-preys system (11). However, the hunting delay of mid-level predator to prey and the delay in mid-level predator maturation in model (11) have been ignored. Thus the model (13) investigated by us has different biological implications. Moreover, the corresponding characteristic equation of system (13) is different and the results in Cui and Yan [22] cannot be applicable to system (13). This implies that the results of this paper are essentially new and complement previously known results.*

4. **Numerical Examples.** In this section, we present some numerical results of system (13) to verify the analytical predictions obtained in the previous section. From Section 3, we may determine the direction of a Hopf bifurcation and the stability of the bifurcation periodic solutions. Let us consider the following special case of system (13).

$$\begin{cases} \dot{x}_1(t) = x_1(t)[1 - 0.8x_1(t) - 0.4x_2(t-\tau_1)], \\ \dot{x}_2(t) = x_2(t)[-0.4 + 0.8x_1(t-\tau_2) - 0.5x_3(t)], \\ \dot{x}_3(t) = x_3(t)[-0.6 + 1.2x_2(t)]. \end{cases} \tag{71}$$

Obviously, $ae_2c_1 - d_2cc_1 - d_1be_2 = 0.384 > 0$ which implies that the hypothesis (H2) is fulfilled and then system (71) has a unique positive equilibrium $E^*(1, 0.5, 0.8)$. By direct computation by means of Matlab 7.0, we get $\omega_0 \approx 0.8541$, $\tau_0 \approx 1.55$, $\lambda'(\tau_0) \approx 1.2437 - 3.4122i$, $c_1(0) \approx -2.9542 - 22.2355i$, $\mu_2 \approx 0.5642$, $\beta_2 \approx -4.4636$, $T_2 \approx 22.1327$. We obtain the conditions indicated in Theorem 2.1 are satisfied. Furthermore, it follows that $\mu_2 > 0$ and $\beta_2 < 0$. Choose $\tau_1 = 0.65$, $\tau_2 = 0.65$, and then $\tau = \tau_1 + \tau_2 = 1.3 < \tau_0 \approx 1.5512$. Thus, the positive equilibrium $E^*(1, 0.5, 0.8)$ is stable when $\tau < \tau_0$ which is illustrated by the computer simulations (see Figure 1). When τ passes through the critical value $\tau_0 \approx 1.55$, the positive equilibrium $E^*(1, 0.5, 0.8)$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcates from the positive equilibrium $E^*(1, 0.5, 0.8)$. Choose $\tau_1 = 0.8, \tau_2 = 0.86$, and then $\tau = \tau_1 + \tau_2 = 1.66 > \tau_0 \approx 1.5512$. Since $\mu_2 > 0$ and $\beta_2 < 0$, the direction of the Hopf bifurcation is $\tau > \tau_0$, and these bifurcating periodic solutions from the positive equilibrium $E^*(1, 0.5, 0.8)$ at τ_0 are stable which can be shown in Figure 2.

Remark 4.1. *Since $a = 1$, $b = 0.8$, $c = 0.4$, $d_1 = 0.4$, $c_1 = 0.8$, $e_1 = 0.5$, $d_2 = 0.6$, $e_2 = 1.2$, then all coefficients of system (71) satisfy the parameter conditions of system (12) and have actual biological significance, and we think that the system (71) is a convincing practical example.*

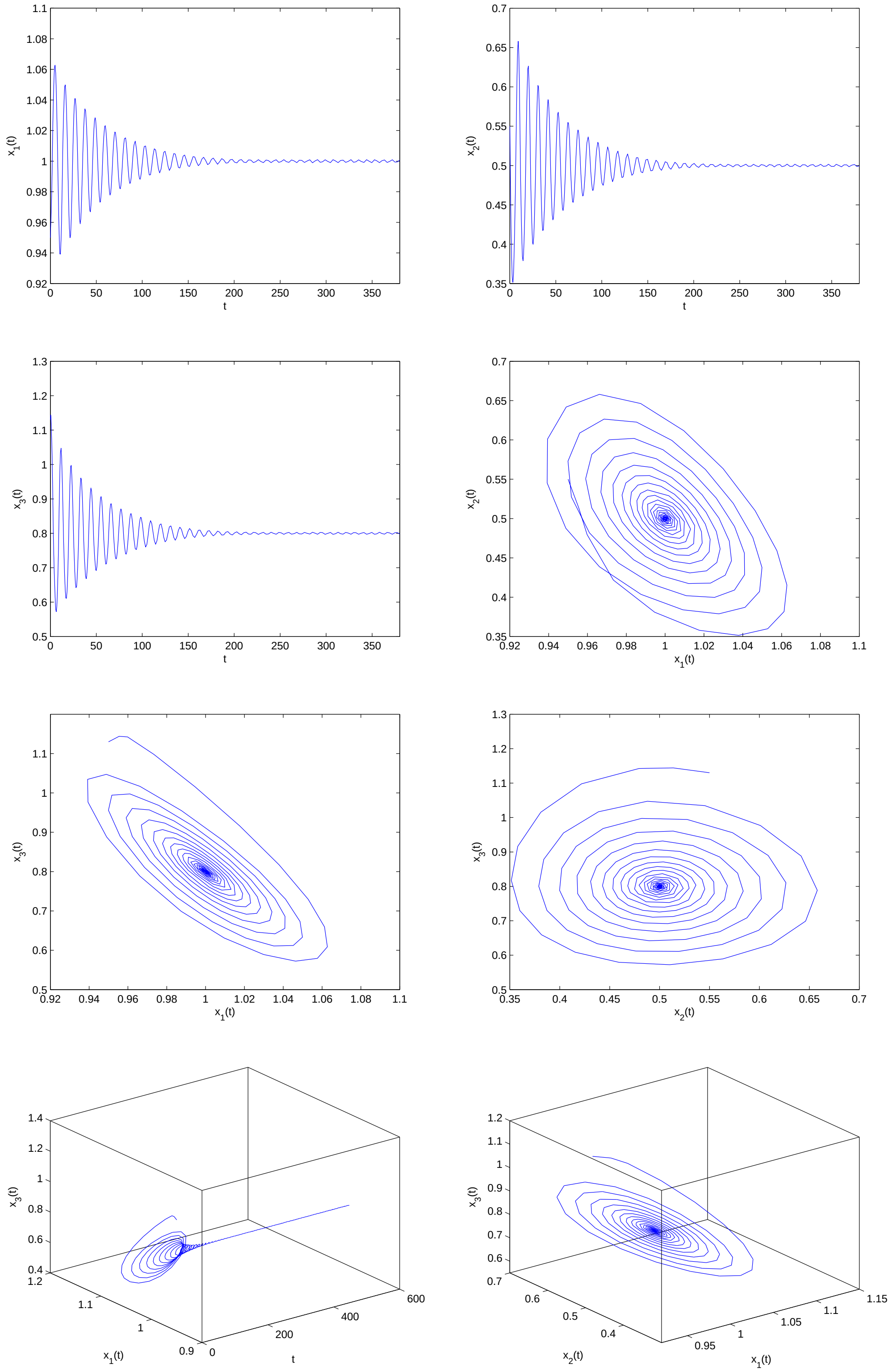

FIGURE 1. Dynamical behavior of system (71): times series of x_i ($i = 1, 2, 3$). A Matlab simulation of the asymptotically stable positive equilibrium to system (71) with $\tau_1 = 0.65$, $\tau_2 = 0.65$ and $\tau_1 + \tau_2 = \tau = 1.3 < \tau_0 \approx 1.5512$. The initial value is $(0.95, 0.55, 1.13)$.

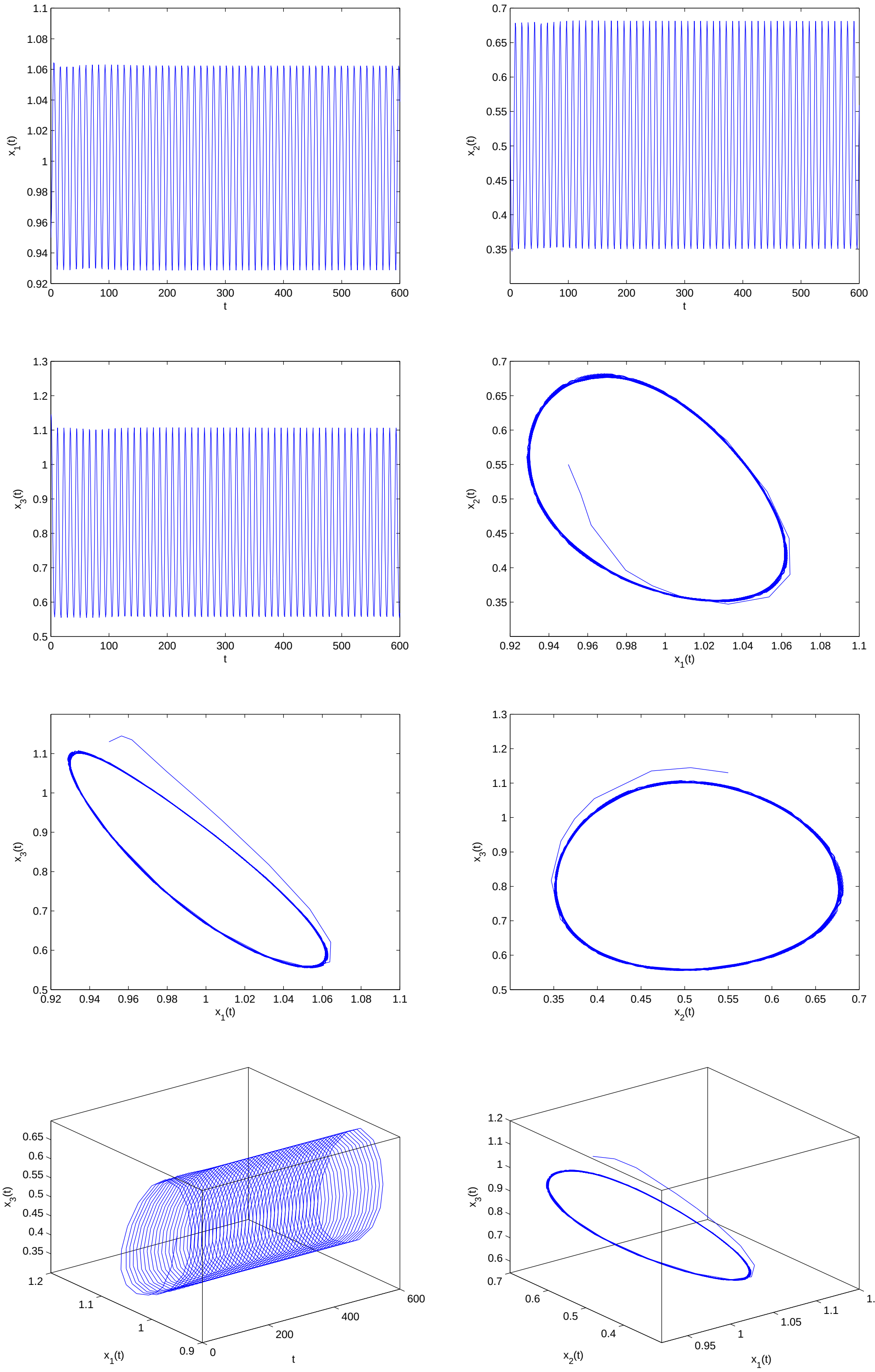

FIGURE 2. Dynamical behavior of system (71): times series of x_i ($i = 1, 2, 3$). A Matlab simulation of a periodic solution to system (71) with $\tau_1 = 0.8, \tau_2 = 0.86$ and $\tau_1 + \tau_2 = \tau = 1.66 > \tau_0 \approx 1.5512$. The initial value is $(0.95, 0.55, 1.13)$.

5. **Conclusions.** In this paper, we have investigated local stability of the positive equilibrium $E^*(x_1^*, x_2^*, x_3^*)$ and local Hopf bifurcation in a three-species Lotka-Volterra food chain model with two discrete delays. We have showed that if the conditions (H1), (H2), (H3) and $r_2 > 0$ hold, the positive equilibrium $E^*(x_1^*, x_2^*, x_3^*)$ of system (13) is asymptotically stable for all $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. We have also showed that, if the conditions (H1), (H2), (H3) and (H4) hold, as the delay sum τ increases, the equilibrium loses its stability and a sequence of Hopf bifurcations occurs at the positive equilibrium $E^*(x_1^*, x_2^*, x_3^*)$; in other words, a family of periodic orbits bifurcates from the positive equilibrium $E^*(x_1^*, x_2^*, x_3^*)$. The direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying the normal form theory and the center manifold theorem.

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